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Operating Characteristics of Log-Normalizer for Weibull and Log-Normal Inputs

Albert H. Nuttall
Surface Ship Sonar Department

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Naval Underwater Systems Center
Newport, Rhode Island/New London, Connecticut

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Preface

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<p>The false alarm and detection probabilities of a log-normalizer, subject to either log-normal or Weibull input statistics, are derived for general input signal and noise strengths and number of normalizer samples, N. Plots of the exceedance distribution function versus the threshold, as well as the receiver operating characteristics (i. e., detection probability P_D vs. false alarm probability P_F) are plotted for $N = \infty, 64, 32, 16$ and for various values of the normalizer input deflection statistic d. In addition, simulation results, based on 8.4 million trials, are superposed for purposes of confirming or rejecting the theoretical results.</p> <p>Plots of the exceedance distribution function are carried out on the extremes of the distribution, to the point where the tail probabilities are ...</p>					
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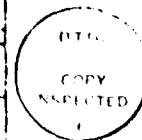
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- > 1E-6. The receiver operating characteristics vary over the range of (P_F, P_D) equal to $(1E-6, 1E-6)$ through $(.5, .99)$. It is found that the theoretical analysis for the log normal input is exact for all N , whereas the approximate theoretical analysis for the Weibull input is sufficiently accurate only for large N , and not on the tails of the distribution.

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LIST OF SYMBOLS

N	Number of normalizer samples, (2)
d	Normalizer input deflection, (29),(30)
P_D	Detection probability, (6)
P_F	False alarm probability, (6)
EDF	Exceedance Distribution Function
ROC	Receiver Operating Characteristic
x_n	Input to Log-Normalizer, figure 1
y_n	Normalizer input, figure 1, (1)
z	Normalizer output, figure 1, (4),(28)
y_0	Candidate signal-bearing sample, (4)
$\hat{\mu}(y)$	Sample mean of normalizer input, (2)
$\hat{\sigma}(y)$	Sample standard deviation of normalizer input, (3)
T	Threshold at normalizer output, (5)
w_n	Fundamental random variables, (7)
a, b	Arbitrary scaling and power law, (7)
p_w	Probability density function of random variable w , (8)
Q_w	Exceedance distribution function of random variable w , (9)
Φ	Cumulative distribution function of normalized Gaussian random variable, (12)
ϕ	Probability density function of normalized Gaussian random variable, (12)
\tilde{w}_n	log-distorted version of w_n , (16)
$\mu(\tilde{w})$	Mean of \tilde{w} , (17)
$\sigma(\tilde{w})$	Standard deviation of \tilde{w} , (17)
v_n	Normalized random variable, (17)
α, β	Constants, (19),(20)

LIST OF SYMBOLS (Cont'd)

$\hat{\mu}(v)$	Sample mean of normalized random variable, (24)
$\hat{\sigma}(v)$	Sample standard deviation of normalized random variable, (24)
v_0	Normalized signal-bearing random variable, (25)
π, ν	Constants, (27)
r	Normalizer input scaling parameter, (29),(30)
H_0	Hypothesis that signal is absent in y_0 , (31)
m	Auxiliary constant = $(N-1)/2$, (36)
σ_N^2	Auxiliary variance = $r^2 + 1/N$, (37)
d'_r	Modified deflection parameter, (38),(39)
T'_r	Modified threshold, (38),(39)
$\bar{\Phi}$	Inverse Φ function, (44)
t	Sample random variable, (46)
γ	Euler's constant = .57721, (49)
$\mu(t)$	Mean of random variable t , (50),(53)
$\sigma(t)$	Standard deviation of random variable t , (50),(54)
h_1, h_2	Auxiliary constants, (52)
ρ	Normalized correlation coefficient, (54),(55)
$\{x_n\}_1^N$	Sequence x_1, x_2, \dots, x_N

OPERATING CHARACTERISTICS OF LOG-NORMALIZER
FOR WEIBULL AND LOG-NORMAL INPUTS

INTRODUCTION

The detection of the presence of a weak signal of unknown location and strength in background noise of unknown strength is often accomplished by comparing a candidate signal-bearing detection sample of the observed process with a local estimate of the background level based on N samples of the (hopefully) noise-only process. The local neighborhood can be time, space, or frequency, depending on the application. In order to obtain a stable estimate of the background level, the number of samples, N , should be large; however, if the background is nonstationary, nonhomogeneous, or nonwhite, or if decision and processing time is at a premium, N should be kept as small as reasonably possible. The tradeoff between these conflicting requirements and the dependence on the number of normalizer samples, N , is of interest in this study. Related work is available in [1,2,3].

Since the performance of the normalizer procedure outlined above is adversely affected by the presence of any outliers or noise bursts anywhere in the total of $N+1$ samples used to make a decision about signal presence or absence in the candidate sample, some form of limiting device

should precede the normalization. The particular combination that we consider in detail here is depicted in figure 1, where \ln is the natural

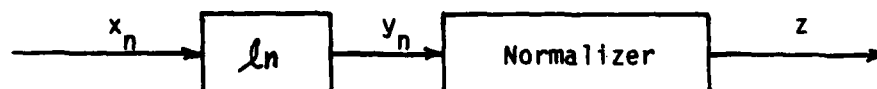


Figure 1. Log-Normalizer

logarithm. The logarithmic device tends to saturate at large input amplitudes and suppress their effect on the normalizer output z . The input sequence of random variables, $\{x_n\}$, (the detector output sequence), is presumed to be statistically independent and limited to positive values, giving logarithmic output

$$y_n = \ln(x_n) . \quad (1)$$

The particular normalizer we consider here is described as follows: call y_0 the candidate signal-bearing sample at the normalizer input, and let y_1, y_2, \dots, y_N be the N noise-only basis samples employed to extract an estimate of the background level at the normalizer input. Despite the notation, these N samples can (and probably will) surround the candidate sample y_0 in location, whether that be time, space, frequency, etc. The sample mean of the normalizer input noise-only samples is

$$\hat{\mu}(y) = \frac{1}{N} \sum_{n=1}^N y_n , \quad (2)$$

while the corresponding sample standard deviation is defined as

$$\hat{\sigma}(y) = \left\{ \frac{1}{N-1} \sum_{n=1}^N [y_n - \hat{\mu}(y)]^2 \right\}^{1/2}. \quad (3)$$

The output of the normalizer in figure 1, that we consider here, is a deflection measure, namely

$$z = \frac{y_0 - \hat{\mu}(y)}{\hat{\sigma}(y)}. \quad (4)$$

The numerator of (4) is an estimate of the difference in means (at the normalizer input) of the candidate signal-plus-noise sample, relative to the noise-only samples; the denominator of (4) is a measure of the inherent fluctuation of the background noise. The dimensionless ratio in (4) eliminates the dependence on absolute levels in favor of relative levels.

The normalizer output z is compared with a threshold T , and a decision made about signal presence in sample y_0 according to the rule

$$\left. \begin{array}{l} z > T: \text{ declare signal present in } y_0 \\ z < T: \text{ declare signal absent in } y_0 \end{array} \right\}. \quad (5)$$

It is desired to evaluate the false alarm probability and the detection probability, that is,

$$P_F = \text{Prob} (z > T \mid \text{signal absent in } y_0),$$

(6)

$$P_D = \text{Prob} (z > T \mid \text{signal present in } y_0).$$

Both of these probabilities in (6) are exceedance distribution functions, that is, probabilities that random variable z is greater than a threshold value T . We will be interested in plots of (6) versus T , for various signal-to-noise ratios, as well as in plots of P_D versus P_F , the latter known as the receiver operating characteristics.

The normalizer input random variables $\{y_n\}_1^N$ are statistically independent and identically distributed, since inputs $\{x_n\}_1^N$ have been presumed to have these properties. When signal is absent in candidate sample y_0 , its probability density function will be taken identical to that of $\{y_n\}_1^N$; however, when signal is present in y_0 , its probability density function can be arbitrary.

CLASSES OF INPUT VARIABLES

The noise-only input samples $\{x_n\}_1^N$ to the log-normalizer in figure 1 will be taken from the class of random variables that can be generated from fundamental independent identically-distributed random variables $\{w_n\}$ according to the rule

$$x_n = a w_n^b \text{ for } 1 \leq n \leq N; \quad a > 0, b > 0, w_n > 0. \quad (7)$$

The probability density function p_w of $\{w_n\}$ is arbitrary; the total class of random variables defined by (7) is that yielded by allowing parameters a and b to be any positive constants (independent of n).

To fix this concept of a class of random variables, consider the case where w_n is a random variable with the fundamental exponential probability density function

$$p_w(u) = \exp(-u) \text{ for } u > 0. \quad (8)$$

Then the exceedance distribution function of w_n is

$$Q_w(u) = \text{Prob}(w > u) = \int_u^\infty dv p_w(v) = \exp(-u) \text{ for } u > 0. \quad (9)$$

It then follows from (7) that the exceedance distribution function of x_n is

$$\begin{aligned} Q_x(u) &= \text{Prob}(x > u) = \text{Prob}(a w^b > u) = \\ &= \text{Prob}\left(w > (u/a)^{1/b}\right) = Q_w\left(\left(\frac{u}{a}\right)^{1/b}\right) = \exp\left[-\left(\frac{u}{a}\right)^{1/b}\right] \quad \text{for } u > 0. \end{aligned} \quad (10)$$

But this is just the exceedance distribution function of a Weibull variate* with shape factor $1/b$ and scaling $(1/a)^{1/b}$. Thus, the class of random variables that can be generated via (7) with arbitrary a, b , from the fundamental exponential probability density function in (8), is the general class of Weibull variates, as given by (10). (If $b = 1/2$, x is a Rayleigh variate, for example.)

As a second case, let w_n be a random variable with the fundamental log-normal exceedance distribution function

$$Q_w(u) = \bar{\Phi}(-\ln u) \quad \text{for } u > 0, \quad (11)$$

where

$$\bar{\Phi}(t) \equiv \int_{-\infty}^t ds (2\pi)^{-1/2} \exp(-s^2/2) \equiv \int_{-\infty}^t ds \phi(s) \quad (12)$$

is the cumulative distribution function of a normalized Gaussian random variable. Then by an analogous procedure to (10), the exceedance distribution function of the random variable x_n generated according to (7)

is

*See appendix A.

$$Q_x(u) = Q_w\left(\left(\frac{u}{a}\right)^{1/b}\right) = \Phi\left(\frac{\ln(a) - \ln(u)}{b}\right) \quad \text{for } u > 0, \quad (13)$$

which is the exceedance distribution function of a general log-normal variate with additive factor $\ln(a)$ and scaling $1/b$. The probability density function corresponding to (13) is

$$p_x(u) = \frac{1}{bu} \phi\left(\frac{\ln(a) - \ln(u)}{b}\right) \quad \text{for } u > 0, \quad (14)$$

where ϕ was defined in (12). Thus, the class of random variables that can be generated via (7) with arbitrary a, b , from the fundamental log-normal exceedance distribution function in (11), is the general class of log-normal variates, as given by (13) and (14).

Returning to the general case for fundamental random variable w_n now, the output of the logarithmic device, (1), is given, upon use of (7), as

$$y_n = \ln(x_n) = \ln a + b \ln w_n = \ln a + b \tilde{v}_n \quad \text{for } 1 \leq n \leq N, \quad (15)$$

where we define

$$\tilde{v}_n = \ln w_n. \quad (16)$$

Let the mean and standard deviation of \tilde{v}_n be denoted by $\mu(\tilde{v})$ and $\sigma(\tilde{v})$, respectively. Then form the normalized random variable

$$v_n = \frac{\tilde{v}_n - \mu(\tilde{V})}{\sigma(\tilde{V})} \quad \text{for } 1 \leq n \leq N, \quad (17)$$

which has mean 0 and standard deviation 1. Substitution of (17) in (15) then yields log output

$$y_n = \alpha + \beta v_n \quad \text{for } 1 \leq n \leq N, \quad (18)$$

where constants

$$\alpha = \ln a + b \mu(\tilde{V}), \quad \beta = b \sigma(\tilde{V}). \quad (19)$$

A direct useful interpretation of these two constants in (19) follows directly from (18); namely, since $\{v_n\}$ are normalized random variables,

$$\alpha = \mu(y), \quad \beta = \sigma(y). \quad (20)$$

These are fundamental statistics of the input to the normalizer in figure 1.

Equations (18) and (19) demonstrate that the output of the logarithmic device in figure 1, for general parameters a, b and random variables $\{w_n\}_1^N$ in transformation (7), can be handled through the linear transformation (18) of a normalized random variable, v_n , with zero mean and unit standard deviation. The new general parameters α, β are given by (19) or (20), where the required statistics are mean

$$\bar{\tilde{v}} = \mu(\tilde{v}) = \mu(\ln w) = \int du \ln(u) p_w(u) , \quad (21)$$

and mean square

$$\overline{\tilde{v}^2} = \overline{(\ln w)^2} = \int du (\ln u)^2 p_w(u) , \quad (22)$$

in terms of the probability density function of input variable w_n in figure 1. Also, except for the specified zero mean and unit standard deviation of v_n in (18), the statistics of v_n are completely arbitrary. Thus, we can use form (18) for the general normalizer input in the following, where α and β are arbitrary constants.

When we now employ (18), the sample quantities in (2) and (3) become

$$\hat{\mu}(y) = \alpha + \beta \hat{\mu}(v) , \quad (23)$$

$$\hat{\sigma}(y) = \beta \hat{\sigma}(v) ,$$

where

$$\hat{\mu}(v) = \frac{1}{N} \sum_{n=1}^N v_n ,$$

$$\hat{\sigma}(v) = \left\{ \frac{1}{N-1} \sum_{n=1}^N [v_n - \hat{\mu}(v)]^2 \right\}^{1/2} , \quad (24)$$

in terms of the normalized random variables $\{v_n\}_1^N$.

As noted in the paragraph following (6), the probability density function of random variable y_0 is arbitrary for the signal-present hypothesis.

Without loss of generality, let

$$y_0 = \eta + \nu v_0, \quad (25)$$

where normalized random variable v_0 has

$$\mu(v_0) = 0, \quad \sigma(v_0) = 1. \quad (26)$$

The constants η and ν absorb the absolute scale of y_0 ; in fact (in analogy with (20)),

$$\eta = \mu(y_0), \quad \nu = \sigma(y_0). \quad (27)$$

When we now combine (23) and (25) in the normalizer output z , as given by (4), there follows

$$z = \frac{d + r v_0 - \hat{\mu}(v)}{\hat{\sigma}(v)}, \quad (28)$$

where constants

$$d = \frac{\eta - \alpha}{\beta}, \quad r = \frac{\nu}{\beta}. \quad (29)$$

Thus, the general output z of the log-normalizer in figure 1, for the general class of inputs (7), can be expressed in the form (28) involving two fundamental constants d , r in (29); an arbitrary normalized random variable v_0 ; and the sample mean and standard deviation of the normalized random variables $\{v_n\}_1^N$ according to (24).

A useful physical interpretation of the constants in (29) is afforded by utilizing (20) and (27), namely

$$d = \frac{\mu(y_0) - \mu(y)}{\sigma(y)}, \quad r = \frac{\sigma(y_0)}{\sigma(y)}. \quad (30)$$

Thus, parameter d measures the deflection criterion at the normalizer input, relative to the standard deviation for signal absent. The parameter r is a scaling quantity reflecting the relative fluctuating strengths at the normalizer input. The fundamental analysis problem is now to evaluate the false alarm and detection probabilities specified by (6), for the output random variable given by (28), where d and r are arbitrary constants, and v_0 and $\{v_n\}_1^N$ are normalized random variables.

CONSTANT FALSE ALARM RATE PROPERTY

The general output of the log-normalizer is given by (28). However, for hypothesis H_0 where signal is absent in candidate signal-bearing sample y_0 , the statistics of normalizer input y_0 are identical to those of $\{y_n\}_1^N$, as noted in the paragraph under (6). In this case, (30) obviously reduces to

$$d = 0, \quad r = 1 \quad \text{under } H_0, \quad (31)$$

and (28) yields

$$z = \frac{v_0 - \hat{\mu}(v)}{\hat{\sigma}(v)} \quad \text{under } H_0, \quad (32)$$

in terms of the independent identically-distributed normalized random variables v_0 and $\{v_n\}_1^N$.

Since v_n in (17) is the normalized random variable corresponding to logarithmic distortion (16) of fundamental random variable w_n , and does not involve a or b , all scale factors involving constants a and b in (7) have disappeared in output z in (32), under hypothesis H_0 . This means that the false alarm probability P_F in (6) cannot depend on a, b ; put another way, the false alarm probability for the log-normalizer of figure 1, subjected to

the class of inputs given by (7), is the same for all members of the class, regardless of the values of a and b . Since the sample mean and sample standard deviation in (32) still depend on N , as seen by reference to (24), the false alarm probability will necessarily be a function of N , as well as depend on the particular probability density function of independent identically-distributed normalized random variables v_0 and $\{v_n\}$. However, in general, there will be no need to investigate the false alarm probability for the general Weibull class in (10), but instead we can confine attention to the fundamental exponential probability density function of w_n as given by (8). Of course, v_n must then be the normalized random variable, as given by (16), (17), (21), and (22). More details on the statistics of Weibull variates and their logarithmically-distorted counterparts are given in appendix A.

A similar statement can be made with regard to the fundamental log-normal exceedance distribution function given by (11). In fact, the logarithmically transformed input, (16), to the normalizer has exceedance distribution function

$$\begin{aligned}
 Q_{\tilde{v}}(u) &= \text{Prob}(\tilde{v} > u) = \text{Prob}(\ln w > u) = \text{Prob}(w > \exp(u)) = \\
 &= Q_w(\exp(u)) = \bar{\Phi}(-u) \quad \text{for all } u.
 \end{aligned}
 \tag{33}$$

But this is the exceedance distribution function of a zero-mean unit-variance Gaussian random variable. Thus, \tilde{v} of (16) is already a normalized random variable, and Gaussian at that. Therefore, decision

variable z in (32) involves a collection of $N+1$ independent identically-distributed zero-mean unit-variance random variables v_0 and $\{v_n\}_1^N$. Again, the false alarm probability can only depend on N , and not on scale factors a and b in (13) and (14). Of course, the detection probability (6), as applied to (28), will depend additionally on parameters d and r in (29) and (30).

In summary, the log-normalizer in figure 1 will possess constant false alarm rate properties, that is, the same false alarm probability for all the members of the class of random variables generated according to (7), regardless of the values of a and b .

PERFORMANCE FOR LOG-NORMAL INPUT x

The problem of interest in this section is the evaluation of detection probability (6) for the decision variable z given by (28), when normalized random variables $\{v_n\}_1^N$ are independent identically-distributed zero-mean unit-variance Gaussian random variables; this is the case discussed in (33) et seq. Although the probability density function of normalized random variable v_0 is arbitrary, we will also take it here to be zero-mean unit-variance Gaussian. Reference to (15), (17), and (25) reveals that this is tantamount to assuming that the normalizer input $\{y_n\}_1^N$ in figure 1 is Gaussian with arbitrary mean and variance, while y_0 is also Gaussian with different arbitrary mean and variance. All these arbitrary parameters are collected together in (28) in the parameters d and r , according to (30). This situation is also equivalent to assuming log-normal excitations at the input of the log-normalizer of figure 1.

