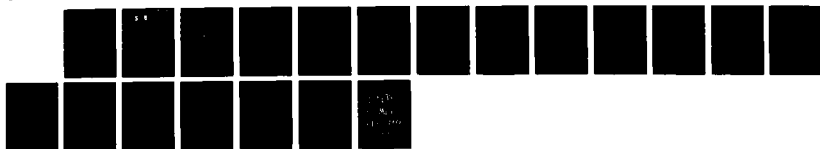
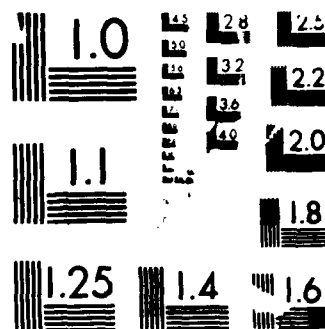


NO-A187 981 PROBABILITY BOUNDS FOR M-SKOROHOD OSCILLATIONS(U) NORTH 1/1
CAROLINA UNIV AT CHAPEL HILL CENTER FOR STOCHASTIC
PROCESSES F AYRAM ET AL DEC 86 TR-173
UNCLASSIFIED AFOSR-TR-87-1884 F49620-85-C-8144 F/G 12/2 NL





MICROCOPY RESOLUTION TEST CHART
 (NBS 1963-A)

AD-A187 981

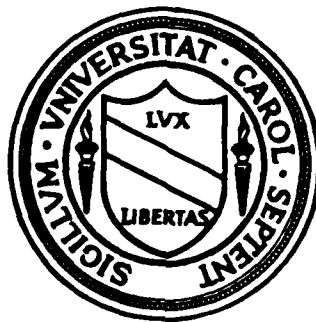
DOCUMENTATION PAGE

MIC FILE COPY

UNCLASSIFIED		1b. RESTRICTIVE MARKINGS	
2a. SECURITY CLASSIFICATION AND STATEMENT OF LIMITATIONS NA		3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; Distribution Unlimited	
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE NA		5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TK-87-1004	
4. PERFORMING ORGANIZATION REPORT NUMBER(S) Technical Report No. 173		5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TK-87-1004	
6a. NAME OF PERFORMING ORGANIZATION University of North Carolina		6b. OFFICE SYMBOL (If applicable)	
6c. ADDRESS (City, State and ZIP Code) Statistics Dept. Phillips Hall 039-A Chapel Hill, NC 27514		7a. NAME OF MONITORING ORGANIZATION AFOSR/NM	
7b. ADDRESS (City, State and ZIP Code) Building 410 Bolling AFB, DC 20332-6448		7c. ADDRESS (City, State and ZIP Code) Building 410 Bolling AFB, DC 20332-6448	
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR		8b. OFFICE SYMBOL (If applicable) NM	
8c. ADDRESS (City, State and ZIP Code) Building 410, Bolling AFB, DC 20332-6448		9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER F49620 85 C 0144	
10. SOURCE OF FUNDING NOS.		11. TITLE (Include Security Classification) Probability bounds for M-Skorohod oscillations	
12. PERSONAL AUTHOR(S) Avram, F. and Taggu, M.S.		13. TIME COVERED FROM 10/1/86 TO 9/30/87	
13a. TYPE OF REPORT Preprint		14. DATE OF REPORT (Yr., Mo., Day) December 1986	
15. PAGE COUNT 16		16. SUPPLEMENTARY NOTATION	
17. COSATI CODES		18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number) Keywords and phrases: Weak convergence, Skorohod topologies, sub-triadditivity.	
19. ABSTRACT (Continue on reverse if necessary and identify by block number) Billingsley developed a widely used method for proving weak convergence with respect to the sup-norm and J_1 -Skorohod topologies, once convergence of the finite-dimensional distributions has been established. Here we show that Billingsley's method works not only for J oscillations, but also for M oscillations. This is done by identifying a common property of the J and M functions, called sub-triadditivity, and then showing that Billingsley's approach in the case of the J function can be adequately modified to apply to any sub-triadditive function.			
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT ☐ DTIC USERS ☐		21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED	
22a. NAME OF RESPONSIBLE INDIVIDUAL Pecoy Ravitch		22b. TELEPHONE NUMBER (Include Area Code) 919-962-2307	
		22c. OFFICE SYMBOL AFOSR/NM	

CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina



PROBABILITY BOUNDS FOR M-SKOROHOD OSCILLATIONS

by

Florin Avram

and

Murad S. Taqqu

Technical Report No. 173

December 1986

PROBABILITY BOUNDS FOR M-SKOROHOD OSCILLATIONS

Florin Avram[†]

Purdue University and University of North Carolina at Chapel Hill

and

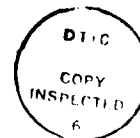
Murad S. Taqqu*

Boston University

Abstract

Billingsley developed a widely used method for proving weak convergence with respect to the sup-norm and J_1 -Skorohod topologies, once convergence of the finite-dimensional distributions has been established. Here we show that Billingsley's method works not only for J oscillations, but also for M oscillations. This is done by identifying a common property of the J and M functions, called sub-triadditivity, and then showing that Billingsley's approach in the case of the J function can be adequately modified to apply to any sub-triadditive function.

or	
21	☒
	☐
	☐
Library Codes	
	Director
	Special



† Research supported by Air Force Office of Scientific Research #F49620 85 C 0144.

