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FIVE DIAGONAL WEIGHTING SCHEME
FOR GEOIDAL PROFILES

Author: Blaha

Naval University Oceanographic Center
2000 North Ocean Drive
Gainesville, Florida 32604

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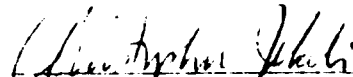
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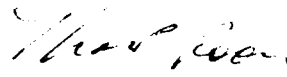
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CHRISTOPHER JEKELI
Contract Manager


THOMAS P. ROONEY, Chief
Geodesy & Gravity Branch

FOR THE COMMANDER


DONALD H. ECKHARDT, Director
Earth Sciences Division

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)															
<p>Rigorous weighting of geoidal observations along profiles, such as encountered, for example, in satellite altimetry, would be a task exceedingly demanding in terms of computer run-time. A profile with m observation points would require an inversion of an $(m \times m)$ variance-covariance matrix, whose entries are correlated and depend almost entirely on the geoidal covariance function. However, if the observations are uniformly distributed, a slight modification of this function can greatly facilitate the formation of the weight matrix of observations, which</p>															

subsequently enters the least-squares adjustment process. In the suggested method this matrix is five-diagonal. Its six distinct entries are known beforehand for any number of observations, which, in addition to the run-time economies, also saves computer storage.

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1. INTRODUCTION

This study is intended to facilitate a gravity field adjustment containing observations of geoid undulations along profiles. The resolution of the adjustment is assumed to be represented by an (n,n) spherical harmonic expansion or its equivalent. This implies that the geoidal detail beyond the (n,n) model enters the realm of observational "errors". It acts as if increasing the variance of the observational noise proper and introducing (or increasing) covariances. Such an added effect can be evaluated with the aid of the geoidal covariance function, and it leads to the final variance covariance matrix associated with geoidal observations. In a straightforward approach this matrix would be inverted, resulting in a fully populated weight matrix.

At first sight, one might object to "mixing" quantities such as the observational noise proper and the neglected geoidal detail, where the latter has the nature of a (neglected) signal. However, their simultaneous treatment is compatible with the collocation principles pioneered by Moritz [1980] and used here in forming the above geoidal variance-covariance matrix. Under the collocation operator $\bar{M} = EM = ME$, where the operator M symbolizes the average over the unit sphere (applied to the signal) and the operator E symbolizes the mathematical expectation (applied to the observational noise and other random effects), the distinction between the original characteristics of such quantities is essentially obliterated.

A typical example of geoidal observations along profiles is offered by satellite altimetry. In considering the thousands of orbital arcs (profiles) entering a global adjustment, a full-scale inversion of the variance-covariance matrices associated with these arcs would be computationally prohibitive. On the other hand, the neglect of all covariances would be unacceptable from the theoretical standpoint. A compromise solution was offered in Chapter 4 of [Blaha, 1984], which resulted in a tri diagonal weight matrix. In that approach, the weight matrix was formed almost as efficiently as in the diagonal case (with neglected covariances), while the corresponding errors in the covariance function were lowered to only about 40% of those characterizing the diagonal case. The standard of comparison was the case called rigorous, where the covariances are properly accounted for and the weight matrix is fully populated. The present analysis is guided by the need to reduce such errors even more significantly.

The basic requirement leading to highly patterned characteristics of the variance covariance and the weight matrices is a uniform spacing of observations along the profiles. With regard to altimetric data, this requirement presents no difficulty. Although the effect of the observational noise proper and other sources of error (e.g., unmodeled tidal variations) was estimated in [Blaha, 1984] and added to the variance computed from the geoidal covariance function, here this effect will be disregarded. For, as is evident from this reference, not only is its estimation very approximate, but it modifies the geoidal variance by only a relatively insignificant amount. However, if desired, this effect - and other modifications such as scaling of the covariance function -- can be included without changing anything in the analysis. Likewise, the same analysis is valid for any (n,n) model describing the adjustable gravity field, for any length of individual geoidal profiles, and for any uniform observational interval.