From (6) and (28), since $\hat{\sigma} > 0$,

$$P_D = \text{Prob}(z > T) = \text{Prob}(d + rv_0 - \hat{\mu}(v) > T \hat{\sigma}(v)) , \quad (34)$$

where $\hat{\mu}(v)$ and $\hat{\sigma}(v)$ are given by (24). It is shown in appendix B that $\hat{\mu}(v)$ and $\hat{\sigma}(v)$ are statistically independent, with probability density functions given by (B-16) and (B-20), respectively, (setting $\mu = 0$, $\sigma = 1$) as

$$p_u(u) = (2\pi/N)^{-1/2} \exp(-\frac{N}{2} u^2) \text{ for all } u \quad (35)$$

and

$$p_\sigma(u) = \frac{2 m^m u^{2m-1} \exp(-mu^2)}{\Gamma(m)} \text{ for } u > 0; \quad m = \frac{N-1}{2}. \quad (36)$$

Now the quantity $d + rv_0 - \hat{\mu}(v)$ in (34) is a Gaussian random variable with mean d and variance $r^2 + 1/N \approx \sigma_N^2$; see (B-16) or (35). Considering $\hat{\sigma}$ fixed for the moment, the conditional detection probability in (34) is then

$$P_{DC} = \int_{T\hat{\sigma}}^{\infty} dt (2\pi\sigma_N^2)^{-1/2} \exp\left[-\frac{(t-d)^2}{2\sigma_N^2}\right] = \Phi\left(\frac{d - T\hat{\sigma}}{\sigma_N}\right), \quad (37)$$

upon use of (12). Averaging this result over the probability density function (36) of $\hat{\sigma}$, we have the unconditional detection probability

$$\begin{aligned} P_D &= \int_0^{\infty} du \Phi\left(\frac{d - Tu}{\sigma_N}\right) \frac{2 m^m u^{2m-1} \exp(-mu^2)}{\Gamma(m)} = \\ &= \int_0^{\infty} dw \frac{w^{N-2} \exp(-w^2/2)}{2^2 \Gamma\left(\frac{N-1}{2}\right)} \Phi(d'_r - T'_r w), \end{aligned} \quad (38)$$

where $N \geq 2$ and

$$\begin{aligned} d'_r &\equiv \frac{d}{\sigma_N} = \frac{d}{\sqrt{r^2 + 1/N}}, \\ T'_r &\equiv \frac{T}{\sqrt{N-1} \sigma_N} = \frac{T}{\sqrt{N-1} \sqrt{r^2 + 1/N}}. \end{aligned} \quad (39)$$

The fundamental parameters upon which P_D depends are

- N, number of normalizer samples;
 - T, threshold at normalizer output;
 - d, deflection criterion (30);
 - r, scaling (30) .
- (40)

However, they show up, in integral result (38), collapsed into the three variables N , d'_r , T'_r .

For signal absent, we have $d=0$ and $r=1$, as noted in (31). Then (38) and (39) reduce to the false alarm probability

$$P_F = \int_0^{\infty} dw \frac{w^{N-2} \exp(-w^2/2)}{2^{\frac{N-3}{2}} \Gamma\left(\frac{N-1}{2}\right)} \Phi\left(-T \sqrt{\frac{N}{N^2-1}} w\right), \quad (41)$$

which depends only on N and T . Thus, given a particular number N of normalizer samples, threshold T can be selected to realize a specified value of false alarm probability P_F . This applies for the complete class of log-normal inputs, (13) or (14), into the log-normalizer in figure 1, and can be achieved without knowledge of a or b .

As $N \rightarrow \infty$, we have

$$\hat{\sigma} \rightarrow 1, \quad \sigma_N \rightarrow r \quad \text{as } N \rightarrow \infty, \quad (42)$$

giving from (37),

$$P_D = \Phi\left(\frac{d-T}{r}\right), \quad P_F = \Phi(-T) \quad \text{for } N = \infty. \quad (43)$$

This yields

$$\tilde{\Phi}(P_D) = \frac{1}{r} \left[d + \tilde{\Phi}(P_F) \right] \quad \text{for } N = \infty, \quad (44)$$

where $\tilde{\Phi}$ is the inverse Φ -function; this last result is useful for plotting receiver operating characteristics on normal-probability paper. It illustrates that for $N = \infty$, those curves are straight lines with slope $1/r$ and offset d/r at $P_F = .5$.

The actual numerical evaluations of false alarm probability (41) and detection probability (38) are undertaken in appendix C. The inputs to the functions considered there are the 3 parameters N, T'_r, d'_r as given by (39), rather than the 4 fundamental parameters N, T, d, r listed in (40). This is no limitation, since for any given values of N, T, d, r , the quantities N, T'_r, d'_r can be easily calculated via (39) and used as inputs to the procedures in appendix C.

PERFORMANCE FOR WEIBULL INPUT *

Here we want to evaluate detection probability (6) for the decision variable z given by (28), when normalized random variables $\{v_n\}_1^N$ are independent identically-distributed zero-mean unit-variance log-distorted Weibull variates. Before we do that, we observe that detection probability (6) can be expressed generally as

$$\begin{aligned}
 P_D &= \text{Prob}(z > T) = Q_z(T) = \text{Prob}\left(\frac{d + rv_0 - \hat{\mu}(v)}{\hat{\sigma}(v)} > T\right) = \\
 &= \text{Prob}\left(v_0 > \frac{t - d}{r}\right) = \int du Q_{v_0}\left(\frac{u - d}{r}\right) p_t(u) ,
 \end{aligned} \tag{45}$$

where we used (28) and defined random variable

$$t = \hat{\mu}(v) + T \hat{\sigma}(v) \tag{46}$$

in terms of the sample quantities in (24). The separation of functions in the last form in (45) is due to the fact that random variables v_0 and t are statistically independent of each other. When signal is absent, then $d=0, r=1$ according to (31), and (45) reduces to false alarm probability

$$P_F = \int du Q_{v_0}(u) p_t(u) . \tag{47}$$

In the special case where $N \rightarrow \infty$, that is, a very large number of samples used in the normalizer, the sample quantities in (24) approach the true mean 0 and standard deviation 1 of normalized random variables $\{v_n\}$, and t tends to the constant T . Then (45) and (47) reduce to

$$P_D = Q_{v_0} \left(\frac{T-d}{r} \right), \quad P_F = Q_{v_0}(T) \quad \text{for } N = \infty. \quad (48)$$

This limiting case can be used as a comparison with practical cases where N is large, but not infinite.

We now specialize the above general results to the case of log-distorted Weibull variates $\{v_n\}_1^N$. Although the probability density function of normalized random variable v_0 is arbitrary, we will also take it here to be a normalized log-distorted Weibull variate. In this case, the exceedance distribution function of random variable v_0 is given by (A-17) as

$$Q_{v_0}(u) = \exp \left[-\exp \left(-\gamma + \frac{\pi}{\sqrt{6}} u \right) \right] \quad \text{for all } u, \quad (49)$$

where $\gamma = .57721$ is Euler's constant. Thus, the detection and false alarm probabilities for $N = \infty$, as given generally by (48), are immediately available upon use of (49).

The probability density function of random variable t defined in (46) is a much more difficult task for finite N . To make any analytic progress, we have had to assume that t is Gaussian; this can be expected to be a fair approximation if the number of normalizer samples N entering the sample

quantities in (24) is large, according to the central limit theorem. However, we can anticipate that the approach of random variable t to normality will be faster near its mean, but considerably slower on the tails. This can lead to a significant bias in the calculation of small false alarm probabilities.

Thus, our assumption is that the probability density function of t in (46) is given by

$$p_t(u) = [2\pi \sigma^2(t)]^{-1/2} \exp\left[-\frac{(u - \mu(t))^2}{2 \sigma^2(t)}\right]. \quad (50)$$

Combining (49) and (50) in (45), the detection probability is given (approximately) by

$$\begin{aligned} P_D &= \int du \exp\left[-\exp\left(-\gamma + \frac{\pi}{\sqrt{6}} \frac{u - d}{r}\right)\right] * \\ &\quad * [2\pi \sigma^2(t)]^{-1/2} \exp\left[-\frac{(u - \mu(t))^2}{2 \sigma^2(t)}\right] = \\ &= (2\pi)^{-1/2} \int dx \exp\left[-\frac{x^2}{2} - \exp(h_1 + h_2 x)\right], \quad (51) \end{aligned}$$

where constants

$$h_1 = -\gamma + \frac{\pi}{\sqrt{6}} \frac{\mu(t) - d}{r}, \quad h_2 = \frac{\pi}{\sqrt{6}} \frac{\sigma(t)}{r}. \quad (52)$$

The false alarm probability follows upon setting $d = 0$, $r = 1$ in (52).

The fundamental parameters in integral result (51) are d and r , along with mean $\mu(t)$ and standard deviation $\sigma(t)$ of random variable t . The

complexity of random variable t , defined by (46) and (24), precludes us from evaluating mean $\mu(t)$ and standard deviation $\sigma(t)$ exactly. However, numerous simulations, each consisting of 100,000 trials, enabled us to extract the following rather accurate rules of thumb for the statistics of t .

First of all, for general definition (46), we have mean

$$\mu(t) = \mu\{\hat{\mu}(v)\} + T \mu\{\hat{\sigma}(v)\} , \quad (53)$$

and variance

$$\sigma^2(t) = \sigma^2\{\hat{\mu}(v)\} + T^2 \sigma^2\{\hat{\sigma}(v)\} + 2T \sigma\{\hat{\mu}(v)\} \sigma\{\hat{\sigma}(v)\} \rho , \quad (54)$$

where ρ is the normalized correlation coefficient between $\hat{\mu}(v)$ and $\hat{\sigma}(v)$.

The simulation results alluded to above, for normalized random variables $\{v_n\}$ in (24) being log-distorted Weibull variates, are given by

$$\begin{aligned} \mu\{\hat{\mu}(v)\} &= 0 & \mu\{\hat{\sigma}(v)\} &\cong 1 - \frac{.39}{N} \\ \sigma^2\{\hat{\mu}(v)\} &= \frac{1}{N} & \sigma^2\{\hat{\sigma}(v)\} &\cong \frac{1.05}{N + 1.5} \\ \rho &\cong -.55 . \end{aligned} \quad (55)$$

Observe the large value of ρ , in contrast with the earlier case of a Gaussian input to the normalizer, where the sample mean and sample standard deviation were not only uncorrelated but in fact independent; see appendix B.

The quantities in (55) depend solely on N ; when used in (53) and (54), it is seen that $\mu(t)$ and $\sigma(t)$ in (54) depend on both N and threshold T . Thus, the totality of fundamental parameters of relevance in detection probability (51) is d, r, N, T , just as for the Gaussian case in (40). Analytic evaluation of integral (51) is impossible; accordingly, numerical integration was employed.

An alternative approximation to the statistics of random variable t of (46) is undertaken in appendix D. Namely, for large threshold values T , where random variable t is dominated by sample standard deviation $\hat{\sigma}(v)$, it might be thought that a χ -approximation would have wider applicability than a Gaussian one. This is indeed true, as will be demonstrated by the numerical results to follow; however, for small false alarm probabilities, the χ -approximation also falls short.

GRAPHICAL RESULTS

In this section, we will present graphical results for the exceedance distribution functions and receiver operating characteristics for both the Gaussian and the log-distorted Weibull inputs to the normalizer in figure 1. This corresponds, respectively, to log-normal and Weibull inputs (that is, detector outputs) to the logarithmic device in figure 1. The theoretical results of the previous two sections are augmented by simulation results, each based upon $2^{23} = 8.4$ million trials of decision variable (normalizer output) z given by (28) and (24).

The scaling parameter r , that is, the ratio of standard deviations in (30), is taken at 1 for all these results, in order to keep the number of plots at a reasonable level. The deflection parameter d in (30) is varied from 0 to values large enough to sweep out the important range of detection and false alarm probabilities of interest. The number of normalizer samples, N , is taken at the values ∞ , 64, 32, 16, which appears to cover the most important range of practical use. Threshold value T in exceedance probabilities (6) is allowed to vary widely, so that the full range of detection and false alarm probabilities can be observed.

GAUSSIAN INPUT TO NORMALIZER

The exceedance distribution function (EDF), defined in (6), for $N=\infty$ is plotted in figure 2 versus threshold T , for deflection parameter d taking on values

$$d = 0(1)9 = 0,1,2,3,4,5,6,7,8,9 . \quad (56)$$

The arrow on the figure indicates the direction of increasing d ; thus $d=0$, which is the false alarm probability, corresponds to the curve at the lower left. The results in figure 2 are based on (43); since the ordinate is according to a normal probability rule, these curves are perfectly straight lines. Exceedance probabilities ranging from $1E-6$ to $.999999$ are covered when threshold T is varied over the range $-5,5$.

When N takes on the values 64, 32, 16, the corresponding results are displayed in figures 3, 4, 5, respectively. These graphs were obtained from (38)-(41), implemented by the procedures in appendix C. The exceedance distribution function for $N=64$ in figure 3 is fairly close to that for $N=\infty$ in figure 2; however, by the time the number of normalizer samples N has decreased to 16 in figure 5, significant curvature has developed in the results.