* Research supported by the National Science Foundation grant ECS-8696-090 at

Boston University.

AMS 1980 subject classifications. Primary 60B10, secondary 60F17.

Key words and phrases Weak convergence, Skorohod topologies, sub-triadditivity.

1. Statement of results

Billingsley (1968) developed a widely used method for proving weak convergence with respect to the sup-norm and J_1 -Skorohod topologies, once convergence of the finite-dimensional distributions has been established. The idea is to replace the evaluation of probabilities of large oscillations by the evaluation of the probability of large increments at fixed given times. Here we show that Billingsley's method can be made to work not only for J oscillations, but also for M oscillations. We also investigate a limiting case.

We use these results in Avram and Taqqu (1987) to study the weak convergence to the Lévy α -stable process of normalized sums of moving averages that have at least two non-zero coefficients. In that paper, we show that J_1 -weak convergence does not hold and we provide sufficient conditions for M_1 convergence.

Let

$$(1.1a) \quad J(x_1, x_2, x_3) = \min(|x_2 - x_1|, |x_3 - x_2|).$$

$$(1.1b) \quad M(x_1, x_2, x_3) = \text{the distance from } x_2 \text{ to } [x_1, x_3] \\ = \begin{cases} 0 & \text{if } x_2 \in [x_1, x_3] \\ J(x_1, x_2, x_3) & \text{otherwise.} \end{cases}$$

Let H stand for either J or M , set

$$H_Z(t_1, t, t_2) = H(Z(t_1), Z(t), Z(t_2)).$$

and define the H oscillation of $Z(t)$ as

$$(1.2) \quad \omega_\delta^H(Z) = \sup_{\substack{t_1 \leq t \leq t_2 \\ 0 < t_2 - t_1 < \delta}} H_Z(t_1, t, t_2).$$

This ω_δ^H is of interest because, if Z_n are $D[0, 1]$ processes whose finite-dimensional distributions converge to those of a process Z , then the H_1 -weak convergence of Z_n to Z is equivalent to the convergence in probability of $\omega_\delta^H(Z_n)$ to 0, uniformly in n , as δ approaches 0 (see Skorohod (1956), Theorem 3.2.1).

Throughout the paper, we will consider a process $Z(t)$, $0 \leq t \leq 1$, satisfying the assumption

$$(A) \quad P\left\{H_Z(t_1, t, t_2) \geq \epsilon\right\} \leq L\epsilon^{-\nu}(t_2 - t_1)^{1+\beta},$$

for some constants $L > 0$, $\nu > 0$ and $\beta \geq 0$.

We shall also assume that

$$(B) \quad \text{There exists a number } n \text{ so that } Z(t) \text{ is pathwise constant on } \left(\frac{i}{n}, \frac{i+1}{n}\right] \text{ for } i = 0, 1, \dots, n-1.$$

Condition (B) is one of convenience and it will be weakened in the case $\beta > 0$ (see Corollary 1 below.)

The following result states that $P\{\omega_\delta^H(Z) \geq \epsilon\}$ satisfies a bound similar to that of $P\{H_Z(t_1, t, t_2) \geq \epsilon\}$ given in (A).

Theorem 1. *Let H stand for either J or M . Let $Z(t)$ be a process satisfying assumptions (A) and (B). Then, Z also satisfies*

$$(1.3) \quad P\{\omega_\delta^H(Z) \geq \epsilon\} \leq C(\nu, \beta, n)L\epsilon^{-\nu}\delta^\beta,$$

where

$$C(\nu, \beta, n) = \begin{cases} C(\nu, \beta) & \text{if } \beta > 0 \\ C(\nu, \beta)(\ln n)^{2\nu+2} & \text{if } \beta = 0 \end{cases}$$

denotes a constant independent of ϵ, δ and the distribution of Z .

Theorem 1 is proved in this section.

Remarks. (1) Throughout the paper, we adopt the convention that "constants" may depend on ν and β , but not on ϵ, δ and the distribution of Z . Dependence or independence on n will be spelled out in each case.

(2) When $H = J$, Theorem 1 reduces basically to Theorem 12.5 of Billingsley (1968).

