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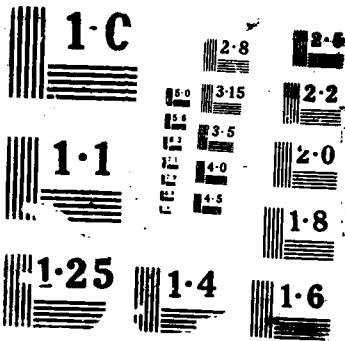
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## EXISTENCE OF SOLUTIONS TO THE CHARNEY MODEL OF THE GULF STREAM

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V. BARCILON†, P. CONSTANTIN‡ AND E.S. TITI‡

**Abstract.** The existence of weak solutions to the equations proposed by Charney (1955) for the inertial model of the Gulf Stream are established by means of the method of artificial viscosity.

**Key words.** Navier-Stokes equations, artificial viscosity, ocean circulation

**AMS(MOS) subject classifications.** 35Q10, 76U05

**1. Introduction.** In this paper, we examine mathematical properties of an equation arising in the theory of ocean circulation. In order to understand the role of this problem in oceanography, a brief review of the subject is necessary. The first successful attempt to provide a mathematical model of the mid-latitude ocean currents was made by Stommel [11] in 1948. He showed conclusively that a Gulf-Stream-like intensification on the western side of an ocean basin could be explained by the so-called  $\beta$ -effect. This is the geophysical terminology for the latitudinal variation of the normal component of the Earth's rotation. Aside from this variable Coriolis force, the other forces which entered into Stommel's model were those due to the pressure gradient, the surface winds and friction. This last force was taken to be proportional to the velocity fields. All the effects of density stratification were neglected by making the assumption that the ocean was homogeneous. Finally, by working with vertical averages, Stommel essentially treated the ocean circulation as a two-dimensional horizontal motion. Somewhat surprisingly, Stommel's *ad hoc* linear model was shown later to provide an accurate description of an experimental set-up [10].

The subsequent work in the field has attempted to overcome the two oversimplifications of Stommel's model. Namely, to take into account inertial forces which are known to be important in Gulf-Stream like boundary layers and to represent more adequately the complex mixing/dissipative processes. These two broad generalizations were initiated by Charney [4] and Munk & Carrier [9] respectively. Although these early papers make use of analytical techniques, the bulk of the recent work in this field has relied on numerical techniques to study the nonlinear problems arising in the mathematical formulation of wind-driven ocean circulation models. The papers by Bryan [3] and Veronis [13],[14] fall

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within this category.

In this paper, we study the Charney's extension of Stommel's model, namely

$$(1.1) \quad \epsilon \Delta \psi + \frac{\partial \psi}{\partial x_1} + R \left( \frac{\partial \psi}{\partial x_1} \frac{\partial \Delta \psi}{\partial x_2} - \frac{\partial \psi}{\partial x_2} \frac{\partial \Delta \psi}{\partial x_1} \right) = f$$

in a bounded domain  $\Omega$  in  $R^2$  with Dirichlet boundary conditions. We prove existence of weak solutions to (1.1). These weak solutions are obtained as limits of solutions of auxiliary equations with artificial viscosity and artificial boundary conditions. Physically, this auxiliary problem corresponds to the addition of side wall friction and stress-free boundary conditions. We derive uniform  $L^\infty$  bounds for the solutions of the auxiliary equations. This procedure cannot yield classical solutions because the artificial boundary conditions give rise to boundary layers in the classical limit. Nevertheless, we expect the solutions to be classical in certain parameter ranges.

The same method was used by Yudovitch [15] in his paper dealing with the time dependent two-dimensional Euler equations. This method cannot be extended to the time independent Euler equations. Actually, it is known [8] that the time independent Euler equations have no solutions for two and three dimensional axisymmetric domains and generic driving forces. However, because of the presence of the bottom friction term  $\epsilon \Delta \psi$ , the method can be used for equation (1.1).

Once the estimates for the stationary problem (1.1) are understood, the time dependent version of this problem

$$(1.2) \quad \frac{\partial \Delta \psi}{\partial t} + \epsilon \Delta \psi + \frac{\partial \psi}{\partial x_1} + R \left( \frac{\partial \psi}{\partial x_1} \frac{\partial \Delta \psi}{\partial x_2} - \frac{\partial \psi}{\partial x_2} \frac{\partial \Delta \psi}{\partial x_1} \right) = f$$

with given initial conditions on  $\Delta \psi$  can be obtained by applying in a straightforward manner the method of Yudovitch. The  $\beta$ -effect term  $\partial \psi / \partial x_1$  does not create any serious difficulties. One can show that (1.2) has unique global solutions.