2. GEOIDAL COVARIANCE FUNCTION AND ITS MODIFICATIONS

The formation of the weight matrix of observations will be carried out in four cases. The first case is represented by the unaltered covariance function. The second case is represented by a modified covariance function as obtained through a fit by two parameters, α_1 and α_2 . The third case is characterized by a more profoundly modified covariance function as obtained through a fit by one parameter, α_1 . And in the fourth case, all covariances are neglected while the variance, computed from the geoidal covariance function, is kept intact.

The representation of the covariance function for geoid undulations is based on the results of [Tscherning and Rapp, 1974]. These results were adopted in [Blaha, 1982] to give the k-th degree variance (in meters square) as follows:

$$\sigma_k^2 = 17981 \times 0.999617^{k+2} / [(k-1)(k-2)(k+24)] .$$

The covariance function for geoid undulations corresponding to the truncation of the spherical-harmonic model at the degree and order (n,n) is then given as

$$D(\psi) \equiv M\{N, N'\} = \sum \sigma_k^2 P_k(\cos\psi) ,$$

where the summation is carried out from the degree n+1 to infinity. The operator M indicates averaging over the unit sphere, N and N' are the geoid undulations at any two points P and P' separated by the spherical distance ψ , and $P_k(\cos\psi)$ are the Legendre polynomials in the argument $\cos\psi$. Since $P_k(1) \equiv 1$, one has for the geoidal variance, $D(0)$, imputable to the neglected degrees (from n+1 to infinity):

$$D(0) = \sum \sigma_k^2 .$$

In practical computations, the degree "infinity" is substituted for by 1000 or some other sufficiently large number.

The truncated model in this analysis is represented by (22,22) spherical-harmonic expansion, i.e., n=22. With this truncation, the geoidal variance is

$$\sigma^2 \equiv D(0) = 11.367 \text{ m}^2 . \tag{1}$$

The observational interval is stipulated to be 0.5°. The geoidal covariances, including the variance, are presented below in a direct form (the units are m^2), as well as in multiples of σ^2 , in which case the entries are denoted "ρ".

ψ :	0	0.5°	1.0°	1.5°	2.0°	2.5°	3.0°	...
D :	11.367	10.661	9.293	7.693	6.047	4.462	3.003	... (2)
ρ :	1	.93789	.81754	.67678	.53198	.39254	.26419	...

For the distances $\psi=3.5^\circ$ through $\psi=8^\circ$ the values of ρ would be .15035, .05296, -.02710, -.09000, -.13627, -.16715, -.18413, -.18888, -.18316, -.16917.

The second case is based on the model

$$\rho_k = \alpha_1 \rho_{k-1} + \alpha_2 \rho_{k-2} \quad (3a)$$

where $k \geq 1$ and where, by definition,

$$\rho_0 = 1, \quad \rho_{(-1)} = \rho_1 \quad (3b,c)$$

The value $\rho_0=1$ in (3b) corresponds to $\rho=1$ for $\psi=0$ in (2). Equations (3a-c) agree with the Yule scheme as is apparent, for example, from (6.21) in [Kendall, 1973]. The third case is characterized by $\alpha_2=0$, while the fourth case is characterized by $\alpha_1=\alpha_2=0$.

From (3a-c), the values of ρ for $k \geq 1$ can be expressed in terms of the parameters α_1 and α_2 as

$$\begin{aligned} \rho_1 &= \alpha_1 / (1 - \alpha_2), & \rho_2 &= \alpha_2 + \alpha_1^2 / (1 - \alpha_2), \\ \rho_3 &= [\alpha_1 / (1 - \alpha_2)] [\alpha_1^2 + 2\alpha_2 - \alpha_2^2], & \dots & \end{aligned}$$

which is valid for all three cases (second, third, and fourth). A better formulation for the second case is given by

$$\rho_k = p^k \sin(k\theta + \beta) / \sin\beta,$$

valid for $k \geq 0$, where

$$p = (-\alpha_2)^{1/2}, \quad \cos\theta = \alpha_1 / 2p, \quad \text{tg}\beta = [(1 - \alpha_2) \text{tg}\theta] / (1 + \alpha_2).$$

This formulation can be derived using an approach similar to that on page 73 of [Kendall, 1973]. From trigonometric considerations it then follows that

$$0 < -\alpha_2 \leq 1, \quad 0 \leq \alpha_1 \leq 2(-\alpha_2)^{1/2} \leq 2. \quad (4a,b)$$

Equations (4a,b) serve to verify the admissibility of α_1 and α_2 .