(3) When $\beta > 0$, $C(\nu, \beta, n)$ is independent of n . Assumption (B) can then be replaced by $Z(t) \in D[0, 1]$ a.s.. Indeed, fix n and divide $[0, 1]$ in 2^n equal parts; then, apply Theorem 1 to the step function approximation $Z_n(t)$ of $Z(t)$ built by using the values of

Z at $i/2^n, i = 1, \dots, 2^n$. Since $Z(t) \in D[0, 1]$ a.s. implies that almost all paths of Z are right-continuous and $\omega_\delta^H(Z_n) \rightarrow \omega_\delta^H(Z)$ a.s., we have.

$$P\{\omega_\delta^H(Z) \geq \epsilon\} = \lim_{n \rightarrow \infty} P\{\omega_\delta^H(Z_n) \geq \epsilon\} \leq C(\nu, \beta) L \epsilon^{-\nu} \delta^\beta,$$

and hence

Corollary 1. *Let H stand for either J or M and let $Z(t)$ be a process satisfying (A), with $\beta > 0$, and with paths in $D[0, 1]$ a.s.. Then*

$$(1.4) \quad P\{\omega_\delta^H(Z) \geq \epsilon\} \leq C(\nu, \beta) L \epsilon^{-\nu} \delta^\beta,$$

where $C(\nu, \beta)$ is a constant.

Theorem 1 is proved by identifying a common property of the J and M functions, which we call *sub-triadditivity*, and then by showing that Billingsley's approach in the case of the J function can be adequately modified to apply to any sub-triadditive function.

First some notation; let

$$[0, 1]_{\leq}^3 = \left\{ (t_1, t, t_2) : 0 \leq t_1 \leq t \leq t_2 \leq 1 \right\}$$

For any function $f : [0, 1]_{\leq}^3 \rightarrow \mathbf{R}^+$, and interval $I = [t_1, t_2]$, we introduce three new functions:

$$(1.5) \quad \bar{f}_I = \bar{f}(t_1, t_2) := \sup_{t \in [t_1, t_2]} f(t_1, t, t_2).$$

$f^* : [0, 1]_{\leq}^3 \rightarrow \mathbf{R}^+$, defined by

$$(1.6) \quad f^*(t_1, t, t_2) := \sup_{\substack{t'_1 \in [t_1, t] \\ t'_2 \in [t, t_2]}} f(t'_1, t, t'_2),$$

and

$$(1.7) \quad \bar{f}_I^* = \bar{f}^*(t_1, t_2) := \sup_{t_1 \leq a \leq b \leq c \leq t_2} f(a, b, c).$$

Billingsley's method consists of showing successively that the bound in assumption (A) leads to similar bounds for probabilities involving $\bar{H}_Z$, $\bar{H}_Z^*$ and $\omega_\delta^H(Z)$. The next theorem

is our basic result. It identifies conditions for the first two steps of the extension, namely $H_Z \rightarrow \bar{H}_Z \rightarrow \bar{H}_Z^*$ and uses the notion of sub-triadditivity defined in Section 3.

Theorem 2. (a) Let $Z(t)$ be a process satisfying assumptions (A) and (B), and let f_Z be a random function: $[0, 1]_-^3 \rightarrow \mathbf{R}^-$ which is a.s. inner sub-triadditive (see (3.1), (3.2) and (3.8) for a definition).

If

$$(1.8) \quad P \left\{ f_Z(t_1, t, t_2) \geq \epsilon \right\} \leq L \epsilon^{-\nu} (t_2 - t_1)^{1+\beta}$$

for some constants $L > 0$, $\nu > 0$ and $\beta \geq 0$, then

$$(1.9) \quad P \left\{ \bar{f}_Z(t_1, t_2) \geq \epsilon \right\} \leq 2K(\nu, \beta, n) L \epsilon^{-\nu} (t_2 - t_1)^{1+\beta}$$

where

$$K(\nu, \beta, n) = \begin{cases} K(\nu, \beta) & \text{if } \beta > 0, \\ K(\nu, \beta)(\log_2 n)^{\nu-1} & \text{if } \beta = 0 \end{cases}$$

(see (2.4) for the exact formula for the constant $K(\nu, \beta, n)$).

(b) If f_Z is in addition outer sub-triadditive (see (3.3), (3.4) for a definition), then

$$(1.10) \quad P \left\{ \bar{f}_Z^*(t_1, t_2) \geq \epsilon \right\} \leq A(\nu, \beta, n) L \epsilon^{-\nu} (t_2 - t_1)^{1+\beta}$$

where

$$A(\nu, \beta, n) = \begin{cases} A(\nu, \beta) & \text{if } \beta > 0 \\ A(\nu, \beta)(\log_2 n)^{2\nu-2} & \text{if } \beta = 0 \end{cases}$$

(see (3.10) for the exact formula for the constant $A(\nu, \beta, n)$.)

Theorem 2 is proved in Section 3.

The last step of the extension $\bar{H}_Z^* \rightarrow \omega_\delta^H(Z)$ always works, as the next Lemma shows.