The problem of describing the set of stationary solutions is still open. We expect the solution to be unique only when  $\epsilon$  is large compared to  $R$

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2. Preliminaries. Let  $\Omega$  be an open bounded set of  $\mathbb{R}^2$  with sufficiently smooth boundary  $\partial\Omega$ . We consider the system

$$(2.1) \quad \begin{cases} \epsilon\omega + \frac{\partial\psi}{\partial x_1} + R \left( \frac{\partial\psi}{\partial x_1} \frac{\partial\omega}{\partial x_2} - \frac{\partial\psi}{\partial x_2} \frac{\partial\omega}{\partial x_1} \right) = f & \text{in } \Omega \\ \Delta\psi = \omega & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega \end{cases}$$

where  $f(x), \epsilon > 0$  and  $R \geq 0$  are given.

In this paper we study the existence of weak solutions of (2.1).

We denote by  $H^s(\Omega)$  the usual Sobolev spaces of order  $s$  and by  $H_0^1(\Omega)$  the closure of  $C_0^\infty(\Omega)$  in the  $H^1(\Omega)$  norm. We define

$$A = -\Delta$$

to be the negative Laplacian with domain  $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ . It is well known that  $A^{-1}$  is a compact linear self-adjoint positive operator in  $L^2(\Omega)$  (cf. [2], [7]). The spectrum of  $A$  consists of an infinite sequence  $0 < \lambda_1 \leq \lambda_2 \leq \dots$  of eigenvalues counted according to their multiplicities;  $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ ; the eigenfunctions  $\{w_n\}$  provide an orthonormal basis in  $L^2(\Omega)$ . Finally, there exists a scale invariant constant  $c_0$  such that  $|\Omega| = c_0 \lambda_1^{-1}$  where  $|\Omega|$  denotes the area of  $\Omega$ . The scalar product and norm in  $L^2(\Omega)$  are denoted by  $(\cdot, \cdot)$  and  $|\cdot|$  respectively. The scalar product in  $H_0^1(\Omega)$ , in view of Poincaré's inequality, is

$$((u, v)) = \int_{\Omega} \nabla u(x) \cdot \nabla v(x) dx \quad \forall u, v \in H_0^1(\Omega),$$

and the corresponding norm is denoted by  $\|\cdot\|$ . The norm in  $L^p(\Omega)$  for  $1 \leq p \leq \infty$  is denoted by  $\|\cdot\|_p$ . It is known (cf [2]) that on  $D(A)$  the  $H^2(\Omega)$  norm is equivalent to the  $|A \cdot|$  norm, i.e. there exists a constant  $c_1 > 0$  such that

$$c_1^{-1} |Au| \leq \|u\|_{H^2(\Omega)} \leq c_1 |Au| \quad \forall u \in D(A).$$

Moreover,  $D(A^{1/2}) = H_0^1(\Omega)$  and  $\|\cdot\| = |A^{1/2} \cdot|$ .

The bilinear bounded operator  $J : H^1(\Omega) \times H^1(\Omega) \rightarrow L^1(\Omega)$  is defined as

$$(2.2) \quad J(\psi, \omega) = \frac{\partial \psi}{\partial x_1} \frac{\partial \omega}{\partial x_2} - \frac{\partial \psi}{\partial x_2} \frac{\partial \omega}{\partial x_1} \quad \forall \quad \psi, \omega \in H^1(\Omega).$$

We need the following inequalities which correspond to various continuity properties of the operator  $J$ .

**Proposition 2.1.** *Let  $s_1, s_2, s_3 \geq 0$  satisfy*

$$s_1 + s_2 + s_3 \geq 1 \quad \text{if } s_i \neq 1 \quad \text{for all } i = 1, 2, 3$$

and

$$s_1 + s_2 + s_3 > 1 \quad \text{if } s_i = 1 \quad \text{for some } i = 1, 2, 3.$$

Then, there exists a scale invariant constant  $c(s_1, s_2, s_3)$  such that

$$(2.3) \quad |(J(\psi, \omega), v)| \leq c(s_1, s_2, s_3) \lambda_1^{(s_1+s_2+s_3)/2} \|\psi\|_{H^{s_1+1}(\Omega)} \cdot \|\omega\|_{H^{s_2+1}(\Omega)} \cdot \|v\|_{H^{s_3}(\Omega)}.$$

The reader is referred to [12] and [5] for the idea of the proof. We also note that

$$(2.4) \quad (J(\psi, \omega), v) = -(J(\omega, \psi), v)$$

for every  $\psi \in H^{s_1+1}(\Omega)$ ,  $\omega \in H^{s_2+1}(\Omega)$  and  $v \in H^{s_3}(\Omega)$ .

