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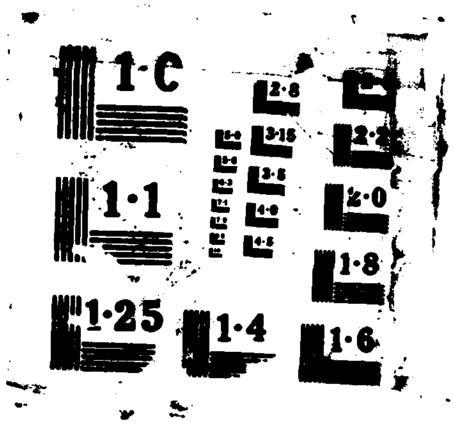
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AIMING CONTROL

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AIMING CONTROL

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ABSTRACT

The problem of aiming control is formulated as the problem of residence time controllability in dynamical systems with stochastic perturbations. The solution is given for linear systems with small, additive, white noise perturbation. It is shown that the existence of the desired aiming controller depends on the relationship between the column spaces of the control and noise matrices. If the former includes the latter, any precision of aiming is possible. If this inclusion does not occur, the precision is bounded, and we give lower and upper estimates of this bound. For each of these cases, aiming controller design techniques are suggested and illustrative examples are considered. The development is based on an asymptotic version of the large deviations theory.

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I. THE PROBLEM

Given a controlled dynamical system with states $x(t) \in I\!R^n$, control $u(t) \in I\!R^m$ and disturbances $\xi(t) \in I\!R^r$, we define the *aiming process specifications* as a pair $\{\Omega, \tau\}$ where $\Omega \subset I\!R^n$ is the domain to which the states $x(t)$ should be confined and τ is the period of the confinement, i.e., $x(t) \in \Omega, \forall \; t \in [t_0, t_0+\tau]$, $t_0 \in I\!R_+$.

For example, in the problem of *telescope pointing* [1], the domain Ω is defined by the size of the film grain, and τ is defined by the time of the exposure. In the *laser beam pointing problem* [2], Ω is defined by the cross-section of the beam and the size of the target, whereas τ is defined by the duration of the pulse. In the *gun pointing problem* [3], Ω is defined by the size of the target and the power of the explosives, whereas τ is defined by the incidence time, i.e., time during which the shell travels in the barrel. In the *robot arm pointing problem* [4], Ω is defined by the relative sizes of the gripper and the object to be manipulated, and τ is defined by the duration of the task. In the *aircraft landing problem* [5], Ω is defined by the parameters of the aircraft and the touchdown area, whereas τ is defined by the landing period. In the *missile terminal guidance problem* [6], Ω is defined by the domain to which the line of sight rate should be confined, and τ is the period of the intercept.

Given a pair $\{\Omega, \tau\}$, the problem of *aiming control* is formulated as the problem of choosing a feedback control law, so as to force the states x to remain, at least on the average, in Ω during period τ, in spite of the disturbances $\xi(t)$ that are acting on the system.

Modern control theory does not offer tools for a direct solution of this problem. Indirect approaches, such as pole placement, LQG design, covariance control, H_2 and H_∞ minimization techniques do not seem to give explicit relationships with the residence time. Therefore, given a relative importance of pointing problems in modern technology, the development of a control theory for *aiming processes* seems desirable.

In this paper such a theory is developed for linear systems with small, additive, stochastic perturbations under the assumption that all states are available for control and the control law is of the form $u = Kx$. Results on output feedback will be considered in the sequel.

The approach developed in this paper is based on the first passage time theory for dynamical systems with random perturbations. Although the fundamentals of this theory have been known for a long time [7], only in recent years a powerful, constructive, asymptotic technique for residence time evaluation has been developed [8], [9]. However, with the exception of [10] and [11] where the stochastic stability problem was addressed, no control-theoretic properties of the residence time have been investigated, and no applications to controller design have been reported.

In the present paper, we apply the ideas and results of [7]-[9] to analysis of controllability properties of the residence time. Specifically, we show that the existence of the desired *aiming controller* depends on the relationship between the column spaces of the control and noise matrices. If the former includes the latter, *any precision of aiming* is possible (strong residence time controllability case). If this inclusion does not occur, the achievable precision is always bounded, and we give lower and upper estimates of this bound (weak residence time controllability case). For each, strong and weak residence time controllability, we give design techniques for *aiming controllers*. These techniques have a peculiarity that they may result in closed loop poles approaching the imaginary axis to ensure the largest possible residence time. This counter-intuitive behavior is explained on the basis of stable pole-zero cancellations in an auxiliary transfer matrix that characterizes the effect of the noise and its derivatives on the residence time.

The structure of the paper is as follows: in Section II mathematical preliminaries are presented, in Section III controllability properties of the residence time are described, Sections IV and V are devoted to aiming controller designs, and in Section VI conclusions are formulated; the proofs are given in Appendices 1-3.

II. PRELIMINARIES

Although the control technique discussed in this paper is presented for linear systems, the mathematical theory on which it is based applies to nonlinear systems as well. Since this theory is not widely known within the control community, we will review it briefly as it applies to nonlinear systems and then prove some new results that will be used in the subsequent sections.

Let $\Omega \subset I\!R^n$ be a bounded domain containing $x = 0$ in its interior and let $\partial\Omega$ be its boundary, which is assumed to be smooth. Consider the following stochastic differential equation

$$dx = f(x)dt + \varepsilon\sigma(x)dw \ , \ \ x(0) = x_0 \in \Omega \ , \tag{2.1}$$

where $x \in I\!R^n$, $f : I\!R^n \to I\!R^n$, $\sigma : I\!R^n \to I\!R^{n \times r}$, $0 < \varepsilon << 1$ and $w \in I\!R^r$ is an r-dimensional standard Brownian motion. Assume $f(\cdot)$ and $\sigma(\cdot)$ satisfy the Lipschitz and growth conditions [12]:

$$\|f(x)-f(y)\| + \|\sigma(x)-\sigma(y)\| \le k\|x-y\| , \quad x,y \in I\!R^n ,$$
$$\|f(x)\| + \|\sigma(x)\| \le k(1+\|x\|) \quad , \quad x,y \in I\!R^n .$$

Then the solution, $x(t)$, of (2.1) is well defined and is a Markov process on $I\!R^n$ with the following infinitesimal generator [9]:

$$L = \sum_{i=1}^{n} f_i(x)\frac{\partial}{\partial x_i} + \varepsilon^2 \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} , \tag{2.2}$$
$$a_{ij}(x) = \frac{1}{2}\,(\sigma(x)\sigma(x)^T)_{ij} .$$

Assume that 0 is an asymptotically stable equilibrium point of $\dot{x} = f(x)$ in Ω, and assume that Ω contains no other ω-limit sets of $\dot{x} = f(x)$. It is indicated in [7] that if $\sigma(0) \ne 0$, then $x(t)$ leaves Ω in finite time with probability one for all $x_0 \in \Omega$. The mean of the first time of exit of $x(t)$ from Ω, i.e.,

$$\bar{\tau}^{\varepsilon}(x_0) = E[\inf\{t : x(t) \in \partial\Omega\} \mid x(0) = x_0 \in \Omega] ,$$

is shown in [7]-[9] to satisfy the following boundary value problem

$$\begin{aligned} L\bar{\tau}^{\varepsilon}(x_0) &= -1 , \quad x_0 \in \Omega , \\ \bar{\tau}^{\varepsilon}(x_0) &= 0 , \quad x_0 \in \partial\Omega . \end{aligned} \tag{2.3}$$

In general it is difficult, or impossible, to find an exact solution of (2.3). However, for sufficiently small ε, asymptotic solution methods have been developed (see [8],[9]). These methods not only give an approximate value of $\bar{\tau}^{\varepsilon}(x_0)$ but also show that, roughly speaking, $\bar{\tau}^{\varepsilon}(x_0) \doteq \hat{\tau}^{\varepsilon} = const$ for all x_0 in Ω outside of a boundary layer around $\partial\Omega$. The value of $\hat{\tau}^{\varepsilon}$ is referred to as the *residence time in* Ω. More precise statements in this regard are given below.