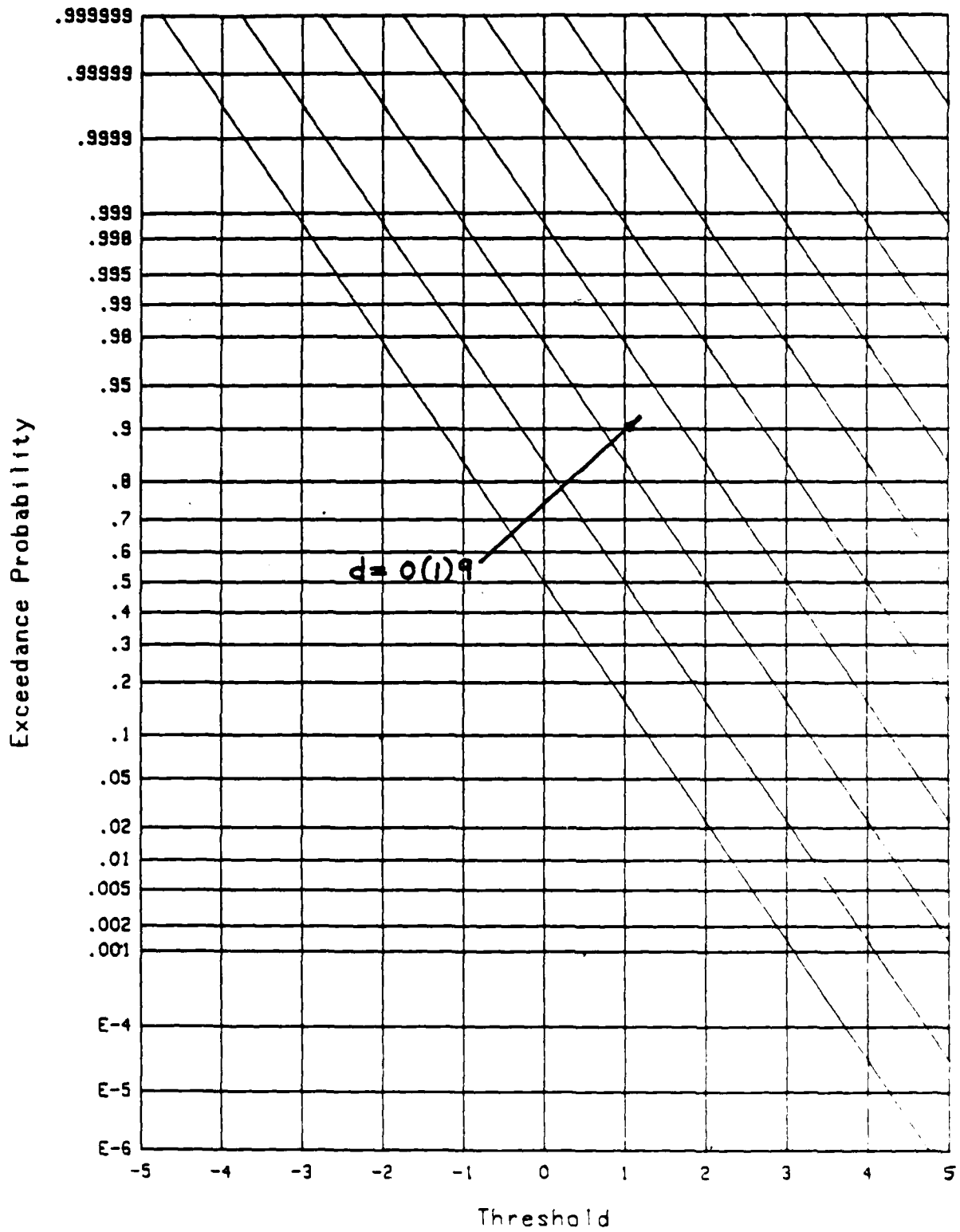


Figure 2. EDF for $N = \infty$; Gaussian

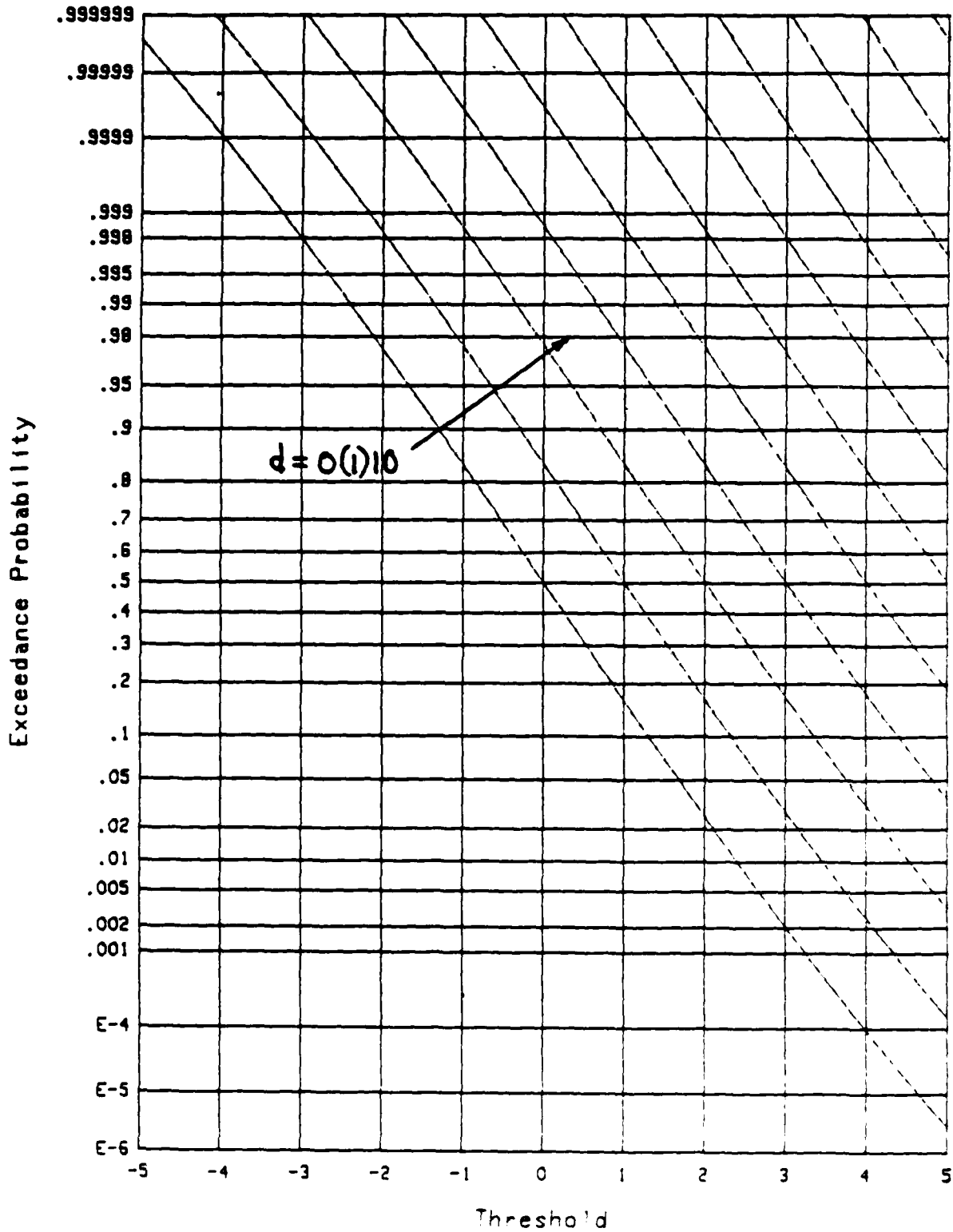


Figure 3. EDF for $N = 64$; Gaussian

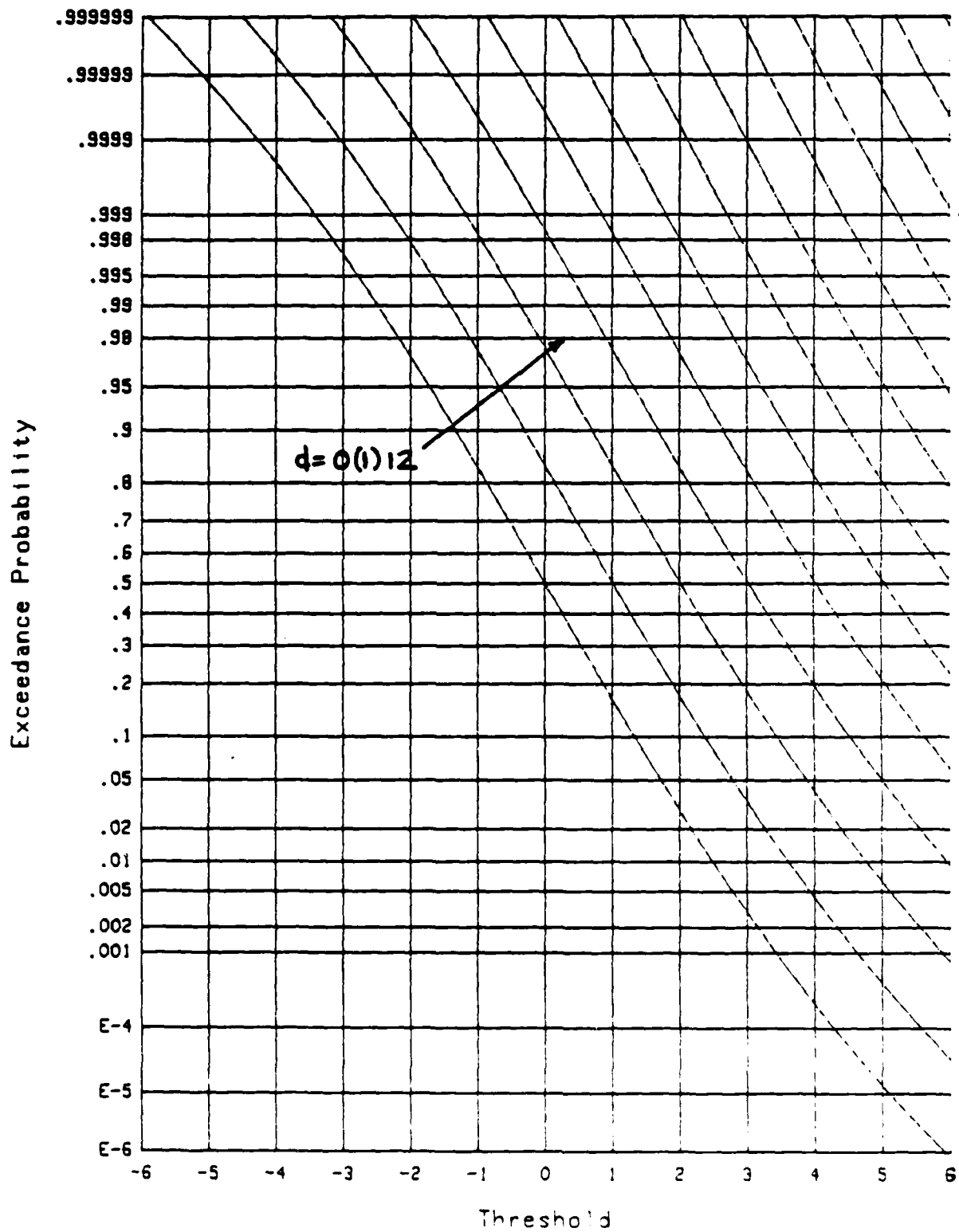


Figure 4. EDF for N = 32; Gaussian

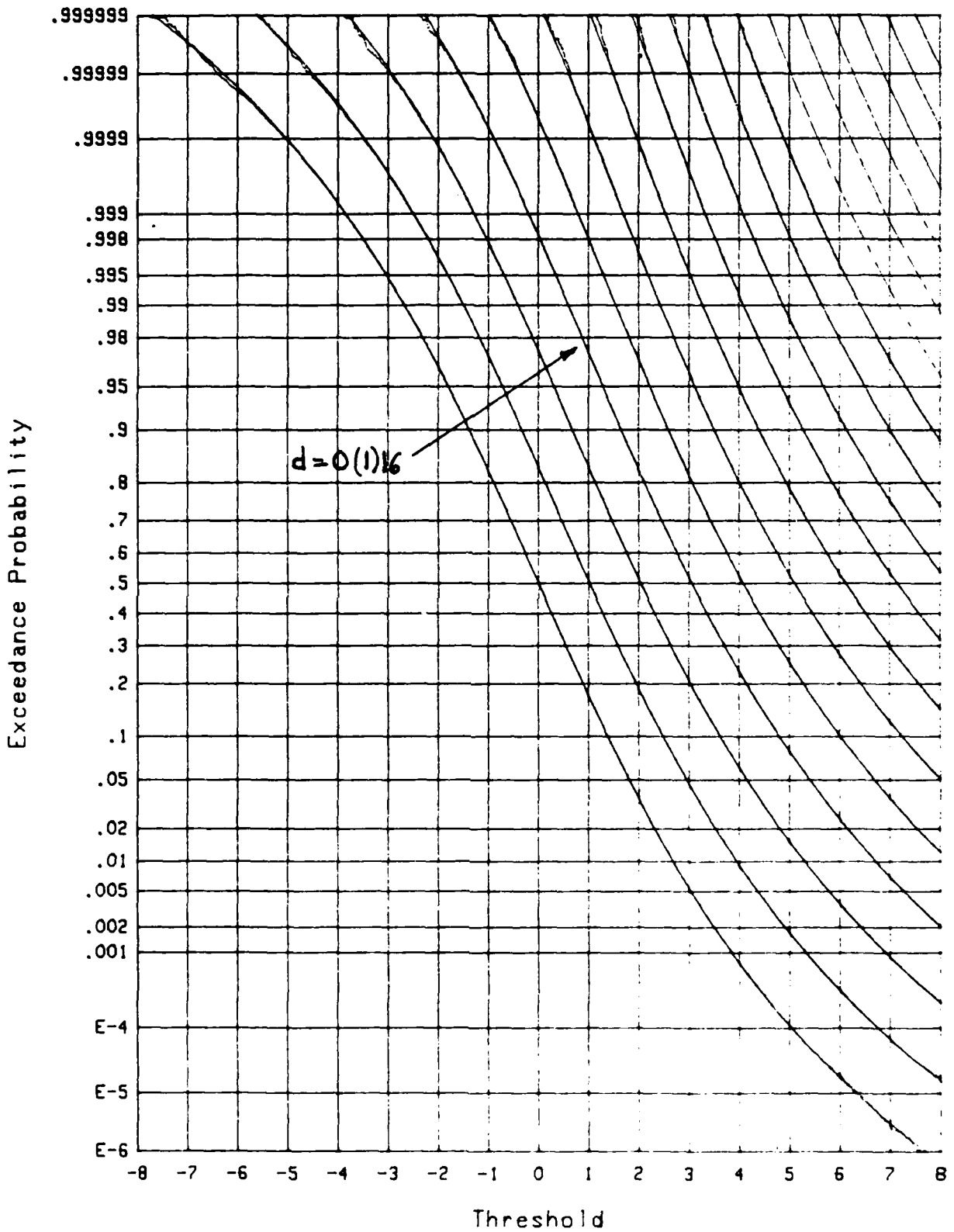


Figure 5. EDF for N = 16; Gaussian

Superposed in figure 5 are 11 simulation results for $d=0(1)10$, each based upon 8.4 million independent trials. Due to the large number of trials, the theoretical and simulation results are indistinguishable, except near the extremes of probability $1E-6$ and $.999999$, where the jagged character of simulation results is manifested. This close agreement of results not only confirms the theoretical analysis but also lends credence to the use of simulation for the estimation of probabilities out on the tails of the distribution, provided that enough trials are conducted.

Figures 3, 4, 5 furnish information which enables the selection of the required threshold T to realize a specified false alarm probability for $N = 64, 32, 16$, respectively. For example, figure 5 with $d = 0$ indicates that to realize a false alarm probability of $1E-5$ for $N = 16$, threshold T in (6) must be chosen as 6.3.

When threshold T is eliminated, and the detection probability plotted versus the false alarm probability, we obtain the receiver operating characteristics (ROC). The result for $N=\infty$ is given in figure 6, where both the abscissa and ordinate are plotted according to a normal probability scale. Deflection parameter d varies over the range

$$d = 0(.5)7.5 = 0, .5, 1, 1.5, \dots, 7, 7.5 . \quad (57)$$

The arrow again points in the direction of increasing d ; thus $d=0$ is the curve on the lower right. These curves are precisely the straight lines indicated by (44) with $r=1$.

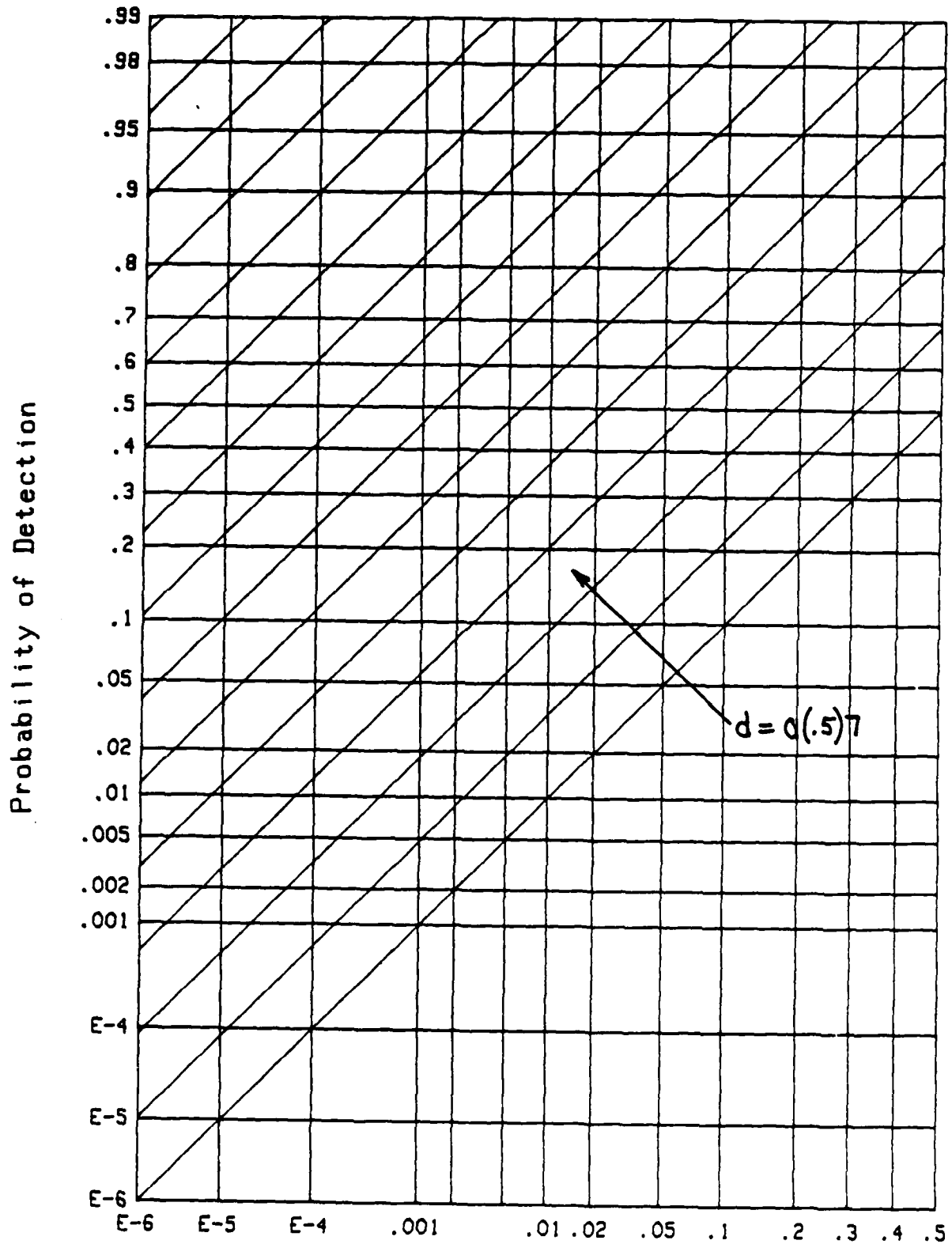


Figure 6. Probability of False Alarm
ROC for $N = \infty$; Gaussian

Corresponding receiver operating characteristics for $N = 64, 32, 16$ are presented in figures 7, 8, 9. The detection and false alarm probabilities both range down to $1E-6$, while the upper limits have been truncated at .99 and .5, respectively. Values beyond these limits can be obtained from the earlier figures 2 through 5.

Superposed in figure 9 are ten simulation results for $d = 1(1)10$. Again, except for the small probability regions like $P_F < 1E-5$, the theoretical and simulation results are indistinguishable and overlay each other. It will be noticed that a characteristic wiggle in the receiver operating characteristics is duplicated for every simulation result, at a constant value of false alarm probability; for example, see the triangular bump in all 10 simulation results at $P_F \approx 1E-6$. The reason for this behavior is that when random variable z in (28) was simulated, the random numbers employed in (24) to generate $\hat{\mu}$ and $\hat{\sigma}$ were not changed when different d values were considered in (28). The reason for this deliberate choice was economy of computer execution time; that is, the time-consuming task of computation of (24) was done once for each trial, and used in (28) for all of the d values of interest. This repeated use of the same $\hat{\mu}, \hat{\sigma}$ values for different d values gives a persistent systematic perturbation to the estimated receiver operating characteristics at a fixed false alarm probability. However, for 8.4 million trials, this bias is small, even for the rare events with probabilities greater than $1E-6$, and was deemed acceptable in light of the greatly increased computer time required for the alternative approach of regeneration of $\hat{\mu}$ and $\hat{\sigma}$.