**Proposition 2.2.** *For every  $\psi \in H^2(\Omega)$ ,  $\omega \in H^1(\Omega)$  and  $v \in H_0^1(\Omega)$  we have*

$$(2.5) \quad (J(\psi, \omega), v) = -(J(\psi, v), \omega).$$

Proof: Because of proposition 2.1 it is sufficient to show that (2.5) holds for  $v \in C_0^\infty(\Omega)$  and  $\psi, \omega \in C^\infty(\Omega)$ . Let  $\vec{u}$  be the vector  $(-\partial\psi/\partial x_2, \partial\psi/\partial x_1)$ ; then one can write

$$\begin{aligned} (J(\psi, \omega), v) &= \int_{\Omega} \nabla \cdot (\omega(x)\vec{u}(x)) v(x) dx \\ &= - \int_{\Omega} (\vec{u}(x) \cdot \nabla v(x)) \omega(x) dx . \end{aligned}$$

The following corollary is an immediate consequence of the above proposition.

**Corollary 2.3.**

$$(2.6) \quad (J(\psi, \omega), \omega) = 0 \quad \forall \psi \in H^2(\Omega), \omega \in H_0^1(\Omega)$$

$$(2.7) \quad (J(\psi, \omega), \omega^p) = 0 \quad \forall \psi \in H^2(\Omega), \omega \in D(A), \quad p = 1, 2, \dots$$

Proof: (2.6) is a direct consequence of (2.5). Since  $H^2(\Omega)$  is a Banach algebra (cf. [1]) one can easily verify that  $\omega^p \in D(A)$ , for  $p = 1, 2, \dots$ , whenever  $\omega \in D(A)$ . We can then use (2.5) to deduce that

$$(J(\psi, \omega), \omega^p) = -(J(\psi, \omega^p), \omega),$$

while from direct computations

$$(J(\psi, \omega^p), \omega) = -p(J(\psi, \omega), \omega), \quad p = 1, 2, \dots$$

which verifies (2.7), in view of (2.6). Let us note that (2.6) also holds for  $\psi \in C^1(\bar{\Omega})$ ,  $\omega \in C^1(\bar{\Omega})$  and  $\nabla\psi$  normal to  $\partial\Omega$ .

**3. A stationary problem with artificial viscosity.** In this section we consider a singular perturbation of the stationary problem (2.1) obtained by adding an artificial viscosity term  $-\nu A\omega$  and imposing homogeneous Dirichlet boundary conditions on  $\omega$ . The artificial viscosity equation provides an approximate solution. Uniform bounds, independent of the viscosity  $\nu$  will enable us to pass to the weak limit. In this process the boundary condition on  $\omega$  is lost. The auxiliary problem is:

$$(3.1a) \quad -\nu\Delta\omega + \epsilon\omega + \partial\psi/\partial x_1 + RJ(\psi, \omega) = f \quad \text{in } \Omega$$

$$(3.1b) \quad -\Delta\psi + \omega = 0 \quad \text{in } \Omega$$

$$(3.1c) \quad \psi = 0, \omega = 0 \quad \text{on } \partial\Omega,$$

where  $\nu$  is given such that  $0 < \nu < \epsilon^3/8$  and  $f \in L^\infty(\Omega)$ .

**3.1 Existence of solutions of the perturbed problem.** We restate the problem thus: find  $(\omega, \psi) \in D(A) \times D(A)$  such that

$$(3.2a) \quad \nu A\omega + \epsilon\omega + \partial\psi/\partial x_1 + RJ(\psi, \omega) = f,$$

$$(3.2b) \quad A\psi + \omega = 0.$$

It is obvious that every solution  $(\omega, \psi)$  of (3.2) is a solution to (3.1) in the distribution sense.

The following *a priori* estimates of the solutions of (3.2) will be needed.