Upon using (3a) and a similar equation for ρ_{k+1} , the solution of these two equations in two unknowns (α_1 and α_2) yields for the second case:

$$\alpha_1 = (\rho_{k-1}\rho_k - \rho_{k-2}\rho_{k+1})/d, \quad \alpha_2 = (\rho_{k-1}\rho_{k+1} - \rho_k^2)/d, \quad (5a,b)$$

$$d = \rho_{k-1}^2 - \rho_{k-2}\rho_k, \quad (5c)$$

where the values for ρ_0 and $\rho_{(-1)}$ are given by (3b,c). For the third case, one has

$$\alpha_1 = \rho_k/\rho_{k-1}, \quad \alpha_2 = 0, \quad (6a,b)$$

where (3b) remains valid, while the fourth case has already been described by

$$\alpha_1 = 0, \quad \alpha_2 = 0. \quad (7a,b)$$

In evaluating the second-case results α_1 and α_2 for a given profile, one could use all of the available ρ 's in (5a-c) and average the results. However, here such averages are used only in the role of initial parametric values in a least-squares adjustment. Upon considering $n+1=17$ values of ρ associated with $\psi=0$ through $\psi=8^\circ$ in (2) and below it, denoted $\rho_0, \rho_1, \rho_2, \dots, \rho_n$, it follows from (5a-c):

$$\bar{\alpha}_1 = 1.70163, \quad \bar{\alpha}_2 = -.75674, \quad (8a,b)$$

where $n=16$ values of α_1 and α_2 have been used to compute either average. (The symbol n need not be confused with the same letter which earlier indicated the degree and order of the spherical-harmonic model.) With the values in (8a,b), equations (4a,b) are satisfied. In the third case, the initial value of α_1 has been computed by averaging only the first six values in (6a,b), resulting in

$$\bar{\alpha}_1 = .80573, \quad \bar{\alpha}_2 = 0. \quad (9a,b)$$

The reason for this limitation stems from the fact that beyond $\psi=3^\circ$ the values of ρ become small and even change the sign.

The adjustment model in both the second and the third cases is general, in the sense that

$$F(X^a, L^a) = 0,$$

where X^a symbolizes the adjusted parameters, α_1 and eventually α_2 , and L^a symbolizes the n adjusted "observations", $\rho_1, \rho_2, \dots, \rho_n$. Since $\rho_0=1$ is a

constant. It is not included among the observables. The linearization of this model yields the familiar matrix equation

$$AX + BV + W = 0$$

where

$$A = F(X^0), \quad B = F(L), \quad W = F(X^0, L^b)$$

and where A and B are evaluated with $X = X^0$ the initial parametric values, and with $L = L^b$ the "observed" values of ρ represented here by the rigorous values in (2) and below it, with the exception of the first ($\rho_0 = 1$). The vector X denotes the adjustment corrections to the initial parametric values, X^0 , and the vector V denotes the residuals. The least squares solution in this general adjustment method is

$$X = (A^T M^{-1} A)^{-1} A^T M^{-1} W, \quad (10)$$

where

$$M = B \Sigma B^T, \quad (11)$$

Σ being the variance covariance matrix of the "observations", taken here as 1.

