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NUSC Technical Report 8121
21 September 1987

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Operating Characteristics for Indicator Or-ing of Incoherently Combined Matched-Filter Outputs

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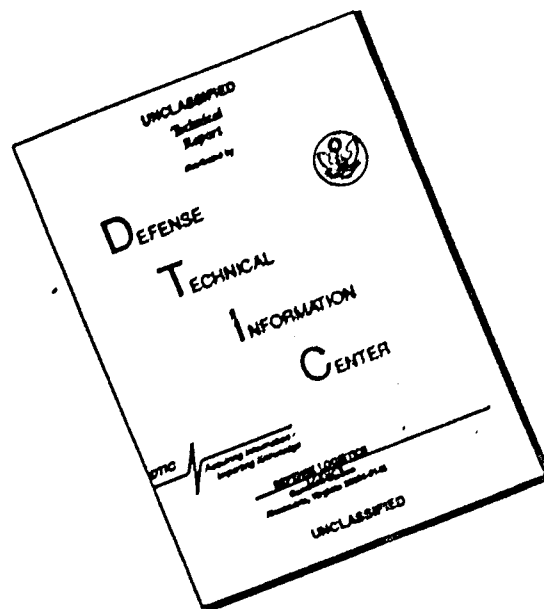


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REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED			1b. RESTRICTIVE MARKINGS	
2a. SECURITY CLASSIFICATION AUTHORITY			3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution is unlimited.	
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE				
4. PERFORMING ORGANIZATION REPORT NUMBER(S) TR 8121			5. MONITORING ORGANIZATION REPORT NUMBER(S)	
6a. NAME OF PERFORMING ORGANIZATION Naval Underwater Systems Center		6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION	
6c. ADDRESS (City, State, and ZIP Code) New London Laboratory New London, CT 06320			7b. ADDRESS (City, State, and ZIP Code)	
8a. NAME OF FUNDING/SPONSORING ORGANIZATION		8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER	
8c. ADDRESS (City, State, and ZIP Code)			10. SOURCE OF FUNDING NUMBERS	
			PROGRAM ELEMENT NO.	PROJECT NO. A75205
			TASK NO.	WORK UNIT ACCESSION NO.
11. TITLE (Include Security Classification) OPERATING CHARACTERISTICS FOR INDICATOR OR-ING OF INCOHERENTLY COMBINED MATCHED-FILTER OUTPUTS				
12. PERSONAL AUTHOR(S) Albert H. Nuttall				
13a. TYPE OF REPORT		13b. TIME COVERED FROM TO	14. DATE OF REPORT (Year, Month, Day) 1987 September 21	15. PAGE COUNT
16. SUPPLEMENTARY NOTATION				
17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)	
FIELD	GROUP	SUB-GROUP	Operating Characteristics Incoherent Combination	
			Indicator Or-ing Detection Probability	
			Matched Filters False Alarm Probability	
19. ABSTRACT (Continue on reverse if necessary and identify by block number)				
The false alarm and detection probabilities for a processor that incoherently combines M matched filter outputs and then subjects these summed quantities to or-ing amongst N channels are derived for general M, N, and signal-to-noise ratios. A probability of correct detection occurs only when the signal channel output exceeds a threshold and all other noise channel outputs. Receiver operating characteristics are plotted for the 40 possible combinations of M = 1(1)10 with N = 1, 10, 100, 1000, for signal-to-noise ratios ranging over values diverse enough to cover false-alarm, detection probability pairs from (.01,.5) to (1E-10,.999). Also, the required signal-to-noise ratio to realize specified false alarm and detection probabilities are plotted versus N, for several values of M. The signal-to-noise ratio parameter employed is related to the total received				
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS			21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED	
22a. NAME OF RESPONSIBLE INDIVIDUAL Albert H. Nuttall			22b. TELEPHONE (Include Area Code) (203) 440-4618	22c. OFFICE SYMBOL Code 3302

18. Subject Terms (Con'td.)

Correct Detection
 Energy Fractionalization
 Multipath

19. Abstract (Cont'd.)

signal energy to Gaussian noise spectral density ratio. This allows for consideration of arbitrary fractionalization of the received signal energy and for investigation of mismatch as well as frequency offset and time desynchronization, if desired. Programs for all procedures are listed.

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FORM CRAB1	☒
FORM TAP	☐
FORM TAP	☐
By	
Date	
Availability Code	
Doc	Availability Code
A-1	



TABLE OF CONTENTS

	Page
LIST OF ILLUSTRATIONS	ii
LIST OF SYMBOLS	v
INTRODUCTION	1
STATISTICS OF FILTER ENVELOPE-SQUARED OUTPUT	7
FALSE ALARM AND DETECTION PROBABILITIES	13
FALSE ALARM PROBABILITY	13
DETECTION PROBABILITY	14
TIGHTNESS OF BOUND	18
GRAPHICAL RESULTS	19
SUMMARY	21
APPENDICES	
A. Q_M -FUNCTION RELATIONSHIPS	A-1
B. TABULATION OF P_{CD} AND $Q_M(d,T)$	B-1
C. PROGRAM LISTING	C-1
REFERENCES	R-1

LIST OF ILLUSTRATIONS

Figure	Page
1. Preprocessing for n-th Channel	3
2. Indicator Or-ing of N Channels	3
3. ROC for $M = 1$, $N = 1$	23
4. ROC for $M = 1$, $N = 10$	24
5. ROC for $M = 1$, $N = 100$	25
6. ROC for $M = 1$, $N = 1000$	26
7. ROC for $M = 2$, $N = 1$	27
8. ROC for $M = 2$, $N = 10$	28
9. ROC for $M = 2$, $N = 100$	29
10. ROC for $M = 2$, $N = 1000$	30
11. ROC for $M = 3$, $N = 1$	31
12. ROC for $M = 3$, $N = 10$	32
13. ROC for $M = 3$, $N = 100$	33
14. ROC for $M = 3$, $N = 1000$	34
15. ROC for $M = 4$, $N = 1$	35
16. ROC for $M = 4$, $N = 10$	36
17. ROC for $M = 4$, $N = 100$	37
18. ROC for $M = 4$, $N = 1000$	38
19. ROC for $M = 5$, $N = 1$	39
20. ROC for $M = 5$, $N = 10$	40
21. ROC for $M = 5$, $N = 100$	41

LIST OF ILLUSTRATIONS (CONT'D)

Figure	Page
22. ROC for $M = 5$, $N = 1000$	42
23. ROC for $M = 6$, $N = 1$	43
24. ROC for $M = 6$, $N = 10$	44
25. ROC for $M = 6$, $N = 100$	45
26. ROC for $M = 6$, $N = 1000$	46
27. ROC for $M = 7$, $N = 1$	47
28. ROC for $M = 7$, $N = 10$	48
29. ROC for $M = 7$, $N = 100$	49
30. ROC for $M = 7$, $N = 1000$	50
31. ROC for $M = 8$, $N = 1$	51
32. ROC for $M = 8$, $N = 10$	52
33. ROC for $M = 8$, $N = 100$	53
34. ROC for $M = 8$, $N = 1000$	54
35. ROC for $M = 9$, $N = 1$	55
36. ROC for $M = 9$, $N = 10$	56
37. ROC for $M = 9$, $N = 100$	57
38. ROC for $M = 9$, $N = 1000$	58
39. ROC for $M = 10$, $N = 1$	59
40. ROC for $M = 10$, $N = 10$	60
41. ROC for $M = 10$, $N = 100$	61
42. ROC for $M = 10$, $N = 1000$	62

LIST OF ILLUSTRATIONS (CONT'D)

Figure	Page
43. Required d Values for $M=1$, $P_{CD} = .5$	63
44. Required d Values for $M=1$, $P_{CD} = .9$	63
45. Required d Values for $M=2$, $P_{CD} = .5$	64
46. Required d Values for $M=2$, $P_{CD} = .9$	64
47. Required d Values for $M=4$, $P_{CD} = .5$	65
48. Required d Values for $M=4$, $P_{CD} = .9$	65
B-1 Comparison of Probabilities	B-2

LIST OF SYMBOLS

M	number of filter outputs added, figure 1
N	number of channels subject to or-ing, figure 2, (2)
$h(\tau)$	impulse response of filter, figure 1
<u> </u>	complex envelope
t_{nm}	sampling time in n -th channel on m -th filter output, figure 1
v_n	summer output of n -th channel, figure 1, (9)
w	maximum output from or-ing device, (2), (3)
$\hat{n}$	channel indication from or-ing device, (2)
$s(t)$	real signal function of time t , (4)
$n(t)$	real noise process, (5)
a, b	real, imaginary parts of signal output, (4)
x, y	real, imaginary parts of noise, (4)
c	complex envelope of filter output, (4)
N_d	double-sided noise spectral density (watts/Hz), (12)
N_o	single-sided noise spectral density ($= 2N_d$), (12)
σ^2	noise power, (14)
d	total output signal-to-noise ratio measure, (15)
E_m	received signal energy in m -th component, (17)
P	cumulative distribution function, (18)
Q_M	Marcum's Q_M - function, (19)
E_n	auxiliary function, (22)
e_n	partial exponential, (23)

LIST OF SYMBOLS (CONT'D)

P_F	false alarm probability, (25)
P_{SD}	probability of signal detection, (26)
P_{AD}	probability of any detection, (27)
P_{CD}	probability of correct detection, (28)
ROC	Receiver Operating Characteristic

OPERATING CHARACTERISTICS FOR INDICATOR OR-ING OF INCOHERENTLY COMBINED MATCHED-FILTER OUTPUTS

INTRODUCTION

When multiple pulses are transmitted, in an effort to detect the presence of a target, the multiple echoes should be optimally processed and combined before a decision is reached. For received signals that are deterministic, except for independent random phases between pulses, the ideal processing consists of matched filtering, envelope detection, and combination according to a $\ln I_0$ rule [1; chapter VII, (1.7)]. Since the receiver input signal-to-noise ratio must be known in order to apply this rule, the slightly suboptimum alternative of combining (adding) squared envelopes is often adopted [1; ch. VII, (1.12)]; this is the situation to be considered here.

In addition, if the target has some movement in the radial direction, causing a Doppler shift of the echoes, a search must be conducted over frequency at the receiver, in order not to miss the received signal energy. For example, suppose a series of M tone bursts at a common center frequency are transmitted and echoed off a moving point target. Since the received center frequency will be unknown, groups of matched filters will be necessary, in order to cover the expected range of frequency shifts. Each one of the possible received center frequencies that must be processed is called a channel.

In figure 1, a block diagram of the processing in the n -th channel is depicted. The M narrowband filters in the n -th channel are indicated by impulse responses $\{h_{nm}(\tau)\}_{m=1}^M$. They are followed by detectors which extract the squared envelopes of the filter outputs. These detector outputs are then sampled at times $\{t_{nm}\}_{m=1}^M$, which should correspond to the times of peak signal at each filter output. The sampled outputs are then added, to yield channel output v_n .

The block diagram in figure 1 is not restricted to a transmitted sequence of M tone bursts at a common center frequency. In fact, due to the general filter impulse responses and sampling times allowed, it encompasses any sequence of orthogonal deterministic signals transmitted at arbitrary time delays and frequency offsets, provided they are known to the receiver. The processor in figure 1 also allows for unknown time delay to the target range and unknown frequency shift due to target movement, by virtue of the sampling times not being optimum, and the filter impulse responses not being matched to each received signal component. An example is afforded by the case where the filters are time-delayed and/or frequency-shifted versions of one another,

$$\underline{h}_{nm}(\tau) = \underline{h}(\tau - \tau_{nm}) \exp(i2\pi f_{nm} \tau), \quad (1)$$

corresponding to a time sequence of frequency-stepped pulses; here $\underline{h}$ is the complex envelope corresponding to impulse response h [1; pages 65-72].

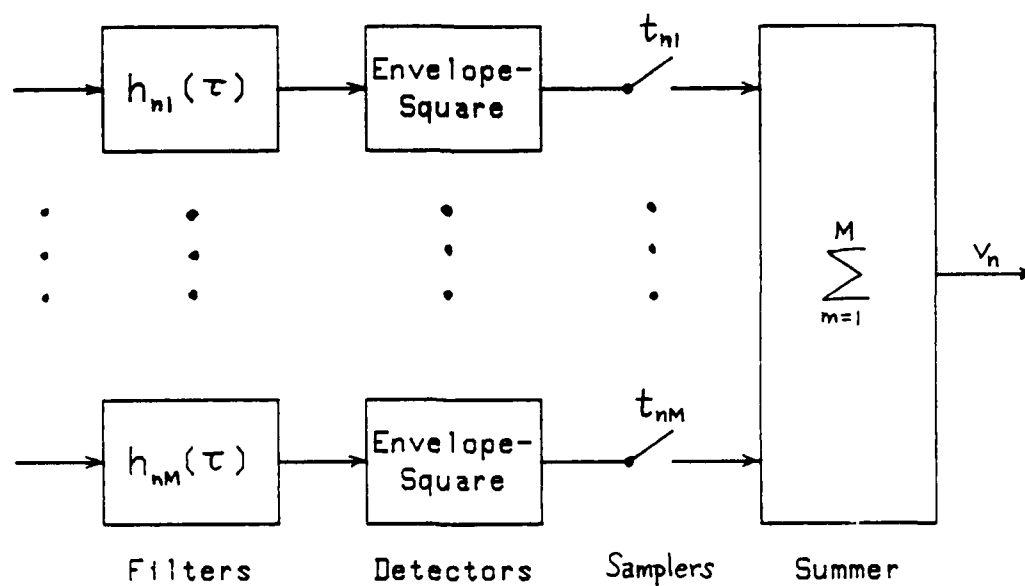


Figure 1. Pre-Processing for n-th Channel

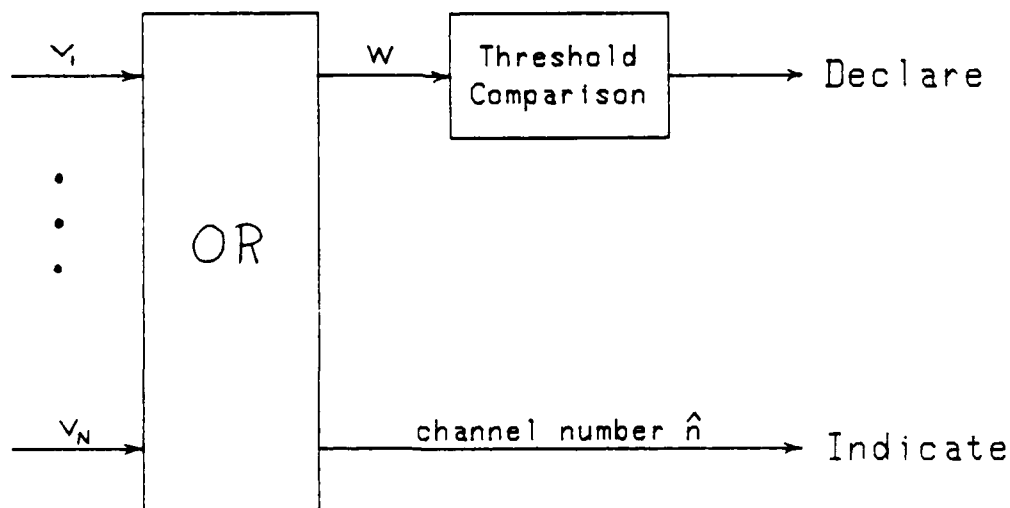


Figure 2. Indicator Or-ing of N Channels

Another instance which is covered by the processing indicated in figure 1 is where the transmitted signal encounters multipath and/or separated target highlight structure. For example, a single transmitted tone burst might be received as four pulses, due to two multipaths and two target highlights. Thus, the number M of filters employed in figure 1 is be interpreted as the total number of received signal components. Some results for the receiver operating characteristics of this processor are given in [1] and [2].