Assume that $f(x)$ and $\sigma(x)$ are twice continuously differentiable in Ω. Furthermore, assume that $f(x)$ and $\sigma(x)$ satisfy the following conditions:

(i) For some constant $\alpha > 0$,

$$\sum_{i,j=1}^{n} a_{ij}(x)\mu_i\mu_j \geq \alpha\sum_{i=1}^{n}\mu_i^2 \quad , x \in \Omega \quad .$$

(ii) If $\dot{x} = f(x)$, $x(0) \in \Omega$, then $x(t) \to 0$ as $t \to \infty$. Furthermore, the Jacobian of $f(x)$ at $x = 0$ is Hurwitz.

(iii) If $n(x)$ is the outward normal of $\partial\Omega$, then $f^T(x)n(x) < 0$ for all $x \in \partial\Omega$.

Conditions (ii) and (iii) imply that Ω is an invariant set of $\dot{x} = f(x)$ with the maximal ω-limit set $\{x = 0\}$.

Assume that the first order partial differential equation

$$\sum_{i=1}^{n} f_i(x)\frac{\partial\phi}{\partial x_i} + \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j} = 0 \quad , \tag{2.4}$$

$$\phi(0) = 0 \quad ,$$

has a strictly positive definite solution in $\overline{\Omega}$ (the closure of Ω) and define the *logarithmic residence time in* Ω by

$$\hat{\phi}(\Omega) = \inf_{x \in \partial\Omega} \phi(x) \quad . \tag{2.5}$$

The following theorem was proven in [9].

Theorem 2.1: Assume that (i) - (iii) hold. Then

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln \bar{\tau}^{\varepsilon}(x_0) = \hat{\phi}(\Omega) \tag{2.6}$$

uniformly on compact subsets of Ω.

Theorem 2.1 states, in particular, that

$$\bar{\tau}^{\varepsilon}(x_0) = C(\varepsilon) e^{\hat{\phi}(\Omega)/\varepsilon^2}(1 + o(1)) \text{ as } \varepsilon \to 0 \quad .$$

If a more precise estimate of $\bar{\tau}^{\varepsilon}(x_0)$ than the logarithmic residence time, $\hat{\phi}(\Omega)$, given by Theorem 2.1 is desired, the preexponential factor $C(\varepsilon)$ can be obtained using the methods of [8]. Indeed, let $z(x)$ be the solution to the equation

$$\begin{gathered} \sum_{i=1}^{n} b_i(x) \frac{\partial z}{\partial x_i} + c(x) z = 0 \quad , \\ z(0) = 1 \quad , \end{gathered} \tag{2.7}$$

where

$$\begin{aligned} b_i(x) &= -2 \sum_{j=1}^{n} a_{ij}(x) \frac{\partial \phi}{\partial x_j} - f_i(x) \quad , \\ c(x) &= -\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j} - 2 \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_j} \frac{\partial \phi}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \quad . \end{aligned}$$

Then it can be proven [13] that $C(\varepsilon)$ satisfies

$$C(\varepsilon) = \frac{-\int_{\Omega} e^{-\phi(x)/\varepsilon^2} z(x) dx}{\int_{\partial\Omega} e^{(\hat{\phi} - \phi(x))/\varepsilon^2} z(x)(f^T(x) n(x)) dS_x} (1 + o(1)) = C_1(\varepsilon)(1 + o(1)) \, . \tag{2.8}$$

Since (2.6) and (2.8) are asymptotic in nature, it is of interest to evaluate the range of ε's for which the constant $\hat{\tau}^{\varepsilon}(\Omega) \triangleq C_1(\varepsilon) e^{\hat{\phi}(\Omega)/\varepsilon^2}$ is indeed close to $\bar{\tau}^{\varepsilon}(x_0)$, $x_0 \in \Omega$. Although a theoretical (extremely concervative) estimate can be derived, to illustrate the situation we give here the following example.

Example 2.1: Consider a scalar system

$$dx = f(x)dt + \varepsilon dw \ , \ x(0) = x_0 \in (-1, 1) \ . \tag{2.9}$$

For this system, (2.3) becomes a two point boundary value problem

$$\begin{gathered} f(x) \frac{d\overline{\tau}^{\varepsilon}}{dx} + \frac{\varepsilon^2}{2} \frac{d^2\overline{\tau}^{\varepsilon}}{dx^2} = -1 \ , x \in (-1, 1) \ , \\ \overline{\tau}^{\varepsilon}(-1) = \overline{\tau}^{\varepsilon}(1) = 0 \ . \end{gathered} \tag{2.10}$$

We divide the analysis into two parts. First we evaluate qualitively for which values of ε function $\overline{\tau}^{\varepsilon}(x_0)$ converges to a constant in the interval $(-1, 1)$ (excluding a small boundary layer), and second we will investigate how close this constant is to the one given in Theorem 2.1.

We consider two cases: $f(x) = -x$ and $f(x) = -x^3$. Even in these simplest situations, it is difficult to find an exact solution for $\overline{\tau}^{\varepsilon}(x_0)$. Thus we solved the two point boundary value problem (2.10) numerically for various values of ε. In Figures 2.1 and 2.2 the solutions are plotted for several values of ε. As it follows from these Figures, $\overline{\tau}^{\varepsilon}(x_0)$ is a "constant" in the interval $(-1, 1)$, if $\varepsilon \le 1/3$ and $\varepsilon \le 1/4$ for $f(x) = -x$ and $f(x) = -x^3$, respectively.

To evaluate quantitively the accuracy of the approximation given by Theorem 2.1, we first solve (2.4) for $f(x) = -x$ and $f(x) = -x^3$ and calculate the logarithmic residence time (2.5). A simple calculation shows that $\phi(x) = x^2$ and $\hat{\phi} = 1$ for $f(x) = -x$ and $\phi(x) = x^4/2$ and $\hat{\phi} = 1/2$ for $f(x) = -x^3$. In Figures 2.3 and 2.4 we have plotted $l(\varepsilon) = |(\varepsilon^2 \ln \overline{\tau}^{\varepsilon}(0) - \hat{\phi})/\hat{\phi}|$ for the two cases. As it follows from these Figures, the approximation is indeed very good: the maximum error is about 13% ($\varepsilon = 1/3$) and 6% ($\varepsilon = 1/4$) for $f(x) = -x$ and $f(x) = -x^3$, respectively.

Finally, we illustrate the calculation of $C(\varepsilon)$. If $f(x) = -x$, then $z(x) = 1$. As it was shown in [8], the integrals appearing in (2.8) can be evaluated by Laplace method in the limit of $\varepsilon \to 0$. This gives (see [8] for details)

$$C_1(\varepsilon) = \frac{\sqrt{2\pi}\,\varepsilon}{4}\,(1 + o(1)) = \hat{C}(\varepsilon)\,(1 + o(1)) \; .$$

Define

$$r(\varepsilon) = \left| \frac{\overline{\tau}^{\varepsilon}(0) - \hat{\tau}_1^{\varepsilon}(\Omega)}{\hat{\tau}_1^{\varepsilon}(\Omega)} \right| , \tag{2.11}$$

where

$$\hat{\tau}_1^{\varepsilon}(\Omega) = \hat{C}(\varepsilon)\, e^{\hat{\phi}(\Omega)/\varepsilon^2} \; .$$

In Figure 2.5 we have plotted $r(\varepsilon)$ as a function of ε in the range (1/10, 1/3). It follows from this figure that $\hat{\tau}_1^{\varepsilon}(\Omega)$ gives a very accurate estimate of $\overline{\tau}^{\varepsilon}(0)$ in the whole range, with a maximum error of about 7% at $\varepsilon = 1/3$.