**Lemma 3.1.** *Let  $(\omega, \psi) \in D(A) \times D(A)$  be a solution of (3.2). Then*

$$(3.3) \quad \nu\|\omega\|^2 + \frac{\epsilon}{4}|\omega|^2 \leq \frac{|f|^2}{\epsilon} K_1(\epsilon, \lambda_1),$$

$$(3.4) \quad \|\psi\|^2 \leq 2|f|^2 K_1(\epsilon, \lambda_1),$$

$$(3.5) \quad |A\psi|^2 \leq \frac{4|f|^2}{\epsilon^2} K_1(\epsilon, \lambda_1),$$

and

$$(3.6) \quad \nu|A\omega| \leq 2|f| + 2\sqrt{2}|f|K_1^{1/2}(\epsilon, \lambda_1) \left[ 1 + \frac{R^2 c^2(1/2, 1/2, 0)c_2^2}{\nu^{3/2}\epsilon^{1/2}} |f|^2 K_1(\epsilon, \lambda_1) \right],$$

where

$$K_1(\epsilon, \lambda_1) = \left( 1 + \frac{1}{\epsilon^2 \lambda_1} \right).$$

**Proof:** Let  $(\omega, \psi) \in D(A) \times D(A)$  be a solution to (3.2). Taking the scalar product of (3.2a) with  $\psi$  we get:

$$(3.7) \quad \nu(A\omega, \psi) + \epsilon(\omega, \psi) + \left( \frac{\partial \psi}{\partial x_1}, \psi \right) + R(J(\psi, \omega), \psi) = (f, \psi).$$

In view of (2.4) and (2.5) and of the fact that  $(\partial \psi / \partial x_1, \psi) = 0$ , (3.7) reduces to

$$\nu(A\omega, \psi) + \epsilon(\omega, \psi) = (f, \psi),$$

or because of (3.2b)

$$\epsilon \|\psi\|^2 \leq \nu |\omega|^2 + |f| |\psi|.$$

Using first Poincaré's inequality we can write

$$\epsilon \|\psi\|^2 \leq \nu |\omega|^2 + |f| \lambda_1^{-1/2} \|\psi\|$$

then by Young's inequality

$$\epsilon \|\psi\|^2 \leq \nu |\omega|^2 + \frac{|f|^2}{2\epsilon \lambda_1} + \frac{\epsilon}{2} \|\psi\|^2.$$

and finally since  $0 \leq \nu \leq \epsilon^3/8$ , we deduce

$$\|\psi\|^2 \leq (2\nu/\epsilon) |\omega|^2 + \frac{|f|^2}{\epsilon^2 \lambda_1} \leq \frac{\epsilon^2}{4} |\omega|^2 + \frac{|f|^2}{\epsilon^2 \lambda_1}.$$

If we take the scalar product of (3.2a) with  $\omega$  and use (2.6) we obtain

$$\begin{aligned} \nu \|\omega\|^2 + \epsilon |\omega|^2 &= -(\partial\psi/\partial x_1, \omega) + (f, \omega) \\ &\leq \|\psi\| |\omega| + |f| |\omega|, \end{aligned}$$

and by Young's inequality

$$\nu \|\omega\|^2 + \frac{\epsilon}{2} |\omega|^2 \leq \frac{\|\psi\|^2}{\epsilon} + \frac{|f|^2}{\epsilon}.$$

From the above inequality and (3.8) we deduce (3.3). From (3.3) and (3.8) we can obtain easily (3.4). Finally, (3.2b) and (3.3) directly imply (3.5).

To establish (3.6) we take the scalar product of (3.2a) with  $A\omega$ , from which we see that

$$\begin{aligned} \nu |A\omega|^2 + \epsilon \|\omega\|^2 &\leq |(\partial\psi/\partial x_1, A\omega)| + R|(J(\psi, \omega), A\omega)| + |(f, A\omega)|, \\ &\leq \|\psi\| |A\omega| + R|(J(\psi, \omega), A\omega)| + |(f, A\omega)|. \end{aligned}$$