When the processing in the n -th channel indicated in figure 1 is completed, the total of N channels that must be considered is subjected to the indicator or-ing depicted in figure 2. Namely, the maximum of the N channel outputs is extracted, along with its identity,

$$w = \max (v_1, v_2, \dots, v_N) = v_{\hat{n}}, \quad (2)$$

and compared with a fixed threshold:

$$\left\{ \begin{array}{l} w < \text{threshold: declare no signal present} \\ w > \text{threshold: declare signal present in channel } \hat{n} \end{array} \right\}. \quad (3)$$

Thus, there are two possible outputs from figure 2, the first being a declaration of no signal present, and the second being a declaration of a signal present along with an indication of which channel contains the signal. (This latter information is useful for identifying the Doppler shift, for example, of a moving target.)

A false alarm occurs when output w in (2) exceeds the threshold, but there is no signal present at the input. On the other hand, a correct detection occurs only when the signal channel output exceeds the threshold and all the noise channel outputs. That is, we insist on accurately identifying the signal channel, in order to achieve a correct detection. The performance characteristics of the processor combination in figures 1 and 2 are of interest, namely the false alarm probability and the probability of correct detection, in terms of M , the number of filter outputs summed, N , the number of channels or-ed, and some signal-to-noise ratio measure at the receiver.

It should be observed that the M received signal components have been presumed to have undergone no fading. The only randomness in the received signals are the independent random phase shifts between components. Some results on fading signals, including partial fading between pulses, are given in [3]; however, or-ing was not considered there.

It is also assumed that the individual signal components are orthogonal with respect to each other, perhaps due to time separation and/or frequency shift. That is, at sampling instant t_{nm} , there is only one signal component contributing, with all the other signal components yielding no output at that filter at that time.

The processor considered in this study has undergone some analysis in the past [4]; however, several significant extensions have been made here. First, a different definition of detection probability has been adopted here, namely one which counts as correct detections only those events for

which the signal channel output exceeds both the threshold and all the other undesired noise channel outputs. Second, results are extended from a sinusoidal signal to arbitrary orthogonal deterministic signals and filters, with arbitrary sampling instants; this allows for analysis of the effects of filter-signal mismatch, Doppler offset, time desynchronization, multiple highlights, etc. Third, a fundamentally different signal-to-noise ratio parameter, d , is used here to characterize performance, namely, a measure of the total received signal energy to noise spectral density ratio, rather than the signal-to-noise ratio per pulse (usually assumed identical for all pulses); this allows for arbitrary fractionalization of the total received signal energy into component pulses. Fourth, the detection probability vs. false alarm probability curves are plotted on normal probability paper with total signal-to-noise ratio, d , as a parameter; this straightens out the curves, makes them nearly equi-spaced in d , and affords easy accurate interpolation in signal-to-noise ratio values. Finally, the current results are extended to much larger values of the number, N , of or-ing channels and much smaller false alarm probabilities P_F ; in particular, values of N up to 1000, and values of P_F as small as $1E-10$, are considered.

STATISTICS OF FILTER ENVELOPE-SQUARED OUTPUT

In this section, we derive the statistical properties of the output of figure 1. Suppose a real narrowband deterministic signal $s(t)$ and a real random noise process $n(t)$ have complex envelopes $\underline{s}(t)$ and $\underline{n}(t)$, respectively. Let the sum of these two processes excite a narrowband filter $h(\tau)$ with complex envelope impulse response $\underline{h}(\tau)$. The complex envelope of the filter output at time t is proportional to

$$c(t) = [\underline{s}(t) + \underline{n}(t)] \bullet \underline{h}(t) = a(t) + ib(t) + x(t) + iy(t), \quad (4)$$

where

$$a(t) + ib(t) = \int d\tau \underline{s}(\tau) \underline{h}(t-\tau) \quad (5)$$

is the deterministic signal output, and

$$x(t) + iy(t) = \int d\tau \underline{n}(\tau) \underline{h}(t-\tau) \quad (6)$$

is the random noise output process. Then the filter squared-envelope output at time t is

$$\begin{aligned} |c(t)|^2 &= |a(t) + ib(t) + x(t) + iy(t)|^2 = \\ &= [a(t) + x(t)]^2 + [b(t) + y(t)]^2. \end{aligned} \quad (7)$$

More generally, for M filters, if signal $s_m(t)$ excites filter $h_m(\tau)$, the m -th filter squared-envelope output at sample time t_m is

$$|c_m(t_m)|^2 = [a_m(t_m) + x_m(t_m)]^2 + [b_m(t_m) + y_m(t_m)]^2 \quad \text{for } 1 \leq m \leq M. \quad (8)$$

Sample times $\{t_m\}_1^M$ can be selected arbitrarily; each individual t_m should be chosen to maximize the size of the m -th signal output, $a_m^2(t_m) + b_m^2(t_m)$.

If we sum these squared-envelope filter output samples, we have channel output

$$\begin{aligned} v &= \sum_{m=1}^M |c_m(t_m)|^2 = \\ &= \sum_{m=1}^M \{ [a_m(t_m) + x_m(t_m)]^2 + [b_m(t_m) + y_m(t_m)]^2 \}. \end{aligned} \quad (9)$$

The signal and noise outputs, given in (5) and (6), apply for an arbitrary complex envelope signal $s_m(t)$ and filter $h_m(\tau)$ in the m -th branch of the receiver. The instantaneous output signal squared-envelope is

$$a_m^2(t) + b_m^2(t) = |a_m(t) + ib_m(t)|^2 = \left| \int d\tau s_m(\tau) h_m(t-\tau) \right|^2, \quad (10)$$

while the instantaneous output noise squared-envelope is

$$x_m^2(t) + y_m^2(t) = |x_m(t) + iy_m(t)|^2 = \left| \int d\tau n(\tau) h_m(t-\tau) \right|^2. \quad (11)$$

Here, we presume that a common broadband noise $n(t)$ excites all the filters $\{h_m(\tau)\}$ in the receiver bank. Observe that if the m -th signal is subject to a random phase shift, according to the factor $\exp(i\theta_m)$, this cancels out of the envelope-squared signal term. Thus, all the results here apply not only to a deterministic signal, but also to one with an arbitrary phase shift. However, no fading of the received signal is allowed in any of the current results.

If the real input noise $n(t)$ is white with double-sided spectral level N_d watts/Hz, then the correlation of complex envelope $\underline{n}(t)$ is [1; ch. II, (3.11) and (6.22)]

$$\overline{\underline{n}(t) \underline{n}^*(t-\tau)} = 4N_d \delta(\tau) = 2N_0 \delta(\tau); \quad (12)$$

N_0 is the single-sided noise spectral density level in watts/Hz. By use of (6), this results in average noise powers for the m -th components, as

$$\overline{x_m^2(t)} = \overline{y_m^2(t)} = 2N_d \int d\tau |h_m(\tau)|^2. \quad (13)$$

We presume that all the filters have the same level (energy); thus, we define

$$\sigma^2 = \overline{x_m^2(t)} = \overline{y_m^2(t)} = 2N_d \int d\tau |h_m(\tau)|^2 \quad \text{for } 1 \leq m \leq M. \quad (14)$$

This is an important restriction; greater generality is given in [2; appendices B and C].

We are now in position to employ the general results listed in appendix A, when the noise is Gaussian. Namely, define, as in (A-1),

$$\begin{aligned}
 d^2 &= \frac{1}{\sigma^2} \sum_{m=1}^M \left[a_m^2(t_m) + b_m^2(t_m) \right] = \\
 &= \frac{1}{\sigma^2} \sum_{m=1}^M \left| \int d\tau \, s_m(\tau) \, h_m(t_m - \tau) \right|^2 = \\
 &= \frac{\sum_{m=1}^M \left| \int d\tau \, s_m(\tau) \, h_m(t_m - \tau) \right|^2}{2N_d \int d\tau \, |h_m(\tau)|^2} .
 \end{aligned} \tag{15}$$

Observe that the absolute level of each filter, h_m , cancels out in this ratio. However, d^2 does depend on the scale of each signal s_m and inversely on noise level N_d .

The maximum value of each term in these ratios is realized by choosing the m -th filter such that its impulse response

$$h_m(\tau) = k \, s_m^*(T_m - \tau), \tag{16}$$

where k is a complex constant selected to guarantee the equal energy requirement in (14), and T_m is a delay inserted for realizability, and by choosing sample time t_m equal to T_m . This is the matched filter to the m -th signal, sampled at the time of peak output. Thus, we have, in the best situation,

$$\max d^2 = \frac{1}{2N_d} \sum_{m=1}^M \int dt |s_m(t)|^2 = \frac{1}{N_d} \sum_{m=1}^M E_m = \frac{E_T}{N_d} = \frac{2E_T}{N_o}, \quad (17)$$

where E_m is the received signal energy in the m -th real signal component $s_m(t)$, and E_T is the total received signal energy over all M paths (branches). Additional interpretations of d^2 are available in (A-21) et seq.

This maximum value of d^2 in (17) is realized only if the receiving filters are the matched filters (16), and if the filter outputs are sampled at the correct time instants. More generally, the generic value of d^2 in (15) allows for arbitrary signals, filters, and sampling instants, thereby affording the possibility of considering losses due to mismatch and desynchronization. The signals can be time-delayed and/or frequency-shifted versions of each other, if desired. A more thorough analysis and comparison is presented in [2; appendices B and C]. The received signals have undergone no fading in any of these considerations; thus the current analysis applies to a deterministic signal, except for random phase.

Reference to (A-2) and (A-6) now allows us to state the exceedance distribution function of channel output v in (9) as

$$\text{Prob}(v > u) = 1 - P_v(u) = Q_M(d, \sqrt{u}/\sigma) \text{ for } u > 0, \quad (18)$$

where the Q_M -function is

$$Q_M(\alpha, \beta) = \int_{\beta}^{\infty} dx \, x \left(\frac{x}{\alpha}\right)^{M-1} I_{M-1}(\alpha x) \exp\left(\frac{x^2 + \alpha^2}{-2}\right). \quad (19)$$

Parameters d and σ in (18) are given by (15) and (14), respectively. These results pertain to the signal-bearing channel; the noise-only channel outputs correspond to setting $d = 0$.

FALSE ALARM AND DETECTION PROBABILITIES

The exceedance distribution function of the processor output v_n for the n -th channel (see figure 1) is given by (18) for signal present in that channel. For those channels with no signal present, the exceedance distribution is

$$1 - P_v^{(0)}(u) = Q_M(0, T) = E_{M-1}(T^2/2) \quad \text{for } u > 0, \quad (20)$$

where we have let

$$T = \sqrt{u} / \sigma \quad (21)$$

for notational convenience, and defined

$$E_n(x) = \exp(-x) e_n(x), \quad (22)$$

where

$$e_n(x) = \sum_{k=0}^n x^k / k! \quad (23)$$

is the partial exponential [5; 6.5.11].

FALSE ALARM PROBABILITY

Since the noises in the N channels subject to or-ing in figure 2 are presumed independent, the probability that all N outputs do not exceed a threshold value u is

$$[P_V^{(0)}(u)]^N = \left[1 - E_{M-1} \left(\frac{u}{2\sigma^2} \right) \right]^N, \quad (24)$$

where cumulative distribution function $P_V^{(0)}$ was obtained from (20).

The false alarm probability is then

$$P_F = 1 - [1 - E_{M-1}(T^2/2)]^N, \quad (25)$$

where we used (21).

DETECTION PROBABILITY

When signal is present in one channel, we have several alternative definitions of a detection probability. For example, we could define the probability of signal detection, P_{SD} , as the probability that the signal channel output exceeds threshold u , disregarding the noise channels completely; then directly from (18) and (21),

$$P_{SD} = Q_M(d, T), \quad (26)$$

which is, of course, independent of N .

However, it is possible that the noise channels could also cause a threshold crossing, even when the signal channel does not. We can then define a probability of any detection, P_{AD} , as the probability that any channel output exceeds the threshold u . This quantity is given by

$$\begin{aligned}
 P_{AD} &= 1 - [P_V^{(0)}(u)]^{N-1} P_V(u) = \\
 &= 1 - [1 - E_{M-1}(T^2/2)]^{N-1} [1 - Q_M(d, T)], \quad (27)
 \end{aligned}$$

by use of (20) and (18). This is the case considered in [4; see (9) and (4)].

The problem with this latter definition is that, since we are interested in knowing which channel contains the signal, the probability P_{AD} contains some (rare) events which indicate the incorrect channel to contain the signal. The best alternative appears to be to define the probability of correct detection, P_{CD} , as the probability that the signal channel output exceeds the threshold u and exceeds all the noise outputs. In this case, the signal will be detected and its channel number correctly indicated. This probability is given by

$$P_{CD} = \int_u^\infty dt p_V(t) [P_V^{(0)}(t)]^{N-1}, \quad (28)$$

where probability density function p_V and cumulative distribution function $P_V^{(0)}$ are given by (A-4) and (A-9), respectively. Substituting these expressions, letting $x = \sqrt{t}/\sigma$, and using (21), there follows the integral result

$$P_{CD} = \int_T^{\infty} dx \, x \left(\frac{x}{d}\right)^{M-1} I_{M-1}(dx) \exp\left(\frac{x^2+d^2}{-2}\right) [1 - E_{M-1}(x^2/2)]^{N-1} . \quad (29)$$

From physical reasoning or mathematical manipulations, it follows that

$$P_{CD} < P_{SD} < P_{AD} \quad \text{for } N > 1. \quad (30)$$

For $N = 1$, no or-ing, all three detection probabilities are equal to $Q_M(d, T)$.

Also, from (29), since the bracketed term is greater than or equal to its value at $x = T$, we have the lower bound

$$P_{CD} > [1 - E_{M-1}(T^2/2)]^{N-1} Q_M(d, T) \quad \text{for } N > 1. \quad (31)$$

Thus we have the tight bounds on the probability of correct detection:

$$[1 - E_{M-1}(T^2/2)]^{N-1} Q_M(d, T) < P_{CD} < Q_M(d, T) . \quad (32)$$

To show how tight these bounds are, recall the false alarm probability in (25), in order to express the bounds as

$$(1 - P_F)^{\frac{N-1}{N}} Q_M(d,T) < P_{CD} < Q_M(d,T) . \quad (33)$$

For small false alarm probabilities,

$$(1 - P_F)^{\frac{N-1}{N}} \approx 1 - P_F \frac{N-1}{N} > 1 - P_F , \quad (34)$$

leading to

$$(1 - P_F) Q_M(d,T) < P_{CD} < Q_M(d,T); \quad (35)$$

thus the bounds in (32) are very tight for small false alarm probabilities.