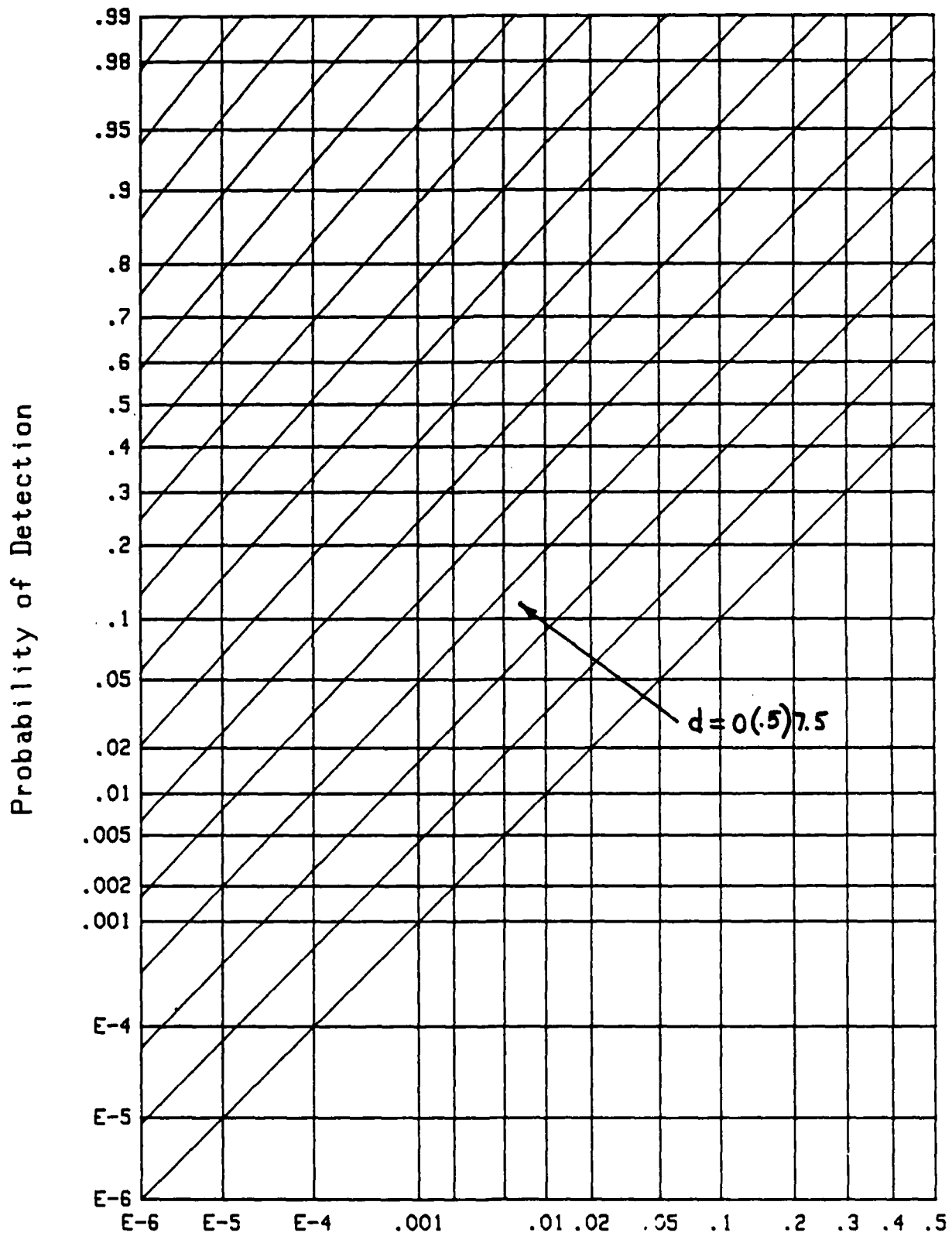


Figure 7. Probability of False Alarm
 ROC for N = 64; Gaussian

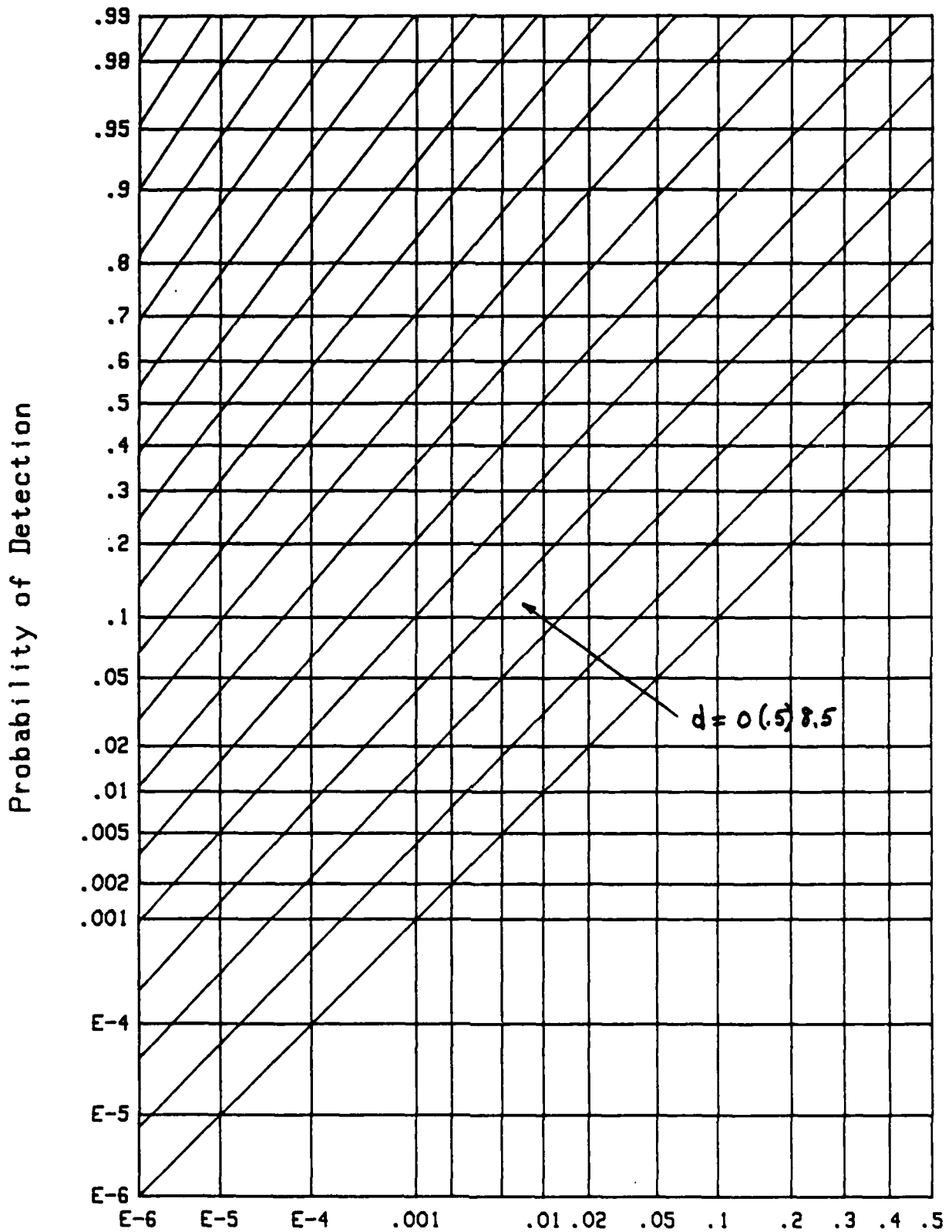


Figure 8. Probability of False Alarm
ROC for $N = 32$; Gaussian

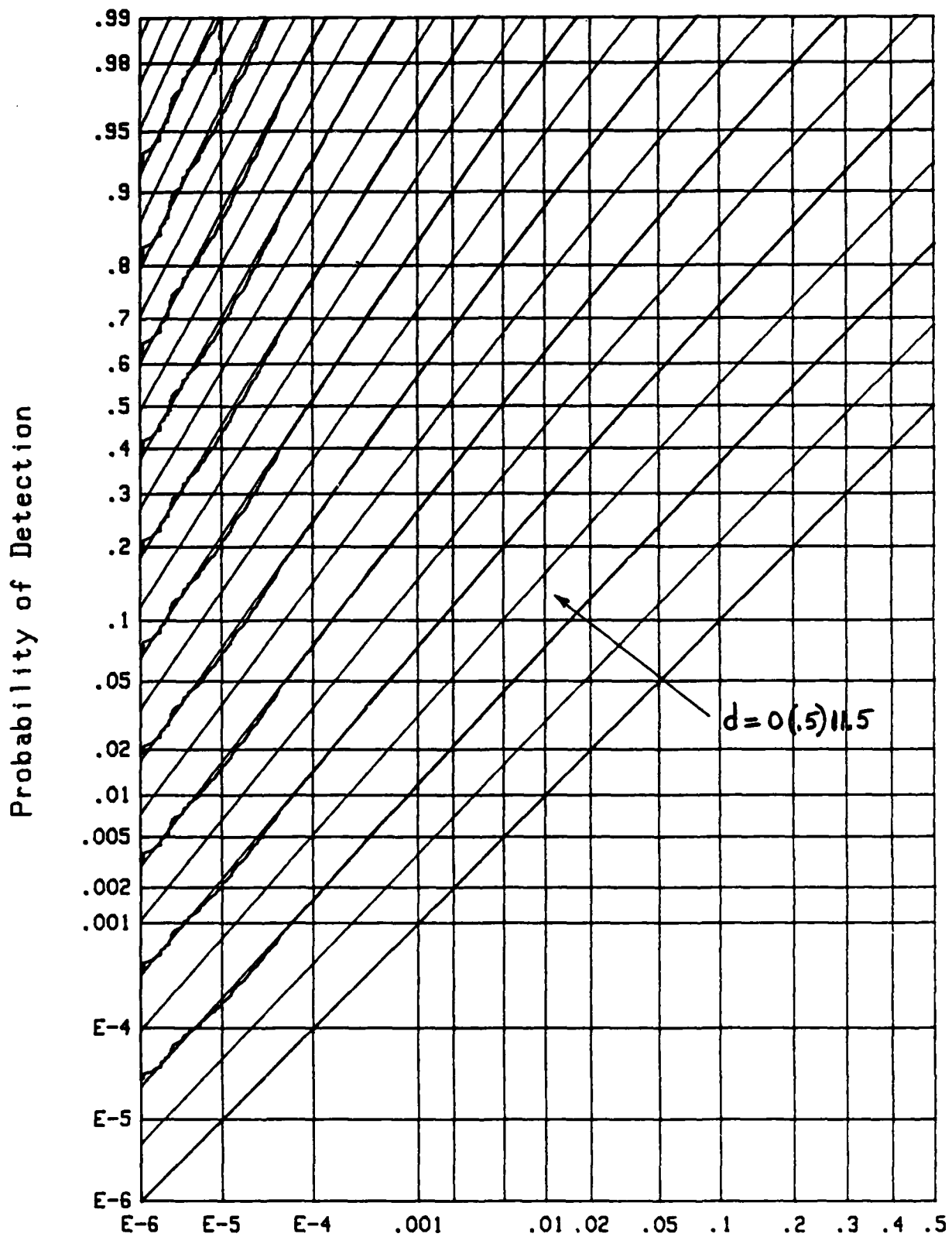


Figure 9. Probability of False Alarm
ROC for N = 16; Gaussian

As an example of the use of figures 6 through 9, the values of deflection d required to realize $P_F = 1E-5$ and $P_D = .5$ are

$$d = 4.3, 4.7, 5.1, 6.2 \quad \text{for } N = \infty, 64, 32, 16, \quad (58)$$

respectively. The cost of reducing N from ∞ to 16 is that d must be increased by the factor $6.2/4.3 = 1.44$; whether this is tolerable depends on the application. The relation of deflection parameter d to any system input signal-to-noise ratio depends on the particular processor form preceding the logarithmic device in figure 1, and must be left to the user and his particular application.

LOG-WEIBULL INPUT TO NORMALIZER

When the input to the normalizer is a log-distorted Weibull variate, the performance is markedly different. The exceedance distribution function for $N = \infty$ is displayed in figure 10 and has a significant curvature when plotted on normal probability paper; these results are based upon the use of (48) and (49). The use of the notation 'Extreme' is explained in (A-7) et seq.

When N is decreased to 64, the corresponding exceedance distribution functions are given in figure 11. Due to the questionable assumptions required in the theoretical analysis of this case and used in (50) et seq., simulation results were also superposed for the values $d = 0(.5)5$. Agreement in the mid-range of probabilities is excellent. At the low end of the probability range, near $1E-6$, the simulation results indicate a systematically lower exceedance probability than predicted by theory.

This erratic trend of the theoretical approximation is continued and accented in figure 12 for $N=32$ and in figure 13 for $N=16$. In fact, in the latter case, for threshold $T=5$, the simulation indicates exceedance probabilities for $d=0$ that are more than 2 orders of magnitude smaller than the theory predicts; see bottom right of figure 13. The discrepancies at the high end of probabilities are also considerable, as seen at the top left of the figure.

Also added to this particular figure is the result of using the χ -approximation for random variable t of (46), as detailed in appendix D. Although the improvement in probability values is over an order of magnitude, there is still another order of magnitude error left in this alternative approximate approach. The reason for the difficulty in the theoretical analysis is two-fold: (1) values of N like 16 or 32 are not large enough for the central limit theorem to have developed substantial accuracy on the tails; (2) the probability density function of a log-distorted Weibull variate, as given by (A-7), is distinctly non-Gaussian on the tails. The decay of (A-7) on the positive tail is much faster than Gaussian, while that on the negative tail is slower, being only exponential.