Remark 2.1: In a recent paper [14], a comparison of the logarithmic residence time $\hat{\phi}(\Omega)$ with the experimental data, obtained by simulating a stochastic system large number of times, has been carried out. This study revealed that, for the second order nonlinear system under investigation, $\hat{\phi}(\Omega)$ gives a very good estimate of $\varepsilon^2 \ln \overline{\tau}^{\varepsilon}(0)$ for even larger values of ε ($\varepsilon = 0.6 - 0.8$ with 10% error). However, in the simulations reported in [14] the initial point $x(0) = x_0$ was selected to belong to a small neighborhood of zero and no qualitative analysis has been carried out. Therefore, the results in [14] can only be interpreted as how well $\hat{\phi}(\Omega)$ estimates the logarithmic first passage time from 0, i.e., $\varepsilon^2 \ln \overline{\tau}^{\varepsilon}(0)$, and nothing can be inferred about the relationship between $\hat{\phi}(\Omega)$ and $\varepsilon^2 \ln \overline{\tau}^{\varepsilon}(x_0)$, $x_0 \in \Omega$. For instance, in Figure 2.6 we have plotted

$l(\varepsilon)$ for $f(x) = -x^3$ in the interval (0.1, 0.8). It follows from this figure that for $\varepsilon = 0.71$, $\hat{\phi}(\Omega)$ is an estimate of $\varepsilon^2 \ln \bar{\tau}^{\varepsilon}(0)$ with 0.4% accuracy. However, for this value of ε, $\bar{\tau}^{\varepsilon}(x_0)$ is not nearly a constant in the interval $(-1, 1)$ as is shown in Figure 2.7 and, therefore, the theory of [8], [9] is not applicable.

In the case of linear systems, the results presented above can be generalized in two directions: first of all equations (2.4) and (2.7) can be solved explicitly and, secondly and more importantly, condition (i) can be weakened considerably to allow for a larger class of noisy systems to be considered. Indeed, consider the equation

$$dx = Axdt + \varepsilon C dw \tag{2.12}$$

and assume that

(i′) the pair (A, C) is completely disturbable, i.e., rank $[C \mid AC \mid \cdots \mid A^{n-1}C] = n$;

(ii′) A is Hurwitz;

(iii′) if $n(x)$ is the outward normal of $\partial\Omega$, then $(Ax)^T n(x) < 0$ for all $x \in \partial\Omega$.

Theorem 2.2: Assume that (i′) and (ii′) hold. Then $\phi(x)$ and $z(x)$ are given by

$$\begin{aligned} \phi(x) &= \frac{1}{2} x^T M x \ , \\ z(x) &\equiv 1 \ , \end{aligned} \tag{2.13}$$

where $M = X^{-1}$ and X is the unique positive definite solution to

$$AX + XA^T + CC^T = 0 \ . \tag{2.14}$$

Proof: See Appendix 1.

Theorem 2.3: Assume that (2.12) satisfies conditions (i′)-(iii′). Then (2.6) holds with $\hat{\phi}(\Omega)$ defined by (2.5) and (2.13).

Proof: See Appendix 1.

The theory described above constitutes the mathematical foundation for the residence time controllability analysis and the aiming control design techniques presented in the subsequent sections.

III. CONTROLLABILITY OF THE RESIDENCE TIME

Consider now a linear, stochastic system with control $u \in \mathbb{R}^m$:

$$dx = (Ax + Bu)dt + \varepsilon C dw \ , \ x(0) = x_0 \in \Omega \ . \tag{3.1}$$

Let $u = Kx$ and let $\bar{\tau}^{\varepsilon}(x_0, K)$ be the mean first exit time of the closed loop system

$$dx = (A + BK)x dt + \varepsilon C dw \ , x(0) = x_0 \in \Omega \ , \tag{3.2}$$

from the domain Ω.

Assume we want to select the controller K such that $\bar{\tau}^{\varepsilon}(x_0, K) \geq T$ for some prescribed $T > 0$ and all $x_0 \in \Omega_1$, where Ω_1 is a subset of Ω which does not contain any boundary points of Ω.

Consider the alternative problem of selecting $\hat{\phi}(\Omega, K) > \phi$ where $\hat{\phi}(\Omega, K)$ is given by

$$\hat{\phi}(\Omega, K) = \inf_{x \in \partial\Omega} \frac{1}{2} x^T M(K) x \ ,$$

$$M(K)(A+BK)+(A+BK)^T M(K)+M(K)CC^T M(K)=0 \quad ,$$

and $\phi > 0$ is some prescribed constant. Then if system (3.2) satisfies the assumptions of Theorem 2.3 we have

$$\varepsilon^2 \ln \overline{\tau}^{\varepsilon}(x_0,K) = \hat{\phi}(\Omega,K) + e(\varepsilon,x_0,K), \quad x_0 \in \Omega \quad ,$$

where $e(\varepsilon,x_0,K) \to 0$ as $\varepsilon \to 0$ uniformly for x_0 belonging to compact subsets of Ω. Thus, since by the choice of K we have $\hat{\phi}(\Omega,K) - \phi > 0$, there exists an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$ we have

$$\varepsilon^2 \ln \overline{\tau}^{\varepsilon}(x_0,K) - \phi = \hat{\phi}(\Omega,K) - \phi + e(\varepsilon,x_0,K) \ge 0$$

or equivalently,

$$\overline{\tau}^{\varepsilon}(x_0,K) \ge e^{\phi/\varepsilon^2} \ge e^{\phi/\varepsilon_0^2}$$

for all $x_0 \in \Omega_1$. Therefore, if the noise intensity, ε, of system (3.2) is less than ε_0 the choice $\phi = \varepsilon^2 \ln T$ guarantees $\overline{\tau}^{\varepsilon}(x_0,K) \ge T$, $x_0 \in \Omega_1$.

Motivated by the above considerations we introduce the following definitions.

Definition 3.1: System (3.1) is said to be weakly residence time controllable (wrt-controllable) if for any bounded $\Omega \subset I\!R^n$, with 0 in its interior, there exists a control $u = Kx$ such that $\hat{\phi}(\Omega,K) > 0$.

Definition 3.2: System (3.1) is said to be strongly residence time controllable (srt-controllable) if for any bounded $\Omega \subset I\!R^n$ $(0 \in \Omega)$ and any $\phi > 0$ there exists $u = Kx$ such that $\hat{\phi}(\Omega,K) \ge \phi$.

In the remainder of the paper we assume that system (3.1) does not contain any modes that are both uncontrollable and undisturbable (see, however, Remark 3.2 below).

Theorem 3.1: System (3.1) is wrt-controllable if and only if (A,B) is stabilizable.

Proof: See Appendix 2.

Theorem 3.2: System (3.1) is srt-controllable if and only if (A,B) is stabilizable and $\mathrm{Im}C \subseteq \mathrm{Im}B$.

Proof: See Appendix 2.