This is very convenient computationally, since it means that we will not have to evaluate the integral in (29) numerically, but need only compute the simpler quantities Q_M and E_{M-1} .

One special case of P_{CD} can be evaluated in closed form: for $d = 0+$, (28) yields

$$\begin{aligned} P_{CD}^{(0)} &= \int_u^\infty dt \, p_V^{(0)}(t) [p_V^{(0)}(t)]^{N-1} = \\ &= \frac{1}{N} \left\{ 1 - [p_V^{(0)}(u)]^N \right\} = \frac{1}{N} P_F , \end{aligned} \quad (36)$$

the latter relation following from (25). This relation agrees with physical reasoning.

TIGHTNESS OF BOUND

To verify the accuracy afforded by using the upper bound $Q_M(d,T)$, instead of the exact result (29) for P_{CD} , a short comparative study of the two quantities was conducted; the numerical results are tabulated in appendix B. False alarm probabilities near the values .1, .01, .001, and detection probabilities near the values .5, .9, .99, .999 were considered, while M took on values 1,10, and N took on values 2,10,100,1000. These ranges of values encompass most of the cases of practical interest; there is no need to consider smaller P_F values, since the discrepancy is even smaller then. It will be observed that for $P_F < .1$ (the only cases plotted here), the differences between the exact P_{CD} and $Q_M(d,T)$ are inconsequential; in particular, see figure B-1.

GRAPHICAL RESULTS

In this section, we plot the analytical results for the false alarm probability (25) and the tight upper bound on the probability of correct detection (33); see (35). The number of filter outputs summed, M , ranges over the values

$$M = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 = 1(1)10, \quad (37)$$

while the number of channel or-ed, N , ranges over the values

$$N = 1, 10, 100, 1000. \quad (38)$$

The parameter, d , on the plots is the generic signal-to-noise ratio defined by (15), for general signals and filters. The 40 combinations corresponding to (37) and (38) are plotted on normal probability paper in figures* 3 through 42. Values of d small enough to encompass the (poor quality) operating point $(P_F, P_{CD}) = (.01, .5)$ have been employed; while at the high quality end, values of d extending up to $(P_F, P_{CD}) = (1E-10, .999)$ have been used. The increment in d is .5 for all the results in figures 3 through 42.

*All the figures are collected together, after the Summary section.

It will be observed that the curves are approximately equispaced in parameter d , thereby allowing for ready accurate interpolation in d , given specified P_F and P_{CD} . The curves, for cases in which $N = 1$, are virtually straight lines, while those for $N = 1000$ have developed significant curvature; nevertheless, the equispaced nature of the results readily accommodates interpolation in all cases.

From these results, it is possible to extract a different type of performance characteristic, namely the required values of d to achieve a specified quality of performance in terms of false alarm probability and detection probability. In figures 43 through 48, these results are plotted for the six combinations of

$$M = 1, 2, 4 \quad \text{with } P_{CD} = .5, .9, \quad (39)$$

while N varies over 1(1)1000, and P_F takes on the values $1E-2$, $1E-4$, $1E-6$, $1E-8$, $1E-10$. (Strictly, only the cases for $N = 1, 10, 100, 1000$ follow from figures 3 through 42; the remaining values of N were obtained directly from (25) and (33).)

The most striking feature of figures 43 through 48 is their slow increase with N , the number of channels subjected to or-ing. Certainly the increase in required d values was anticipated, since or-ing cannot improve performance capability; however, the amount of increase is not very significant. Thus, from figure 43, for $P_F = 1E-10$, d need only increase from 6.71 to 7.67 as N increases from 1 (no or-ing) to $N = 1000$. Greater increases are necessary for the larger P_F values.

SUMMARY

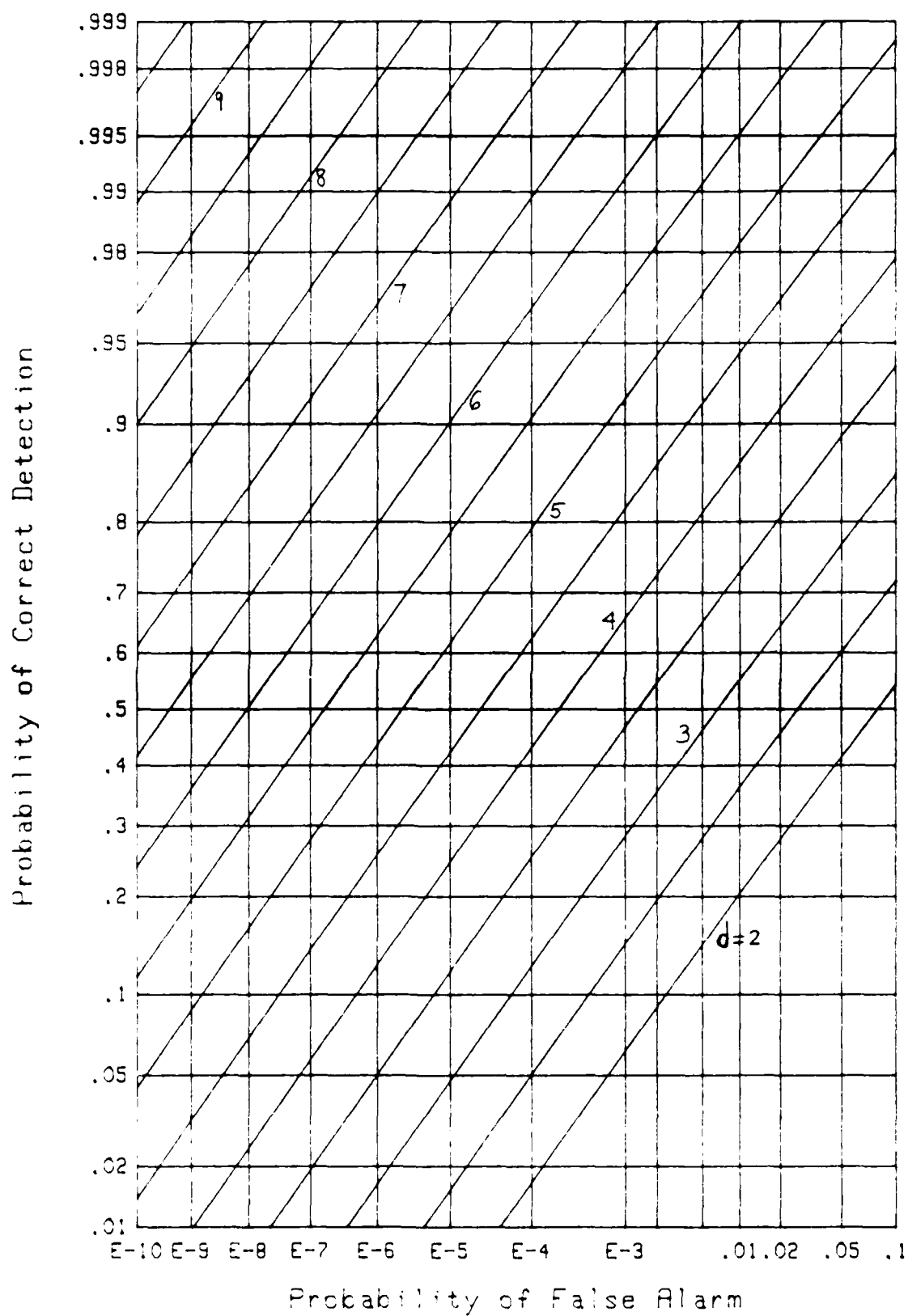
It will be easily observed from the graphical results in figures 3 through 42 that, for a fixed amount of or-ing (fixed N), the performance degrades as M increases. That is, for specified values of P_F and d , the values of P_{CD} decrease as M is increased. Alternatively, to maintain a specified performance pair P_F, P_{CD} , the values of d must be increased as M increases. This is due to the fact that parameter d in (15) or (17) is a total (or output) signal-to-noise ratio measure and that larger M corresponds to increased fractionalization of the received signal energy into more paths or branches. Since the filter-output combination rule is incoherent, namely adding squared envelopes, this fractionalization cannot be made up by summation, and a loss occurs.

On the other hand, if we were to add more paths to a particular system, then both M and d would increase. Whether this results in an improvement or degradation depends on the relative amount of additional energy. Particular cases can be studied quantitatively by referring to figures 3 through 42. In addition, programs for the procedures in this report are listed in BASIC in appendix C, if additional cases of interest to the reader need to be investigated.

The maximum value of d^2 is given by (17) as $2E_T/N_0$; this can be realized only if the matched filters (16) are utilized and if the sampling times are properly selected. If these conditions are not met, the value of d^2 given by (15) must be employed. In any event, the figures are

parameterized by quantity d , regardless of what filters and sampling times are used. Thus a desired value of d for a mismatched situation will require larger signal levels for $\{s_m\}$ in (15) than the values indicated by the ideal, (17). In this manner, the degradation caused by mismatch and/or desynchronization can be quantitatively assessed.

The received signal was assumed to have undergone no fading in the current analysis. Extensions to fading signals, but without or-ing, are available in [3]. This latter reference presumed a fixed threshold for decision variable comparisons (as did this analysis in (3) and (18)); extensions to a variable threshold, based on a finite sample size noise-level estimation procedure, are currently underway. Results on this normalizer in a fading environment will be reported on shortly by the author.