Lemma 1. If

$$P \left\{ \bar{f}_Z^*(t_1, t_2) \geq \epsilon \right\} \leq L \epsilon^{-\nu} (t_2 - t_1)^{1+\beta}$$

then

$$P \left\{ \omega_\delta(f_Z) \geq \epsilon \right\} \leq 2^{1+\beta} L \epsilon^{-\nu} \delta^\beta.$$

where

$$\omega_\delta(f_Z) := \sup_{\substack{t_1 \leq t \leq t_2 \\ 0 \leq t_2 - t_1 \leq \delta}} f_Z(t_1, t, t_2).$$

Proof of Lemma 1. Let $m = \lceil \delta^{-1} \rceil \equiv \lceil \delta^{-1} \rceil + 1$ and partition $[0, 1]$ with t_i , $i = 1, \dots, m$, $t_{i+1} - t_i < \delta$ where $t_0 = 0$ and $t_{m-1} = 1$. Note that $\{\omega_\delta(f_Z) \geq \epsilon\}$ implies $\{\max_{i=0, \dots, m-1} \bar{f}_Z^*(t_i, t_{i+2}) \geq \epsilon\}$. Hence

$$\begin{aligned} P\left\{\omega_\delta(f_Z) \geq \epsilon\right\} &\leq \sum_{i=1}^{m-1} P\left\{\bar{f}_Z^*(t_i, t_{i+2}) \geq \epsilon\right\} \\ &\leq (m-1)L\epsilon^{-\nu}(2\delta)^{1+\beta} \\ &\leq 2^{1+\beta}L\epsilon^{-\nu}\delta^\beta. \end{aligned}$$

Theorem 1 follows now as a particular case of Theorem 2(b), since the functions $J_Z(t_1, t, t_2)$ and $M_Z(t_1, t, t_2)$ are a.s. inner and outer sub-triadditive (see Appendix).

When $\beta = 0$, a better bound than in (1.3) may be obtained, if Z satisfies a stronger assumption than (A), namely

$$(A') \quad P\left\{H_Z^*(t_1, t, t_2) > \epsilon\right\} \leq L\epsilon^{-\nu}(t_2 - t_1)^{1-\beta},$$

where

$$(1.11) \quad H_Z^*(t_1, t, t_2) := \sup_{\substack{t'_1 \in (t_1, t) \\ t'_2 \in (t, t_2)}} H_Z(t'_1, t, t'_2),$$

and where H stands for either J or M . Because the function H_Z^* is inner sub-triadditive (see Appendix), one can apply Theorem 2(a). This yields

$$P\{\bar{H}_Z^*(t_1, t_2) \geq \epsilon\} \leq 2[\log_2(4n)]^{\nu+1}L\epsilon^{\nu+1}$$

where we used (2.4) to evaluate $K(\nu, 0, n)$. Applying Lemma 1, we obtain

Theorem 3. Let H stand for either J or M , and let $Z(t)$ satisfy assumption (A') with $\beta = 0$, and assumption (B). Then

$$(1.12) \quad P\left\{\omega_\delta^H(Z) \geq \epsilon\right\} \leq 4[\log_2(4n)]^{\nu+1}L\epsilon^{-\nu}.$$

The paper is organized as follows:

In Section 2 we give a general formulation of the classical method of bisection used in Billingsley (1968).

In Section 3 we define sub-triadditivity and prove Theorem 2. using the method of bisection.

The sub-triadditivity of J , M , and f_Z^* is established in the Appendix.

2. The Bisection Method

The bisection method seems to have originated with Menchoff (1923) in the context of a maximal inequality for partial sums of orthogonal random variables (see Stout (1974), Th. 2.3.1). It is used by Billingsley (1968) to establish a more general maximal inequality. The following proposition is a formalization of the bisection method.

Consider a random field g_I indexed by subintervals I of $[0, 1]$. For every $I \subset [0, 1]$, denote by I' the first half of I , and by I'' the second half of I . Let $m(\cdot)$ denote Lebesgue measure.

Proposition 1. *Suppose that a random field g_I satisfies the inequality*

$$(2.1) \quad g_I \leq \max(g_{I'}, g_{I''}) + h_I$$

where

$$(2.2) \quad P\{h_I \geq \epsilon\} \leq M\epsilon^{-\nu}(m(I))^{1+\beta},$$

with M constant, $\nu > 0$, $\beta \geq 0$. Suppose also that there exists an integer n such that for every $J \subset [0, 1]$ with $m(J) < \frac{1}{n}$, one has

$$(2.3) \quad P\{g_J \geq \epsilon\} \leq M\epsilon^{-\nu}(m(J))^{1+\beta}.$$

Then, for every $I \subset [0, 1]$,

$$P\{g_I \geq \epsilon\} \leq M K(\nu, \beta, n)\epsilon^{-\nu}(m(I))^{1+\beta}$$

where

$$(2.4) \quad K(\nu, \beta, n) = \begin{cases} [1 - 2^{-\beta, (\nu+1)}]^{-(\nu+1)} & \text{if } \beta > 0 \\ [\log_2(4n)]^{\nu+1} & \text{if } \beta = 0. \end{cases}$$

Remark. In general, we would expect $M' = MK(\nu, \beta, n) \approx nM$, since $[0, 1]$ is composed of n intervals on which the bound (2.3) holds. Proposition 1, however, shows that, in fact, when $\beta > 0$, M' can be made independent of n , and even for $\beta = 0$ the growth is at most logarithmic in n .