By using (2.3) for  $s_1 = s_2 = 1/2$  and  $s_3 = 0$ , we get

$$\nu |A\omega|^2 + \epsilon \|\omega\|^2 \leq (\|\psi\| + R c(1/2, 1/2, 0) \|\psi\|_{H^{3/2}(\Omega)} \|\omega\|_{H^{3/2}(\Omega)} + |f|) |A\omega|.$$

and by an interpolation inequality

$$\nu |A\omega|^2 + \epsilon \|\omega\|^2 \leq (\|\psi\| + Rc(1/2, 1/2, 0) c_2 \|\psi\|^{1/2} \|A\psi\|^{1/2} \|\omega\|^{1/2} |A\omega|^{1/2} + |f|) |A\omega|,$$

and hence

$$\frac{\nu}{2} |A\omega|^2 + \epsilon \|\omega\|^2 \leq (\|\psi\| + \frac{R^2}{2\nu} c^2(1/2, 1/2, 0) c_2 \|\psi\| \|A\psi\| \|\omega\| + |f|) |A\omega|.$$

Making use of (3.3), (3.4) and (3.5) yields (3.6).

**Lemma 3.2.** *There exists at least one solution to problem (3.2). Moreover, every solution of (3.2) satisfies (3.3)-(3.6).*

**Proof:** One approach is to use the Galerkin approximation method based on the eigenfunctions of the operator  $A$  as in Constantin & Foias [5] and Temam [12] to show the existence of a solution to (3.2). The crucial point in this approach is to establish similar *a priori* estimates to the ones in (3.5)-(3.6) for the approximate solution.

Another approach is by the Leray-Schauder degree theory. Indeed, problem (3.2) is equivalent to

$$\begin{pmatrix} \psi \\ \omega \end{pmatrix} + \begin{bmatrix} A^{-1} & 0 \\ 0 & A^{-1} \end{bmatrix} K(\psi, \omega) = \begin{pmatrix} A^{-1}f \\ 0 \end{pmatrix}$$

where

$$K(\psi, \omega) = \begin{pmatrix} \nu^{-1}[\epsilon\omega + \partial\psi/\partial x_1 + R J(\psi, \omega)] \\ \omega \end{pmatrix}.$$

By Rellich's Lemma and (2.3), one can show that the nonlinear mapping

$$K : D(A) \times D(A) \rightarrow L^2(\Omega) \times L^2(\Omega)$$

is compact. Therefore, making use of the *a priori* estimates (3.5)-(3.6), we can conclude that (3.9) has at least one solution.

**Remark 3.3:** If  $f \in C^\infty(\bar{\Omega})$ , then every solution  $(\omega, \psi) \in D(A) \times D(A)$  of (3.2) satisfies (3.1) in the classical sense, namely  $\psi, \omega \in C^\infty(\bar{\Omega})$ .

**3.2 Uniform  $L^\infty$  bounds for the artificial viscosity problem** In (3.3), (3.4) and (3.5) we gave estimates for  $|\omega|$ ,  $\|\psi\|$  and  $|A\psi|$  which were independent of  $\nu$ . In this section we shall derive uniform (i.e. independent of  $\nu$ )  $L^\infty$  estimates for every solution of (3.2) and for  $0 < \nu < \epsilon^3/8$ . We recall first the following known result in potential theory (see e.g. [6]).

**Theorem 3.4.** Let  $G(x, y)$  denote the Green function of the Laplacian operator in the domain with Dirichlet boundary conditions. Then, there exists two scale invariant constants  $c_3, c_4$  such that:

$$(3.10) \quad |G(x, y)| \leq c_3 (1 + |\log(\lambda_1^{1/2}|x - y|)|) \quad \forall x, y \in \Omega, x \neq y,$$

and

$$(3.11) \quad \left| \frac{\partial G(x, y)}{\partial x_k} \right| \leq \frac{c_4}{|x - y|}, \quad \forall k = 1, 2, \quad x, y \in \Omega, x \neq y.$$

Moreover since

$$\phi(x) = \int_{\Omega} G(x, y) \Delta \phi(y) dy,$$

it follows that

$$(3.12) \quad \frac{\partial \phi(x)}{\partial x_k} = \int_{\Omega} \frac{\partial G(x, y)}{\partial x_k} \Delta \phi(y) dy.$$

**Theorem 3.5 (Uniform  $L^\infty$  bounds).** Let  $(\omega, \psi) \in D(A) \times D(A)$  be a solution of (3.2) (or equivalently a solution of (3.1)). Then:

$$(3.13) \quad \|\omega\|_\infty \leq \frac{2}{\epsilon} \|f\|_\infty + \frac{64 c_4 c_0^{1/2}}{\epsilon^3 \lambda_1^{1/2}} |f| K_1^{1/2}(\epsilon, \lambda_1)$$

and

$$(3.14) \quad \|\nabla\psi\|_\infty \leq 4\sqrt{2\pi}c_4 c_0^{1/4} \lambda_1^{-1/4} \sqrt{\|\omega\|_\infty |\omega|}.$$

Note that (3.3) provides a uniform upper bound for  $|\omega|$ .