For any wrt-controllable system there exists a maximal constant $\phi^*(\Omega)$ such that any logarithmic residence time $\hat{\phi}(\Omega,K) < \phi^*(\Omega)$ is realizable by a choice of K. The constant $\phi^*(\Omega)$, referred to as *maximum achievable precision of aiming*, can be characterized as follows: Let Q_γ be the positive definite solution of the Ricatti equation

$$A^T Q_\gamma + Q_\gamma A + I - \frac{1}{\gamma} Q_\gamma B B^T Q_\gamma = 0 \ , \quad \gamma > 0 \ , \tag{3.3}$$

and define K^γ and X_γ by

$$K^\gamma = -\frac{1}{\gamma} B^T Q_\gamma \ , \tag{3.4}$$

$$(A+BK^\gamma)X_\gamma + X_\gamma (A+BK^\gamma)^T + CC^T = 0 \ . \tag{3.5}$$

Let $\mathrm{Tr}X_0 = \lim_{\gamma \to 0} \mathrm{Tr}X_\gamma$ (this limit exists since $\mathrm{Tr}X_\gamma$ is nonincreasing and bounded below as $\gamma \to 0$ [15]) and $\lambda_0 = \inf_{\gamma \geq 0} \lambda_{\max}(X_\gamma)$.

Theorem 3.3: Let $r = \min_{x \in \partial\Omega} \|x\|$, $R = \max_{x \in \partial\Omega} \|x\|$, $\lambda_1 = n/\mathrm{Tr}X_0$ and $\lambda_2 = 1/\lambda_0$.

Then

$$\phi_l(\Omega) = \frac{\lambda_2 r^2}{2} \leq \phi^*(\Omega) \leq \frac{\lambda_1 R^2}{2} = \phi_u(\Omega) \ . \tag{3.6}$$

Proof: See Appendix 2.

Remark 3.1: To obtain the lower bound $\phi_l(\Omega)$, the computation of X_γ for a sequence of γ's converging to zero is necessary. A simpler but more conservative lower bound of $\phi^*(\Omega)$ is

$$\phi_{l1}(\Omega) = \frac{r^2}{2\mathrm{Tr}X_0} .$$

Evaluation of a sufficiently accurate estimate of ϕ_{l1} and ϕ_u requires only the solution of (3.3), (3.5) for one small value of γ.

Remark 3.2: If system (3.1) does have modes that are both uncontrollable and undisturbable, then by a change of coordinates, $x = \bar{P}\bar{x}$, it can be reduced to the form:

$$\begin{bmatrix} d\bar{x}_1 \\ d\bar{x}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} dt + \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} udt + \varepsilon \begin{bmatrix} \bar{C}_1 \\ 0 \end{bmatrix} dw$$

where the deterministic subsystem, $\dot{\bar{x}}_2 = \bar{A}_{22}\bar{x}_2$, contains the uncontrollable, undisturbable part of the system. In this case we would consider the aiming control problem for the lower dimensional system $(\bar{A}_{11}, \bar{B}_1, \bar{C}_1)$ in the bounded domain $\bar{\Omega}_1 = \{\bar{x}_1 \mid \bar{P}\bar{x} \in \Omega\}$ and for which Theorems 3.1 - 3.3 apply.

IV. CONTROLLER DESIGN A: SRT-CONTROLLABILITY CASE

Consider again (3.1) and assume that either wrt- or srt-controllability takes place. Then with a control $u_0 = K_0 x$, which renders (3.1) stable and $(A + BK_0, C)$ disturbable, the logarithmic residence time of the closed loop system

$$dx = A_0 x dt + \varepsilon C dw \quad , \qquad A_0 = A + BK_0 \tag{4.1}$$

in a bounded domain $\Omega \subset I\!R^n$ is given by

$$\hat{\phi}(\Omega,K_0) = \inf_{x\in\partial\Omega} \phi_0(x) = \inf_{x\in\partial\Omega} \frac{1}{2} x^T M_0 x \quad ,$$

$$A_0^T M_0 + M_0 A_0 + M_0 C C^T M_0 = 0 \quad . \tag{4.2}$$

This residence time can be increased (or, more generally, changed) by an appropriate modification of K_0. The modification, however, depends on whether srt- or wrt-controllability takes place. In this section the former is addressed.

Theorem 4.1: Assume that system (3.1) is srt-controllable. Then the control

$$u^{\alpha} = (K_0 - \frac{\alpha-1}{2} H C^T M_0)x \triangleq K^{\alpha} x \quad , \tag{4.3}$$

where H is given by $C = BH$, applied to (3.1) results in a logarithmic residence time in Ω given by

$$\hat{\phi}(\Omega,K^{\alpha}) = \alpha\hat{\phi}(\Omega, K_0) \quad .$$

Proof: Follows directly from the sufficiency part of the proof of Theorem 3.2.

From Theorem 4.1 we obtain:

Design Procedure 4.1: (i) Select K_0 such that $A_0 = A + BK_0$ is Hurwitz and (A_0,C) is completely disturbable. Calculate M_0 and $\hat{\phi}(\Omega,K_0)$ given by (4.2).

(ii) For any desired logarithmic residence time $\phi > 0$ calculate

$$\alpha = \frac{\phi}{\hat{\phi}(\Omega,K_0)} \quad .$$

With $K^{\alpha} = K_0 - \frac{\alpha-1}{2} H C^T M_0$ the closed loop system $dx = (A+BK^{\alpha})xdt + \varepsilon C dw$ has logarithmic residence time $\hat{\phi}(\Omega,K^{\alpha}) = \phi$.

It is of interest to investigate the location of the closed loop poles defined by residence time controller (4.3). For this purpose, introduce an $r \times r$ transfer matrix

$$G_0(s) = C^T M_0(sI - A_0)^{-1}C \tag{4.4}$$

where M_0 is defined by (4.2). Assume that C has full rank (to avoid trivial situations) and define

$$\det G_0(s) = \frac{Z(s)}{\det(sI - A_0)} \quad . \tag{4.5}$$

Let $z_1, \ldots, z_\rho$, $\rho < n$, be the zeroes of $Z(s)$.

Theorem 4.2: As $\alpha \to \infty$, ρ closed loop poles of

$$dx = (A + B(K_0 - \frac{\alpha - 1}{2} HC^T M_0))xdt + \varepsilon C dw \tag{4.6}$$

converge to zeroes of $Z(s)$ while the remaining $n - \rho$ poles converge to $-\infty$.

Proof: See Appendix 3.

Theorem 4.2 states, in particular, *that a feedback* which ensures a very large residence time may place the closed loop poles arbitrarily close to the imaginary axis if $Z(s)$ has purely imaginary zeroes. The explanation of this counter-intuitive phenomenon is the following: Each zero of $\det G_0(s)$ corresponds to a differentiation of the external input, i.e., noise. The derivatives of white noise have a very strong disturbing power. That is why it is advantageous to place the closed loop poles so that the zeroes of $G_0(s)$ are cancelled.

Example 4.1: Consider system (3.1) with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad .$$

Clearly (A, B, C) is completely controllable and disturbable and $\mathrm{Im}C = \mathrm{Im}B$. Therefore, conditions of Theorems 3.2 and 4.1 are satisfied and a controller of the form (4.3) can be employed to obtain any desired residence time. Choosing

$$K_0 = [-1, -1, -2]$$

results in

$$M_0 = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

and

$$\begin{aligned} G_0(s) &= C^T M_0 (sI - (A + BK_0))^{-1} C \\ &= \frac{4(1+2s^2)}{s^3+2s^2+s+1} \end{aligned}$$

Therefore, controller (4.3) becomes

$$K^\alpha = K_0 - \frac{\alpha-1}{2} HC^T M_0 = [-\alpha, -1, -2\alpha] \quad .$$

By Theorem 4.2, as $\alpha \rightarrow \infty$, two of the closed loop poles converge to the zeroes of $G_0(s)$ at $s = \pm j\sqrt{2}/2$, while the third pole converges to $-\infty$. Furthermore, $\hat{\phi}(B(0,1), K^\alpha)$ is a linear function of α, $\hat{\phi}_\alpha = 0.382\alpha$.