Proof. We split $I \subset [0, 1]$ in two halves, split each half again, and so on, until, after

$$k = \lceil \log_2[n \cdot m(I)] \rceil \leq \log_2[nm(I)] + 1 = \log_2 2[nm(I)] \leq \log_2(2n)$$

splittings, we end up with intervals of size less than $\frac{1}{n}$.

Let I_j denote any interval which appeared at the $k - j$ splitting: thus $I_k = I$, and I_0 is some interval such that $m(I_0) < \frac{1}{n}$. We show now by induction on j that

$$(2.5) \quad P\left\{g_{I_j} \geq \epsilon\right\} \leq c_j M \epsilon^{-\nu} m(I_j)^{1-\beta},$$

where the sequence c_j is given by

$$(2.6) \quad c_0 = 1, \quad (c_j)^{\frac{1}{\nu+1}} = 1 + \left(\frac{c_{j-1}}{2^\beta}\right)^{\frac{1}{\nu+1}}.$$

For $j = 0$, this is just assumption (2.3). For $j \geq 1$, let I'_j and I''_j denote the two halves of I_j . Then

$$\begin{aligned} P\left\{\max(g_{I'_j}, g_{I''_j}) > \epsilon\right\} &\leq 2P\left\{g_{I'_j} > \epsilon\right\} \\ &\leq 2c_{j-1}M\epsilon^{-\nu}m(I'_j)^{1+\beta} \\ &= \frac{c_{j-1}}{2^\beta}M\epsilon^{-\nu}m(I_j)^{1+\beta}. \end{aligned}$$

Relation (2.6) now follows from the fact, that if X, Y, Z are random variables satisfying $0 \leq X \leq Y + Z$, $P\{Z \geq \epsilon\} \leq M\epsilon^{-\nu}$ and $P\{Y \geq \epsilon\} \leq cM\epsilon^{-\nu}$, then

$$\begin{aligned} (2.7) \quad P\{X \geq \epsilon\} &\leq \inf_{0 \leq \lambda \leq 1} \left[P\{Y \geq (1-\lambda)\epsilon\} + P\{Z \geq \lambda\epsilon\} \right] \\ &\leq \inf_{0 \leq \lambda \leq 1} M\epsilon^{-\nu} \left[c(1-\lambda)^{-\nu} + \lambda^{-\nu} \right] \\ &= M\epsilon^{-\nu} (1 + c^{\frac{1}{\nu+1}})^{\nu+1} \end{aligned}$$

Using (2.6) with $\beta = 0$ yields

$$(c_k)^{\frac{1}{\nu-1}} = 1 + (c_{k-1})^{\frac{1}{\nu-1}} = \dots = k + 1,$$

and

$$c_k = (k + 1)^{\nu-1} \leq (\log_2 4n)^{\nu-1}.$$

On the other hand, if $\beta > 0$, then

$$(c_k)^{\frac{1}{\nu-1}} \leq \sum_{j=0}^{\infty} 2^{-\beta j (\nu-1)} = \left[1 - 2^{-\beta (\nu-1)} \right]^{-1},$$

yielding (2.4). ■

3. Inner and outer sub-triadditivity

We focus at first on *deterministic* functions H defined on

$$[0, 1]_{\leq}^3 = \left\{ (x_1, x_2, x_3) : 0 \leq x_1 \leq x \leq x_2 \leq 1 \right\}$$

or defined on all of $\mathbb{R}^3$.

Definition. A function $H : [0, 1]_{\leq}^3 \rightarrow \mathbb{R}^+$ is called *inner sub-triadditive* if it satisfies

$$(3.1) \quad H(x_1, x, x_2) \leq H(x_1, x, y) + H(x_1, y, x_2)$$

whenever $x_1 \leq x \leq y \leq x_2$, and

$$(3.2) \quad H(x_1, x, x_2) \leq H(y, x, x_2) + H(x_1, y, x_2)$$

whenever $x_1 \leq y \leq x \leq x_2$.

A function $H : \mathbb{R}^3 \rightarrow \mathbb{R}^+$ is called *inner sub-triadditive* if it satisfies (3.1) and (3.2) for any reals x_1, x, x_2, y .