Figure 3. ROC for $M=1$, $N=1$

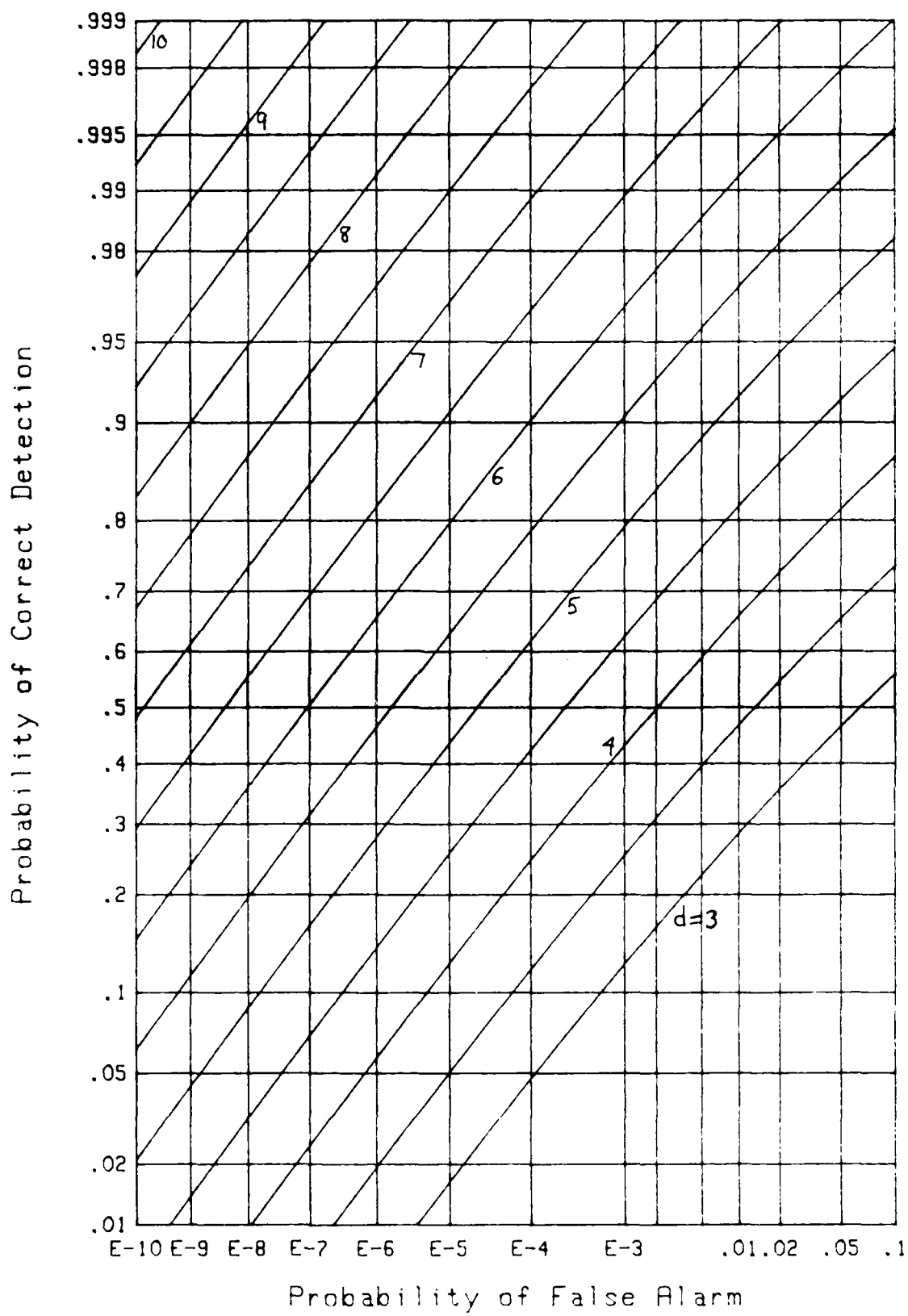
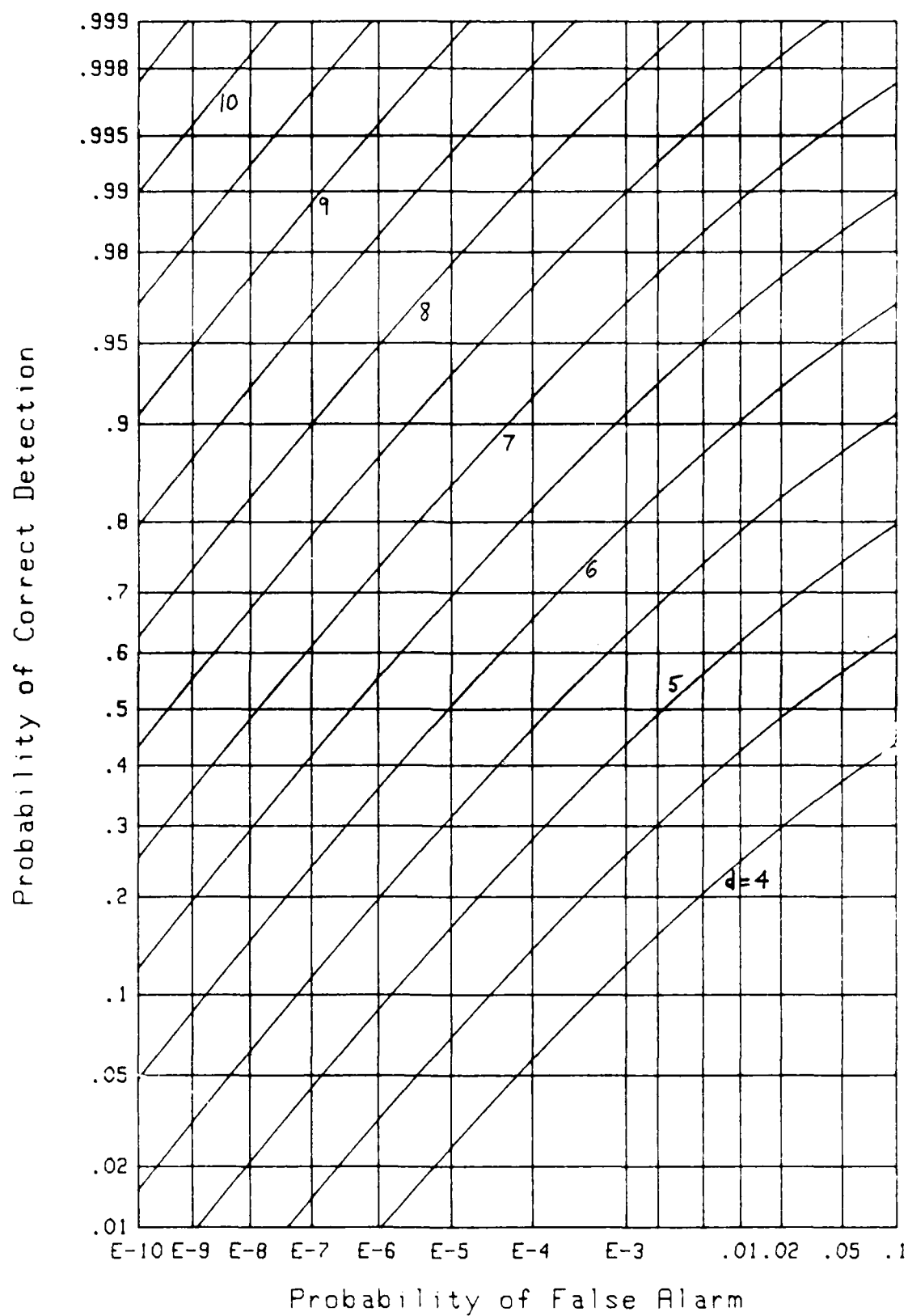
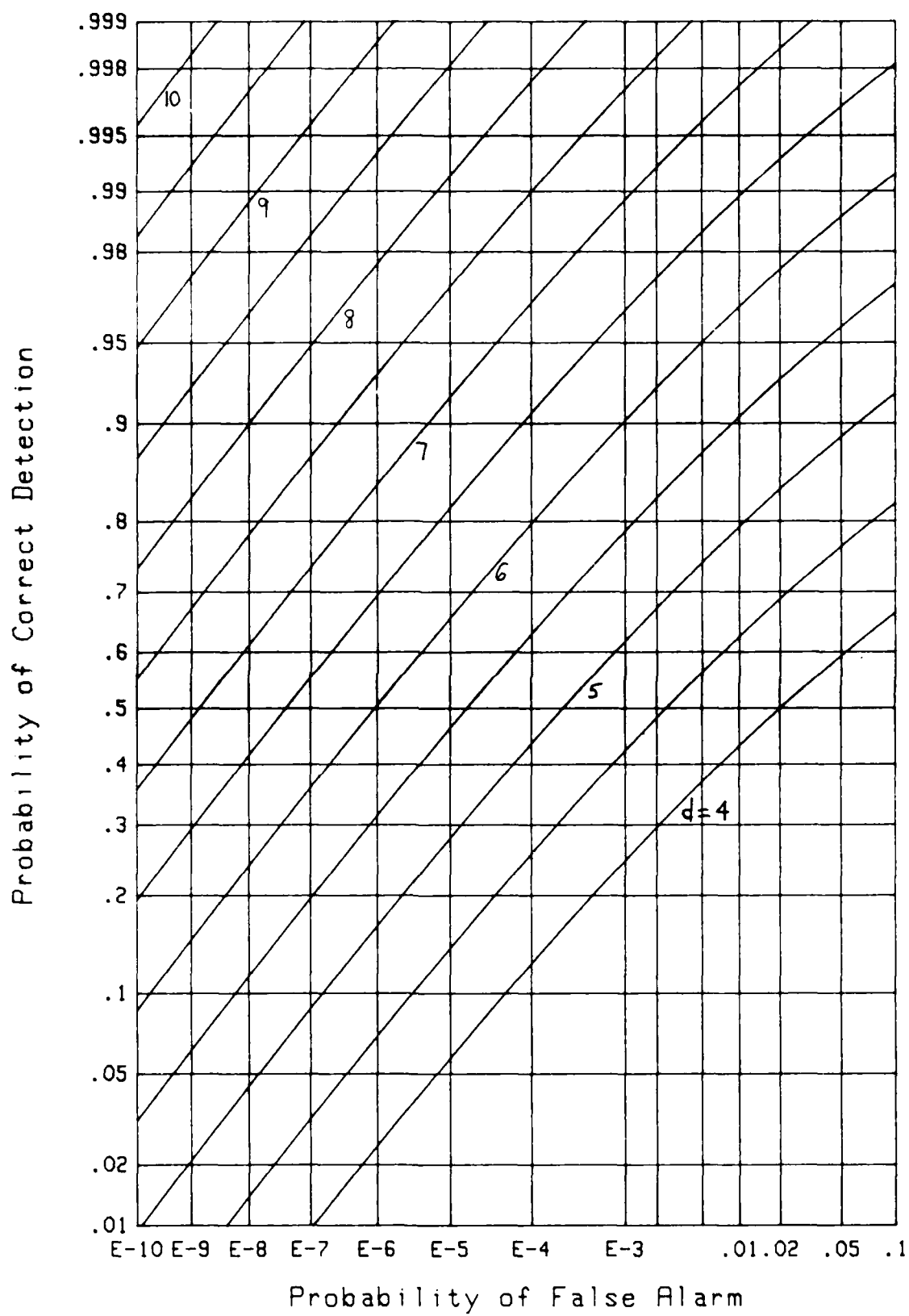
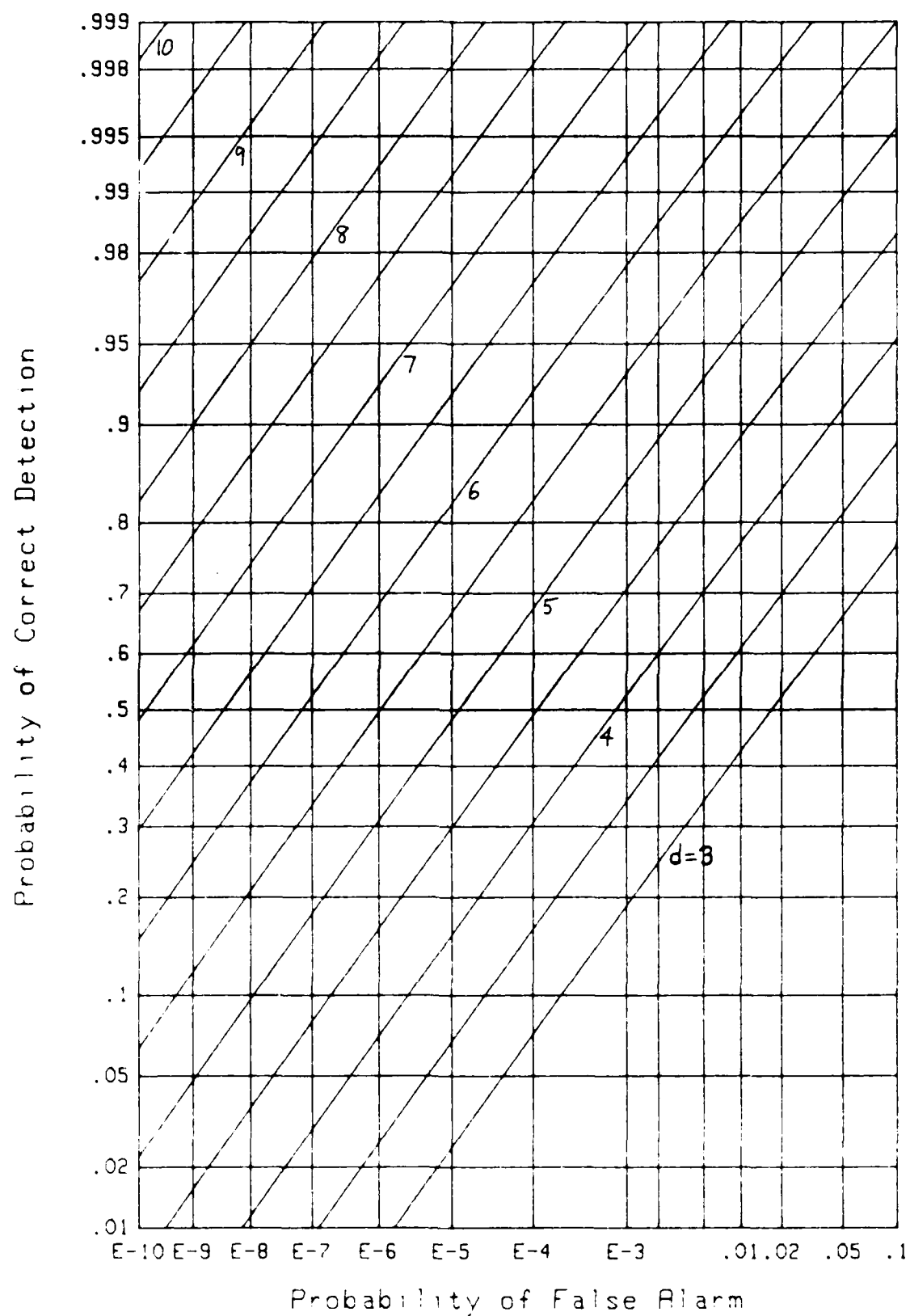
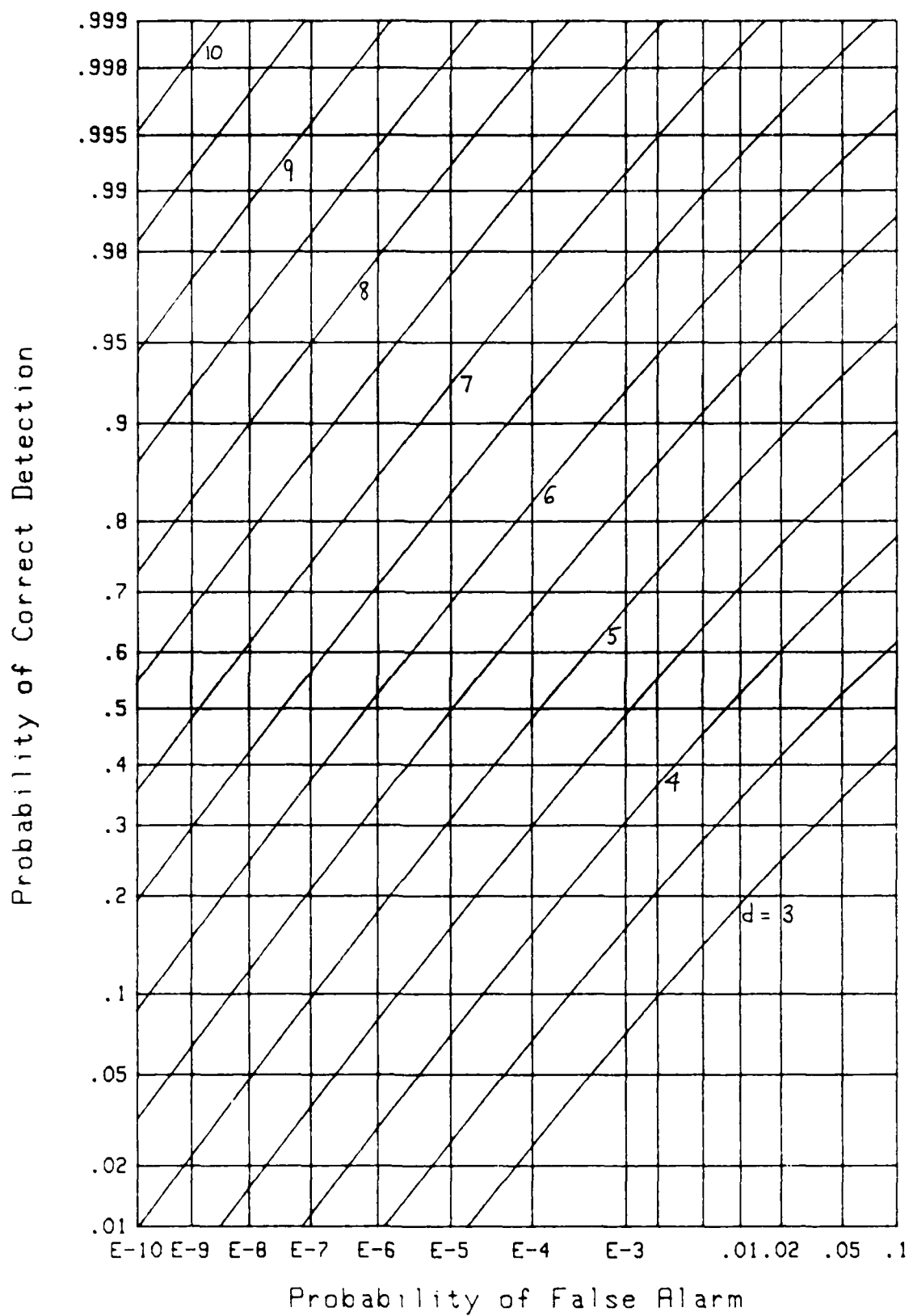