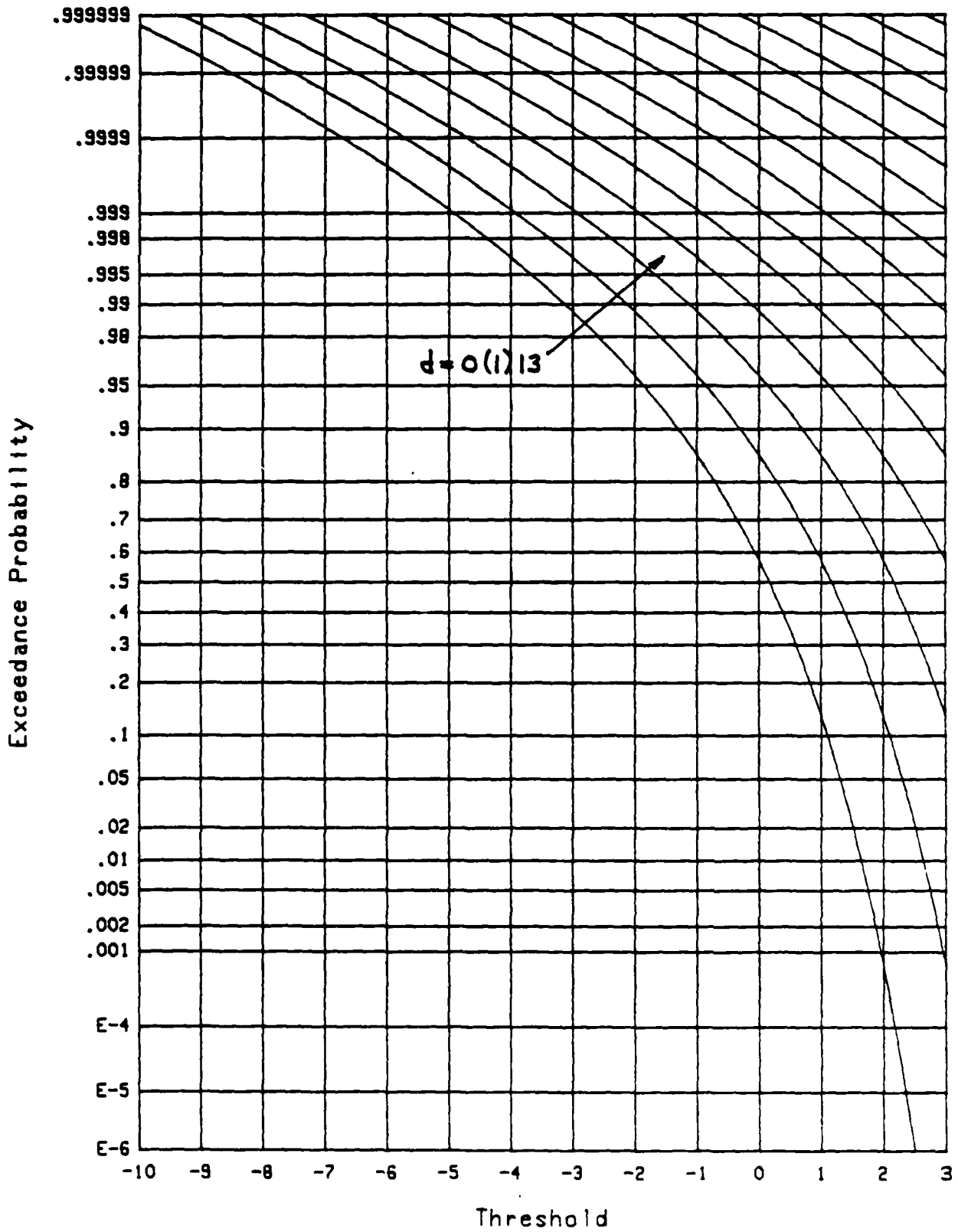


Figure 10. EDF for $N = \infty$; Extreme

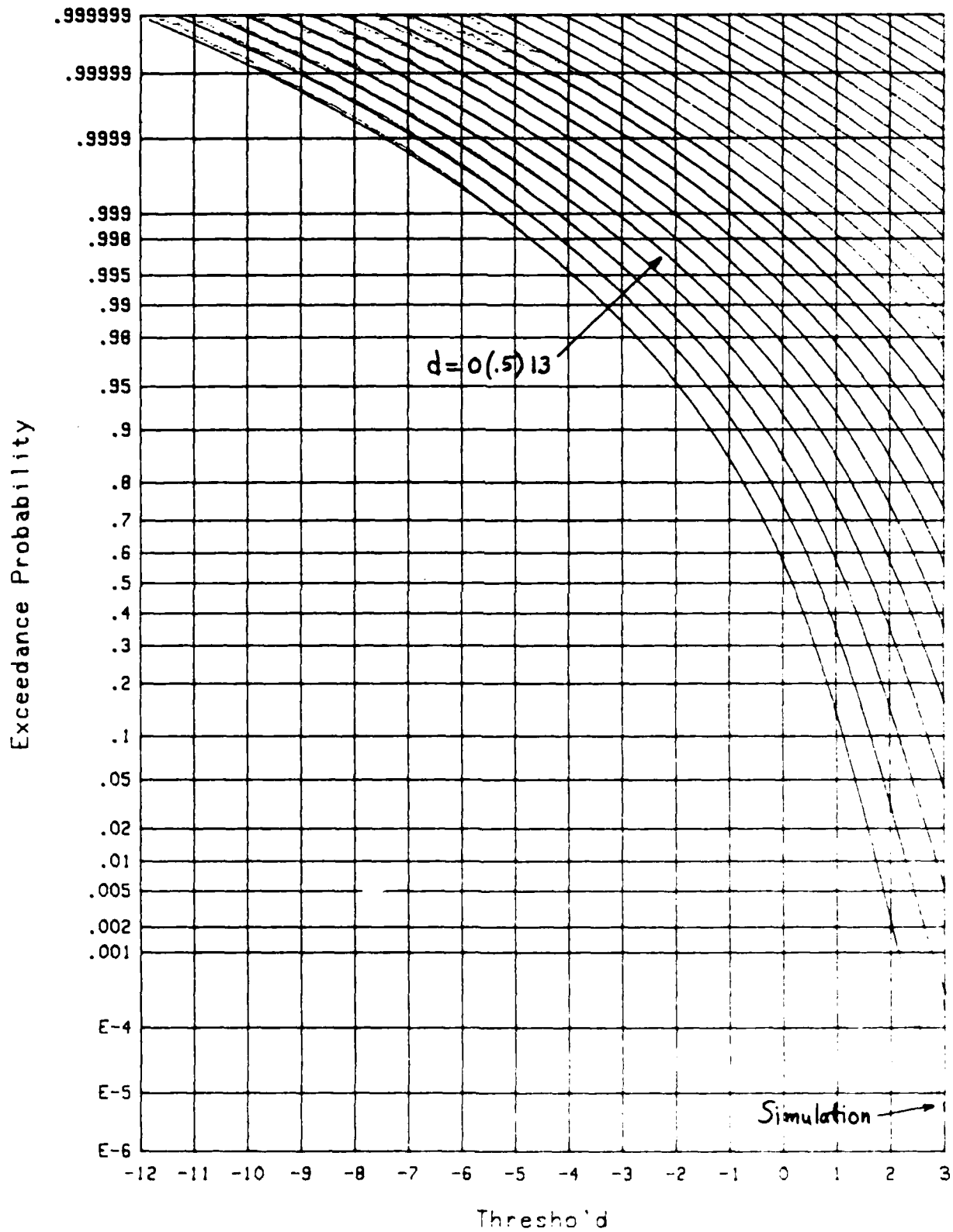


Figure 11. EDF for N = 64; Extreme

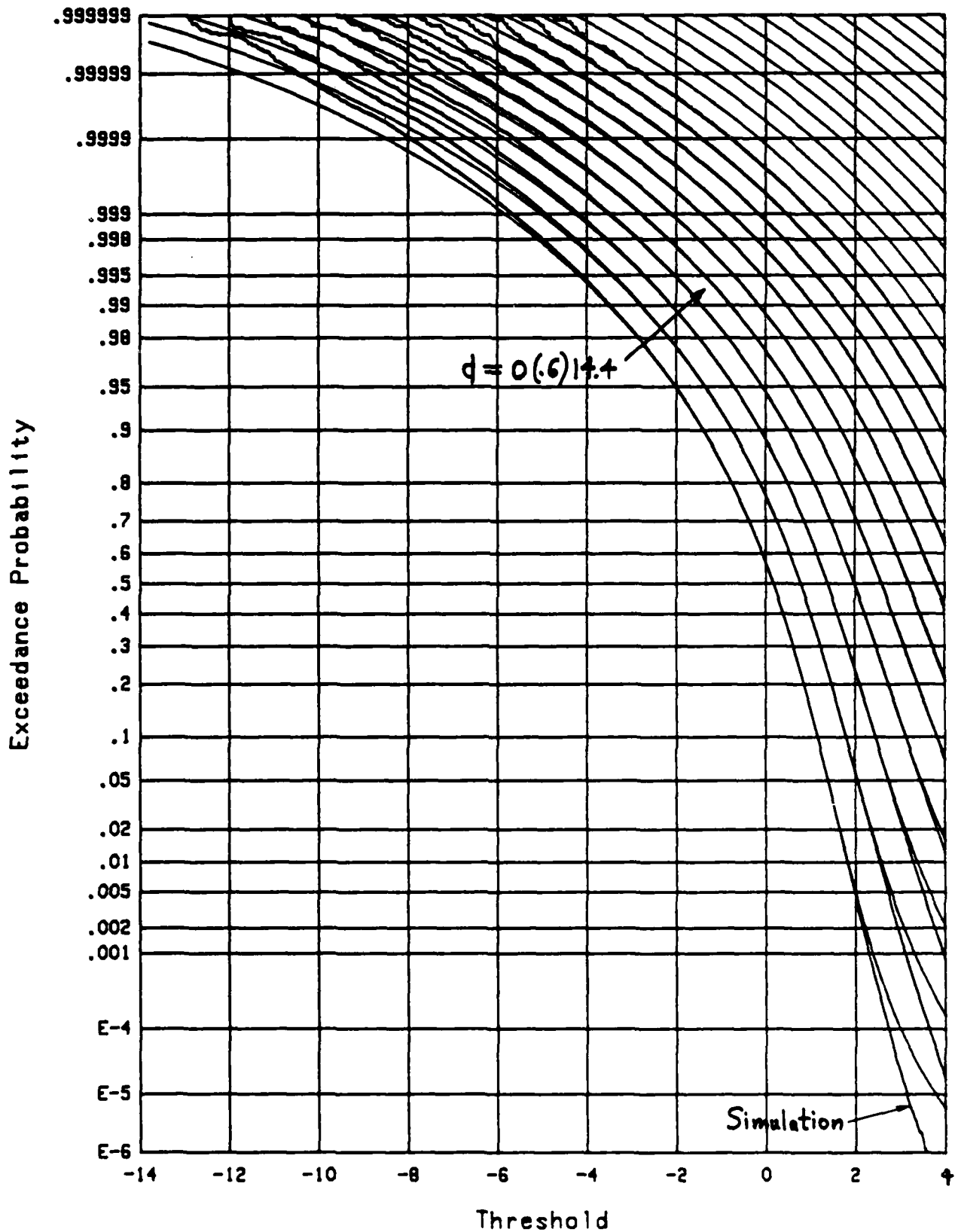


Figure 12. EDF for $N = 32$; Extreme

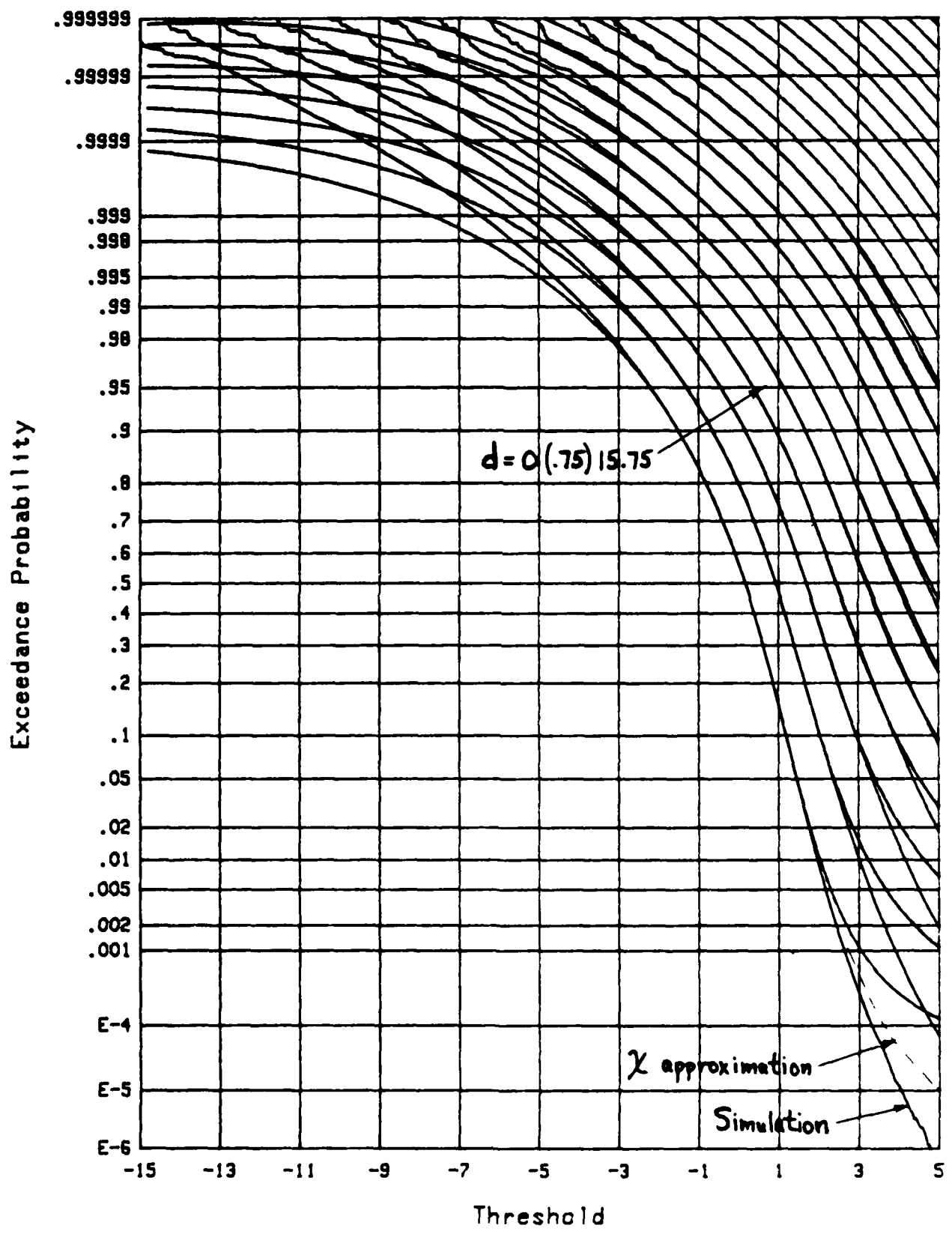


Figure 13. EDF for N = 16; Extreme

The receiver operating characteristics for $N = \infty$ are given in figure 14, while those for $N = 64, 32, 16$ are given in figures 15 through 17, respectively. The discrepancy between theory and simulation becomes progressively larger as N decreases, reaching the point in figure 17 where the theory is entirely invalid for false alarm probabilities less than approximately .001. The reason for the severe dip of the theoretical curves to the left of each figure is the inadequacy of the false alarm probability approximation, it being much too large for the larger threshold values; see bottom right of figure 13. On the other hand, the simulation results in these figures are all based on 8.4 million independent trials, making them trustworthy well down near the $1E-6$ level of probability plotted here.

As an example of the use of figures 14 through 17, the values of d required to realize probabilities $P_F = 1E-5$ and $P_D = .5$ are

$$d = 2.2, 2.6, 2.9, 3.8 \quad \text{for } N = \infty, 64, 32, 16, \quad (59)$$

respectively. The latter three values are extracted from the simulation results in figures 15 through 17. Direct comparison of the absolute levels in (59) with the corresponding Gaussian results in (58) is not valid, because the shapes of the input probability density functions in the two cases are markedly different and are more important than the deflection criterion, defined by (30).

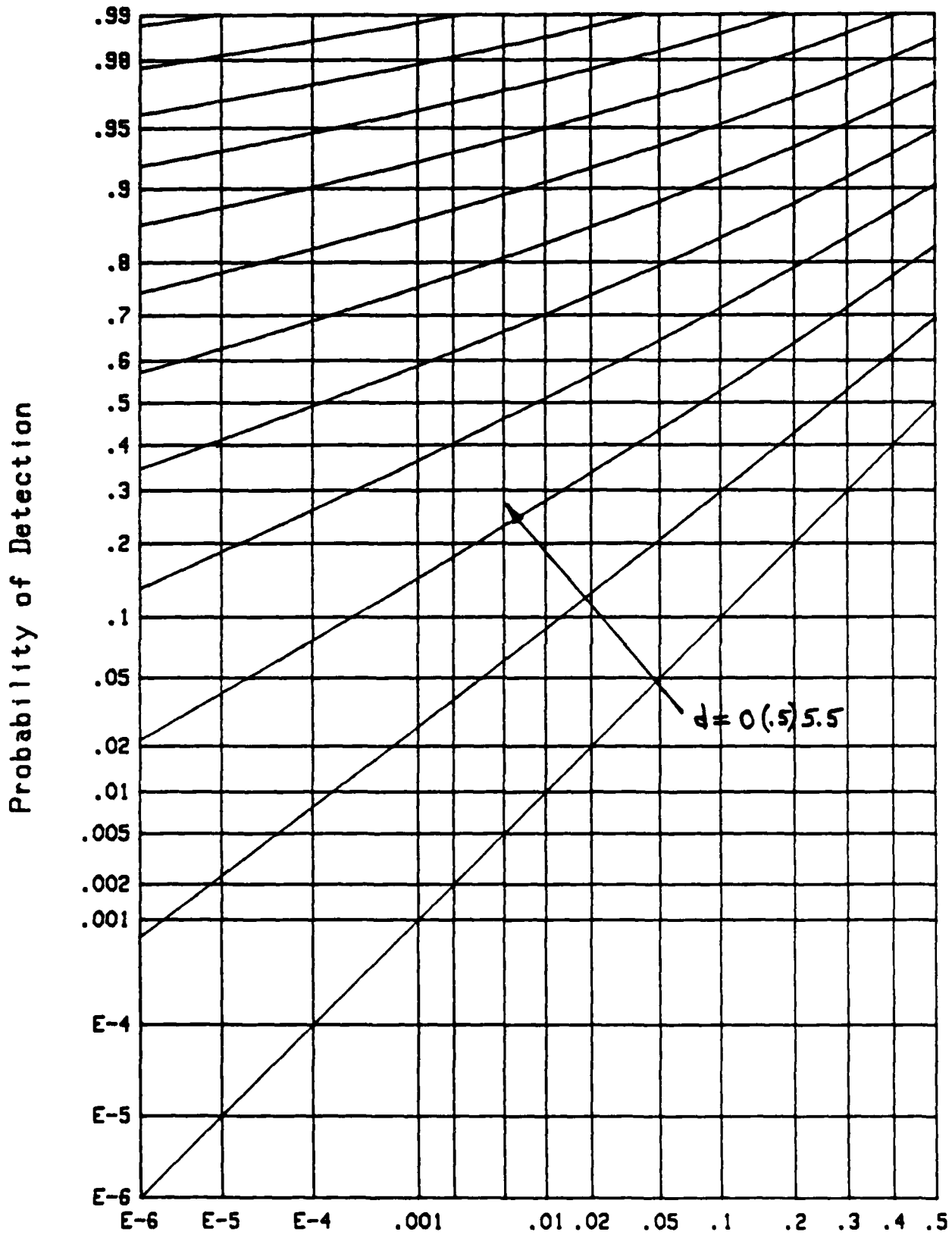


Figure 14. Probability of False Alarm
ROC for $N = \infty$; Extreme

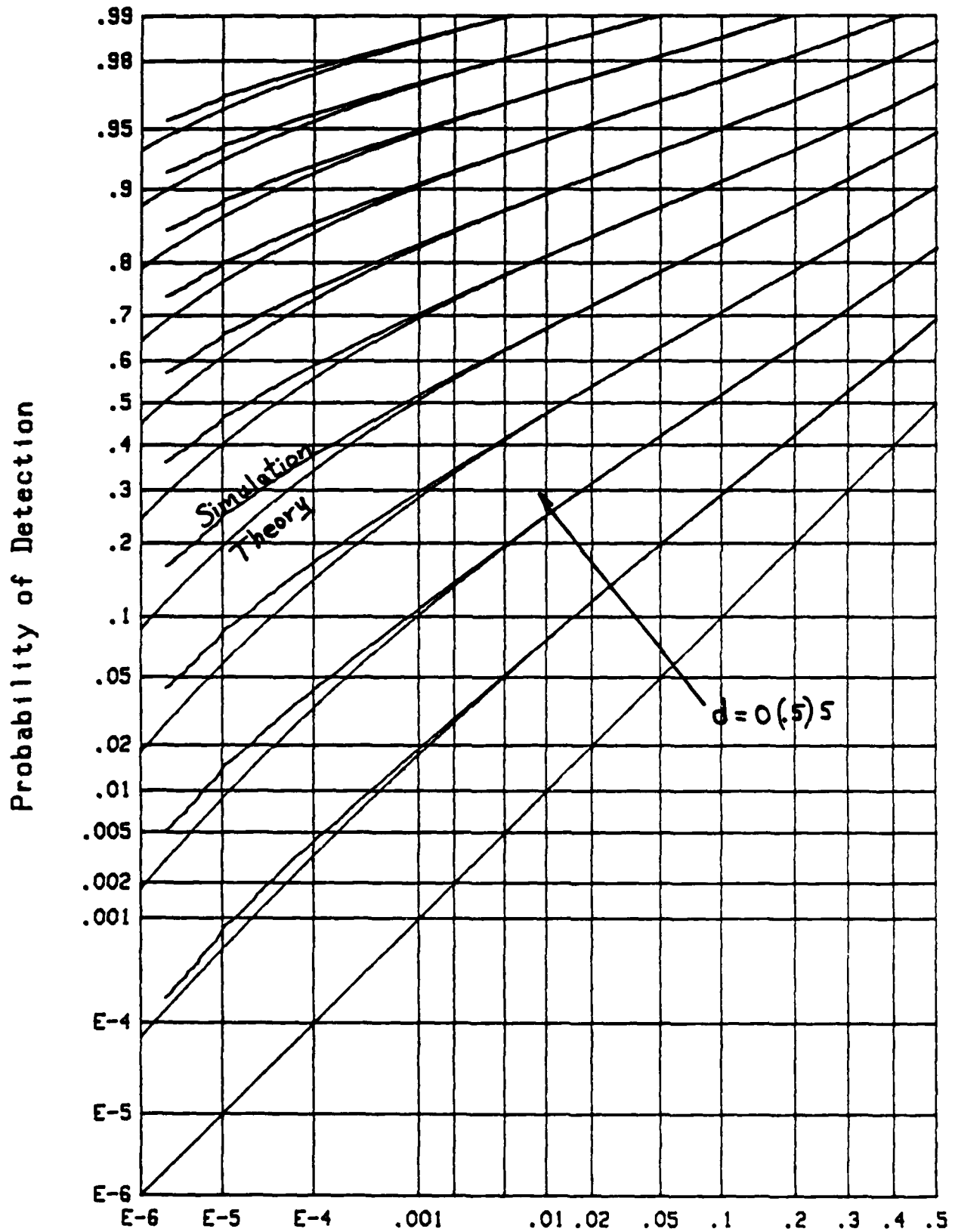


Figure 15. Probability of False Alarm
ROC for $N = 64$; Extreme

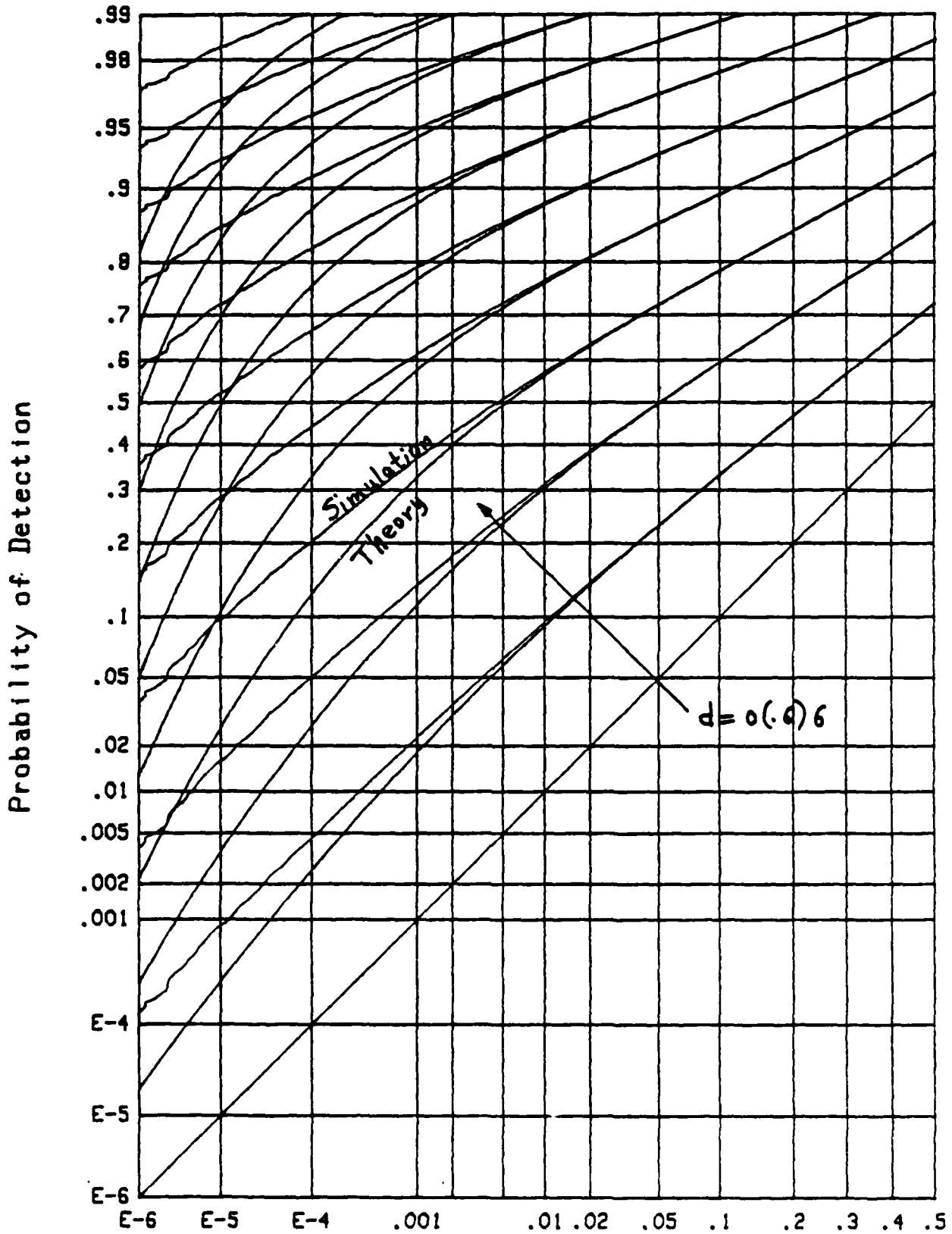


Figure 16. Probability of False Alarm
ROC for N = 32; Extreme

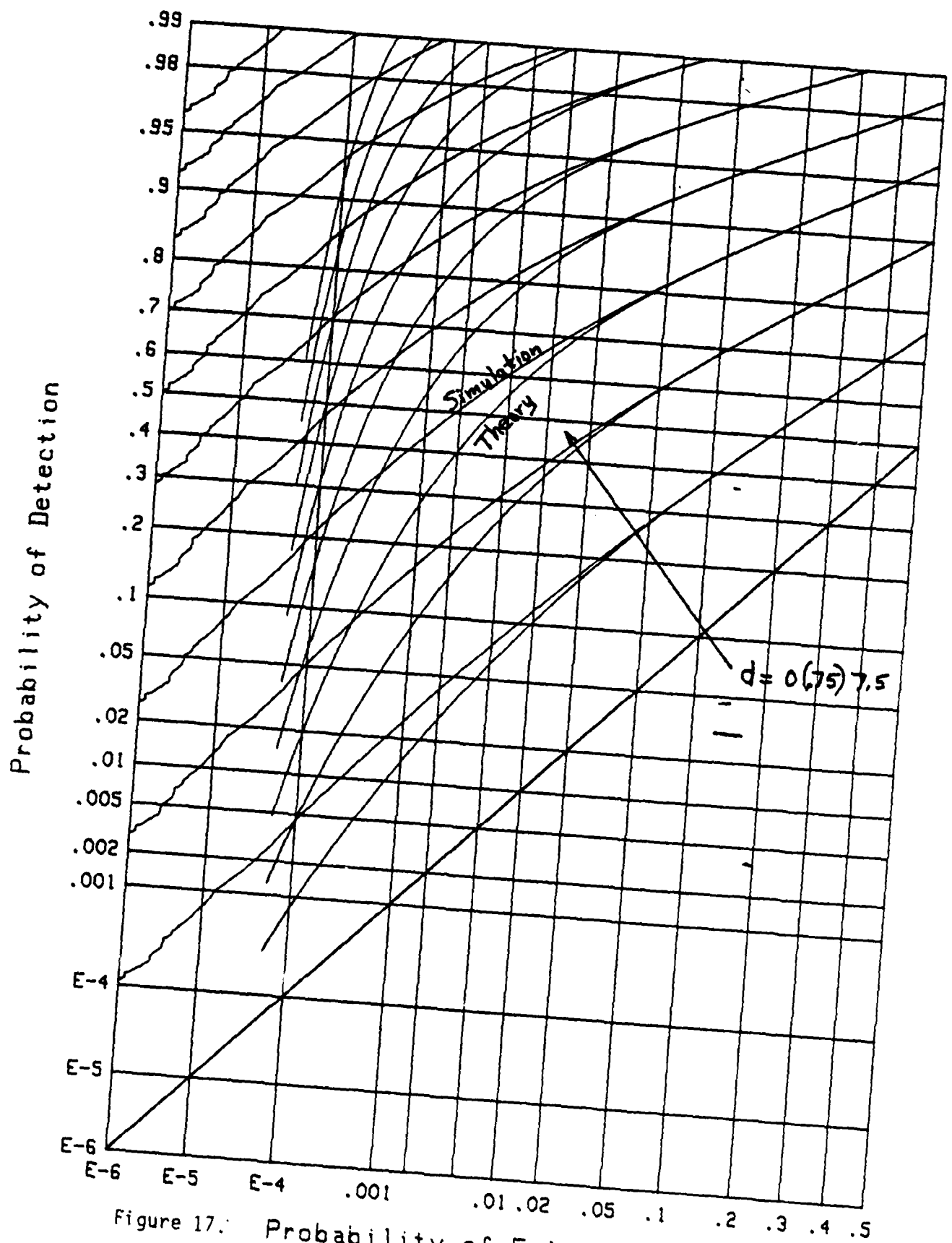


Figure 17. Probability of False Alarm
ROC for N = 16; Extreme

CONCLUSION

The performance of the normalizer with a Gaussian input is capable of exact analysis in terms of integrals which are readily evaluated via recursions. The main reason that this fortuitous situation obtains is the statistical independence of the sample mean and sample standard deviation for Gaussian random variables. However, for other inputs to the normalizer, these sample statistics are highly correlated with each other and create an untractable analysis problem.

An acceptable alternative in this latter case is simulation with a large number of trials. Here 8.4 million trials were employed, which allowed for estimation of tail probabilities in the $1E-6$ range. If the false alarm probability could be evaluated theoretically, then simulation would only need to be conducted for the detection probability P_D . And if P_D were of interest only in the range (.5, .99) say, then as few as 10,000 trials would suffice for a decent estimate. However, it appears that, in general, even the analysis for the false alarm probability involves some unmanageable statistical relations.