Definition. A function $H : [0, 1]_{\leq}^3 \rightarrow \mathbb{R}^+$ is called *outer sub-triadditive* if it satisfies

$$(3.3) \quad H(x_1, x, x_2) \leq H(x_1, x, y) + H(x, x_2, y)$$

whenever $x_1 \leq x \leq x_2 \leq y$, and

$$(3.4) \quad H(x_1, x, x_2) \leq H(y, x, x_2) + H(y, x_1, x)$$

whenever $y \leq x_1 \leq x \leq x_2$.

A function $H : \mathbb{R}^3 \rightarrow \mathbb{R}^+$ is called *outer sub-triadditive* if it satisfies (3.3) and (3.4) for any x_1, x, x_2, y in $\mathbb{R}$.

J and M are examples of functions that are both inner and outer sub-triadditive (see Appendix). We show next why inner or outer sub-triadditivity are useful properties. But first, some notation.

If $I = [t_1, t_2]$ is an interval, let t_I denote the middle point in I , and let $I' = [t_1, t_I]$, $I'' = [t_I, t_2]$ denote the two half intervals of I .

Lemma 2. (a) If the function $f : [0, 1]_{\mathbb{Z}}^3 \rightarrow \mathbb{R}_+$ is inner sub-triadditive, then

$$(3.5) \quad \bar{f}_I \leq \max \left\{ \bar{f}_{I'}, \bar{f}_{I''} \right\} + f(t_1, t_I, t_2).$$

(b) If, moreover, f is outer sub-triadditive, then

$$(3.6) \quad f_I^* \leq \max \left\{ \bar{f}_{I'}^*, \bar{f}_{I''}^* \right\} + f_{I'} + f_{I''} + f(t_1, t_I, t_2).$$

Proof (a) If $t \leq t_I$, then by (3.1),

$$\begin{aligned} f(t_1, t, t_2) &\leq f(t_1, t, t_I) + f(t_1, t_I, t_2) \\ &\leq \bar{f}_{I'} + f(t_1, t_I, t_2), \end{aligned}$$

while if $t \geq t_I$, by (3.2) we have similarly

$$f(t_1, t, t_2) \leq \bar{f}_{I''} + f(t_1, t_I, t_2).$$

(b) If $t_1 < a < b < t_I < c < t_2$, then

$$\begin{aligned} f(a, b, c) &\leq f(a, b, t_I) + f(a, t_I, c) \text{ (by inner sub-triadditivity)} \\ &\leq f(a, b, t_I) + f(t_1, t_I, c) + f(t_1, a, t_I) \text{ (by outer sub-triadditivity)} \\ &\leq f(a, b, t_I) + f(t_1, a, t_I) + f(t_1, t_I, t_2) + f(t_I, c, t_2) \text{ (by outer sub-triadditivity)} \\ &\leq \bar{f}_{I'}^* + \bar{f}_{I'} + \bar{f}_{I''} + f(t_1, t_I, t_2) \\ &\leq \text{R.H.S. of (3.6)}. \end{aligned}$$

In the same way, we get

$$(3.7) \quad f(a, b, c) \leq R.H.S. \text{ of } (3.6)$$

when $t_1 < a < t_I < b < c < t_2$, and since (3.7) is obvious when $a, b, c \in I'$ or $a, b, c \in I''$, (3.6) holds. ■

We consider now an a.s. random sub-triadditive function of a process Z , denoted f_Z . Recall that we assume that the process Z satisfies assumption (B). We will always assume that random sub-triadditive functions f_Z also satisfy

$$(3.8) \quad f_Z(t_1, t, t_2) = f_Z(t'_1, t', t'_2)$$

whenever (t_1, t'_1) , (t, t') and (t_2, t'_2) are pairs of points for which Z is constant on the interval between them.

Note that both H_Z and H_Z^* satisfy (3.8), where H is either J or M .

We will now prove Theorem 2.

Proof of Theorem 2. (a) The result follows from Proposition 1 of Section 2, applied to the functions $g_I = \bar{f}_Z(t_1, t_2)$ and $h_I = f_Z(t_1, t_I, t_2)$. Indeed, (2.1) holds by Lemma 2a. For (2.3), note that if $J = [t_1, t_2]$ is such that $t_2 - t_1 < \frac{1}{n}$, then by Assumption (B) on Z and by Assumption (3.8) on f_Z we have

$$\begin{aligned} P\left\{g_J \geq \epsilon\right\} &= P\left\{\bar{f}_Z(t_1, t_2) \geq \epsilon\right\} \\ &= P\left\{\max[f_Z(t_1, t_1, t_2), f_Z(t_1, t_2, t_2)] \geq \epsilon\right\} \\ &\leq P\left\{f_Z(t_1, t_1, t_2) \geq \epsilon\right\} + P\left\{f_Z(t_1, t_2, t_2) \geq \epsilon\right\} \\ &\leq 2L\epsilon^{-\nu}(t_2 - t_1)^{1+\beta}. \end{aligned}$$

Thus (2.3) holds with $M = 2L$, and clearly (2.2) holds also with $M = 2L$. We get then by Proposition 1

$$P\left\{\bar{f}_Z(I) \geq \epsilon\right\} \leq 2K(\nu, \beta, n)L\epsilon^{-\nu}(t_2 - t_1)^{1+\beta},$$

with $K(\nu, \beta, n)$ given by (2.4).