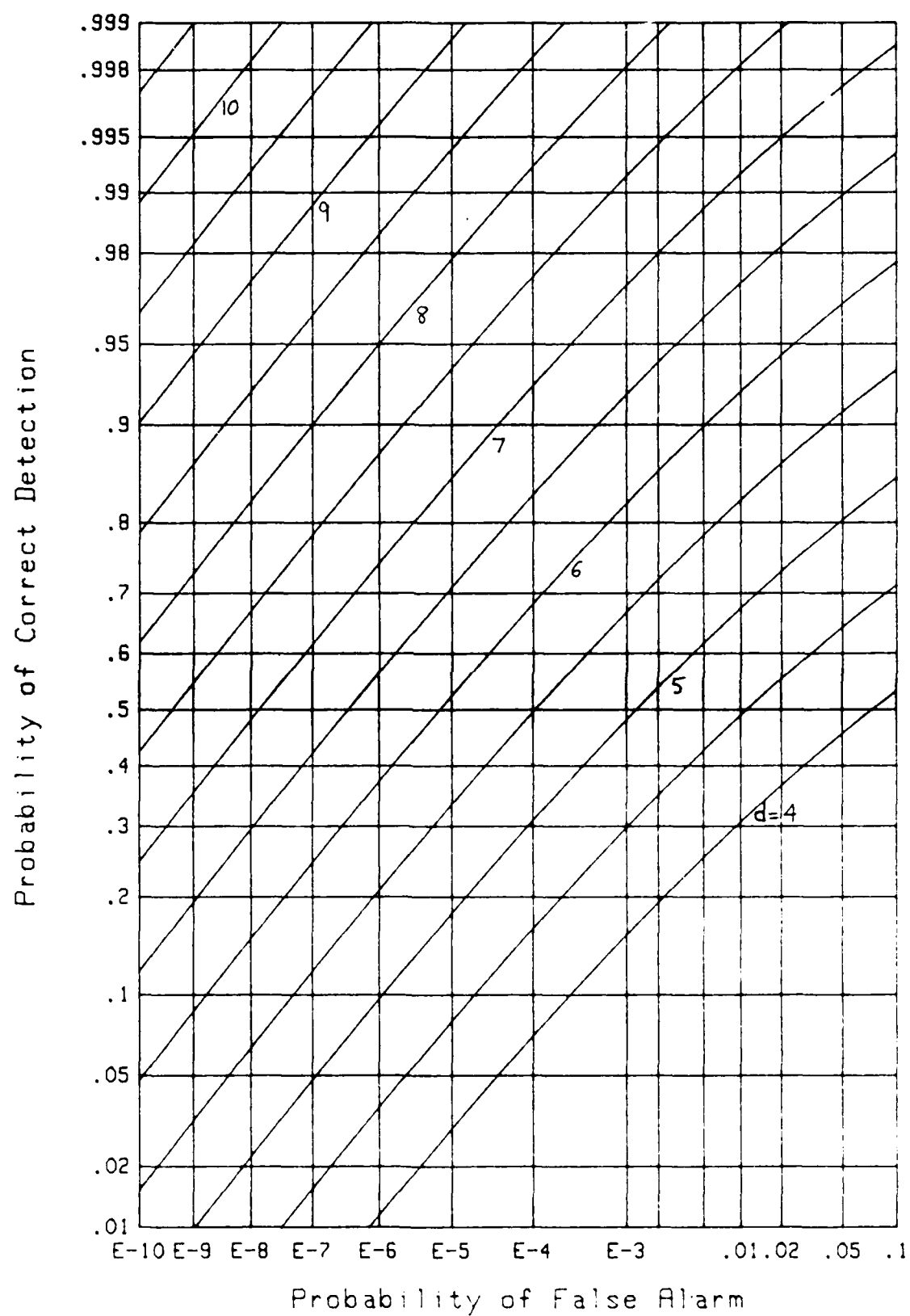
Figure 4. ROC for $M=1$, $N=10$

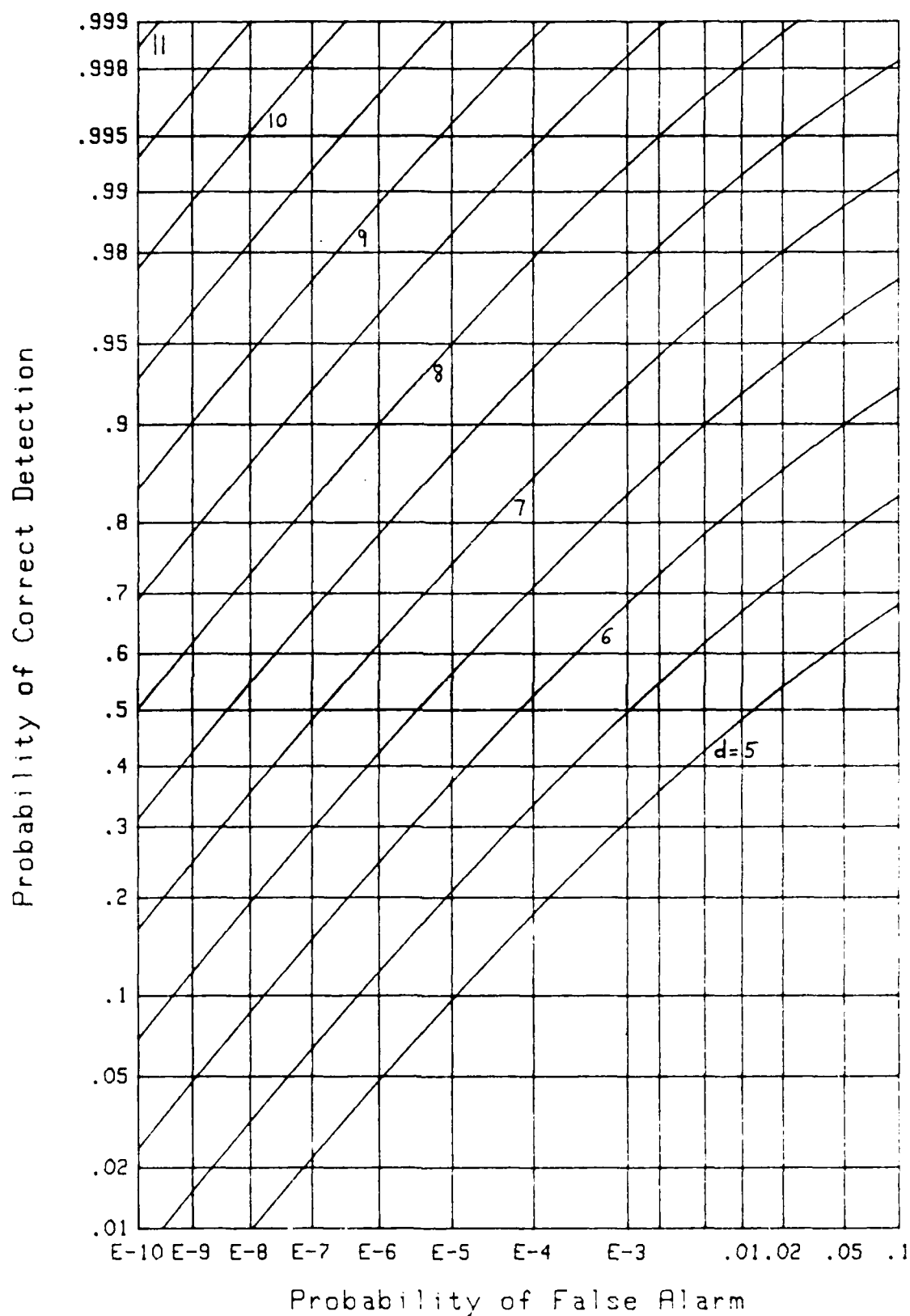
Figure 6. ROC for $M=1$, $N=1000$

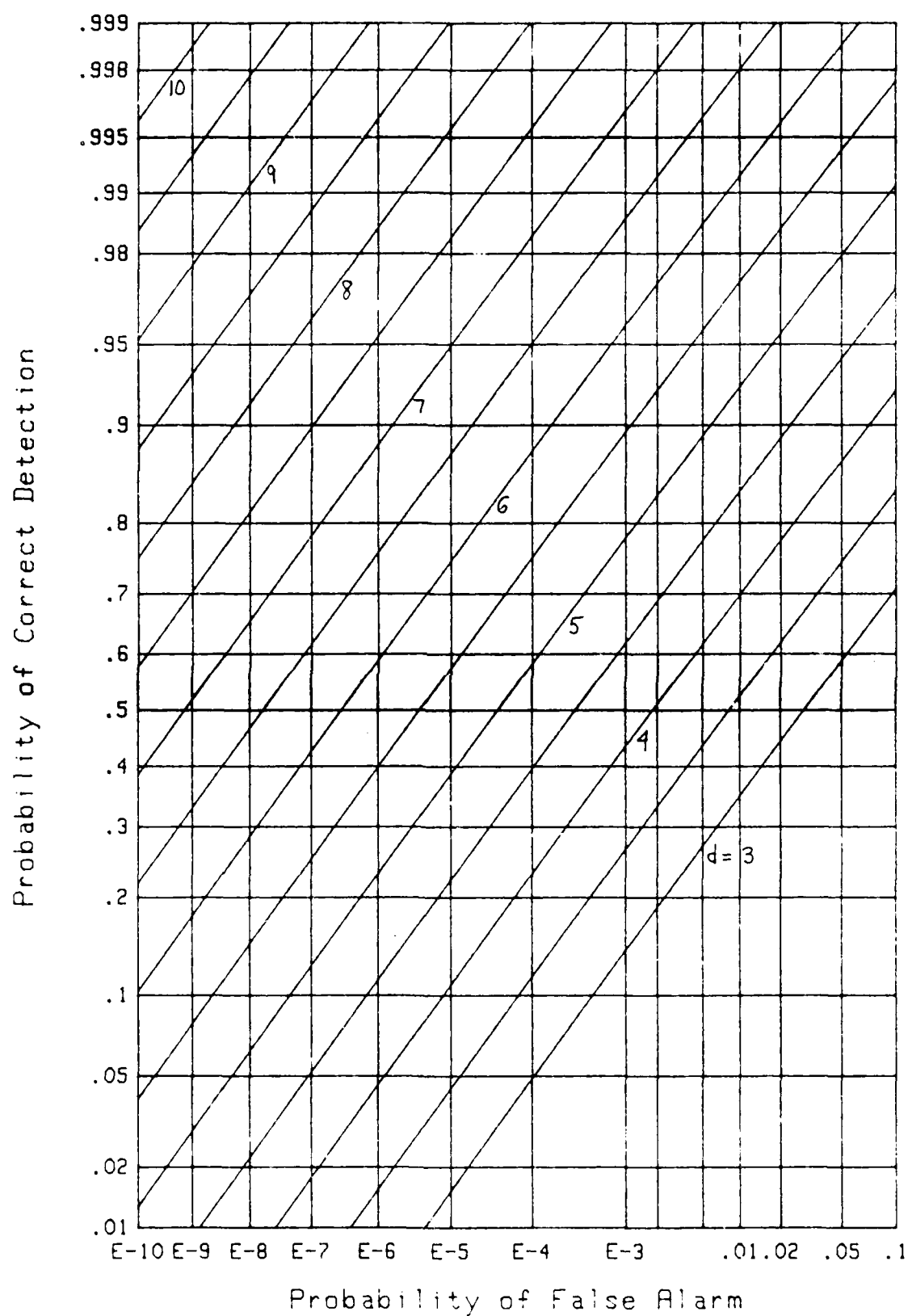
Figure 5. ROC for $M=1$, $N=100$

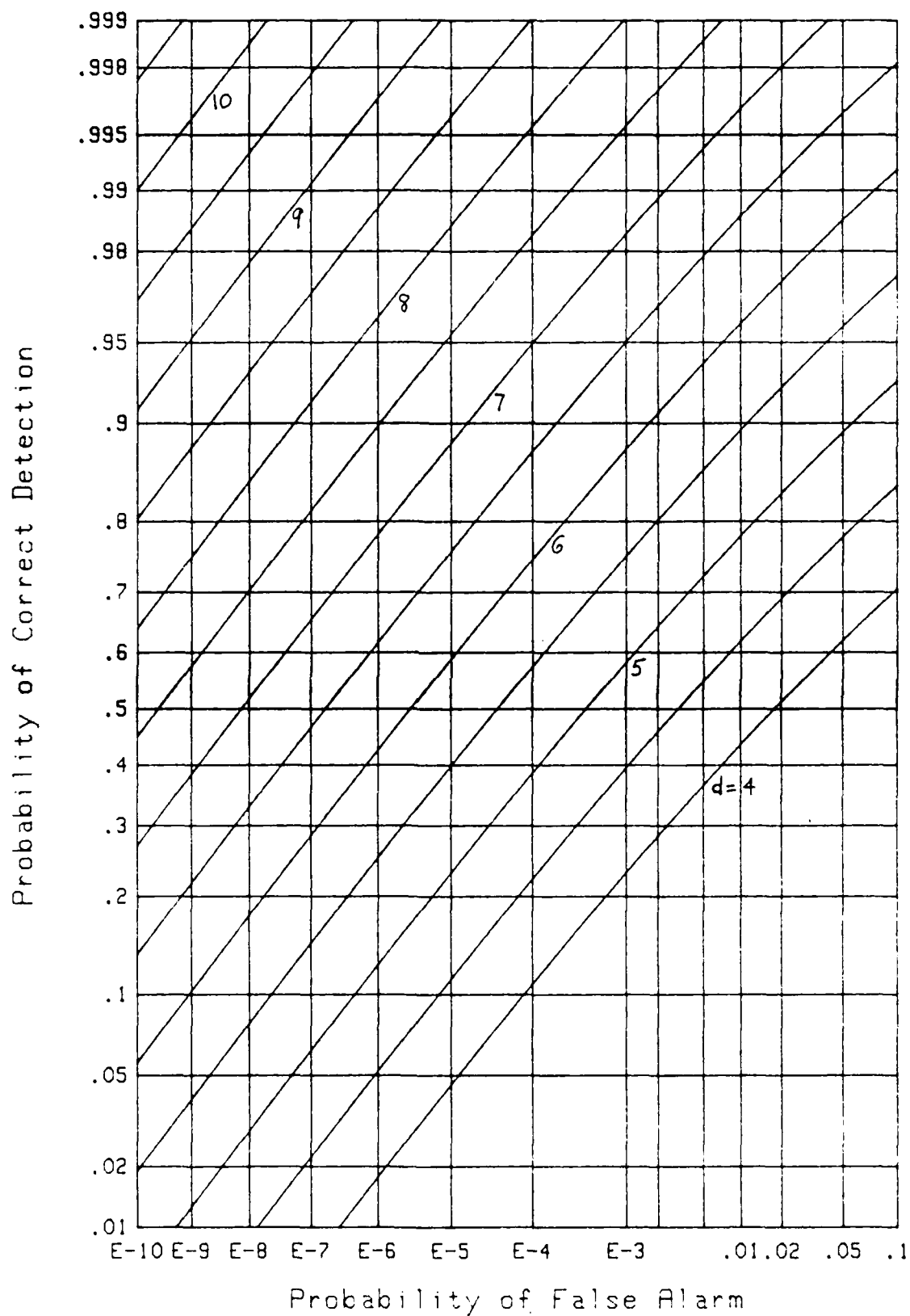