APPENDIX A. WEIBULL VARIATES

The exceedance distribution function of a general Weibull random variable x is given by [8; p. 52]

$$Q_x(u) = \text{Prob}(x > u) = \exp\left[-\left(\frac{u}{a}\right)^{1/b}\right] \quad \text{for } u > 0; a > 0, b > 0. \quad (\text{A-1})$$

The corresponding probability density function of x is

$$p_x(u) = -Q'_x(u) = \frac{u^{\frac{1}{b}-1}}{a^{\frac{1}{b}} b} \exp\left[-\left(\frac{u}{a}\right)^{1/b}\right] \quad \text{for } u > 0. \quad (\text{A-2})$$

The ν -th moment of x is

$$\overline{x^\nu} = \int du u^\nu p_x(u) = a^\nu \Gamma(1 + b\nu) \quad \text{for } \nu > -1/b. \quad (\text{A-3})$$

The characteristic function of x is not available in closed form, for general b ; however

$$\overline{\exp(i\xi x)} = (1 - i\xi a)^{-1} \quad \text{for } b = 1. \quad (\text{A-4})$$

The normalized cumulants of x , for general b , are independent of a ; however, they do not approach zero as either $b \rightarrow 0$ or $b \rightarrow \infty$. Therefore x does not tend to Gaussian as the shape parameter b is changed.

LOG-DISTORTED WEIBULL VARIATE

As indicated in (1) and (15), we are interested in the log-distorted random variable

$$y = \ln x , \quad (A-5)$$

where x is a Weibull variate with probability density function (A-2). The exceedance distribution function of y is

$$\begin{aligned} Q_y(u) &= \text{Prob}(y > u) = \text{Prob}(\ln x > u) = \text{Prob}(x > \exp(u)) = \\ &= Q_x(\exp(u)) = \exp \left[- \frac{\exp(u/b)}{a^{1/b}} \right] \quad \text{for all } u , \end{aligned} \quad (A-6)$$

where (A-1) was employed. The corresponding probability density function of random variable y is

$$p_y(u) = -Q'_y(u) = \frac{1}{b a^{1/b}} \exp \left[\frac{u}{b} - \frac{\exp(u/b)}{a^{1/b}} \right] \quad \text{for all } u , \quad (A-7)$$

which is a form of the probability density function for extreme values; see [4; (14.65)]. We will refer to (A-7) as an extreme value probability density function here.

The characteristic function of random variable y is

$$\begin{aligned} f_y(i\xi) &= \overline{\exp(i\xi y)} = \overline{\exp(i\xi \ln x)} = \overline{x^{i\xi}} = \\ &= a^{i\xi} \Gamma(1 + i\xi b), \end{aligned} \quad (\text{A-8})$$

the last step by use of (A-3); this is a generalization of [4; page 344, exercise 14.4]. The actual numerical evaluation of the characteristic function in (A-8) for real ξ is best accomplished by employing (A-7):

$$\begin{aligned} f_y(i\xi) &= \overline{\exp(i\xi y)} = \int du \exp(i\xi u) p_y(u) = \\ &= \frac{1}{b} \frac{1}{a^{1/b}} \int_{-\infty}^{+\infty} du \exp(i\xi u) \exp\left[\frac{u}{b} - \frac{\exp(u/b)}{a^{1/b}}\right]. \end{aligned} \quad (\text{A-9})$$

This can be efficiently and accurately evaluated by use of a fast Fourier transform; the integrand decays very rapidly as $u \rightarrow \pm \infty$.

In anticipation of getting the cumulants of random variable y , we have from (A-8),

$$\ln f_y(i\xi) = i\xi \ln a + \ln \Gamma(1 + i\xi b). \quad (\text{A-10})$$

Now from [5: (6.1.33) and section 23.2],

$$\ln \Gamma(1+z) = -\gamma z + \sum_{n=2}^{\infty} (-1)^n \mathfrak{J}(n) z^n/n, \quad (\text{A-11})$$

where $\gamma = .57721$ is Euler's constant and

$$\mathfrak{J}(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}. \quad (\text{A-12})$$

In particular, $\mathfrak{J}(2) = \pi^2/6$.

There then follows, from (A-10) and (A-11), the cumulants of random variable y as

$$\chi_y(n) = \left\{ \begin{array}{ll} \ln a - b\gamma & \text{for } n=1 \\ (-1)^n \mathfrak{J}(n)(n-1)! b^n & \text{for } n \geq 2 \end{array} \right\}. \quad (\text{A-13})$$

In particular, the variance of y is $\chi_y(2) = b^2 \pi^2/6$. For $n \geq 2$, the normalized cumulant of y is

$$\frac{\chi_y(n)}{[\chi_y(2)]^{n/2}} = \left(-\frac{\sqrt{6}}{\pi}\right)^n \mathfrak{J}(n)(n-1)!, \quad (\text{A-14})$$

which is independent of both a and b ; thus random variable y does not approach Gaussian as a and/or b approach any limits whatsoever. These results generalize [4; page 344, exercise 14.4].

NORMALIZED LOG-DISTORTED WEIBULL VARIATE

If $a=b=1$ in (7), then $x_n = w_n$, and it then follows from (16) and (1) that $\tilde{v}_n = \ln w_n = \ln x_n = y_n$. In this case, we can use (A-6) with $a=b=1$ to obtain the exceedance distribution function of \tilde{v}_n as

$$Q_{\tilde{v}}(u) = \exp[-\exp(u)] \quad \text{for all } u. \quad (\text{A-15})$$

Additionally, there follows from (A-13)

$$\mu(\tilde{v}) = \chi_y(1) = -\gamma = -.57721,$$

$$\sigma(\tilde{v}) = (\chi_y(2))^{1/2} = \pi/\sqrt{6}. \quad (\text{A-16})$$

We are now in position to determine the exceedance distribution function of the normalized log-distorted Weibull random variable v_n defined in (17), namely,

$$\begin{aligned} Q_v(u) &= \text{Prob}(v > u) = \text{Prob}(\tilde{v} > \mu(\tilde{v}) + \sigma(\tilde{v})u) = \\ &= Q_{\tilde{v}}(\mu(\tilde{v}) + \sigma(\tilde{v})u) = \exp\left[-\exp\left(-\gamma + \frac{\pi}{\sqrt{6}} u\right)\right] \quad \text{for all } u. \end{aligned} \quad (\text{A-17})$$

LOG-NORMAL VARIATES

For completeness, we list here the v -th moment of log-normal variate x with probability density function as given by (14) and (12):

$$\begin{aligned} \overline{x^v} &= \int du u^v p_x(u) = \frac{1}{b} \int_0^{\infty} \frac{du}{u} u^v \exp\left(\frac{\ln(a) - \ln(u)}{b}\right) = \\ &= \frac{1}{\sqrt{2\pi} b} \int_{-\infty}^{+\infty} dt \exp\left[vt - \frac{1}{2} \left(\frac{t - \ln(a)}{b}\right)^2\right] = \exp\left[v \ln a + \frac{1}{2} v^2 b^2\right] \end{aligned} \quad (\text{A-18})$$

APPENDIX B. INDEPENDENCE OF SAMPLE MEAN AND SAMPLE VARIANCE
FOR GAUSSIAN RANDOM VARIABLES

Let $\{x_n\}_1^N$ be independent identically-distributed Gaussian random variables with mean and variance

$$\overline{x_n} = \mu, \quad \overline{(x_n - \mu)^2} = \sigma^2 \quad \text{for all } n. \quad (\text{B-1})$$

Define sample mean

$$m = \frac{1}{N} \sum_{n=1}^N x_n, \quad (\text{B-2})$$

and sample variance

$$v = g \sum_{n=1}^N (x_n - m)^2, \quad (\text{B-3})$$

where scale factor $g = 1/N$ or $1/(N-1)$ typically. ($N \geq 2$ required.)

We have, in vector notation

$$m = \frac{1}{N} \mathbf{1}^T \mathbf{x} = \frac{1}{N} [1 \ 1 \ \dots \ 1] [x_1 \ x_2 \ \dots \ x_N]^T, \quad (\text{B-4})$$

and

$$v = g \left[\sum_{n=1}^N x_n^2 - m^2 N \right] = g \left[\mathbf{x}^T \mathbf{x} - \frac{1}{N} \mathbf{x}^T \mathbf{1} \mathbf{1}^T \mathbf{x} \right] = g \mathbf{x}^T \mathbf{Q} \mathbf{x}, \quad (\text{B-5})$$

where

$$Q = I - \frac{1}{N} \mathbf{1} \mathbf{1}^T. \quad (\text{B-6})$$

The joint characteristic function of m and v is

$$\begin{aligned} f(\xi, \theta) &= \overline{\exp(i\xi m + i\theta v)} = \\ &= \overline{\exp\left[i\xi \frac{1}{N} \mathbf{1}^T X + i\theta g X^T Q X\right]}. \end{aligned} \quad (\text{B-7})$$

Now the joint probability density function of vector X is

$$\begin{aligned} p(X) &= \prod_{n=1}^N (\sqrt{2\pi} \sigma)^{-1} \exp\left[-\frac{(x_n - \mu)^2}{2\sigma^2}\right] = \\ &= (2\pi \sigma^2)^{-N/2} \exp\left[-\frac{1}{2\sigma^2} (X^T X - 2\mu \mathbf{1}^T X + \mu^2 N)\right]. \end{aligned} \quad (\text{B-8})$$

Therefore

$$\begin{aligned} f(\xi, \theta) &= (2\pi \sigma^2)^{-N/2} \int dX \exp\left[-\frac{1}{2\sigma^2} (X^T X - 2\mu \mathbf{1}^T X + \mu^2 N) + \right. \\ &\quad \left. + i \frac{\xi}{N} \mathbf{1}^T X + i\theta g X^T Q X\right]. \end{aligned} \quad (\text{B-9})$$

Now we use [6; (B-1)]

$$\int dX \exp\left[-\frac{1}{2} X^T M X + L^T X\right] = \left[\frac{(2\pi)^N}{\det M}\right]^{1/2} \exp\left[\frac{1}{2} L^T M^{-1} L\right] \quad (\text{B-10})$$

with identifications

$$M = \frac{1}{\sigma^2} I - i2\theta g Q, \quad L = \left(\frac{\mu}{\sigma^2} + i \frac{\xi}{N}\right) \mathbf{1}. \quad (\text{B-11})$$

Using the definition of Q in (B-6), there follows

$$M = \left(\frac{1}{\sigma^2} - i2\theta g \right) I + \frac{i2\theta g}{N} \mathbf{1} \mathbf{1}^T \quad (B-12)$$

Now from [6; (21) and (22)],

$$\det M = \frac{1}{\sigma^{2N}} (1 - i2\sigma^2\theta g)^{N-1},$$

$$M^{-1} = \frac{\sigma^2}{1 - i2\sigma^2\theta g} \left[I - \frac{i2\theta g\sigma^2}{N} \mathbf{1} \mathbf{1}^T \right] \quad (B-13)$$

There follows

$$L^T M^{-1} L = N \sigma^2 \left(\frac{u}{\sigma^2} + i \frac{v}{N} \right)^2 \quad (B-14)$$

and

$$f(\mathbf{r}, \theta) = \exp \left[i\mathbf{r}^T \mu - \frac{1}{2} \mathbf{r}^T \Sigma \frac{\sigma^2}{N} \mathbf{r} \right] (1 - i2\sigma^2\theta g)^{\frac{1-N}{2}} \quad (B-15)$$

Since this joint characteristic function factors, it follows that sample statistics m and v are statistically independent. Also the probability density functions are obviously

$$p_m(u) = \frac{1}{\sqrt{2\pi} \sigma/\sqrt{N}} \exp \left[-\frac{(u - \mu)^2}{2 \sigma^2/N} \right] \quad \text{for all } u \quad (B-16)$$

and

$$p_v(u) = \frac{u^{\frac{N-3}{2}} \exp \left(\frac{-u}{2\sigma^2 g} \right)}{\Gamma \left(\frac{N-1}{2} \right) (2\sigma^2 g)^{\frac{N-1}{2}}} \quad \text{for } u > 0 \quad (B-17)$$

Thus sample mean m is Gaussian with mean μ and variance σ^2/N ; while sample variance v is chi-squared of $N-1$ degrees of freedom with mean $\sigma^2(N-1)g$ and variance $2\sigma^4(N-1)g^2$. For the typical choice of gain $g = 1/(N-1)$, this implies that v has mean σ^2 and variance $2\sigma^4/(N-1)$, and therefore

$$\lim_{N \rightarrow \infty} p_v(u) = \delta(u - \sigma^2) . \quad (B-18)$$

The sample standard deviation

$$s = \sqrt{v} \quad (B-19)$$

has probability density function

$$p_s(u) = 2u p_v(u^2) = \frac{2u^{N-2} \exp(-u^2/\beta)}{\Gamma\left(\frac{N-1}{2}\right) \beta^{\frac{N-1}{2}}} \quad \text{for } u > 0 \quad (B-20)$$

where

$$\beta = \frac{2\sigma^2}{N-1} \quad \text{for } g = 1/(N-1) . \quad (B-21)$$

The k -th moment of s is

$$\overline{s^k} = \int_0^{\infty} du u^k p_s(u) = \int_0^{\infty} du \frac{2u^{N-2+k} \exp(-u^2/\beta)}{\Gamma\left(\frac{N-1}{2}\right) \beta^{\frac{N-1}{2}}} = \frac{\Gamma\left(\frac{N+k-1}{2}\right) \beta^{k/2}}{\Gamma\left(\frac{N-1}{2}\right)} . \quad (B-22)$$

In particular,

$$\overline{s^2} = \sigma^2 , \quad (B-23)$$

and [5; (6.1.47)]

$$\begin{aligned}
\bar{s} &= \sigma \left(\frac{2}{N-1}\right)^{1/2} \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)} = \\
&= \sigma \left(\frac{2}{N-1}\right)^{1/2} \left(\frac{N - \frac{3}{2}}{2}\right)^{1/2} \left[1 + \frac{1/16}{\left(N - \frac{3}{2}\right)^2} + O\left(N - \frac{3}{2}\right)^{-3}\right] = \\
&= \sigma \left[1 - \frac{1/4}{N - \frac{9}{8}} + O\left(N - \frac{9}{8}\right)^{-3}\right] \left[1 + \frac{1/16}{\left(N - \frac{3}{2}\right)^2} + O\left(N - \frac{3}{2}\right)^{-3}\right] = \\
&= \sigma \left[1 - \frac{1/4}{N - \frac{9}{8}} + \frac{1/16}{\left(N - \frac{3}{2}\right)^2} + O(N^{-3})\right] = \\
&= \sigma \left[1 - \frac{1/4}{N - \frac{7}{8}} + O(N^{-3})\right] \quad \text{as } N \rightarrow \infty .
\end{aligned}
\tag{B-24}$$

APPENDIX C. PROBABILITY RECURSIONS FOR GAUSSIAN CASE

The detection and false alarm probabilities are given in integral form in (38) and (41). These integrals have already been encountered in [1; appendix E], and evaluated in a recursive fashion. We will modify those results somewhat, in order to better suit the current forms.

First, we have, from (41) and (39),

$$P_F = \int_0^{\infty} dw \frac{w^{N-2} \exp(-w^2/2)}{2^{\frac{N-3}{2}} \Gamma\left(\frac{N-1}{2}\right)} \Phi(-T_1' w) \equiv P_F(N, T_1') \quad (C-1)$$

Define

$$x_1 = (1 + T_1'^2)^{-1}, \quad \text{where } T_1' = T_1 \sqrt{\frac{N}{N^2 - 1}} \quad (C-2)$$

Then from [1; (E-17)], using identifications (that is, replacements from there to here)

$$r \rightarrow \frac{T_1'}{(1 + T_1'^2)^{1/2}}, \quad K \rightarrow N-2, \quad (C-3)$$

there follows the simple result

$$P_F(N, T'_1) = \frac{1}{2} - \frac{1}{\pi} \arctan(T'_1) - \frac{1}{\pi} T'_1 x_1 \sum_{k=0}^{\frac{N-2}{2}} b_k x_1^k \quad \text{for } N=2,4,6,\dots, \quad (C-4)$$

where

$$b_0 = 1, \quad b_k = b_{k-1} \frac{k}{k + \frac{1}{2}} \quad \text{for } k \geq 1. \quad (C-5)$$

A program for this false alarm probability is given in appendix E under the name FNPf246, where T'_1 is represented by variable T_p .

Also,

$$P_F(N, T'_1) = \frac{1}{2} - \frac{1}{2} T'_1 \sqrt{x_1} \sum_{k=0}^{\frac{N-3}{2}} a_k x_1^k \quad \text{for } N=3,5,7,\dots, \quad (C-6)$$

where

$$a_0 = 1, \quad a_k = a_{k-1} \frac{k - \frac{1}{2}}{k} \quad \text{for } k \geq 1. \quad (C-7)$$

These results are very tractable and efficient forms for recursive computer evaluation. A program for (C-6) is given in appendix E under the name FNPf357, where T'_1 is represented by variable T_p .

The current form for detection probability $P_D = P_D(N, T'_r, d'_r)$ in (38) is identical to [1; (E-1)] if we make replacements

$$d_T \rightarrow d'_r, \quad r \rightarrow \frac{T'_r}{(1 + T'^2_r)^{1/2}}, \quad K \rightarrow N-2. \quad (C-8)$$

(The curves in [1] are not directly applicable here because they employed the fundamental parameter $d_T \rightarrow d'_r$, which is $d/(r^2 + 1/N)^{1/2}$ here; however, the recursions derived there are immediately useable.) We can then use [1; (E-8)] to develop an expression for P_D , in terms of the auxiliary sequence $\{g(K)\}$ defined in [1; (E-7)]. In particular, [1; (E-9)] yields, with

$$x_r = (1 + T'^2_r)^{-1}, \quad (C-9)$$

the result

$$g(0) = T'_r \sqrt{x_r} \exp\left(-\frac{1}{2} d'^2_r x_r\right) \Phi(d'_r T'_r \sqrt{x_r}); \quad (C-10)$$

[1; (E-13)] yields

$$g(1) = T'_r x_r \left[\frac{1}{\pi} \exp\left(-d'^2_r/2\right) + \left(\frac{2}{\pi}\right)^{1/2} d'_r g(0) \right]; \quad (C-11)$$

and [1; (E-12)] yields

$$g(K) = x_r \left[h(K) g(K-1) + \frac{K-1}{K} g(K-2) \right] \quad \text{for } K \geq 2, \quad (C-12)$$

with definition

$$h(K) = \frac{1}{\sqrt{2}} T'_r d'_r \frac{\Gamma\left(\frac{K+1}{2}\right)}{\Gamma\left(\frac{K}{2} + 1\right)}. \quad (C-13)$$

Then we can also use

$$\begin{aligned}
 h(0) &= T_r' d_r' \left(\frac{\pi}{2}\right)^{1/2}, & h(1) &= T_r' d_r' \left(\frac{2}{\pi}\right)^{1/2}, \\
 h(K) &= h(K-2) \frac{K-1}{K} & \text{for } K &\geq 2.
 \end{aligned}
 \tag{C-14}$$

Finally, $P_D = P_D(N, T_r', d_r')$ is given by [1; (E-8)] as

$$P_D = \left\{ \begin{array}{l} \Phi(d_r') - \sum_{\substack{K=0 \\ K \text{ even}}}^{N-3} g(K) \quad \text{for } N=3,5,7, \dots \\ P_{D2} - \sum_{\substack{K=1 \\ K \text{ odd}}}^{N-3} g(K) \quad \text{for } N=4,6,8, \dots \end{array} \right\}, \tag{C-15}$$

where P_{D2} is the value of detection probability P_D for $N = 2$. Observe that the input parameters to P_D are N, T_r', d_r' , rather than the four fundamental parameters in (40); that is, N, T, d, r are collapsed into N, T_r', d_r' according to (39). Programs for (C-15) are furnished in appendix E under the names FNPd357 and FNPd246, respectively, where T_r' and d_r' are represented by variables T_p and D_p .