In the second case, the general model follows from (3a) as

$$\rho_k - \alpha_1 \rho_{k-1} - \alpha_2 \rho_{k-2} = 0, \quad (12)$$

where the superscript "a" has been omitted. For $k \geq 3$, the k th row in the matrix A, the three nonzero elements in the k th row of the matrix B, and the k th element in the vector W are, respectively,

$$[-\rho_{k-1} \quad \rho_{k-2}], \quad [-\bar{\alpha}_2 \quad \bar{\alpha}_1 \quad 1], \quad \rho_k - \bar{\alpha}_1 \rho_{k-1} - \bar{\alpha}_2 \rho_{k-2}, \quad (13a, b, c)$$

where the superscript "b" in conjunction with the ρ 's has also been omitted. The element "1" in (13b) is located at the diagonal of B. For $k = 1$, equations (13a, c) become

$$[1 \quad \rho_1], \quad [1 \quad \bar{\alpha}_2 \quad 0 \quad 0], \quad \rho_1(1 - \bar{\alpha}_2) - \bar{\alpha}_1, \quad (13'a, b, c)$$

whereas for $k = 2$ they become

$$[-\rho_1 \quad 1], \quad [-\bar{\alpha}_1 \quad 1 \quad 0], \quad \rho_2 - \bar{\alpha}_1 \rho_1 - \bar{\alpha}_2, \quad (13''a, b, c)$$

the dimensions of A , B , and w are seen to be $(n \times 2)$, $(n \times n)$, and $(n \times 1)$, respectively. The third case can be described in a perfect analogy to the second, except that the second column in A is nonexistent and that $\alpha_2 = 0$.

Applying the solution X in (10), one can take advantage of the fact that the matrix M can be inverted analytically. Since B is a regular matrix (square and nonsingular), it follows from (11) that

$$w = \frac{1}{\alpha_1} \sum_{i=1}^n 1_{i-1}$$

where 1_{i-1} is simplified by $\sum 1_{i-1}$. Due to the patterned characteristics of B in (13a) and (13b), and (13'b), it is readily verified that the matrix B^{-1} can be formed according to the following algorithm. Initially, a "dummy" column is formed whose first element is $d_1 = 1$, second element is $d_2 = \alpha_1$, and any further element is $d_i = \alpha_1 d_{i-1} + \alpha_2 d_{i-2}$, $i=3,4,\dots,n$. Next, the first column in B^{-1} is formed as the dummy column times the factor $1/(1-\alpha_2)$. Finally, starting at the diagonal, the second column in B^{-1} consists of the first $n-1$ values of the dummy column. Similarly, starting at the diagonal, the third column in B^{-1} consists of the first $(n-2)$ values of the dummy column, and so on, until the diagonal element "1" in the last column is filled in. The elements above the diagonal in these columns are zero, i.e., the matrix B^{-1} is lower-triangular. The third case again follows by setting $\alpha_2 = 0$.

A nearly perfect convergence in the iterated adjustment (10) has been achieved in four iterations in both the second and the third cases. The results for the second case are

$$\alpha_1 = 1.69244, \quad \alpha_2 = .76172, \quad (14a,b)$$

satisfying (4a,b). The $n=17$ values of the adjusted "observations" ρ corresponding to (2) and its extension are

$$\begin{aligned} \rho(\text{second}) = & 1.96068, .86417, .73080, .57858, .42255, .27442, \\ & 14258, .03228, .05397, .11594, .15510, .17419, \\ & 17667, .16641, .14690, .12194. \end{aligned} \quad (15a)$$

Compared to (2) and its extension, the root-mean-square of the errors in $\rho(\text{second})$ is

$$\text{rms}(\text{second}) = .02989 \quad (15b)$$

Similarly, the third case yields

$$\alpha_1 = .80243 , \quad \alpha_2 = 0 , \quad (16a,b)$$

and

$$\rho(\text{third}) = 1, .80243, .64389, .51668, .41460, .33268, .26695, \\ .21421, .17189, .13793, .11068, .08881, .07126, .05719, \quad (17a)$$

$$.04589, .03682, .02955 ;$$

$$\text{rms}(\text{third}) = .16907 . \quad (17b)$$

The fourth case is trivial, namely

$$\rho(\text{fourth}) = 1, 0, 0, 0, \dots, 0 ; \quad (18a)$$

$$\text{rms}(\text{fourth}) = .40040 . \quad (18b)$$

In terms of the root-mean-square, the error in the third case amounts to approximately 42% of the fourth-case error, in line with the result achieved in [Blaha, 1984] for the (14,14) truncation. However, the error in the second case is much smaller, amounting to under 8% of the fourth-case error. The second-case error is only about 18% of the third-case error. Such a significant reduction indicates that no further refinements are likely to be necessary in the future. Indeed, the covariance function itself is only approximate, and, furthermore, the values it produces are indicative of a global situation, not that pertinent to a particular geoidal profile.