V. CONTROLLER DESIGN B: SRT/WRT-CONTROLLABILITY CASE

When $\mathrm{Im}C \not\subseteq \mathrm{Im}B$, the residence time contoller design is not as simple as in Theorem 4.1. One possible approach is to generalize Design Procedure 4.1 in an iterative manner. However, this approach is inferior to the approach presented below and will not be pursued here. Our second approach for the design of an aiming controller is based on the Ricatti equation (3.3). As before, we select an initial control $u_0 = K_0 x$. Define a controller

$$u^{\gamma} = K^{\gamma}x = (K_0 - \frac{1}{\gamma}B^TQ_{\gamma})x \quad , \quad \gamma > 0 \quad , \tag{5.1}$$

where Q_{γ} is the positive definite solution of

$$A_0^TQ_{\gamma} + Q_{\gamma}A_0 + I - \frac{1}{\gamma}Q_{\gamma}BB^TQ_{\gamma} = 0 \quad . \tag{5.2}$$

The logarithmic residence time of system (3.1) with the stabilizing control (5.1) in a bounded $\Omega \subset I\!R^n$ is given by

$$\hat{\phi}(\Omega, K^{\gamma}) = \inf_{x \in \partial\Omega} \frac{1}{2}x^TM_{\gamma}x \tag{5.3}$$

where

$$(A + BK^{\gamma})^TM_{\gamma} + M_{\gamma}(A + BK^{\gamma}) + M_{\gamma}CC^TM_{\gamma} = 0 \quad . \tag{5.4}$$

Theorem 5.1: Assume (A, B) is stabilizable. Then

(a) if $\mathrm{Im}C \subseteq \mathrm{Im}B$, then $\hat{\phi}(\Omega, K^{\gamma}) \to \infty$ as $\gamma \to 0$;

(b) if $\mathrm{Im}C \not\subseteq \mathrm{Im}B$, then $\sup_{\gamma \geq 0} \hat{\phi}(\Omega, K^{\gamma})$ lies between the upper and lower bounds of Theorem 3.3.

Proof: See Appendix 3.

Remark 5.1: It follows from the proof of Theorem 3.3 that for each $\gamma > 0$,

$$\hat{\phi}(\Omega, K^{\gamma}) \geq \frac{r^2}{2\mathrm{Tr}X_{\gamma}} \quad .$$

Furthermore, $\frac{r^2}{2TrX_{\gamma}}$ is nondecreasing as $\gamma \to 0$ and, thus, is a good lower estimate for design purposes.

Theorem 5.1 suggests the following:

Design Procedure 5.1: (i) Select K_0 such that $(A + BK_0, C)$ is a disturbable pair.

(ii) For a given logarithmic residence time $0 < \phi < \phi_l(\Omega)$ iteratively find a $\gamma > 0$ such that $\hat{\phi}(\Omega, K^\gamma) \geq \phi$.

Remark 5.2: The above design procedure involves a solution of a quadratic (Ricatti) matrix equation. For high order systems the computational effort may be considerable at each iteration. On the other hand, the design procedure outlined in Section IV involves only the solution of one Liapunov equation and, hence, the computational effort is considerably less.

The asymptotic analysis as $\gamma \to 0$ of the poles of the closed loop system

$$dx = (A + BK^\gamma)xdt + \varepsilon C dw \tag{5.5}$$

can be carried out in a similar manner as in Section IV. Indeed, it is shown in [15] that as $\gamma \to 0$,

$$\frac{1}{\sqrt{\gamma}} B^T Q_\gamma \to B^T \overline{Q} \neq 0 \quad .$$

Therefore, a simple analysis of $\det(sI - A - BK^\gamma)$ shows that the closed loop poles that remain finite as $\gamma \to 0$ converge to the zeroes of the $m \times m$ transfer matrix

$$G(s) = B^T \overline{Q}(sI - A_0)^{-1}B \quad . \tag{5.6}$$

Example 5.1: Design Procedure 5.1 (DP5.1) was used to design a controller for the system in Example 4.1. In Fig. 5.1 the logarithmic residence times of the resulting closed loop systems are compared as a function of control effort, $||K||$. The plots reveal that DP5.1 results in a larger residence time for a given control effort as compared with DP4.1. However, bearing in mind that DP4.1 is a one step procedure, it is clear that in many cases the computational advantage may be considerable.

It is easily checked that

$$\lim_{\gamma \to 0} \frac{1}{\sqrt{\gamma}} B^T Q_\gamma = B^T \overline{Q} \tilde{=} [0.456,\ 0.7899,\ 0.456] \quad .$$

Therefore, as $\gamma \to 0$, two of the closed loop poles of system (5.5) converge to the zeroes of (5.6),

$$G(s) = B^T \overline{Q}(sI - A_0)^{-1} B = \frac{0.456s^2 + 0.7899s + 0.456}{s^3 + 2s^2 + s + 1} \quad ,$$

located at $s = -\sqrt{3}/2 \pm j\,1/2$, while the third pole converges to $-\infty$. There is an interesting difference between the two designs in this respect, i.e., to approach an infinite residence time DP 4.1 results in closed loop poles approaching the imaginary axis, whereas DP 5.1 results in poles that have strictly negative real parts.

Example 5.2: Consider the problem of designing a roll attitude regulator for a missile disturbed by random torques. A simple linearized model for the system is [16]

$$\begin{bmatrix} \dot{\delta} \\ \dot{\omega} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 10 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \delta \\ \omega \\ \phi \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u + \varepsilon \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \dot{w} \quad , \tag{5.7}$$

where δ is the aileron deflection, ω is the roll angular velocity, ϕ is the roll angle, u is a command signal to aileron actuators and $\dot{w}$ is white noise.

Clearly $\mathrm{Im}C \not\subseteq \mathrm{Im}B$ and (A, B) is controllable. Therefore, system (5.7) is wrt-controllable and $\phi^*(\Omega)$ is estimated using Theorem 3.3 to satisfy

$$5.36r^2 \le \phi^*(\Omega) \le 15R^2 \quad .$$

In [16] a covariance control approach was used to design a controller for (5.7). The design specifications were

$$E\delta^2 \leq \frac{121\varepsilon^2}{1150}\ ,\ E\omega^2 \leq \frac{81\varepsilon^2}{1150}\ ,\ E\phi^2 \leq \frac{\varepsilon^2}{1150}\ ,$$

and the resulting controller

$$u = -\ 14.565\delta - 24.43\omega - 68.57\phi\ . \tag{5.8}$$

The logarithmic residence time of the closed loop system with control (5.8) in the ball $B(0,\sqrt{2})$ is calculated to be $\hat{\phi} \approx 6.94$.

A residence time controller was designed using Design Procedure 5.1 with a starting value $K_0 = [-\ 3, -\ 6, -\ 4]$. The following results are of importance:

(a) A controller which results in the same logarithmic residence time in $B(0,\sqrt{2})$ as (5.8) is

$$u = -\ 4.912\delta - 6.580\omega - 4.823\phi\ .$$

We note that this controller uses much less control effort than (5.8).