The quantity P_{D2} in (C-15) is evaluated according to the method in [1; appendix F]; an error tolerance and maximum number of terms must also be specified to terminate the infinite sum given by [1; (F-2)].

APPENDIX D. χ -APPROXIMATION FOR RANDOM VARIABLE t

Suppose we assume that the random variable t in (46) is a multiple of a χ -variate with K degrees of freedom; then its probability density function is [7; pages 5-7 for $\nu = 1/2$]

$$p_t(u) = \frac{u^{K-1} \exp(-u^2/(2A^2))}{A^K \frac{K}{2^2}^{-1} \Gamma(\frac{K}{2})} \quad \text{for } u > 0. \quad (D-1)$$

Then the ν -th moment of random variable t is

$$\bar{t}^\nu = \int du u^\nu p_t(u) = \frac{2^{\nu/2} A^\nu \Gamma(\frac{K+\nu}{2})}{\Gamma(\frac{K}{2})}, \quad (D-2)$$

and in particular

$$\bar{t} = A \frac{\sqrt{2} \Gamma(\frac{K+1}{2})}{\Gamma(\frac{K}{2})}, \quad \bar{t}^2 = A^2 K. \quad (D-3)$$

Then the ratio

$$R = \frac{\bar{t}}{\sqrt{\bar{t}^2}} = \frac{\Gamma(\frac{K+1}{2})}{\sqrt{\frac{K}{2}} \Gamma(\frac{K}{2})} = 1 - \frac{1}{4K + \frac{1}{2}} + O(K^{-3}), \quad (D-4)$$

where the last result uses the development in (B-24) with N replaced by $K+1$.

Given a value for ratio R on the left side of (D-4), K can be solved for uniquely, since the ratio involving gamma functions increases monotonically from 0 to 1 as K goes from 0 to $+\infty$. In fact, to a good approximation for large K , the last part of (D-4) gives

$$K \approx \frac{1}{4(1-R)} - \frac{1}{8}. \quad (D-5)$$

Here we are allowing K in probability density function p_t in (D-1) to be arbitrary, that is, not limited to integer values. Then we can solve for the required value of A according to (D-3), as $A^2 = t^2/K$. This procedure fits the assumed probability density function form in (D-1) to specified values of the first two moments of t given by (D-3), as given by simulation results (53)-(55).

If we now employ the χ -approximate probability density function for t given by (D-1) in detection probability result (45), along with (49), we obtain

$$\begin{aligned} P_D &= \int_0^\infty du \exp\left[-\exp\left(-\gamma + \frac{\pi}{\sqrt{6}} \frac{u-d}{r}\right)\right] \frac{u^{K-1} \exp(-u^2/(2A^2))}{A^K 2^{\frac{K}{2}-1} \Gamma\left(\frac{K}{2}\right)} = \\ &= \left[2^{\frac{K}{2}-1} \Gamma\left(\frac{K}{2}\right)\right]^{-1} \int_0^\infty dx x^{K-1} \exp\left[-x^2/2 - \exp(h_1 + h_2 x)\right], \quad (D-6) \end{aligned}$$

where constants

$$h_1 = -\gamma - \frac{\pi}{\sqrt{6}} \frac{d}{r}, \quad h_2 = \frac{\pi}{\sqrt{6}} \frac{A}{r}. \quad (D-7)$$

The numerical evaluation of (D-6) was undertaken for $N = 16$, and is discussed in the Graphical Results section of this report.

APPENDIX E. PROGRAMS

In this appendix, four programs are listed. They are written in BASIC for the Hewlett-Packard 9000 Model 520 Desk Top Computer. Their titles are

EDF - Gaussian,
EDF - Extreme,
Simulation-Extreme,
Plot-Simulation.

The first one computes the exceedance distribution function for a Gaussian input to the normalizer of figure 1, for $N=16$ (line 10) and for $d = 0(1)12$ (lines 960-970). This program is heavily based on the results of appendix C.

The second program computes the approximate exceedance distribution function for a log-distorted Weibull input to the normalizer of figure 1, for $N=16$ (line 10), $r=1$ (line 20), and $d = 0(.75)7.5$ (line 1070). It is based on numerical integration of (51) via Simpson's rule.

The third program simulates the normalizer output (28) and (24) for a log-distorted Weibull input, for $N=16$ (line 10), $d = 0(.75)7.5$ (line 20), $r= 1$ (line 30), and $2^{23} = 8.4$ million trials (line 40). The range of

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values in z is $(-15,5)$, which is divided into 1000 bins; see lines 60, 90, 100. The resultant histogram is then summed on the upper tail to yield the exceedance distribution function. The fourth program plots these simulation results for the exceedance distribution function vs threshold T .

Table E-1. EDF - Gaussian

```

10  Ns=16                ! NUMBER OF SAMPLES
20  X1=-7                ! THRESHOLD
30  X2=7                 ! LIMITS
40  DIM A$(30),B$(30)
50  DIM Xlabel$(1:30),Ylabel$(1:30)
60  DIM Xcoord(1:30),Ycoord(1:30)
70  DIM Xgrid(1:30),Ygrid(1:30)
80  DOUBLE Lx,Ly,Nx,Ny,I,Ns,It
90  !
100 A$="Threshold"
110 B$="Exceedance Probability"
120 !
130 Lx=15
140 REDIM Xlabel$(1:Lx),Xcoord(1:Lx)
150 DATA -7,-6,-5,-4,-3,-2,-1,0,1,2,3,4,5,6,7
160 READ Xlabel$(*)
170 DATA -7,-6,-5,-4,-3,-2,-1,0,1,2,3,4,5,6,7
180 READ Xcoord(*)
190 !
200 Ly=27
210 REDIM Ylabel$(1:Ly),Ycoord(1:Ly)
220 DATA E-6,E-5,E-4,.001,.002,.005,.01,.02,.05
230 DATA .1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.98,.99
240 DATA .995,.998,.999,.9999,.99999,.999999
250 READ Ylabel$(*)
260 DATA 1.E-6,1.E-5,1.E-4,.001,.002,.005,.01,.02,.05
270 DATA .1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.98,.99
280 DATA .995,.998,.999,.9999,.99999,.999999
290 READ Ycoord(*)
300 !
310 Nx=15
320 REDIM Xgrid(1:Nx)
330 DATA -7,-6,-5,-4,-3,-2,-1,0,1,2,3,4,5,6,7
340 READ Xgrid(*)
350 !
360 Ny=27
370 REDIM Ygrid(1:Ny)
380 DATA 1.E-6,1.E-5,1.E-4,.001,.002,.005,.01,.02,.05
390 DATA .1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.98,.99
400 DATA .995,.998,.999,.9999,.99999,.999999
410 READ Ygrid(*)
420 !
430 FOR I=1 TO Lx
440   Ycoord(I)=FNIn(Phi+Ycoord(I))
450 NEXT I
460 FOR I=1 TO Ny
470   Xgrid(I)=FNIn(Phi+Xgrid(I))
480 NEXT I
490 Y1=Ygrid(1)
500 Y2=Ygrid(Ny)

```

```

510 GINIT 180./240.           ! VERTICAL PAPER
520 PLOTTER IS 505,"HPGL"
530 PRINTER IS 505
540 LIMIT PLOTTER 505,0.,180.,0.,240. ! 1 GDU = 2 mm
550 VIEWPORT 20.,120.,19.,132.
560 ! VIEWPORT 22.,85.,59.,122.      ! TOP OF PAPER
570 ! VIEWPORT 22.,85.,19.,62.      ! BOTTOM OF PAPER
580 WINDOW X1,X2,Y1,Y2
590 ! PRINT "VS2"
600 FOR I=1 TO Nx
610 MOVE Xgrid(I),Y1
620 DRAW Xgrid(I),Y2
630 NEXT I
640 FOR I=1 TO Ny
650 MOVE X1,Ygrid(I)
660 DRAW X2,Ygrid(I)
670 NEXT I
680 PENUP
690 LDIR 0
700 CSIZE 2.3,.5
710 LORG 5
720 Y=Y1-(Y2-Y1)*.02
730 FOR I=1 TO Lx
740 MOVE Xcoord(I),Y
750 LABEL Xlabel$(I)
760 NEXT I
770 CSIZE 3.,.5
780 MOVE .5*(X1+X2),Y1-.06*(Y2-Y1)
790 LABEL A$
800 CSIZE 2.3,.5
810 LORG 8
820 X=X1-(X2-X1)*.01
830 FOR I=1 TO Lx
840 MOVE X,Ycoord(I)
850 LABEL Ylabel$(I)
860 NEXT I
870 LDIR PI/2.
880 CSIZE 3.,.5
890 LORG 5
900 MOVE X1-.15*(X2-X1),.5*(Y1+Y2)
910 LABEL B$
920 ! PRINT "VS36"

```

```

930     Dx=(X2-X1)/100.
940     F1=SQR(Ns/(Ns+1))
950     F2=SQR(Ns/(Ns*Ns-1))
960     FOR I=0 TO 12
970         Ds=I
980         Dp=Ds*F1
990         FOR It=0 TO 100
1000            T=X1+Dx*It
1010            Tp=T*F2
1020            IF Dp>0. THEN 1080
1030            IF Ns MODULO 2=0 THEN 1060
1040            Pd=FNPf357(Ns,Tp)
1050            GOTO 1120
1060            Pd=FNPf246(Ns,Tp)
1070            GOTO 1120
1080            IF Ns MODULO 2=0 THEN 1110
1090            Pd=FNPd357(Ns,Tp,Dp)
1100            GOTO 1120
1110            Pd=FNPd246(Ns,Tp,Dp)
1120            IF Pd<=0. THEN 1160
1130            IF Pd>=1. THEN 1160
1140            Y=FNInuphi(Pd)
1150            PLOT T,Y
1160            NEXT It
1170            PENUP
1180            NEXT I
1190            PAUSE
1200            PRINTER IS CRT
1210            PLOTTER 505 IS TERMINATED
1220            END
1230            !
1240            DEF FNInuphi(X)
1250                IF X=.5 THEN RETURN 0.
1260                P=MIN(X,1.-X)
1270                T=-LOG(P)
1280                T=SQR(T+T)
1290                P=1.+T*(1.432788+T*(.189269+T*(.001308+T*(.000001))))
1300                P=T*(2.515517+T*(.802853+T*(.010328+T*(.000001))))
1310                IF X<.5 THEN P=-P
1320                RETURN P
1330            FNEED
1340            !
1350            DEF FNPf246(DOUBLE N,REAL Tp)
1360                DOUBLE Ks
1370                Pf=.5-ATN(Tp)/PI
1380                IF N=2 THEN RETURN Pf
1390                K=1.+(1.+Tp)*Tp
1400                S=B=.5
1410                FOR Ks=1 TO N-2
1420                    B=B*(K+Ks)/(Ks+.5)
1430                    S=S+B
1440                NEXT Ks
1450                Pf=Pf-Tp*(1.-S)/PI
1460                RETURN Pf
1470            FNEED
1480            !

```

```

1490 DEF FNPf357(DOUBLE N,REAL Tp)      | N=3,5,7,...
1500 DOUBLE Ks                          | INTEGER
1510 X=1./((1.+Tp*Tp)
1520 S=Ax=1.
1530 FOR Ks=1 TO (N-3)/2
1540 Ax=Ax*X*((Ks-.5)/Ks
1550 S=S+Ax
1560 NEXT Ks
1570 Pf=.5*(1.-Tp*SQR(X)*S)
1580 RETURN Pf
1590 FNEND
1600 |
1610 DEF FNPd246(DOUBLE N,REAL Tp,Dp)   | N=2,4,6,...
1620 Error=1.E-15                       | Tolerance in Pd for N=2
1630 Nterms=500                          | Number of terms for N=2
1640 DOUBLE Ks                           | INTEGER
1650 X=1./((1.+Tp*Tp)
1660 Rk=SQR(X)
1670 Tsq=Tp*Rk
1680 Dsq=Dp*Dp
1690 R=SQR(2.-PI)
1700 A1=A0=EXP(-.5*Dsq/PI)
1710 A=A1+Dp/R
1720 B1=.5*PI-ATN(Tp)
1730 B=Rk
1740 Pd=A1+B1+A*B
1750 FOR Ks=2 TO Nterms
1760 F=FLT(Ks-1)
1770 T=Dsq*A1/F
1780 A1=A
1790 A=T
1800 Rk=Rk+Tsq
1810 T=(Rk+F+B1)/Ks
1820 B1=B
1830 B=T
1840 P=A+B
1850 Pd=Pd+P
1860 IF P =Error+Pd THEN 1890
1870 NEXT Ks
1880 PRINT "500 TERMS IN FNPd246"
1890 IF N=2 THEN RETURN Pd
1900 G1=Tsq*EXP(-.5*Dsq/PI)+FNPf357(Dp+Tsq)
1910 G=Tp*(K+A0+R+Dp+G1)
1920 F1=Tp+Dp/R
1930 F=Tp+Dp+R
1940 Pd=Pd-G
1950 IF N=4 THEN RETURN Pd
1960 FOR Ks=2 TO N-3
1970 R=(K-1)/Ks
1980 T=F1+R
1990 F1=F
2000 F=T
2010 T=T*(F+G+R+G1)
2020 G1=G
2030 G=T
2040 IF Ks MODULO 2=1 THEN F3=F3-G
2050 NEXT Ks
2060 RETURN F3
2070 FNEND
2080 |

```

```

2090 DEF FNPd357(DOUBLE N,REAL Tp,Dp) ! N=3,5,7,...
2100 DOUBLE Ks ! INTEGER
2110 X=1./(<1.+Tp*Tp>)
2120 Tsq=Tp*SQR(X)
2130 D2=.5*Dp*Dp
2140 G=Tsq*EXP(-D2*X)*FNPhi(Dp*Tsq)
2150 Pd=FNPhi(Dp)-G
2160 IF N=3 THEN RETURN Pd
2170 R=SQR(2./PI)
2180 G1=G
2190 G=Tp*X*(EXP(-D2)/PI+R*Dp*G1)
2200 F1=Tp*Dp/R
2210 F=Tp*Dp+R
2220 FOR Ks=2 TO N-3
2230 R=(Ks-1)/Ks
2240 T=F1+R
2250 F1=F
2260 F=T
2270 T=X*(F+G+R*G1)
2280 G1=G
2290 G=T
2300 IF Ks MODULO 2=0 THEN Pd=Pd-G
2310 NEXT Ks
2320 RETURN Pd
2330 FNEND
2340 !
2350 DEF FNPhi(X) ! HART, page 140. #5708 : #5725
2360 Y=ABS(X)+.70710678118654746
2370 SELECT Y
2380 CASE <8.
2390 P=1631.76026875371470+Y+.456.261458706092631+Y+.66.0827622119485451+
+10.0648589749095425+Y+.564189586761813614+Y
2400 P=3723.50798155480672+Y+.7113.66324695404987+Y+.6758.21696411048589+
+4032.26701083004974+Y+P
2410 Q=7542.47951019347576+Y+.2968.00490148230872+Y+.817.622186304544077+
+153.077710750362216+Y+.17.8394984391395565+Y
2420 Q=3723.50798155480654+Y+.11315.1920813544055+Y+.15302.5159994020425+
+13349.3465612844574+Y+Q
2430 Phi=.5*EXP(-Y/Y)+P/Q
2440 CASE <26.6
2450 P=2.97886562639399289+Y+.7.40974060596474179+Y+.6.12020985010963054+
+5.01904372678426746+Y+.1.27536664472996595+Y+.564189581547755074
2460 Q=3.36907520698275277+Y+.9.60896532719278787+Y+.17.0914407474660040+
+12.0489519278551290+Y+.9.39603401623505415+Y+.2.26052352078732697+
2470 Phi=.5*EXP(-Y/Y)+P/Q
2480 CASE ELSE
2490 Phi=0.
2500 END SELECT
2510 IF .0. THEN Phi=1.-Phi
2520 RETURN Phi
2530 FNEND

```

Table E-2. EDF - Extreme

```

10  Ns=16          ! NUMBER OF SAMPLES
20  Rs=1.         ! RATIO r
30  X1=-15
40  X2=5
50  Gamma=.577215664902
60  C1=-Gamma
70  C2=PI/SQR(6.)
80  F=SQR(.5/PI)
90  A=-8.        ! LIMITS ON
100 B=8.         ! INTEGRAL
110 Mean_mucv=0.
120 Var_mucv=1./Ns
130 Mean_sigcv=1.-.39/Ns
140 Var_sigcv=1.05/(Ns+1.5)
150 Rho=-.55
160 COM H1,H2
170 DIM A$(30),B$(30)
180 DIM Xlabel$(1:30),Ylabel$(1:30)
190 DIM Xcoord(1:30),Ycoord(1:30)
200 DIM Xgrid(1:30),Ygrid(1:30)
210 DOUBLE Lx,Ly,Nx,Ny,I,Ns,It
220 I
230 A$="Threshold"
240 B$="Exceedance Probability"
250 I
260 Lx=11
270 REDIM Xlabel$(1:Lx),Xcoord(1:Lx)
280 DATA -15,-13,-11,-9,-7,-5,-3,-1,1,3,5
290 READ Xlabel$(*)
300 DATA -15,-13,-11,-9,-7,-5,-3,-1,1,3,5
310 READ Xcoord(*)
320 I
330 Ly=27
340 PEDIM Ylabel$(1:Ly),Ycoord(1:Ly)
350 DATA E-6,E-5,E-4,.001,.002,.005,.01,.02,.05,.1,.2,.3,.4,.5
360 DATA .6,.7,.8,.9,.95,.98,.99
370 DATA .995,.998,.999,.9999,.99999,.999999
380 READ Ylabel$(*)
390 DATA 1.E-6,1.E-5,1.E-4,.001,.002,.005,.01,.02,.05,.1,.2,.3,.4,.5
400 DATA .6,.7,.8,.9,.95,.98,.99
410 DATA .995,.998,.999,.9999,.99999,.999999
420 READ Ycoord(*)
430 I
440 Nx=11
450 REDIM Xgrid(1:Nx)
460 DATA -15,-13,-11,-9,-7,-5,-3,-1,1,3,5
470 READ Xgrid(*)
480 I
490 Ny=27
500 PEDIM Ygrid(1:Ny)
510 DATA 1.E-6,1.E-5,1.E-4,.001,.002,.005,.01,.02,.05,.1,.2,.3,.4,.5
520 DATA .6,.7,.8,.9,.95,.98,.99
530 DATA .995,.998,.999,.9999,.99999,.999999
540 READ Ygrid(*)
550 I

```

```

560   FOR I=1 TO Ly
570   Ycoord(I)=FNInuphi(Ycoord(I))
580   NEXT I
590   FOR I=1 TO Ny
600   Ygrid(I)=FNInuphi(Ygrid(I))
610   NEXT I
620   Y1=Ygrid(1)
630   Y2=Ygrid(Ny)
640   GINIT 180./240.          | VERTICAL PAPER
650   PLOTTER IS 505,"HPGL"
660   PRINTER IS 505
670   LIMIT PLOTTER 505,0.,180.,0.,240. | 1 GDU = 2 mm
680   VIEWPORT 20.,120.,19.,132.
690   | VIEWPORT 22.,85.,59.,122.      | TOP OF PAPER
700   | VIEWPORT 22.,85.,19.,52.      | BOTTOM OF PAPER
710   WINDOW X1,X2,Y1,Y2
720   | PRINT "VS2"
730   FOR I=1 TO Nx
740   MOVE Xgrid(I),Y1
750   DRAW Xgrid(I),Y2
760   NEXT I
770   FOR I=1 TO Ny
780   MOVE X1,Ygrid(I)
790   DRAW X2,Ygrid(I)
800   NEXT I
810   PENUP
820   CSIZE 2.3,.5
830   LORG 5
840   Y=Y1-(Y2-Y1)*.02
850   FOR I=1 TO Lx
860   MOVE Xcoord(I),Y
870   LABEL Xlabel$(I)
880   NEXT I
890   CSIZE 3.,.5
900   MOVE .5*(X1+X2),Y1-.06*(Y2-Y1)
910   LABEL A$
920   CSIZE 2.3,.5
930   LORG 8
940   X=X1-(X2-X1)*.01
950   FOR I=1 TO Lx
960   MOVE X,Ycoord(I)
970   LABEL Ylabel$(I)
980   NEXT I
990   LDIP PI 2.
1000  CSIZE 3.,.5
1010  LORG 5
1020  MOVE .15*(X2-X1)+.25*(Y1+Y2)
1030  LABEL B$
1040  PRINT "VS05"
1050  D = .2*(Y1+Y2)
1060  DIM P(0:100)