On the other hand, it may be beneficial to repeat the above adjustment for varying lengths of the profile, as well as for a different observational interval. For example, the current results (14a,b) may be deemed satisfactory for a profile length of 6°-10°, whereas for a different length the final values of α_1 and α_2 may slightly change. But the parameters adjusted by the general method for any length of the profile are preferred to the approximate values obtained by averaging. In the situation under scrutiny, the approximate values in (8a,b) would produce an rms error over three times as large as that presented in (15b), obtained with the adjusted values (14a,b).

3. WEIGHT MATRICES

The variance covariance matrix Σ of $m=n+1$ observations can be written as

$$\Sigma = \sigma^2 \begin{bmatrix} 1 & \rho_1 & \rho_2 & \rho_3 & \dots & \rho_n \\ \rho_1 & 1 & \rho_1 & \rho_2 & \dots & \rho_{n-1} \\ \rho_2 & \rho_1 & 1 & \rho_1 & \dots & \rho_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \rho_n & \rho_{n-1} & \rho_{n-2} & \dots & \dots & 1 \end{bmatrix} \quad (19)$$

where the dimensions of Σ are $(m \times m)$. Equation (19) stems from the general form of the correlation coefficient ρ symbolized by $\rho_{xy} = \sigma_{xy} / (\sigma_x \sigma_y)$. Here $\sigma_x = \sigma_y = \sigma$, and

$$\rho_k = \sigma_k / \sigma^2 = D(\psi) / D(0) \quad ,$$

where the values of D and ρ have been listed in (2). If the geoidal observations are uniformly distributed, the values of ρ can be modeled by (3a-c), for example, resulting in great computational savings in the formation of the weight matrix P , where

$$P = \Sigma^{-1} \quad , \quad (20)$$

as will be discussed next.

If the coefficients ρ_k fulfilled (3a-c) exactly, the matrix (19) could be inverted analytically, resulting in the weight matrix P for the second case presented below, where the factor q is given by

$$q = (1 - \alpha_2) / \{ (1 + \alpha_2) [(1 - \alpha_2)^2 - \alpha_1^2] \} \quad . \quad (21a)$$

$$P = (1/\sigma^2)q \begin{bmatrix} 1 & -\alpha_1 & -\alpha_2 & 0 & 0 & 0 & \dots & 0 \\ -\alpha_1 & 1+\alpha_1^2 & -\alpha_1(1-\alpha_2) & -\alpha_2 & 0 & 0 & \dots & 0 \\ -\alpha_2 & -\alpha_1(1-\alpha_2) & 1+\alpha_1^2+\alpha_2^2 & -\alpha_1(1-\alpha_2) & -\alpha_2 & 0 & \dots & 0 \\ 0 & -\alpha_2 & -\alpha_1(1-\alpha_2) & 1+\alpha_1^2+\alpha_2^2 & -\alpha_1(1-\alpha_2) & -\alpha_2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & \dots & \dots & \dots & \dots & 1 \end{bmatrix}$$

(21b)

The entries in all the rows except the first two and the last two repeat in the fashion indicated. The last two rows are symmetric with respect to the first two. The matrix P in the third case would be formed exactly as in (21a,b), except that α_2 would be replaced by zero. And the trivial fourth case would have both α_1 and α_2 replaced by zero, resulting in $P=(1/\sigma^2)I$.