**Proof:** Let  $\delta > 0$ . We denote by  $B_\delta(z)$  the ball in  $\mathbb{R}^2$  that is centered at  $z$  with radius  $\delta$ . From (3.1a) and (3.13) we have

$$\nabla\psi(x) = \int_\Omega \nabla_x G(x, y) \omega(y) dy,$$

which on account of (3.11) implies

$$|\nabla\psi(x)| \leq 2c_4 \int_\Omega \frac{|\omega(y)|}{|x-y|} dy,$$

or

$$|\nabla\psi(x)| \leq 2c_4 \left\{ \int_{\Omega \cap B_\delta(x)} \frac{|\omega(y)|}{|x-y|} dy + \int_{\Omega \setminus B_\delta(x)} \frac{|\omega(y)|}{|x-y|} dy \right\}.$$

Using Hölder's inequality, we see that for odd integers  $p \geq 3$

$$|\nabla\psi(x)| \leq 2c_4 \left\{ \|\omega\|_{p+1} \left( \int_{B_\delta(x)} |x-y|^{-(p+1)/p} dy \right)^{p/p+1} + \frac{|\omega|}{\delta} |\Omega|^{1/2} \right\},$$

i.e.

$$|\nabla\psi(x)| \leq 2c_4 \left\{ \|\omega\|_{p+1} (2\pi)^{p/p+1} \delta^{(p-1)/(p+1)} + \frac{|\omega|}{\delta} |\Omega|^{1/2} \right\}.$$

and therefore

$$(3.15) \quad \|\nabla\psi\|_\infty \leq (4\pi c_4) \|\omega\|_{p+1} \delta^{p-1/p+1} + 2c_4 |\Omega|^{1/2} \frac{|\omega|}{\delta}.$$

In order to get an estimate for  $\|\omega\|_{p+1}$ , we multiply (3.1a) by  $\omega^p(x)$  and integrate over  $\Omega$ . This leads to

$$-\nu \int_\Omega \Delta\omega(x) \omega^p(x) dx + \epsilon \int_\Omega \omega^{p+1}(x) dx + R(J(\psi, \omega), \omega^p) = -(\partial\psi/\partial x_1, \omega^p) + (f, \omega)$$

or, after integrating by parts and using (2.7), to

$$p\nu \int_\Omega |\nabla\omega(x)|^2 \omega^{p-1}(x) dx + \epsilon \int_\Omega \omega^{p+1}(x) dx + R(J(\psi, \omega), \omega^p) = -(\partial\psi/\partial x_1, \omega^p) + (f, \omega).$$

Using Hölder's inequality as well as the fact that  $p$  is an odd integer, we conclude that

$$\epsilon \|\omega\|_{p+1}^{p+1} \leq \|\nabla \psi\|_{\infty} \|\omega\|_p^p + \|f\|_{p+1} \|\omega\|_{p+1}^p$$

or

$$(3.16) \quad \epsilon \|\omega\|_{p+1}^{p+1} \leq (\|\nabla \psi\|_{\infty} |\Omega|^{1/p+1} + \|f\|_{p+1}) \|\omega\|_{p+1}^p.$$

Substituting (3.15) in (3.16) we see that

$$(3.17) \quad \epsilon \|\omega\|_{p+1} \leq \|f\|_{p+1} + (4\pi c_4) |\Omega|^{1/(p+1)} \delta^{(p-1)/(p+1)} \|\omega\|_{p+1} + 2c_4 |\Omega|^{(p+3)/2(p+1)} \frac{|\omega|}{\delta}.$$

If we choose  $\delta = (\epsilon/8\pi c_4) |\Omega|^{-1/(p-1)}$ , then (3.17) implies that

$$(3.19) \quad \|\omega\|_{p+1} \leq \frac{2}{\epsilon} \|f\|_{p+1} + \frac{4\pi c_4}{\epsilon} \left( \frac{8\pi c_4}{\epsilon} |\Omega|^{1/2} \right)^{p+1/p-1} |\Omega|^{1/p+1} |\omega|.$$

Passing to the limit as  $p \rightarrow \infty$ , we conclude that

$$(3.19) \quad \|\omega\|_{\infty} \leq \frac{2}{\epsilon} \|f\|_{\infty} + \frac{32\pi c_4^2}{\epsilon^2} |\Omega|^{1/2} |\omega|.$$

After replacing  $|\Omega|$  in the above expression by  $c_0 \lambda_1^{-1}$  and using (3.3), we obtain (3.13).