(b) A controller that uses about the same control effort as (5.8) is

$$u = -\ 42.19\delta - 42.28\omega - 40.31\phi\ .$$

However, the closed loop system with this control has logarithmic residence time in $B(0,\sqrt{2})$ of $\hat{\phi} \approx 10.19$. The largest achievable logarithmic residence time in $B(0,\sqrt{2})$ using Design Procedure 5.1 is exactly the lower bound of $\phi^*(\Omega)$ calculated above with $r = \sqrt{2}$. When this bound is approached, two closed loop poles of system (5.5) converge to zeroes of

$$G(s) = B^T\bar{Q}(sI-A_0)^{-1}B \doteq [0.46\ 0.46\ 0.46] \begin{bmatrix} \dfrac{s(s+1)}{p(s)} \\ \dfrac{10s}{p(s)} \\ \dfrac{10}{p(s)} \end{bmatrix} = \frac{0.46(s^2+11s+10)}{s^3+7s^2+116s+80}$$

located at $s = -1, -10$, while the third closed loop pole converges to infinity.

VI. CONCLUSIONS

In this paper, a new problem of *aiming control* is formulated and solved for linear systems with small additive white noise. A generalization for wide-band noise processes is straightforward (using the techniques of [17]) and does not change the conclusions of this paper.

Among these conclusions, the following are of prime importance: The properties of the *aiming controllers* strongly depend on the relationship between the column spaces of the control and noise matrices. If the former includes the latter, any *precision of aiming* is achievable and the desired controller can be obtained in a one step procedure. If this inclusion does not occur, the *achievable precision* is limited and an *aiming controller* is designed by an iterative procedure.

The main advantage of the developed approach is that it is based directly on *aiming process specifications* and does not involve an informal choice of any parameter (poles, weighting matrices or covariances, for example).

The results presented in this paper may also be viewed from a perspective of stochastic system stability. Indeed, stability in moments, probability or almost surely does not exclude large deviations of the state vector from the equilibrium point. Therefore, controllers designed in this paper may be viewed as mechanisms for ensuring certain types of stochastic stability along with guaranteeing a desired residence time.

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APPENDIX 1

Proof of Theorem 2.2: In this case equation (2.4) becomes

$$\left[\frac{\partial\phi}{\partial x}\right]^T Ax + \frac{1}{2}\left[\frac{\partial\phi}{\partial x}\right]^T CC^T\frac{\partial\phi}{\partial x} = 0 \ , \qquad \text{(A1.1)}$$

$$\phi(0) = 0 \ .$$

We have to show that $\phi(x)$ given by (2.13) satisfies (A1.1), i.e.,

$$x^T(MA + \frac{1}{2}MCC^TM)x = 0 \ . \qquad \text{(A1.2)}$$

Since (A, C) is disturbable and A is Hurwitz, (2.14) has a unique positive definite solution X. Let $M = X^{-1}$ and rewrite (2.14) in terms of M

$$MA + A^TM + MCC^TM = 0$$

or

$$M(A + \frac{1}{2} CC^T M) + (A + \frac{1}{2} CC^T M)^T M = 0 \quad . \tag{A1.3}$$

(A1.3) shows that the matrix $M(A + \frac{1}{2} CC^T M)$ is skew symmetric which in turn implies that (A1.2) holds.

Next we show that $z(x) \equiv 1$. Equation (2.7) now becomes

$$(CC^T Mx + Ax)^T \frac{\partial z}{\partial x} + (\frac{1}{2} \mathrm{Tr}(CC^T M) + \mathrm{Tr} A)z = 0 \quad . \tag{A1.4}$$

Let S be a unitary transformation that diagonalizes M. Let $\hat{A} = S^T AS$, $\hat{M} = S^T MS$, $D = S^T CC^T S$. Then we have

$$0 = (MA + A^T M + MCC^T M) = (\hat{M}\hat{A} + \hat{A}^T \hat{M} + \hat{M}D\hat{M}) \quad .$$

This implies that $2\hat{m}_{ii} \, \hat{a}_{ii} + \hat{m}_{ii}^2 d_{ii} = 0$, $i = 0, 1, \cdots ,n$. Therefore

$$\sum_{i=1}^{n} (\hat{a}_{ii} + \frac{1}{2} \hat{m}_{ii} d_{ii}) = 0 \quad . \tag{A1.5}$$

Equation (A1.5) can equivalently be written as

$$0 = \mathrm{Tr}(\hat{A} + \frac{1}{2} D\hat{M}) = \mathrm{Tr}(A + \frac{1}{2} CC^T M)$$

which in turn implies that (A1.4) becomes

$$(CC^T Mx + Ax)^T \frac{\partial z}{\partial x} = 0 \quad . \tag{A1.6}$$

Thus, $z(x) \equiv 1$ is a solution to (A1.6) which satisfies the boundary conditions.

Q.E.D

Proof of Theorem 2.3: By assumptions (i′) and (ii′) $x(t)$, the solution to (2.12), has a transition probability density [18]

$$p^{\varepsilon}(t,x,y)=\frac{1}{(2\pi\varepsilon^2)^{n/2}(\det X(t))^{1/2}}\; e^{\frac{-\phi(t,x,y)}{\varepsilon^2}} \quad , \tag{A1.7}$$

where

$$\phi(t,x,y)=\frac{(y-e^{At}x)^T X^{-1}(t)(y-e^{At}x)}{2} \quad ,$$

$$X(t)=\int_0^t e^{As}CC^T e^{A^Ts}ds \quad .$$

Furthermore, $X=\lim_{t\to\infty}X(t)$ satisfies (2.14). Let

$$P(t,x,A)=\int_A p^{\varepsilon}(t,x,y)dy$$

be the transition function for $x(t)$ and let Δ be a subset of $\partial\Omega$ and $0<\delta \ll 1$ be given. Define a set W_Δ to consist of points y such that y lies on the normal $n(\gamma)$ for some $\gamma \in \Delta$ and such that the distance from y to γ is less than or equal to δ. Now, by Lemma 4 in [19] we have

$$\int_{\partial\Omega} P(\tau^{\varepsilon},x_0,d\gamma)\int_0^1 P(s,\gamma,W_\Delta)ds = P(W_\Delta)\overline{\tau}^{\varepsilon}(x_0)+0(e^{-1/\varepsilon}) \quad , \tag{A1.8}$$

where

$$P(W_\Delta)=\lim_{t\to\infty} P(t,x_0,W_\Delta)$$
$$=\int_{W_\Delta} p^{\varepsilon}(y)dy$$

and

$$p^{\varepsilon}(y)=\frac{1}{(2\pi\varepsilon^2)^{n/2}(\det X)^{1/2}}\; e^{-\phi(y)/\varepsilon^2}$$

where $\phi(y)$ is given by (2.13). Choose $\Delta=\partial\Omega$, then each $\gamma\in\partial\Omega$ belongs to $W=W_{\partial\Omega}$ and therefore

$$P(0,\gamma,W) = 1 \quad .$$

Using assumption (iii′), the continuity of $P(s,\gamma,W)$ and the exponential character of $p^{\varepsilon}(t,x,y)$ we have for some $\tilde{\delta} > 0$

$$P(s,\gamma,W) \geq 1 - e^{\frac{-\tilde{\delta}}{2\varepsilon^2 s}} \quad .$$

Therefore,

$$\int_0^1 P(s,\gamma,W)ds \geq 1 - \int_0^1 e^{\frac{-\tilde{\delta}}{2\varepsilon^2 s}} \geq 1 - e^{\frac{-\tilde{\delta}}{2\varepsilon^2}} \quad .$$

This gives

$$\int_{\partial\Omega} P(\tau^{\varepsilon},x_0,d\gamma)\int_0^1 P(s,\gamma,W)ds$$

$$= \int_{\partial\Omega} P(\tau^{\varepsilon},x_0,d\gamma)(1 + 0(e^{-1/\varepsilon^2})) = 1 + 0(e^{-1/\varepsilon^2}) \quad .$$

Here, we have used the fact that since τ^{ε} is an exit-time,

$$\int_{\partial\Omega} P(\tau^{\varepsilon},x_0,d\gamma) = 1 \quad .$$

Then we have from (A1.8)

$$1 = P(W)\overline{\tau}^{\varepsilon}(x_0) + 0(e^{-1/\varepsilon}) \quad . \tag{A1.9}$$

Next we use Laplace method to evaluate $P(W)$ in the limit as $\varepsilon \to 0$, i.e.,

$$P(W) = e^{-\hat{\phi}(\Omega)/\varepsilon^2}\tilde{C}(\varepsilon)\,(1 + o(1)) \text{ as } \varepsilon \to 0 \quad , \tag{A1.10}$$

where $\tilde{C}(\varepsilon)$ is a constant depending on $\phi(x)$ and the boundary $\partial\Omega$. Furthermore, $\tilde{C}(\varepsilon)$ grows not faster than ε^{-n} as $\varepsilon \to 0$. Whence, (2.6) follows directly from (A1.9) and (A1.10).