In the practical procedure suggested herein the coefficients ρ given by the geoidal covariance function are modified to conform to (3a-c). As (21b) indicates, the weight matrix P is then five-diagonal, all the other entries being zero. With $\alpha_2=0$ this matrix would be tri-diagonal, and with $\alpha_1=\alpha_2=0$ it would be diagonal. Accordingly, the first case will also be called "rigorous", while the second, third, and fourth cases will also be called "five-diagonal", "tri-diagonal", and "diagonal", respectively. In [Brown and Trotter, 1969], the tri-diagonal and the five-diagonal cases were referred to as the first-order and the second order autoregressive processes, respectively. It is noted that for the purpose of that reference, the factor q was omitted. Although the present approach could be extended to a seven-diagonal case, etc., the five-diagonal weight matrix is deemed completely satisfactory for geoidal profiles. According to the discussion at the close of Chapter 2, the errors committed by modifying the covariance function in this case are negligible.

The validity of (21a,b) can be confirmed upon performing

$$P\Sigma = I , \quad (22)$$

where the entries ρ_k in (19) are taken from (3a-c). In considering the first row of P, the first row of (22) follows from the following three identities:

$$q(1-\alpha_1\rho_1-\alpha_2\rho_2) = 1 , \quad (1-\alpha_2)\rho_1 - \alpha_1 = 0 ,$$

$$\rho_{i+2} - \alpha_1\rho_{i+1} - \alpha_2\rho_i = 0 ,$$

where $i \geq 0$. The second and the third identities correspond to (3a-c). The other elements in (22) could be verified in a similar fashion. For example, the element (2,2) is the consequence of

$$q[-\alpha_1\rho_1 + (1+\alpha_1^2) - \alpha_1(1-\alpha_2)\rho_1 - \alpha_2\rho_2] = 1 ,$$

and further diagonal elements (except for the last two) are the consequence of

$$q[-\alpha_2\rho_2 - \alpha_1(1-\alpha_2)\rho_1 + (1+\alpha_1^2 + \alpha_2^2) - \alpha_1(1-\alpha_2)\rho_1 - \alpha_2\rho_2] = 1 .$$

The same confirmation is valid also with $\alpha_2=0$ and with $\alpha_1=\alpha_2=0$.

4. SIMPLE NUMERICAL EXAMPLE

In this example, $m=17$ observations of geoid undulations are considered distributed in 0.5° -intervals spanning a 8° -profile. From (2) and its extension, one forms the rigorous (17×17) variance-covariance matrix as

$$\Sigma = 11.367 \begin{bmatrix} 1 & .93789 & .81754 & \dots & -.16917 \\ .93789 & 1 & .93789 & \dots & -.18316 \\ .81754 & .93789 & 1 & \dots & -.18888 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -.16917 & -.18316 & -.18888 & \dots & 1 \end{bmatrix}$$

The inversion of this matrix yields a fully populated weight matrix P , computed with the aid of the Choleski algorithm but not listed here.

The variance-covariance matrix modified according to the second-case principle would follow from (15a). However, it is not needed explicitly since the five-diagonal weight matrix is given directly by (21a,b) with the numerical values of α_1 and α_2 presented in (14a,b). Similarly, the tri-diagonal weight matrix of the third case follows from (21a,b) with α_1 and $\alpha_2=0$ given by (16a,b). Finally, the diagonal weight matrix of the fourth case is $P=(1/11.367)I$.

The 17 observations considered are used to adjust one parameter, taken here as a point-mass magnitude. The point mass itself is chosen to have a central location with respect to the profile, and to be located at the depth $d=704\text{km}$ beneath the earth's surface represented by a sphere of radius $R=6371\text{km}$. Thus, the central angle ψ varies between 0 and 4° on either side of the point mass. The mathematical model for geoid undulations is written as

$$N_i = (1/G)(\text{kM})_j / s_{ij} \quad i=1,2,\dots,m,$$

where G is the average value of gravity adopted as 980gal ; $(\text{kM})_j = x$ is the unknown parameter, i.e., the gravitational constant times the mass of the point mass "j"; and s_{ij} is the distance between the observational point "i" and the point mass "j". This distance is computed as

$$s_{ij} = \{[(R-d)\sin\psi_{ij}]^2 + [R-(R-d)\cos\psi_{ij}]^2\}^{1/2} ,$$

where ψ_{ij} is the angle ψ corresponding to the above points i and j .