In order to derive (3.14), we observe that (3.15) holds for every  $\delta > 0$ . Therefore

$$(3.20) \quad \|\nabla \psi\|_{\infty} \leq (4\pi c_4) \|\omega\|_{\infty} \delta + 2c_4 |\Omega|^{1/2} \frac{|\omega|}{\delta}.$$

Minimizing the right hand side, we deduce that

$$\|\nabla \psi\|_{\infty} \leq 4\sqrt{2\pi} c_4 |\Omega|^{1/4} \sqrt{\|\omega\|_{\infty} |\omega|},$$

and after replacing  $|\Omega|$  by  $c_0 \lambda_1^{-1}$ , we obtain (3.14).

**4. Existence of weak solutions.** Let  $\{\nu_k\}_{k=1}^{\infty}$  be an arbitrary sequence of real numbers,  $\nu_k \in (0, \epsilon^3/8)$  for  $k = 1, 2, \dots$ , and such that  $\nu_k \rightarrow 0$  as  $k \rightarrow \infty$ . Let  $(\omega_k, \psi_k)$  be a sequence of solutions to problem (3.1) corresponding to  $\nu = \nu_k$  for  $k = 1, 2, \dots$  respectively. By Lemma 3.2, we know that such sequences exist.

**Theorem 4.1.** *There exists a  $\psi \in D(A)$  and an  $\omega \in L^\infty(\Omega)$  that are weak solutions of the stationary problem (2.1), i.e.*

$$(4.1) \quad \begin{cases} \epsilon \omega + \partial\psi/\partial x_1 + R \nabla \cdot (\omega \bar{u}) = f & \text{in } \Omega \\ \Delta\psi = \omega & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega \end{cases}$$

where  $\bar{u} = (-\partial\psi/\partial x_2, \partial\psi/\partial x_1)$ . Furthermore,  $\psi$  and  $\omega$  satisfy:

$$(4.2) \quad |\omega| \leq \frac{2|f|}{\epsilon} K_1^{1/2}(\epsilon, \lambda_1),$$

$$(4.3) \quad \|\omega\|_\infty \leq \frac{2}{\epsilon} \|f\|_\infty + \frac{64 c_4 c_0^{1/2}}{\epsilon^3 \lambda_1^{1/2}} |f| K_1^{1/2}(\epsilon, \lambda_1),$$

$$(4.4) \quad \|\nabla\psi\|_\infty \leq 4\sqrt{2\pi} c_4 c_0^{1/4} \lambda_1^{-1/4} \sqrt{\|\omega\|_\infty |\omega|}.$$

Proof: Let  $(\omega_k, \psi_k)$  be as mentioned above. From (3.5) we know that

$$|A\psi_k| \leq \frac{2|f|}{\epsilon} K_1^{1/2}(\epsilon, \lambda_1).$$

Therefore, there exists a subsequence, say  $\{\psi_{k_1}\}$ , which converges weakly in  $H^2(\Omega)$  to  $\psi \in H^2(\Omega)$ , and by virtue of Rellich's Lemma,  $\psi_{k_1}$  converges strongly to  $\psi \in H^1(\Omega)$ , and hence  $\psi \in D(A)$ . From (3.13) we know that

$$\|\omega_{k_1}\|_\infty \leq \frac{2}{\epsilon} \|f\|_\infty + \frac{64 c_4 c_0^{1/2}}{\epsilon^3 \lambda_1^{1/2}} |f| K_1^{1/2}(\epsilon, \lambda_1),$$

and therefore

$$(4.5) \quad \|\omega_{k_1}\|_p \leq |\Omega|^{1/p} \left( \frac{2}{\epsilon} \|f\|_\infty + \frac{64 c_4 c_0^{1/2}}{\epsilon^3 \lambda_1^{1/2}} |f| K_1^{1/2}(\epsilon, \lambda_1) \right) \quad \forall p = 1, 2, \dots$$