Q.E.D.

APPENDIX 2

Proof of Theorem 3.1: For the necessity we note that with $\Omega = B(0,1)$ wrt-controllability implies that there exists a control $u = Kx$ such that $\hat{\phi}(B(0,1),K) = \frac{1}{2}\lambda_{min}(M(K)) > 0$. Thus, $M(K) > 0$ and it follows from a standard Liapunov theorem that $(A+BK)$ is Hurwitz. To prove the sufficiency we first note that it follows from the proof of Theorem 5 in [20] that there exists a stabilizing feedback $u_0 = K_0x$ such that the closed loop system with this control is completely disturbable (in fact, this is true for almost any stabilizing feedback matrix). Now the rest of the proof follows directly from Theorem 2.3.

Q.E.D.

Proof of Theorem 3.2: We prove the necessity and sufficiency of the condition $\text{Im}C \subseteq \text{Im}B$.

Necessity: Let $x = \hat{P}\hat{x}$ be a change of coordinates which maps system (3.1) into the form:

$$d\hat{x} = \left(\begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \hat{x} + \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} u \right) dt + \varepsilon \begin{bmatrix} \hat{C}_1 \\ \hat{C}_2 \end{bmatrix} dw \ , \tag{A2.1}$$

where $\hat{B}_1$ has full row rank. With a stabilizing control $u = \hat{K}\hat{x}$, the logarithmic residence time of system (A2.1) is determined by $\hat{M} = \hat{X}^{-1}$ where $\hat{X}$ is the positive definite solution of

$$(\hat{A} + \hat{B}\hat{K})\hat{X} + \hat{X}(\hat{A} + \hat{B}\hat{K})^T + \hat{C}\hat{C}^T = 0 \ . \tag{A2.2}$$

Writing equation (A2.2) in a compatible block form with equation (A2.1) results in the following equation for the (2,2) block

$$\hat{A}_{21}\hat{X}_{12} + \hat{A}_{22}\hat{X}_{22} + \hat{X}_{22}\hat{A}_{22}^T + \hat{X}_{12}^T\hat{A}_{21}^T + \hat{C}_2\hat{C}_2^T = 0 \quad . \tag{A2.3}$$

From (A2.3) we obtain

$$2\|\hat{X}_{12}^T\hat{A}_{21}^T + \hat{X}_{22}\hat{A}_{22}^T\|_2 \geq \|\hat{C}_2\hat{C}_2^T\|_2 \quad , \tag{A2.4}$$

which gives

$$\begin{aligned} \frac{1}{2}\ \|\hat{C}_2\hat{C}_2^T\|_2 &\leq \|(\hat{X}_{12}^T\ \hat{X}_{22})\|_2\ \left\|\begin{bmatrix}\hat{A}_{21}^T \\ \hat{A}_{22}^T\end{bmatrix}\right\|_2 \\ &\leq \|\hat{X}\|_2 \left\|\begin{bmatrix}\hat{A}_{21}^T \\ \hat{A}_{22}^T\end{bmatrix}\right\|_2 \quad . \end{aligned} \tag{A2.5}$$

Finally, since $\|\hat{X}\|_2 = \lambda_{\max}(\hat{X})$, $\|\hat{C}_2\hat{C}_2^T\|_2 = \lambda_{\max}(\hat{C}_2\hat{C}_2^T)$ and $\left\|\begin{bmatrix}\hat{A}_{21}^T \\ \hat{A}_{22}^T\end{bmatrix}\right\|_2 =$ $(\lambda_{\max}(\hat{A}_{21}\hat{A}_{21}^T + \hat{A}_{22}\hat{A}_{22}^T))^{1/2}$, we have using (A2.5)

$$\lambda_{\max}(\hat{X}) \geq \frac{\lambda_{\max}(\hat{C}_2\hat{C}_2^T)}{2(\lambda_{\max}(\hat{A}_{21}\hat{A}_{21}^T + \hat{A}_{22}\hat{A}_{22}^T))^{1/2}} \quad ,$$

which in turn implies that if $\hat{C}_2 \neq 0$,

$$\lambda_{\min}(\hat{M}) \leq \frac{2(\lambda_{\max}(\hat{A}_{21}\hat{A}_{21}^T + \hat{A}_{22}\hat{A}_{22}^T))^{1/2}}{\lambda_{\max}(\hat{C}_2\hat{C}_2^T)} = \lambda^* \quad .$$

Therefore, the logarithmic residence time of systems (A2.1) in a bounded Ω is bounded by

$$\hat{\phi}(\Omega,\hat{K}) \leq \frac{R^2\lambda^*}{2} \quad , \quad R^2 = \max_{x\in\partial\Omega} x^Tx \quad .$$

This completes the proof of the necessity.

Sufficiency: The proof is by construction. Select a stabilizing control $u_0 = K_0x$ such that $(A + BK_0C)$ is disturbable. Then the closed loop system

$$dx = (A + BK_0)xdt + \varepsilon C dw$$

has a logarithmic residence time in a bounded Ω given by

$$\hat{\phi}(\Omega, K_0) = \inf_{x \in \partial\Omega} \frac{1}{2} x^T M_0 x \quad , \tag{A2.6}$$

$$(A + BK_0)^T M_0 + M_0(A + BK_0) + M_0 C C^T M_0 = 0 \quad . \tag{A2.7}$$

Define a feedback matrix K^α by

$$K^\alpha = K_0 - \frac{\alpha - 1}{2} H C^T M_0 , \ \alpha > 0 \quad , \tag{A2.8}$$

where H is determined by the relationship $C = BH$. Then, if $A + BK^\alpha$ is Hurwitz, the closed loop system has logarithmic residence time in Ω determined by

$$(A + BK^\alpha)^T M + M(A + BK^\alpha) + MCC^T M = 0 \quad . \tag{A2.9}$$

Substituting (A2.8) into (A2.9) and rearranging gives

$$\begin{aligned} &(A + BK_0)^T M + M(A + BK_0) + MCC^T M \\ &\quad - \frac{\alpha - 1}{2} MCC^T M_0 - \frac{\alpha - 1}{2} M_0 C C^T M = 0 \quad . \end{aligned} \tag{A2.10}$$

A simple check shows that $M = \alpha M_0$ is a positive definite solution of (A2.10) which in turn implies that $A + BK^\alpha$ is Hurwitz. Thus, the logarithmic residence time of the closed loop system with control (A2.8) is

$$\hat{\phi}(\Omega, K^\alpha) = \alpha \hat{\phi}(\Omega, K_0) \quad . \tag{A2.11}$$

Furthermore, it follows from eq. (A2.11) that $\hat{\phi}(\Omega, K^\alpha) \to \infty$ as $\alpha \to \infty$.