```

```

1070 FOR Ds=0. TO 7.5 STEP .75      ! DEFLECTION d
1080 FOR It=0 TO 100
1090 J=It
1100 Th=X2-Dx*It                    ! THRESHOLD
1110 Mut=Mean_mucv+Th*Mean_sigcv
1120 Temp=Var_mucv+Th*Th*Var_sigcv
1130 Temp=Temp+2.*Th*SQR(Var_mucv*Var_sigcv)*Rho
1140 Sigmat=SQR(Temp)
1150 H1=C1+C2*(Mut-Ds)/Rs
1160 H2=C2*Sigmat/Rs
1170 Sa=FNS(A)
1180 Sb=FNS(B)
1190 PRINT "FNS(A) = ";Sa;"    FNS(B) = ";Sb
1200 DOUBLE N,K,J
1210 N=2
1220 H=(B-A)*.5
1230 S=(Sa+Sb)*.5
1240 V=1.E10
1250 T=0.
1260 FOR K=1 TO N-1 STEP 2
1270 T=T+FNS(A+H*K)
1280 NEXT K
1290 S=S+T
1300 Vo=V
1310 V=(S+T)*H*2./3.
1320 IF ABS(V-Vo)<1.E-9 THEN 1360
1330 N=N+N
1340 H=H*.5
1350 GOTO 1250
1360 Pd=F*V
1370 IF Pd>.9999995 THEN 1400
1380 P(It)=FNInuphi(Pd)
1390 NEXT It
1400 FOR It=0 TO J-1
1410 PLOT X2-Dx*It,P(It)
1420 NEXT It
1430 PENUP
1440 NEXT Ds
1450 PAUSE
1460 PRINTER IS CRT
1470 PLOTTER 505 IS TERMINATED
1480 END
1490 !
1500 DEF FNInuphi(X)      ! AMS 55, 26.2.23
1510 IF X=.5 THEN RETURN 0.
1520 P=MIN(X,1.-X)
1530 T=-LOG(P)
1540 T=SQR(T+T)
1550 P=1.+T*(1.432788+T*(1.189269+T*(.001308)))
1560 P=T*(2.515517+T*(.802853+T*(.010328))) P
1570 IF X<.5 THEN P=-P
1580 RETURN P
1590 FNEND
1600 !
1610 DEF FNS(X)
1620 COM H1,H2
1630 T=.5*(X+1)+EXP*(H1+H2*(X))
1640 IF T 100. THEN RETURN 0.
1650 RETURN EXP*(T)
1660 FNEND

```

Table E-3. Simulation - Extreme

```

10  Ns=16          ! NUMBER OF SAMPLES
20  DATA 0, .75, 1.5, 2.25, 3., 3.75, 4.5, 5.25, 6., 6.75, 7.5 ! DEFLECTION d
30  Rs=1.         ! SCALING r
40  Nt=2^23      ! NUMBER OF TRIALS
50  A$="E-16-75to75-1-23" ! FILE NAME
60  Nb=1000      ! NUMBER OF BINS
70  Mean=-.577215664902 ! FOR Vt, LOG OF EXPONENTIAL
80  Sigma=PI/SQR(6.)   ! RANDOM VARIABLE
90  Zmin=-15.       ! MINIMUM Z VALUE
100 Zmax=5.        ! MAXIMUM Z VALUE
110 DOUBLE Ns, Nt, Nb, Kt, Ks, J ! INTEGERS
120 DIM P0(1000), P1(1000), P2(1000), P3(1000), P4(1000), P5(1000)
130 DIM P6(1000), P7(1000), P8(1000), P9(1000), P10(1000), Edf(1:11000)
140 READ D0, D1, D2, D3, D4, D5, D6, D7, D8, D9, D10
150 A0=-Mean/Sigma
160 A1=1./Sigma
170 F=1./Ns
180 G=1./SQR(Ns-1)
190 Dz=(Zmax-Zmin)/Nb
200 FOR Kt=1 TO Nt          ! SIMULATION
210 S1=S2=0.
220 FOR Ks=1 TO Ns
230 Vt=LOG(-LOG(RND))      ! UN-NORMALIZED RANDOM VARIABLE.
240 V=A1*Vt+A0            ! ZERO MEAN, UNIT VARIANCE RV
250 S1=S1+V
260 S2=S2+V*V
270 NEXT Ks
280 Mc=S1*F                ! SAMPLE MEAN
290 Sc=SQR(S2-Ns*Mc*Mc)+G ! SAMPLE STANDARD DEVIATION
300 Vt=LOG(-LOG(RND))
310 V=A1*Vt+A0
320 C=Rs*V-Mc
330 Z=(D0+C)/Sc
340 J=INT((Z-Zmin)/Dz)
350 IF J=0 THEN J=0
360 IF J=Nb THEN J=Nb
370 P0(J)=P0(J)+1.
380 Z=(D1+C)/Sc
390 J=INT((Z-Zmin)/Dz)
400 IF J=0 THEN J=0
410 IF J=Nb THEN J=Nb
420 P1(J)=P1(J)+1.
430 Z=(D2+C)/Sc
440 J=INT((Z-Zmin)/Dz)
450 IF J=0 THEN J=0
460 IF J=Nb THEN J=Nb
470 P2(J)=P2(J)+1.
480 Z=(D3+C)/Sc
490 J=INT((Z-Zmin)/Dz)
500 IF J=0 THEN J=0
510 IF J=Nb THEN J=Nb
520 P3(J)=P3(J)+1.

```

```

530 Z=(D4+C)/Sc
540 J=INT((Z-Zmin)/Dz)
550 IF J<0 THEN J=0
560 IF J>Nb THEN J=Nb
570 P4(J)=P4(J)+1.
580 Z=(D5+C)/Sc
590 J=INT((Z-Zmin)/Dz)
600 IF J<0 THEN J=0
610 IF J>Nb THEN J=Nb
620 P5(J)=P5(J)+1.
630 Z=(D6+C)/Sc
640 J=INT((Z-Zmin)/Dz)
650 IF J<0 THEN J=0
660 IF J>Nb THEN J=Nb
670 P6(J)=P6(J)+1.
680 Z=(D7+C)/Sc
690 J=INT((Z-Zmin)/Dz)
700 IF J<0 THEN J=0
710 IF J>Nb THEN J=Nb
720 P7(J)=P7(J)+1.
730 Z=(D8+C)/Sc
740 J=INT((Z-Zmin)/Dz)
750 IF J<0 THEN J=0
760 IF J>Nb THEN J=Nb
770 P8(J)=P8(J)+1.
780 Z=(D9+C)/Sc
790 J=INT((Z-Zmin)/Dz)
800 IF J<0 THEN J=0
810 IF J>Nb THEN J=Nb
820 P9(J)=P9(J)+1.
830 Z=(D10+C)/Sc
840 J=INT((Z-Zmin)/Dz)
850 IF J<0 THEN J=0
860 IF J>Nb THEN J=Nb
870 P10(J)=P10(J)+1.
880 NEXT Kt
890 MAT P0=P0/(Nt)
900 MAT P1=P1/(Nt)
910 MAT P2=P2/(Nt)
920 MAT P3=P3/(Nt)
930 MAT P4=P4/(Nt)
940 MAT P5=P5/(Nt)
950 MAT P6=P6/(Nt)
960 MAT P7=P7/(Nt)
970 MAT P8=P8/(Nt)
980 MAT P9=P9/(Nt)
990 MAT P10=P10/(Nt)
1000 S0=S1=S2=S3=S4=S5=S6=S7=S8=S9=S10=0.

```

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```

1010   FOR J=Nb TO 0 STEP -1
1020   S0=S0+P0(J)
1030   IF S0=0. THEN 1060
1040   IF S0>=1. THEN 1070
1050   P0(J)=FNInuphi(S0)
1060   NEXT J
1070   P0(J)=0.
1080   FOR J=Nb TO 0 STEP -1
1090   S1=S1+P1(J)
1100   IF S1=0. THEN 1130
1110   IF S1>=1. THEN 1140
1120   P1(J)=FNInuphi(S1)
1130   NEXT J
1140   P1(J)=0.
1150   FOR J=Nb TO 0 STEP -1
1160   S2=S2+P2(J)
1170   IF S2=0. THEN 1200
1180   IF S2>=1. THEN 1210
1190   P2(J)=FNInuphi(S2)
1200   NEXT J
1210   P2(J)=0.
1220   FOR J=Nb TO 0 STEP -1
1230   S3=S3+P3(J)
1240   IF S3=0. THEN 1270
1250   IF S3>=1. THEN 1280
1260   P3(J)=FNInuphi(S3)
1270   NEXT J
1280   P3(J)=0.
1290   FOR J=Nb TO 0 STEP -1
1300   S4=S4+P4(J)
1310   IF S4=0. THEN 1340
1320   IF S4>=1. THEN 1350
1330   P4(J)=FNInuphi(S4)
1340   NEXT J
1350   P4(J)=0.
1360   FOR J=Nb TO 0 STEP -1
1370   S5=S5+P5(J)
1380   IF S5=0. THEN 1410
1390   IF S5>=1. THEN 1420
1400   P5(J)=FNInuphi(S5)
1410   NEXT J
1420   P5(J)=0.
1430   FOR J=Nb TO 0 STEP -1
1440   S6=S6+P6(J)
1450   IF S6=0. THEN 1480
1460   IF S6>=1. THEN 1490
1470   P6(J)=FNInuphi(S6)
1480   NEXT J
1490   P6(J)=0.
1500   FOR J=Nb TO 0 STEP -1
1510   S7=S7+P7(J)
1520   IF S7=0. THEN 1550
1530   IF S7>=1. THEN 1560
1540   P7(J)=FNInuphi(S7)
1550   NEXT J
1560   P7(J)=0.

```

! P0(J)=Prob(Z0>=Zmin+J*Dz)

```

1570   FOR J=Nb TO 0 STEP -1
1580   S8=S8+P8(J)
1590   IF S8=0. THEN 1620
1600   IF S8>=1. THEN 1630
1610   P8(J)=FNInvphi(S8)
1620   NEXT J
1630   P8(J)=0.
1640   FOR J=Nb TO 0 STEP -1
1650   S9=S9+P9(J)
1660   IF S9=0. THEN 1690
1670   IF S9>=1. THEN 1700
1680   P9(J)=FNInvphi(S9)
1690   NEXT J
1700   P9(J)=0.
1710   FOR J=Nb TO 0 STEP -1
1720   S10=S10+P10(J)
1730   IF S10=0. THEN 1760
1740   IF S10>=1. THEN 1770
1750   P10(J)=FNInvphi(S10)
1760   NEXT J
1770   P10(J)=0.
1780   FOR J=1 TO Nb
1790   Edf(J)=P0(J)
1800   Edf(J+Nb)=P1(J)
1810   Edf(J+Nb*2)=P2(J)
1820   Edf(J+Nb*3)=P3(J)
1830   Edf(J+Nb*4)=P4(J)
1840   Edf(J+Nb*5)=P5(J)
1850   Edf(J+Nb*6)=P6(J)
1860   Edf(J+Nb*7)=P7(J)
1870   Edf(J+Nb*8)=P8(J)
1880   Edf(J+Nb*9)=P9(J)
1890   Edf(J+Nb*10)=P10(J)
1900   NEXT J
1910   CREATE DATA A$,396
1920   ASSIGN #1 TO A$
1930   PRINT #1;Edf(*)
1940   ASSIGN #1 TO *
1950   PAUSE
1960   END
1970   !
1980   DEF FNInvphi(X)
1990   IF X=.5 THEN RETURN 0.
2000   P=MIN(X,1.-X)
2010   T=-LOG(P)
2020   T=SQR(T+T)
2030   P=1.+T*(1.432788+T*(.189269+T*(-.001308))
2040   P=T*(2.515517+T*(.802853+T*(-.010328)) P
2050   IF X<.5 THEN P=-P
2060   RETURN P
2070   FNEND

```

AMS 55, 26.2.23

Table E-4. Plot - Simulation

```

10  A$="E-16-75to75-1-23"
20  Zmin=-15.
30  Zmax=5.
40  X1=-15.
50  X2=5.
60  Nb=1000
70  ASSIGN #1 TO A$
80  READ #1;Edf(*)
90  ASSIGN #1 TO *
100 DIM Edf(1:11000)
110 DIM A$(30),B$(30)
120 DIM Xlabel$(1:30),Ylabel$(1:30)
130 DIM Xcoord(1:30),Ycoord(1:30)
140 DIM Xgrid(1:30),Ygrid(1:30)
150 DOUBLE Nb,Lx,Ly,Nx,Ny,I,K      ! INTEGERS
160 !
170 A$="Threshold"
180 B$="Exceedance Probability"
190 !
200 Lx=11
210 REDIM Xlabel$(1:Lx),Xcoord(1:Lx)
220 DATA -15,-13,-11,-9,-7,-5,-3,-1,1,3,5
230 READ Xlabel$(*)
240 DATA -15,-13,-11,-9,-7,-5,-3,-1,1,3,5
250 READ Xcoord(*)
260 !
270 Ly=27
280 REDIM Ylabel$(1:Ly),Ycoord(1:Ly)
290 DATA E-6,E-5,E-4,.001,.002,.005,.01,.02,.05
300 DATA .1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.98,.99
310 DATA .995,.998,.999,.9999,.99999,.999999
320 READ Ylabel$(*)
330 DATA 1.E-6,1.E-5,1.E-4,.001,.002,.005,.01,.02,.05
340 DATA .1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.98,.99
350 DATA .995,.998,.999,.9999,.99999,.999999
360 READ Ycoord(*)
370 !
380 Nx=11
390 REDIM Xgrid(1:Nx)
400 DATA -15,-13,-11,-9,-7,-5,-3,-1,1,3,5
410 READ Xgrid(*)
420 !
430 Ny=27
440 REDIM Ygrid(1:Ny)
450 DATA 1.E-6,1.E-5,1.E-4,.001,.002,.005,.01,.02,.05
460 DATA .1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.98,.99
470 DATA .995,.998,.999,.9999,.99999,.999999
480 READ Ygrid(*)
490 !

```

```

500   FOR I=1 TO Ly
510   Ycoord(I)=FNInuphi(Ycoord(I))
520   NEXT I
530   FOR I=1 TO Ny
540   Ygrid(I)=FNInuphi(Ygrid(I))
550   NEXT I
560   Y1=Ygrid(1)
570   Y2=Ygrid(Ny)
580   GINIT 180./240.           ! VERTICAL PAPER
590   PLOTTER IS 505,"HPGL"
600   PRINTER IS 505
610   LIMIT PLOTTER 505,0.,180.,0.,240. ! 1 GDU = 2 mm
620   VIEWPORT 20.,120.,19.,132.
630   ! VIEWPORT 22.,85.,59.,122.      ! TOP OF PAPER
640   ! VIEWPORT 22.,85.,19.,62.      ! BOTTOM OF PAPER
650   WINDOW X1,X2,Y1,Y2
660   ! PRINT "VS2"
670   FOR I=1 TO Nx
680   MOVE Xgrid(I),Y1
690   DRAW Xgrid(I),Y2
700   NEXT I
710   FOR I=1 TO Ny
720   MOVE X1,Ygrid(I)
730   DRAW X2,Ygrid(I)
740   NEXT I
750   PENUP
760   LDIR 0
770   CSIZE 2.3,.5
780   LORG 5
790   Y=Y1-(Y2-Y1)*.02
800   FOR I=1 TO Lx
810   MOVE Xcoord(I),Y
820   LABEL Xlabel$(I)
830   NEXT I
840   CSIZE 3.,.5
850   MOVE .5*(X1+X2),Y1-.06*(Y2-Y1)
860   LABEL A$
870   CSIZE 2.3,.5
880   LORG 8
890   X=X1-(X2-X1)*.01
900   FOR I=1 TO Ly
910   MOVE X,Ycoord(I)
920   LABEL Ylabel$(I)
930   NEXT I
940   LDIR PI/2.
950   CSIZE 3.,.5
960   LORG 5
970   MOVE X1-.15*(X2-X1),.5*(Y1+Y2)
980   LABEL B$
990   ! PRINT "VS36"

```

```
1000 Dz=(Zmax-Zmin)/Nb
1010 FOR I=1 TO Nb
1020 T=Edf(I)
1030 IF T=0. THEN 1050
1040 PLOT Zmin+Dz*I,T
1050 NEXT I
1060 PENUP
1070 FOR I=1 TO Nb
1080 T=Edf(I+Nb)
1090 IF T=0. THEN 1110
1100 PLOT Zmin+Dz*I,T
1110 NEXT I
1120 PENUP
1130 K=Nb*2
1140 FOR I=1 TO Nb
1150 T=Edf(I+K)
1160 IF T=0. THEN 1180
1170 PLOT Zmin+Dz*I,T
1180 NEXT I
1190 PENUP
1200 K=Nb*3
1210 FOR I=1 TO Nb
1220 T=Edf(I+K)
1230 IF T=0. THEN 1250
1240 PLOT Zmin+Dz*I,T
1250 NEXT I
1260 PENUP
1270 K=Nb*4
1280 FOR I=1 TO Nb
1290 T=Edf(I+K)
1300 IF T=0. THEN 1320
1310 PLOT Zmin+Dz*I,T
1320 NEXT I
1330 PENUP
1340 K=Nb*5
1350 FOR I=1 TO Nb
1360 T=Edf(I+K)
1370 IF T=0. THEN 1390
1380 PLOT Zmin+Dz*I,T
1390 NEXT I
1400 PENUP
1410 K=Nb*6
1420 FOR I=1 TO Nb
1430 T=Edf(I+K)
1440 IF T=0. THEN 1460
1450 PLOT Zmin+Dz*I,T
1460 NEXT I
1470 PENUP
```

```

1480 K=Nb*7
1490 FOR I=1 TO Nb
1500 T=Edf(I+K)
1510 IF T=0. THEN 1530
1520 PLOT Zmin+Dz*I,T
1530 NEXT I
1540 PENUP
1550 K=Nb*8
1560 FOR I=1 TO Nb
1570 T=Edf(I+K)
1580 IF T=0. THEN 1600
1590 PLOT Zmin+Dz*I,T
1600 NEXT I
1610 PENUP
1620 K=Nb*9
1630 FOR I=1 TO Nb
1640 T=Edf(I+K)
1650 IF T=0. THEN 1670
1660 PLOT Zmin+Dz*I,T
1670 NEXT I
1680 PENUP
1690 K=Nb*10
1700 FOR I=1 TO Nb
1710 T=Edf(I+K)
1720 IF T=0. THEN 1740
1730 PLOT Zmin+Dz*I,T
1740 NEXT I
1750 PENUP
1760 PAUSE
1770 PRINTER IS CRT
1780 PLOTTER 505 IS TERMINATED
1790 END
1800 !
1810 DEF FNInuph(X) AMS 55, 26.2.23
1820 IF X=.5 THEN RETURN 0.
1830 P=MIN(X,1.-X)
1840 T=-LOG(P)
1850 T=SQR(T+T)
1860 P=1.+T*(1.432788+T*(1.189269+T*(.001308)))
1870 P=T*(2.515517+T*(1.802853+T*(.010328))) P
1880 IF X<.5 THEN P=-P
1890 RETURN P
1900 FNEND

```

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