(b) The result follows again from Proposition 1, this time applied to

$$g_I = \bar{f}_Z^*(t_1, t_2)$$

and

$$h_I = \bar{f}_Z(t_1, t_1) + \bar{f}_Z(t_1, t_2) + f_Z(t_1, t_1, t_2).$$

Relation (2.1) holds by Lemma 2b.

We check now (2.2).

$$\begin{aligned} P\left\{h_I \geq \epsilon\right\} &\leq \inf_{0 \leq \lambda \leq 1} \left[P\left\{\bar{f}_Z(I') > \left(\frac{1-\lambda}{2}\right)\epsilon\right\} + P\left\{f_Z(I'') \geq \left(\frac{1-\lambda}{2}\right)\epsilon\right\} \right. \\ &\quad \left. + L(\lambda\epsilon)^{-\nu}(t_2 - t_1)^{1-\beta} \right] \\ &\leq \inf_{0 \leq \lambda \leq 1} \left[\epsilon(1-\lambda)^{-\nu} + \lambda^{-\nu} \right] L\epsilon^{-\nu}(t_2 - t_1)^{1-\beta}. \end{aligned}$$

The last step holds by part (a), with

$$c = 2^{1+1-\nu-1-\beta} K(\nu, \beta, n) = 2^{1-\nu-\beta} K(\nu, \beta, n).$$

As in (2.7), we then get

$$\begin{aligned} (3.9) \quad P\left\{h_I \geq \epsilon\right\} &\leq \left(1 + 2^{1-\beta/(\nu+1)} K(\nu, \beta, n)^{\frac{1}{\nu+1}}\right)^{\nu+1} L\epsilon^{-\nu}(t_2 - t_1)^{1-\beta} \\ &\leq K(\nu, \beta, n) \left[1 + 2^{-\beta/(\nu+1)}\right]^{\nu+1} L\epsilon^{-\nu}(t_2 - t_1)^{1-\beta}, \end{aligned}$$

since $K(\nu, \beta, n) \geq 1$.

To check (2.3), let $J = [t_1, t_2]$ be such that $t_2 - t_1 \leq \frac{1}{n}$. Then, as in part (a).

$$P\left\{\bar{f}_J^* \geq \epsilon\right\} \leq 2L\epsilon^{-\nu}(t_2 - t_1)^{1-\beta}.$$

Hence (2.2) and (2.3) hold with

$$M = K(\nu, \beta, n) \max\left\{2, [1 + 2^{1-\beta/(\nu+1)}]^{\nu+1}\right\}.$$

Applying Proposition 1, we get

$$P\left\{\bar{f}_Z^*(I) \geq \epsilon\right\} \leq A(\nu, \beta, n) L\epsilon^{-\nu}(t_2 - t_1)^{1+\beta}$$

where

$$(3.10) \quad A(\nu, \beta, n) = K^2(\nu, \beta, n) \max \left\{ 2, [1 + 2^{-\beta/(\nu+1)}]^{\nu+1} \right\} \\ = \begin{cases} [1 - 2^{-\beta/(\nu+1)}]^{-(2\nu+2)} \max \{ 2, [1 + 2^{-\beta/(\nu+1)}]^{\nu+1} \} & \text{if } \beta > 0 \\ 3^{\nu+1} (\log_2(4n))^{2\nu+2} & \text{if } \beta = 0. \end{cases}$$

This concludes the proof. ■

Appendix

Lemma A1. *The deterministic functions J and M defined in (1.1) are inner and outer sub-triadditive.*

Proof. (a) *Inner sub-triadditivity*

Since J and M are symmetric in x_1, x_2 , it is enough to check (3.1). For J , we must show that

$$(A.1) \quad |x - x_1| \wedge |x - x_2| \leq |x - x_1| \wedge |x - y| + |x_1 - y| \wedge |y - x_2|$$

holds.

If $|x - x_1| < |x - y|$, then

$$\text{R.H.S. of (A.1)} \geq |x - x_1| \geq \text{L.H.S. of (A.1)}.$$

If $|x - y| < |x - x_1|$, then either the R.H.S. of (A.1) equals $|x - y| + |x_1 - y| \geq |x - x_1|$, or the R.H.S. of (A.1) equals $|x - y| + |y - x_2| \geq |x - x_2|$. Hence (A.1) holds.