The 17 observations of geoid undulations (in meters) are simulated in a symmetrical manner, the first and the last being zero and the middle one directly above the point mass being 5m, as follows:

0.000, 0.975, 1.913, 2.778, 3.536, 4.157, 4.619, 4.904, 5.000,
4.904, 4.619, 4.157, 3.536, 2.778, 1.913, 0.975, 0.000 .

These values represent the elements of the vector called ϵ . The units are chosen such that x is expressed in terms of 10^{-8} times the earth's km (the latter is $3.986005 \times 10^{14} \text{ m}^3/\text{s}^2$). The design matrix A has the dimensions (17x1). Its elements, symmetric about the middle, are

.49635, .51232, .52748, .54143, .55370, .56384, .57144, .57615, .57775,
.57615, .57144, .56384, .55370, .54143, .52748, .51232, .49635 .

The parametric least-squares adjustment yields

$$x = (A^T P A)^{-1} A^T P \epsilon ,$$

$$\sigma_x^2 = (A^T P A)^{-1} ;$$

the symbol x denotes here the least-squares estimate of the point-mass magnitude. In agreement with the previous classification, this adjustment entails four cases. The main results are grouped in Table 1, which is self-explanatory. This table shows the error in the tri-diagonal solution as being 46% of the error in the diagonal solution. The error in the sigma of the tri-diagonal solution amounts to about 13% of the error in the sigma of the diagonal solution.

The error in the five-diagonal solution is much smaller, amounting to less than 23% of the error in the tri-diagonal solution, and thus to merely about 10% of the error in the diagonal solution. With regard to the error in the sigma of the five-diagonal solution, these two numbers would be replaced by 15% and 2%, respectively. Although this example is quite specialized, it nevertheless illustrates that the five-diagonal setup is likely to match the rigorous setup

much more closely than would be possible with the tri-diagonal setup developed in [Blaha, 1984].

Case	Name	x	Error	σ_x	Error
1	Rigorous	3.292	--	3.560	--
2	Five-diagonal	3.049	-0.243	3.599	+0.039
3	Tri-diagonal	4.367	+1.075	3.821	+0.261
4	Diagonal	5.622	+2.330	1.499	-2.061

Table 1

Least-squares solutions and their sigmas, together with their errors, in the four weighting schemes; the first case is an errorless standard.

5. CONCLUSION

Rigorous weighting of geoidal observations along profiles, such as encountered, for example, in satellite altimetry, would be a task exceedingly demanding in terms of computer run-time. A profile with n observation points would require an inversion of an $(n \times n)$ variance-covariance matrix, Σ , whose entries are correlated and depend almost entirely on the geoidal covariance function. However, if the observations are uniformly distributed, a slight modification of this function can greatly facilitate the formation of the weight matrix of observations, $P = \Sigma^{-1}$, which subsequently enters the least-squares adjustment process. In the suggested method this matrix is five diagonal. Its six distinct entries are known beforehand for any number of observations, which, in addition to the run-time economies, also saves computer storage.

The pertinent modification of the covariance function is nearly negligible, especially if one considers that this function itself is only approximate, used here for a global description of the effects of the geoidal detail unexpressible by the adjustment. A five to six times more profound modification of the covariance function (in terms of the root-mean square) leads to a tri-diagonal weight matrix. A similar relationship has been noted in an adjustment of a single point-mass magnitude via simulated geoidal observations, in the sense that the point-mass error associated with the tri-diagonal weighting scheme has been four to five times larger than the error associated with the five diagonal weighting scheme. Since the five diagonal approach entails a marked improvement in accuracy over its tri-diagonal counterpart, and since both approaches are almost equally simple and offer nearly the same computer economies, the former is preferred to the latter, and represents, in fact, the desired algorithm for the adjustment of geoidal profiles as sought in the present study.

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