Q.E.D.

Proof of Theorem 3.3: It follows from the results of [15], [21] that for each $\gamma > 0$, K^γ defined by (3.4) is a stabilizing control and that for any other stabilizing K

$$\mathrm{Tr}X_\gamma(I+\frac{1}{\gamma}Q_\gamma BB^TQ_\gamma)\le \mathrm{Tr}X(K)(I+\gamma K^TK) \quad ,$$

$$(A+BK)X(K)+X(K)(A+BK)^T+CC^T=0 \quad .$$

Furthermore, it is shown in [15] that $\mathrm{Tr}X_\gamma$ is a nonincreasing function of γ as $\gamma\to 0$ and

$$\lim_{\gamma\to 0}\mathrm{Tr}X_\gamma(I+\frac{1}{\gamma}Q_\gamma BB^TQ_\gamma)=\lim_{\gamma\to 0}\mathrm{Tr}X_\gamma=\mathrm{Tr}X_0 \quad .$$

Therefore, for all stabilizing K,

$$\mathrm{Tr}X_0\le \mathrm{Tr}X(K) \quad . \tag{A2.12}$$

Define

$$\lambda_{\max}(\hat{X})=\inf_K\left[\lambda_{\max}(X(K))\mid A+BK \text{ is Hurwitz}\right] \quad , \tag{A2.13}$$

and note that, by the proof of Theorem 3.2, $\lambda_{\max}(\hat{X})>0$ when $\mathrm{Im}C\not\subseteq\mathrm{Im}B$. Therefore, (A2.13) gives

$$0<\lambda_{\max}(\hat{X})\le\lambda_0 \quad , \tag{A2.14}$$

and from (A2.12) we have

$$\mathrm{Tr}X_0\le\mathrm{Tr}\hat{X}\le n\lambda_{\max}(\hat{X}) \quad . \tag{A2.15}$$

The residence time in Ω for system (3.1) with a stabilizing K is determined by

$$\hat{\phi}(\Omega,K)=\inf_{x\in\partial\Omega}\frac{1}{2}x^TM(K)x \quad ,$$

$$(A+BK)^TM(K)+M(K)(A+BK)+M(K)CC^TM(K)=0 \quad .$$

Therefore,

$$\hat{\phi}(\Omega,K)\ge\frac{1}{2}\lambda_{\min}(M(K))\inf_{x\in\partial\Omega}x^Tx=\frac{1}{2}\lambda_{\min}(M(K))r^2 \quad ,$$

and, using (A2.14),

$$\sup_{K\in\mathbf{K}} \hat{\phi}(\Omega,K) \geq \sup_{K\in\mathbf{K}} \frac{1}{2}\lambda_{\min}(M(K))r^2$$

$$= \frac{r^2}{2\lambda_{\max}(\bar{X})} \geq \frac{r^2}{2\lambda_0} = \frac{r^2\lambda_2}{2} \quad . \tag{A2.16}$$

Similarly, (A2.15) gives

$$\sup_{K\in\mathbf{K}} \hat{\phi}(\Omega,K) \leq \frac{R^2}{2\lambda_{\max}(\bar{X})} \leq \frac{nR^2}{2\mathrm{Tr}X_0} = \frac{R^2\lambda_1}{2} \quad . \tag{A2.17}$$

Finally, (3.6) follows from the bounds (A2.16) and (A2.17).

Q.E.D.

APPENDIX 3

Proof of Theorem 4.2: The characteristic equation of the closed loop system (4.6) is

$$\det(\lambda I_n - A_0 + \frac{\alpha-1}{2} BHC^T M_0) = 0 \quad . \tag{A3.1}$$

To simplify notation let $\frac{1}{\gamma} = \frac{\alpha-1}{2}$ and $BH = C$. Then (A3.1) can rewritten as follows

$$\begin{aligned} 0 &= \det(\lambda I_n - A_0 + \frac{1}{\gamma} CC^T M_0) \\ &= \det(\lambda I_n - A_0)\det(I_n + \frac{1}{\gamma} CC^T M_0(\lambda I_n - A_0)^{-1}) \\ &= \det(\lambda I_n - A_0)\det(I_r + \frac{1}{\gamma} C^T M_0(\lambda I_n - A_0)^{-1} C) \\ &= (\frac{1}{\gamma})^r \det(\lambda I_n - A_0)\det(\gamma I_r + G_0(\lambda)) \quad . \end{aligned} \tag{A3.2}$$

When $\gamma \to 0$ (i.e., $\alpha \to \infty$), ρ of the roots of (A3.2) converge to the zeroes of

$$0 = \det(\lambda I_n - A_0)\det G_0(\lambda) = Z(\lambda)$$

(for a proof see e.g. [22]) and the remaining $n - \rho$ roots converge to infinity.

Q.E.D.

Proof of Theorem 5.1: To prove statement (a) of the theorem assume, without loss of generality, that B has the form

$$B = \left[\frac{B_1}{0}\right] \quad . \tag{A3.3}$$

where B_1 has full row rank. Let $\lim\limits_{\gamma \to 0} Q_\gamma = Q_0$ (this limit exists because Q_γ is a nonincreasing function of γ as $\gamma \to 0$ [15]). Then Q_0 satisfies [23]

$$A_0^T Q_0 + Q_0 A_0 + I - \lim_{\gamma \to 0} \frac{1}{\gamma} Q_\gamma B B^T Q_\gamma = 0 \quad . \tag{A3.4}$$

The last term in (A3.4) when rewritten in a compatible block form with (A3.3) is

$$\begin{aligned} &\lim_{\gamma \to 0} \frac{1}{\gamma} Q_\gamma B B^T Q_\gamma \\ &= \lim_{\gamma \to 0} \frac{1}{\gamma} \begin{bmatrix} Q_\gamma^{11} B_1 B_1^T Q_\gamma^{11} & Q_\gamma^{11} B_1 B_1^T Q_\gamma^{12} \\ (Q_\gamma^{12})^T B_1 B_1^T Q_\gamma^{11} & (Q_\gamma^{12})^T B_1 B_1^T Q_\gamma^{12} \end{bmatrix} \quad . \end{aligned} \tag{A3.5}$$

Since the limit in equation (A3.5) exists and $B_1 B_1^T$ is nonsingular we must have

$$\lim_{\gamma \to 0} Q_\gamma^{11} = 0 \quad .$$

Therefore,

$$\begin{aligned} \lim_{\gamma \to 0} \mathrm{Tr} C^T Q_\gamma C &= \lim_{\gamma \to 0} \mathrm{Tr} H^T B^T Q_\gamma B H \\ &= \mathrm{Tr}(H^T \lim_{\gamma \to 0} B_1^T Q_\gamma^{11} B_1 H) = 0 \quad , \end{aligned}$$

and the relations

$$\begin{aligned} \mathrm{Tr} X_\gamma (I + \frac{1}{\gamma} Q_\gamma B B^T Q_\gamma) &= \mathrm{Tr} Q_\gamma C C^T \\ &= \mathrm{Tr} C^T Q_\gamma C \quad , \end{aligned}$$

$$\lambda_{max}(X_\gamma) \le \mathrm{Tr} X_\gamma$$

imply that $\lambda_{max}(X_\gamma) \to 0$ as $\gamma \to 0$. Therefore, $M_\gamma = X_\gamma^{-1} \to \infty$ as $\gamma \to 0$ which completes the proof of part (a).

The proof of (b) follows directly from the proof of Theorem 3.3.

Q.E.D.

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