Now for M , we check again (3.1) in different cases:

$$(a) \quad x \in [x_1, x_2]; \text{ then } M(x_1, x, x_2) = 0.$$

If $x \notin [x_1, x_2]$, w.l.o.g., let $x < x_1 < x_2$; we have the following subcases:

$$(b) \quad y < x < x_1 < x_2; \text{ then}$$

$$M(x_1, x, x_2) = x_1 - x \leq x_1 - y = M(x_1, y, x_2).$$

(c) $x < y < x_1 < x_2$: then

$$M(x_1, x, x_2) = x_1 - x = (y - x) + (x_1 - y) = M(x_1, x, y) + M(x_1, y, x_2).$$

(d) $x < x_1 < y$: then

$$M(x_1, x, x_2) = x_1 - x = M(x_1, x, y).$$

(b) *Outer sub-triadditivity*

Since J, M are symmetric in x_1, x_2 , it is enough to check (3.3). For J , we must show that

$$(A.2) \quad |x - x_1| \wedge |x - x_2| \leq |x - x_1| \wedge |x - y| + |x_2 - x| \wedge |x_2 - y|$$

holds. The only case different from part (a) is when $|x - y| < |x - x_1|$ and $|x_2 - x| < |x_2 - y|$. In this case,

$$\text{R.H.S. of (A.2)} = |x - y| + |x_2 - x| > |x_2 - y| > |x_2 - x| > \text{L.H.S. of (A.2)}.$$

For $H = M$, we assume, w.l.o.g. $x < x_1 < x_2$. If $y < x < x_1 < x_2$,

$$M(x_1, x, x_2) = x_1 - x \leq x_2 - x = M(x, x_2, y).$$

If $x < y < x_1 < x_2$,

$$M(x_1, x, x_2) = x_1 - x \leq (y - x) + (x_2 - y) = M(x_1, x, y) + M(x, x_2, y).$$

If $x < x_1 < y$,

$$M(x_1, x, x_2) = x_1 - x_2 = M(x_1, x, y). \quad \blacksquare$$

Lemma A2. If $H : \mathbb{R}^3 \rightarrow \mathbb{R}^+$ is inner sub-triadditive, then the function $H_Z^* : [0, 1]_{\leq}^3 \rightarrow \mathbb{R}^+$ defined by

$$H_Z^*(t_1, t, t_2) = \sup_{\substack{t'_1 \in [t_1, t] \\ t'_2 \in [t, t_2]}} H(Z(t'_1), Z(t), Z(t'_2))$$

is inner sub-triadditive.

Proof. Since $H_Z^*(t_1, t, t_2)$ is defined only for ordered triples $t_1 < t < t_2$, we have to check that

$$(A.3a) \quad \text{If } t_1 < t < u < t_2, \text{ then } H_Z^*(t_1, t, t_2) \leq H_Z^*(t_1, t, u) + H_Z^*(t_1, u, t_2)$$

and

$$(A.3b) \quad \text{If } t_1 < u < t < t_2, \text{ then } H_Z^*(t_1, t, t_2) \leq H_Z^*(u, t, t_2) + H_Z^*(t_1, u, t_2).$$

Since the proofs are similar, we show only (A.3a). Also, for convenience, assume that the sup is obtained, that is,

$$H_Z^*(t_1, t, t_2) = H(Z(t'_1), Z(t), Z(t'_2)),$$

for some t'_1, t'_2 , with $t_1 \leq t'_1 \leq t \leq t'_2 \leq t_2$.

Let now $t \leq u \leq t_2$; if $u \geq t'_2$, then, obviously, $H_Z^*(t_1, t, t_2) = H_Z^*(t_1, t, u)$, and (A.3a) holds. Suppose, hence $u \leq t'_2$; by the inner sub-triadditivity of H ,

$$\begin{aligned} H(Z(t'_1), Z(t), Z(t'_2)) &\leq H(Z(t'_1), Z(t), Z(u)) + H(Z(t'_1), Z(u), Z(t'_2)) \\ &\leq H_Z^*(t_1, t, u) + H_Z^*(t_1, u, t_2). \end{aligned}$$

The result follows by taking sup in the L.H.S. ■

References

- [1] Avram, F. and Taqqu, M.S. (1987). Weak convergence of moving averages to the Lévy stable motion. Preprint.
- [2] Billingsley, P. (1968). *Convergence of probability measures*. Wiley & Sons, New York.
- [3] Menchoff, D. (1923). Sur les séries de fonctions orthogonales I. *Fund. Math.* **4**, 82-105.
- [4] Skorohod, A.V. (1956). Limit theorems for stochastic processes. *Theory of Prob. Appl.* **1** (1956), 261-290.
- [5] Stout, W.F. (1974). *Almost Sure Convergence*. Academic Press, New York.

Florin Avram

Department of Statistics

Purdue University

West Lafayette, IN 47907

Murad S. Taqqu

Department of Mathematics

Boston University

Boston, MA 02215

END
FILMED
FEB. 1988
DTIC