Figure 7. ROC for $M=2$, $N=1$

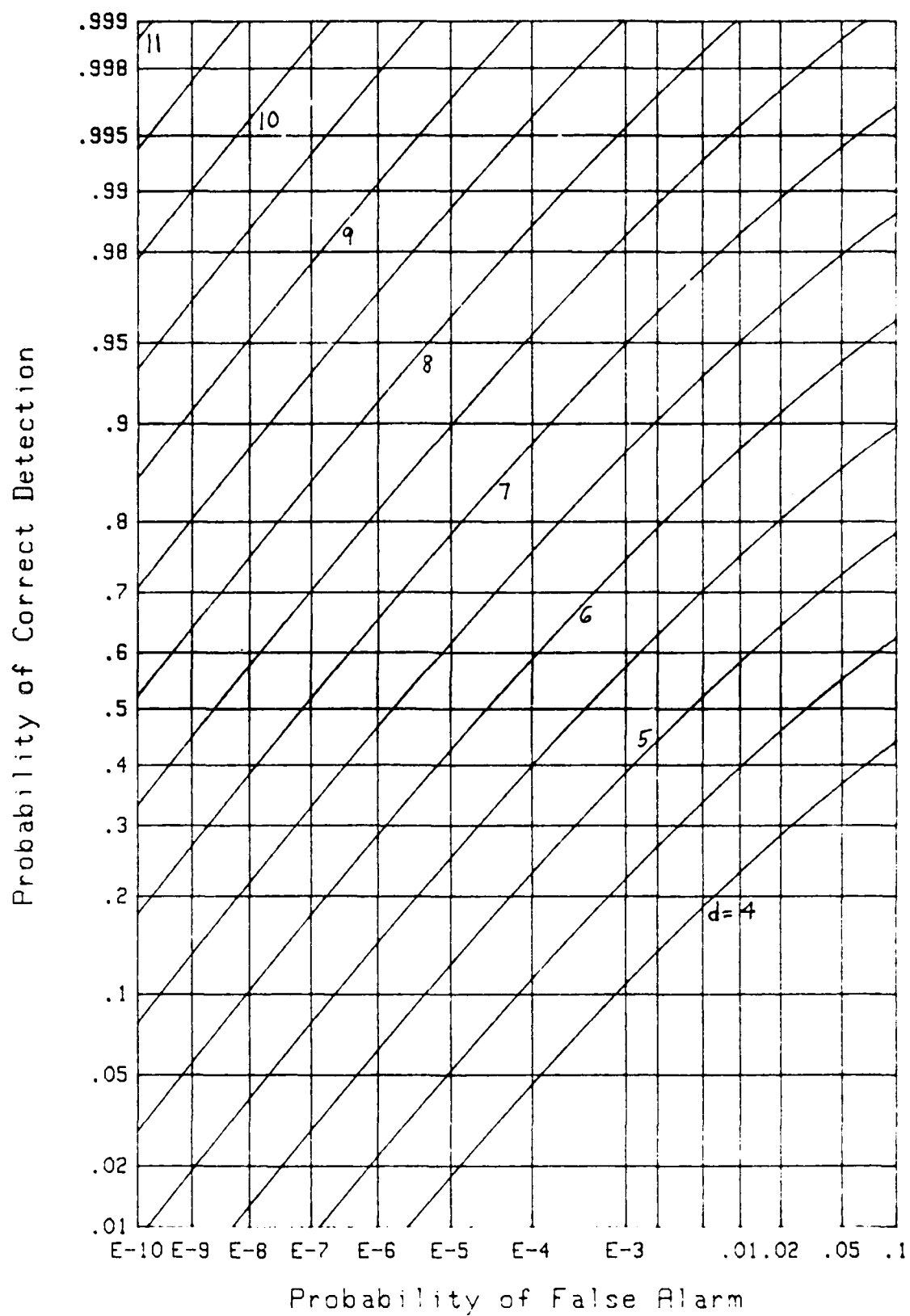
Figure 8. ROC for $M=2$, $N=10$

Figure 9. ROC for $M=2$, $N=100$

Figure 10. ROC for $M=2$, $N=1000$

Figure 11. ROC for $M=3$, $N=1$

Figure 12. ROC for $M=3$, $N=10$