By means of (4.5) we can inductively find, for every  $p = 2, 3, \dots$  a subsequence  $\{\omega_{k_p}\}$  of  $\{\omega_k\}$  which converges weakly in  $L^p(\Omega)$  to the same limit  $\omega \in L^p(\Omega)$  for every  $p = 2, 3, \dots$ . In particular, we have

$$\|\omega\|_p \leq \liminf_{k_p \rightarrow \infty} \|\omega_{k_p}\| \quad \forall p = 2, 3, \dots$$

which by (4.5) entails that

$$(4.6) \quad \|\omega\|_p \leq |\Omega|^{1/p} \left( \frac{2}{\epsilon} \|f\|_\infty + \frac{64 c_4 c_0^{1/2}}{\epsilon^3 \lambda_1^{1/2}} |f| K_1^{1/2}(\epsilon, \lambda_1) \right) \quad \forall p = 2, 3, \dots$$

Passing to the limit, we deduce (4.3). The derivations of (4.2) and (4.4) follow simply from (3.3) and (3.14). We shall omit their proofs.

In conclusion, the diagonal subsequence  $\{\omega_{k_h}\}$  converges weakly to  $\omega \in L^p(\Omega)$  for every  $p = 2, 3, \dots$ , and  $\{\psi_{k_h}\}$  converges weakly to  $\psi \in H^2(\Omega)$  and strongly in  $H^1(\Omega)$ . Because  $\nabla \psi_{k_h}$  converges strongly in  $L^2(\Omega)$  to  $\nabla \psi$  and  $\omega_{k_h}$  converges weakly in  $L^2(\Omega)$ , it follows that the product  $(\nabla \psi_{k_h})(\omega_{k_h})$  converges to  $(\nabla \psi)(\omega)$  in distribution sense. Since  $\{\omega_{k_h}, \psi_{k_h}\}$  solves (3.1) for  $\nu = \nu_{k_h}$ , by passing to the limit we can verify that  $(\omega, \psi)$  solves (4.1).

Up to now, the parameter  $R$  did not enter at all in our estimates. In the next theorem, we give an upper bound to the "diameter" of the set of stationary solutions of problem 4.1.

**Theorem 4.3.** *Let  $(\psi_1, \omega_1)$  and  $(\psi_2, \omega_2)$  be two weak solutions to problem (4.1) satisfying (4.2)-(4.4). Then*

$$(4.7) \quad \|\psi_1 - \psi_2\| \leq \frac{4R}{\epsilon} |f|^2 K_1(\epsilon, \lambda_1) \left( \frac{2}{\epsilon} \|f\|_\infty + \frac{64 c_4 c_0^{1/2}}{\epsilon^3 \lambda_1^{1/2}} |f| K_1^{1/2}(\epsilon, \lambda_1) \right).$$

Proof: From (4.1) we have:

$$\epsilon \Delta(\psi_1 - \psi_2) + \frac{\partial}{\partial x_1}(\psi_1 - \psi_2) + R \operatorname{div}(\omega_1 \vec{u}_1 - \omega_2 \vec{u}_2) = 0,$$

or

$$\epsilon \Delta(\psi_1 - \psi_2) + \frac{\partial}{\partial x_1}(\psi_1 - \psi_2) + R \operatorname{div}(\omega_1(\vec{u}_1 - \vec{u}_2) + (\omega_1 - \omega_2)\vec{u}_2) = 0.$$

We form the scalar product with  $(\psi_1 - \psi_2)$  and, since  $\psi_1 - \psi_2 \in H_0^1(\Omega)$ , we can integrate by parts and obtain:

$$(4.8) \quad \epsilon \|\psi_1 - \psi_2\|^2 + R(\vec{u}_2(\omega_1 - \omega_2), \nabla(\psi_1 - \psi_2)) = 0.$$

From (4.8) we get:

$$\epsilon \|\psi_1 - \psi_2\|^2 \leq R \|\omega_1 - \omega_2\|_\infty |\bar{u}_2| \|\psi_1 - \psi_2\|,$$

and hence

$$(4.9) \quad \|\psi_1 - \psi_2\| \leq \frac{R}{\epsilon} \|\omega_1 - \omega_2\|_\infty \|\psi_2\|.$$

As a consequence of Lemma 3.1 and Theorem 4.1,

$$\|\psi_2\| \leq 2|f|^2 K_1(\epsilon, \lambda_1)$$

Substituting this expression and (4.3) in (4.9) we arrive at (4.7).

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