Figure 13. ROC for $M=3$, $N=100$

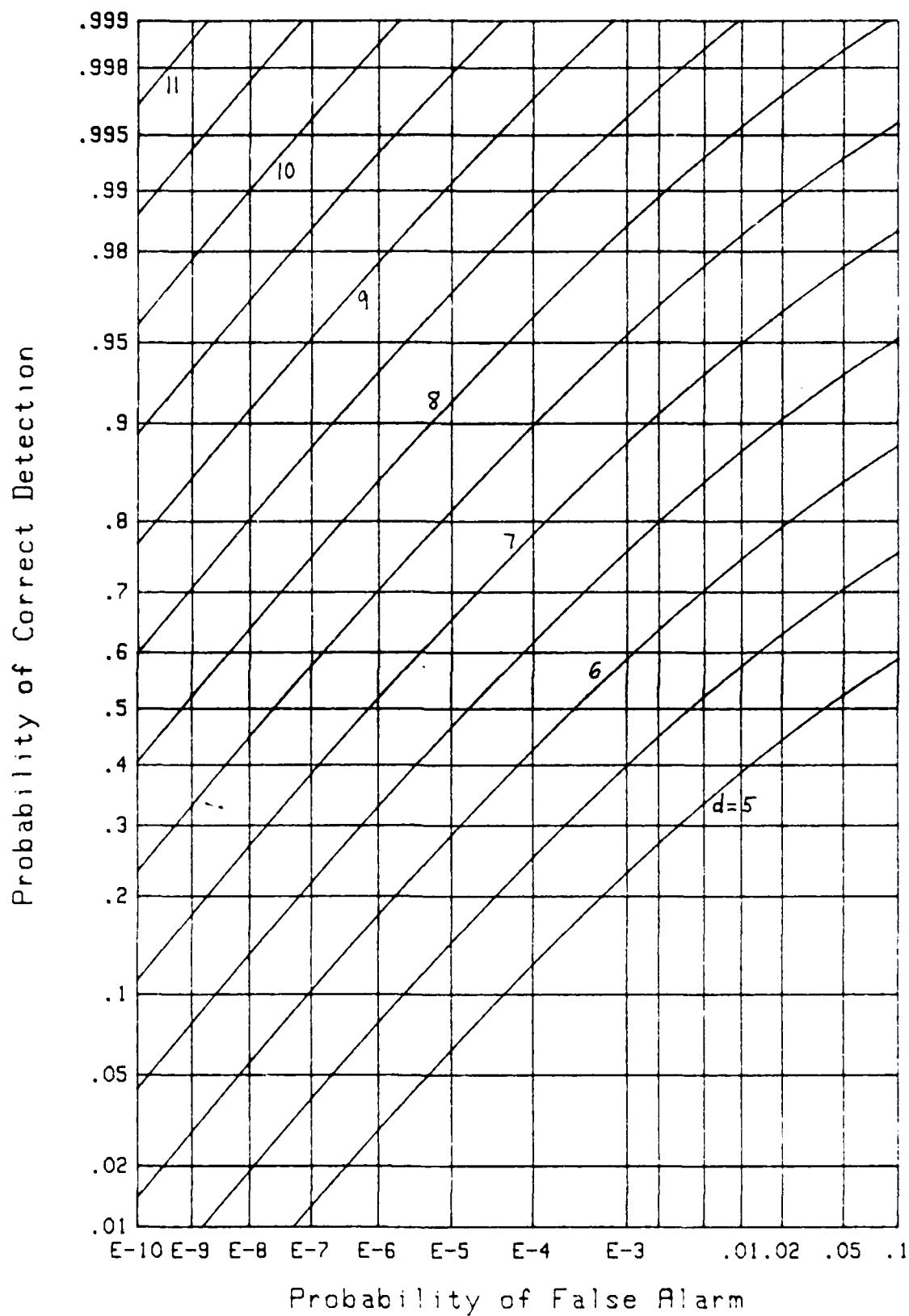
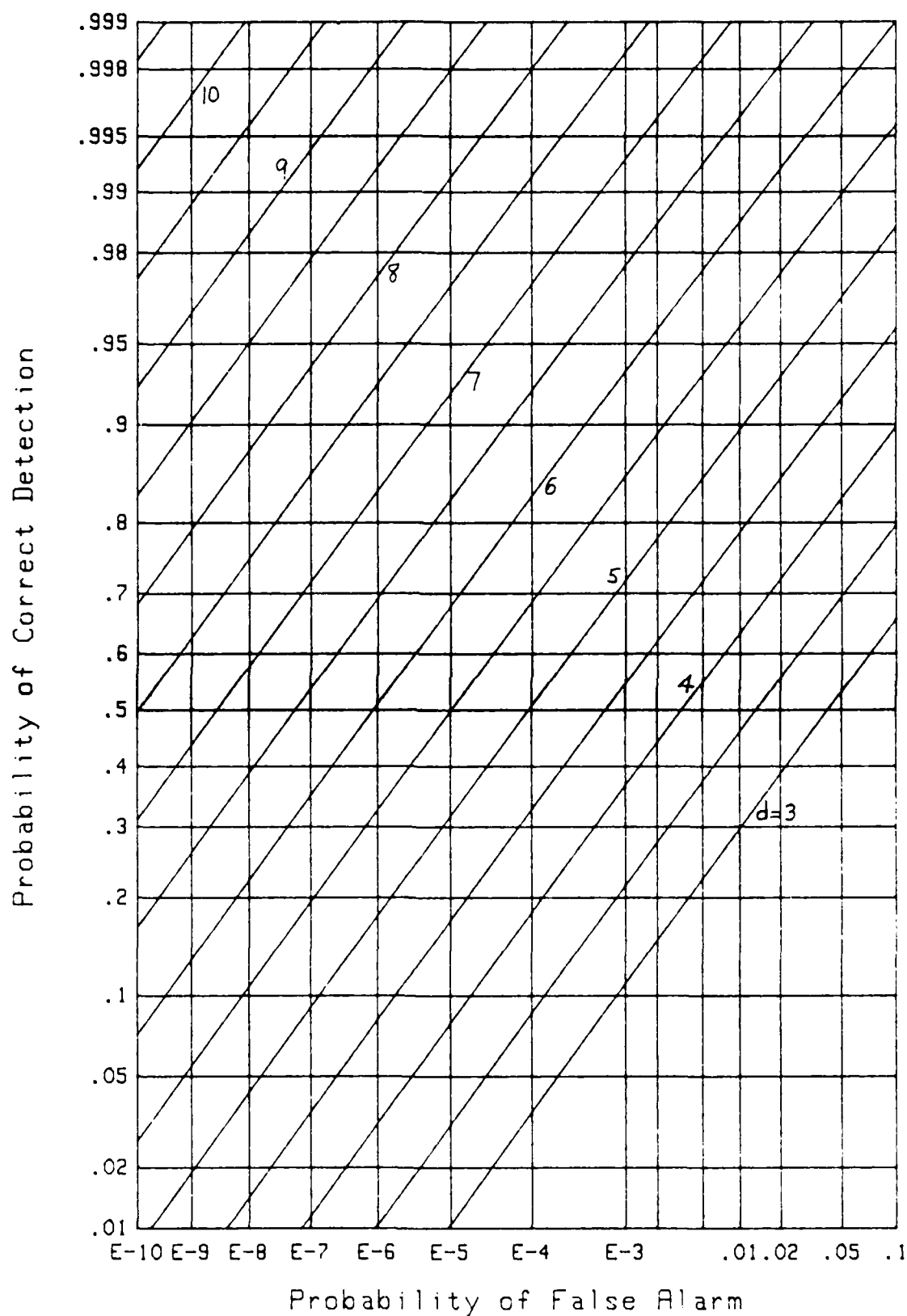
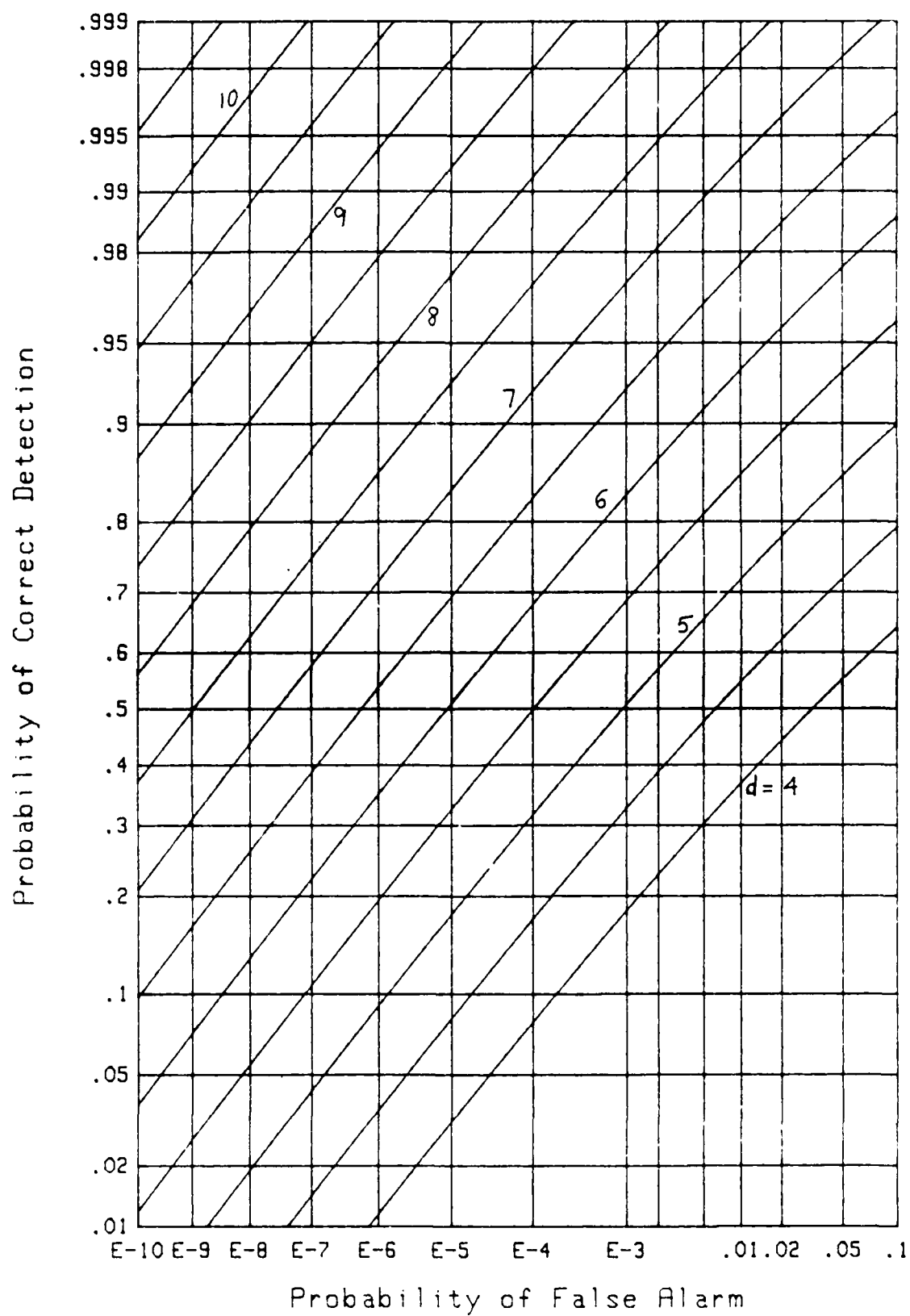
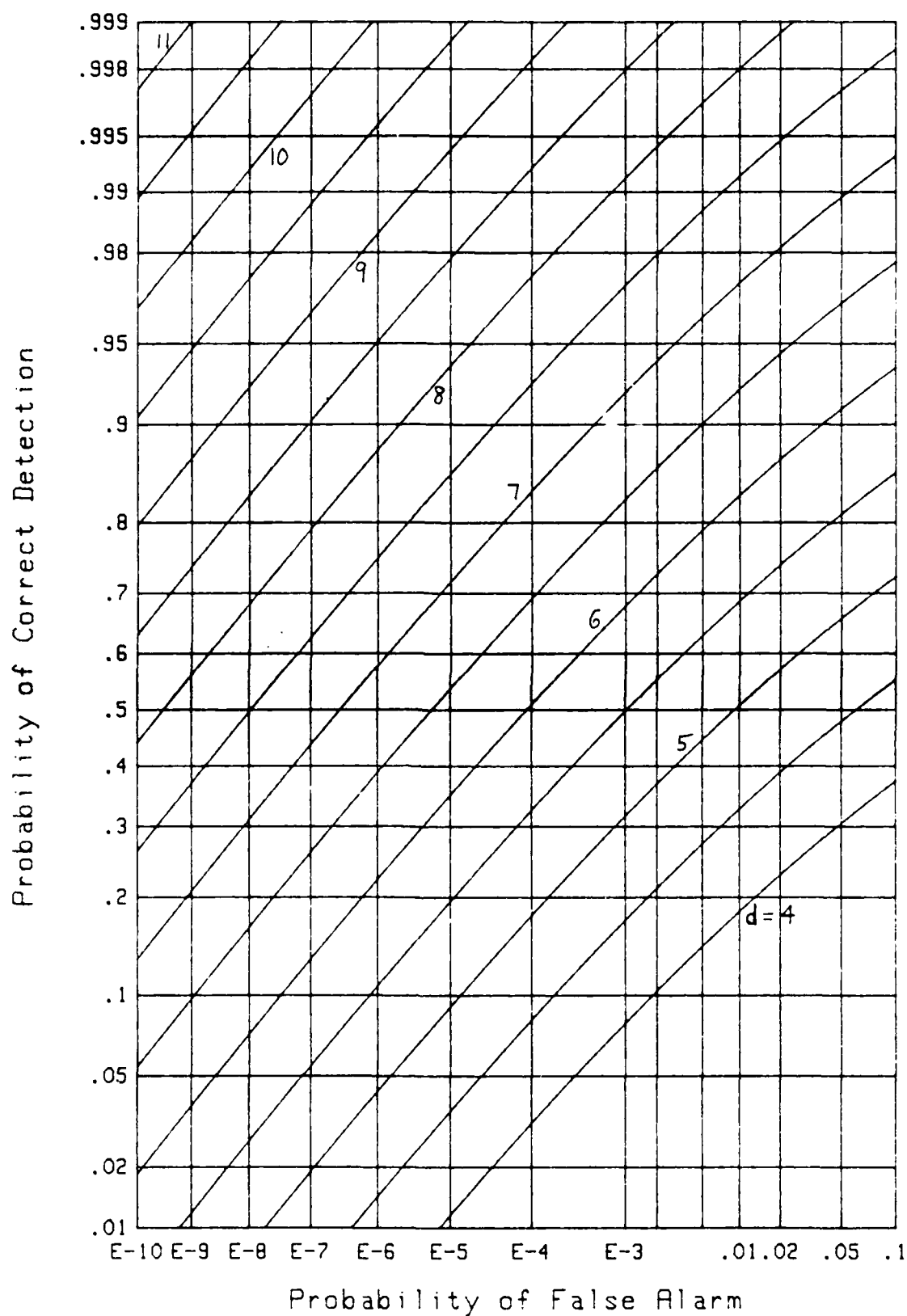
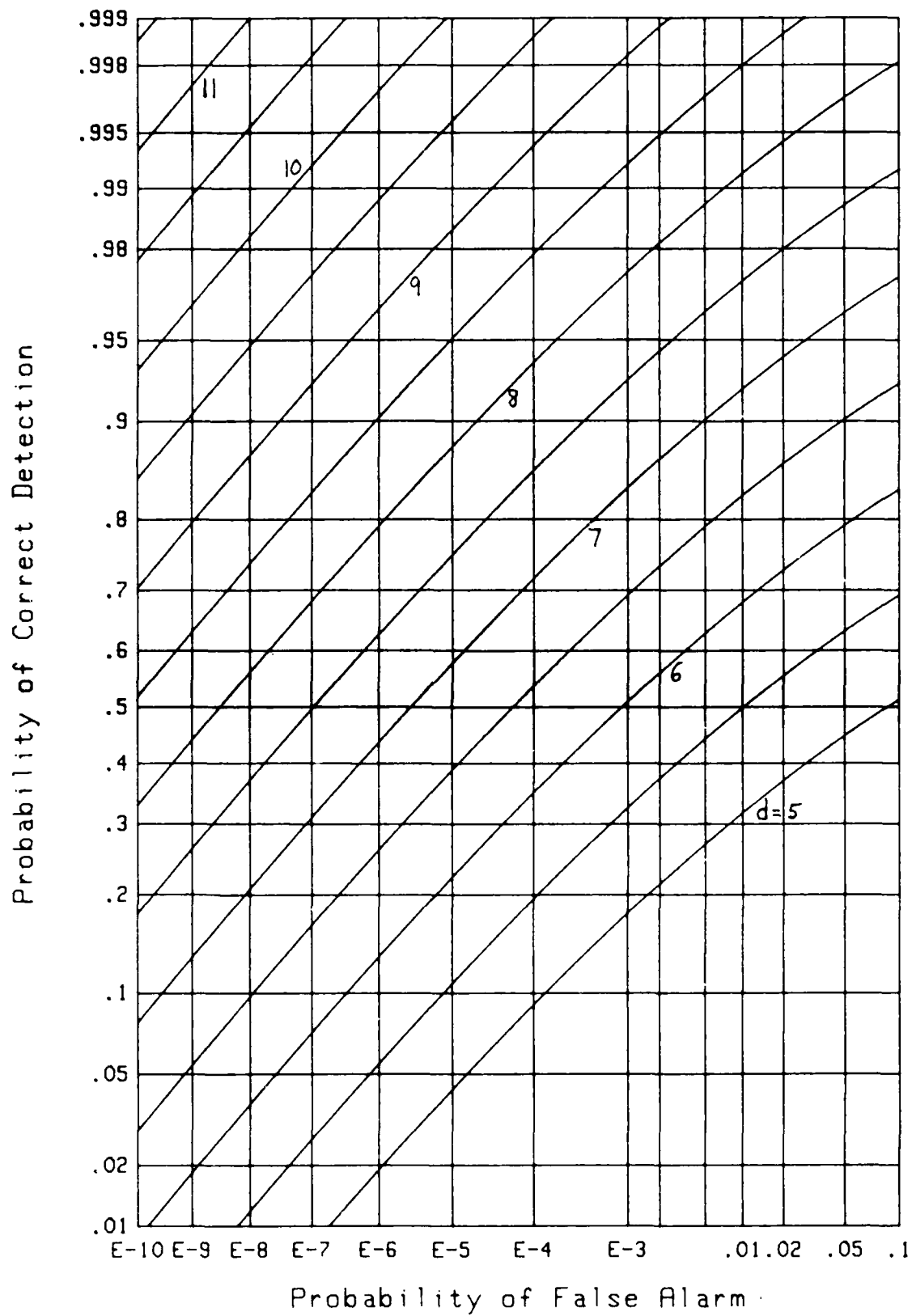


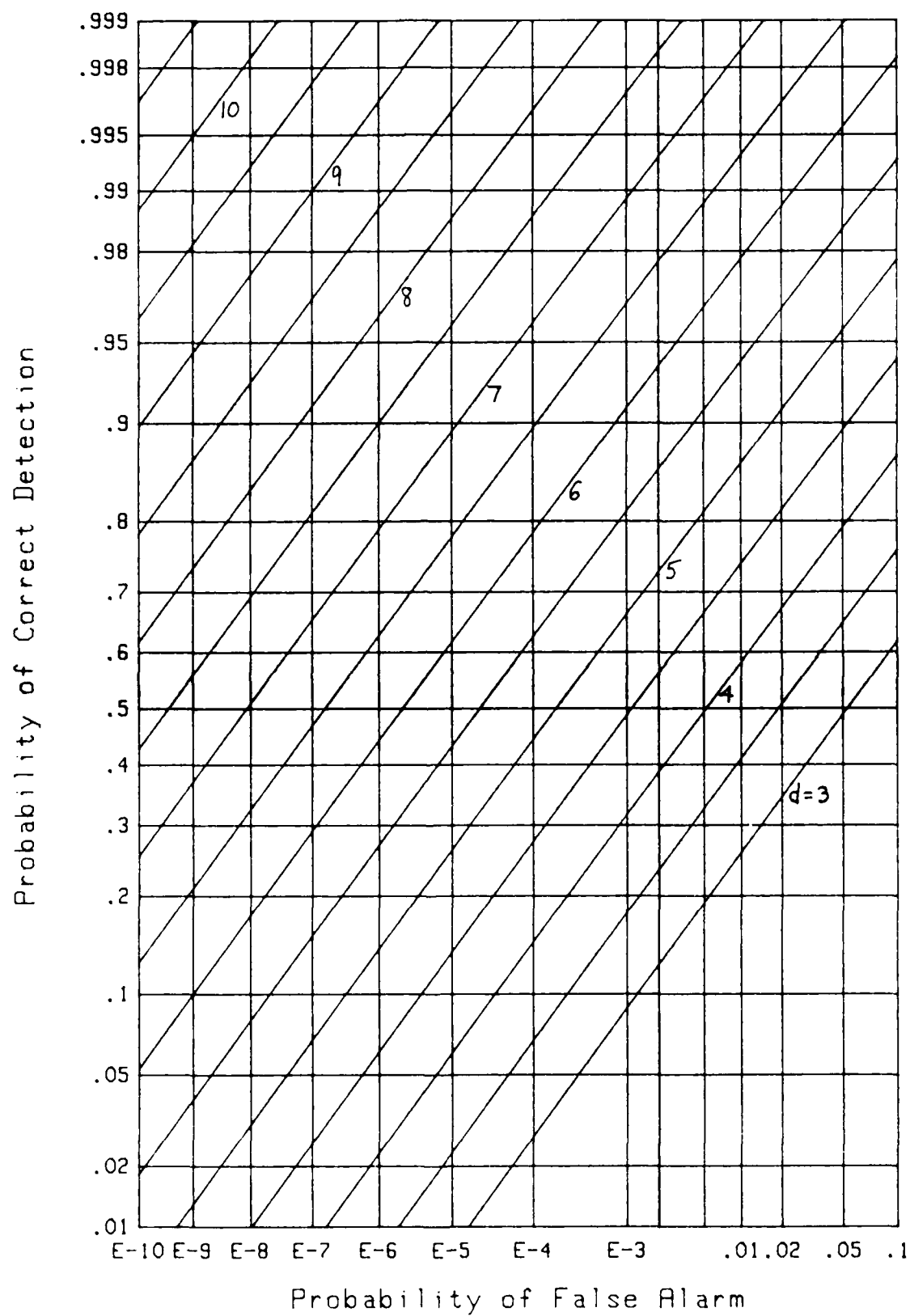
Figure 14. ROC for M=3, N=1000

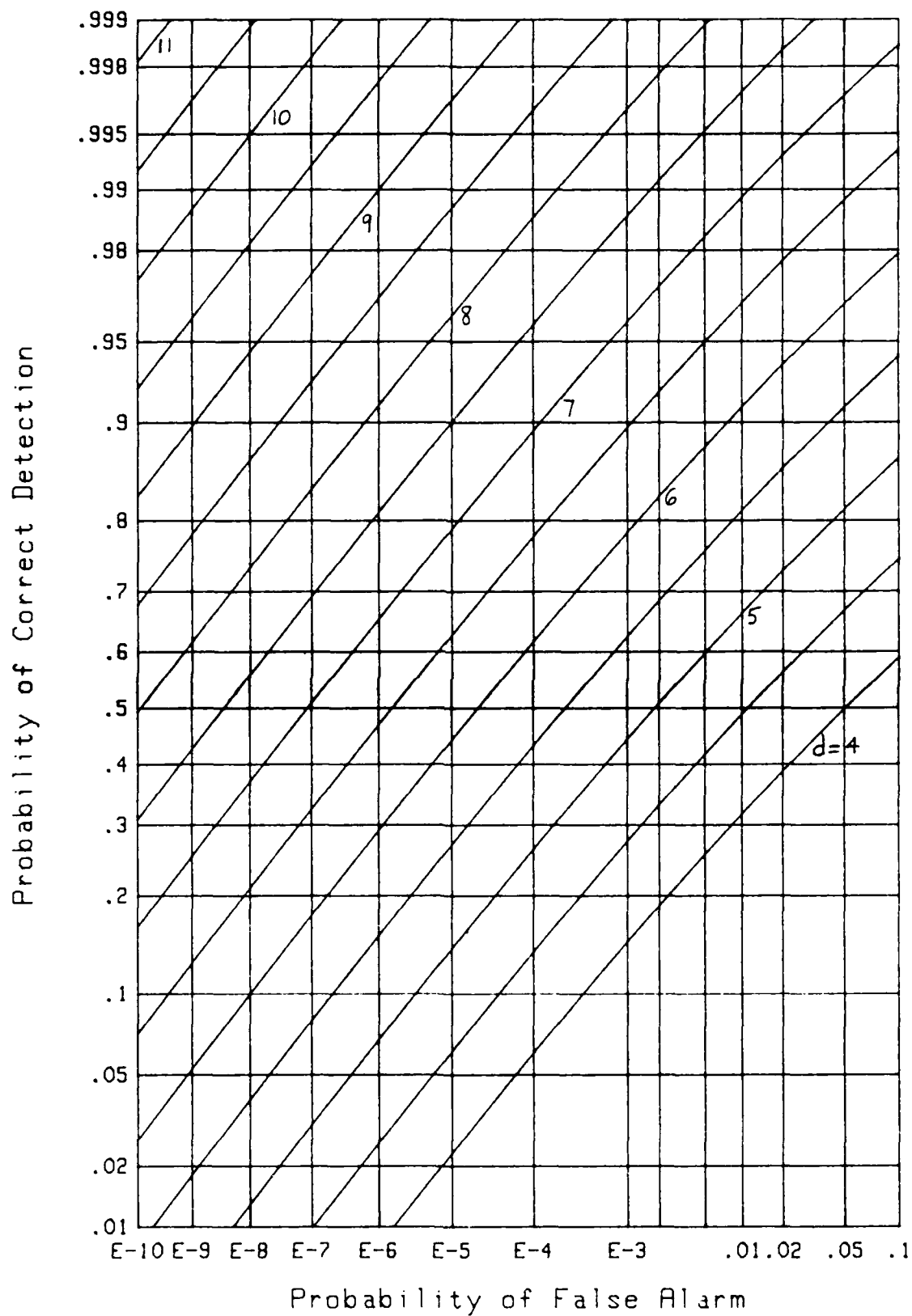
Figure 15. ROC for $M=4$, $N=1$

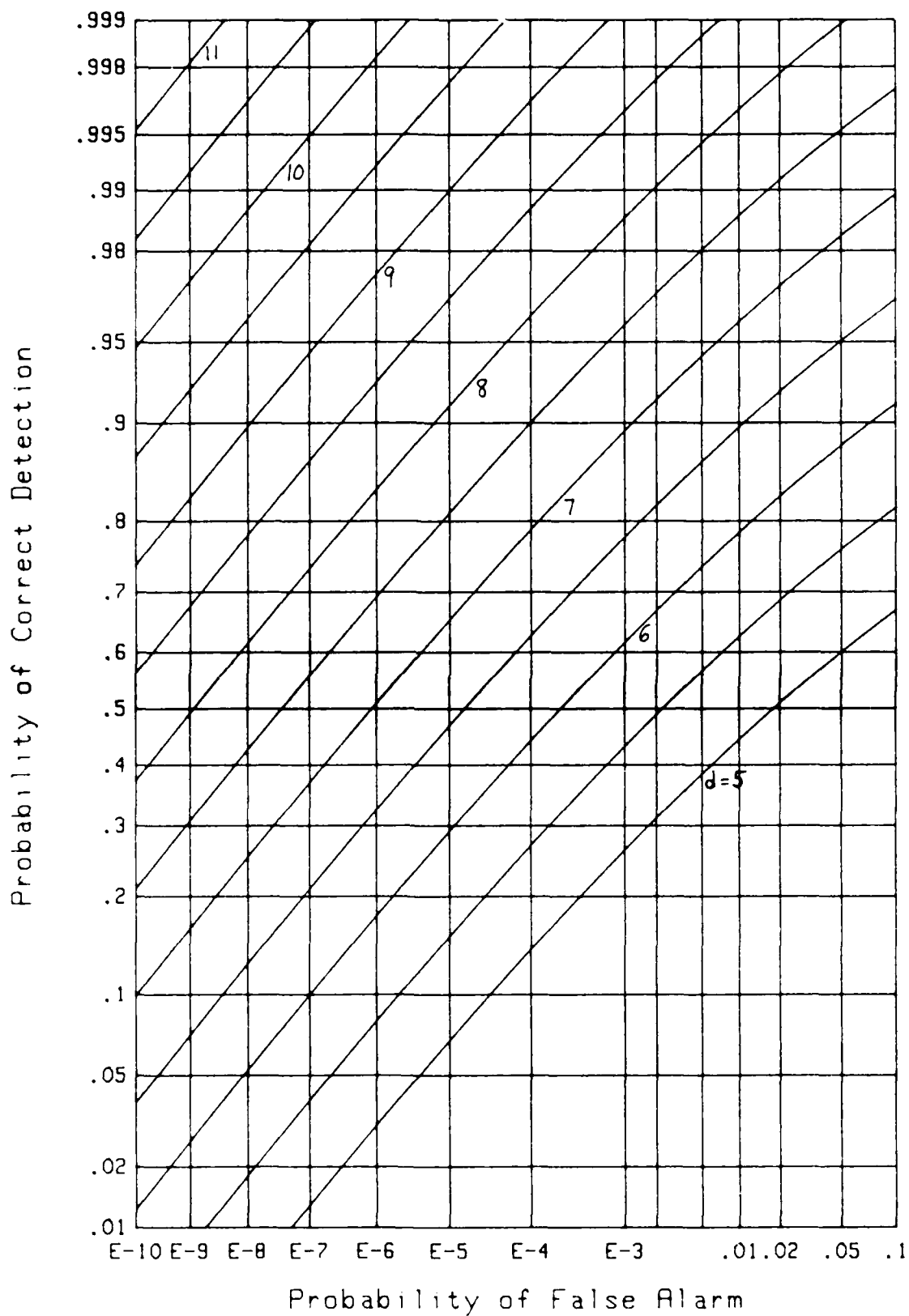
Figure 16. ROC for $M=4$, $N=10$

Figure 17. ROC for $M=4$, $N=100$

Figure 18. ROC for $M=4$, $N=1000$

Figure 19. ROC for $M=5$, $N=1$

Figure 20. ROC for $M=5$, $N=10$

Figure 21. ROC for $M=5$, $N=100$

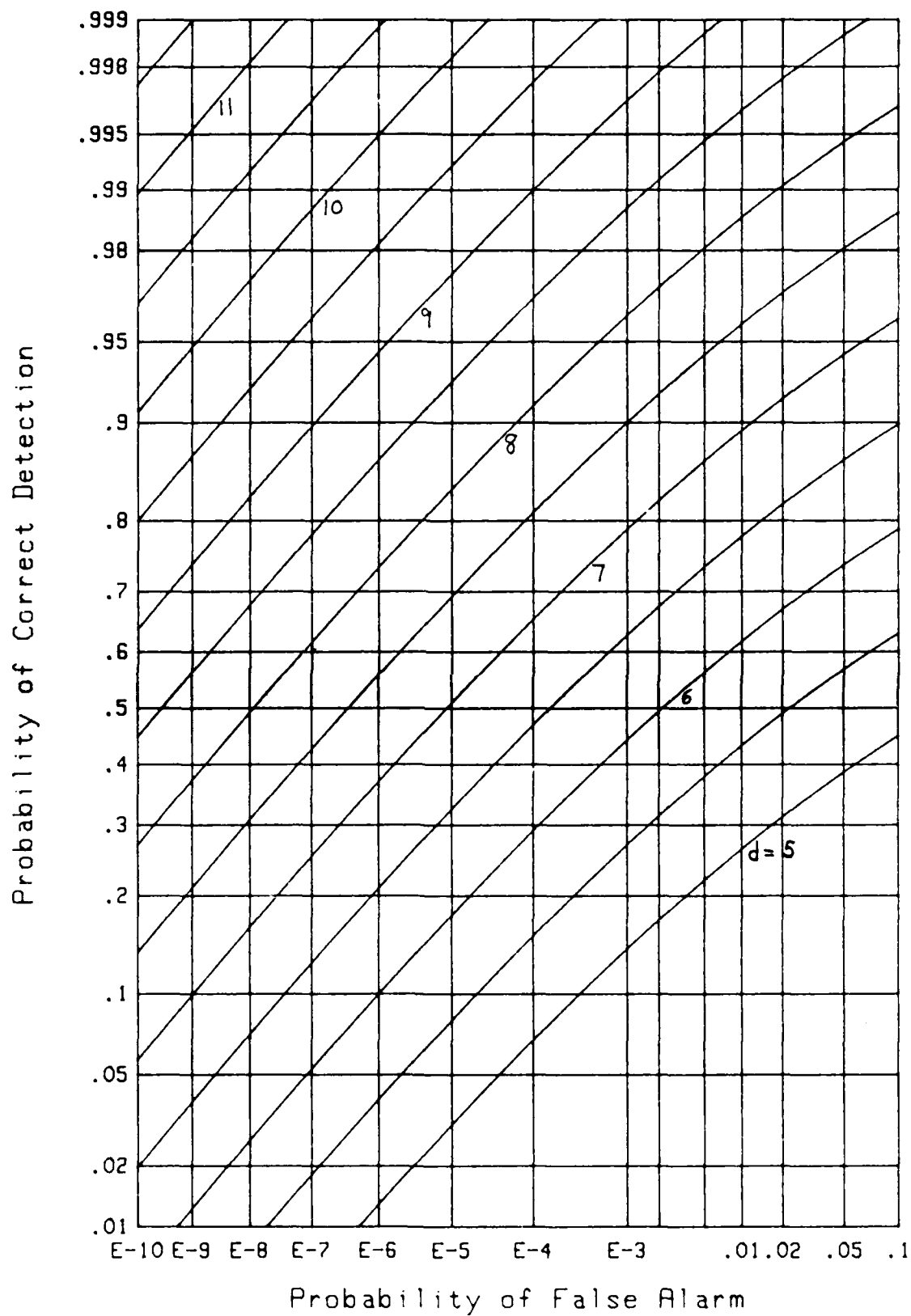
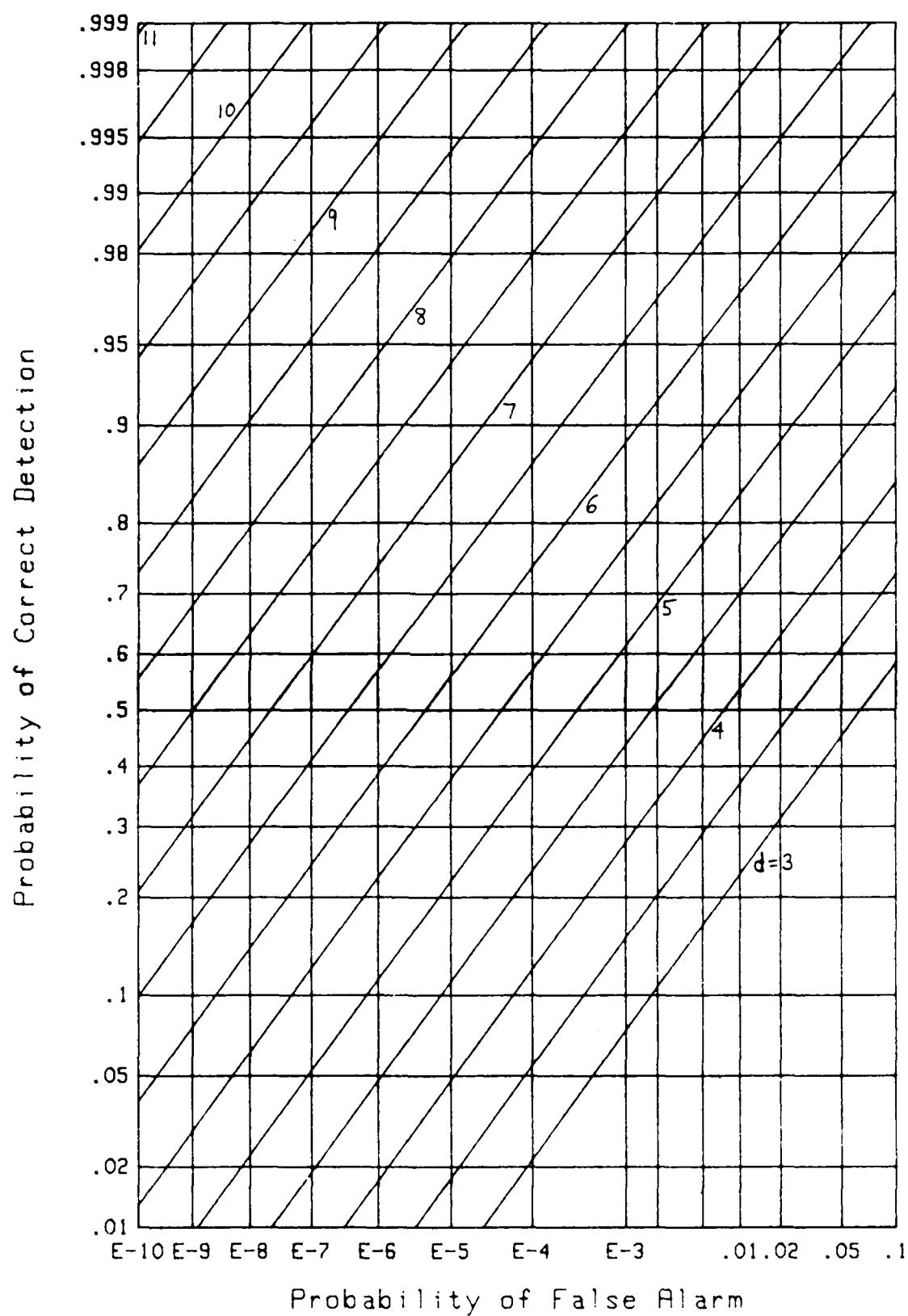
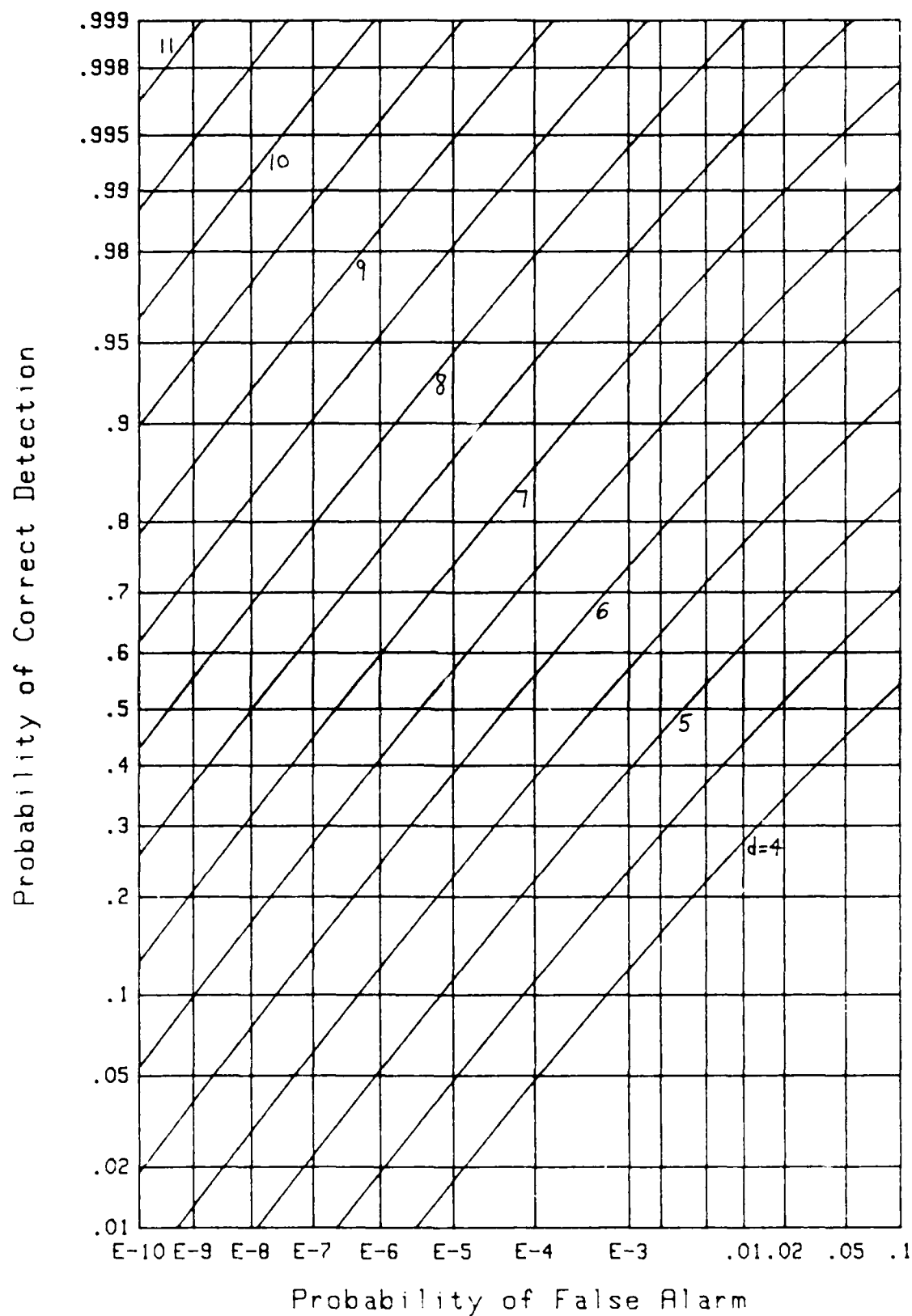
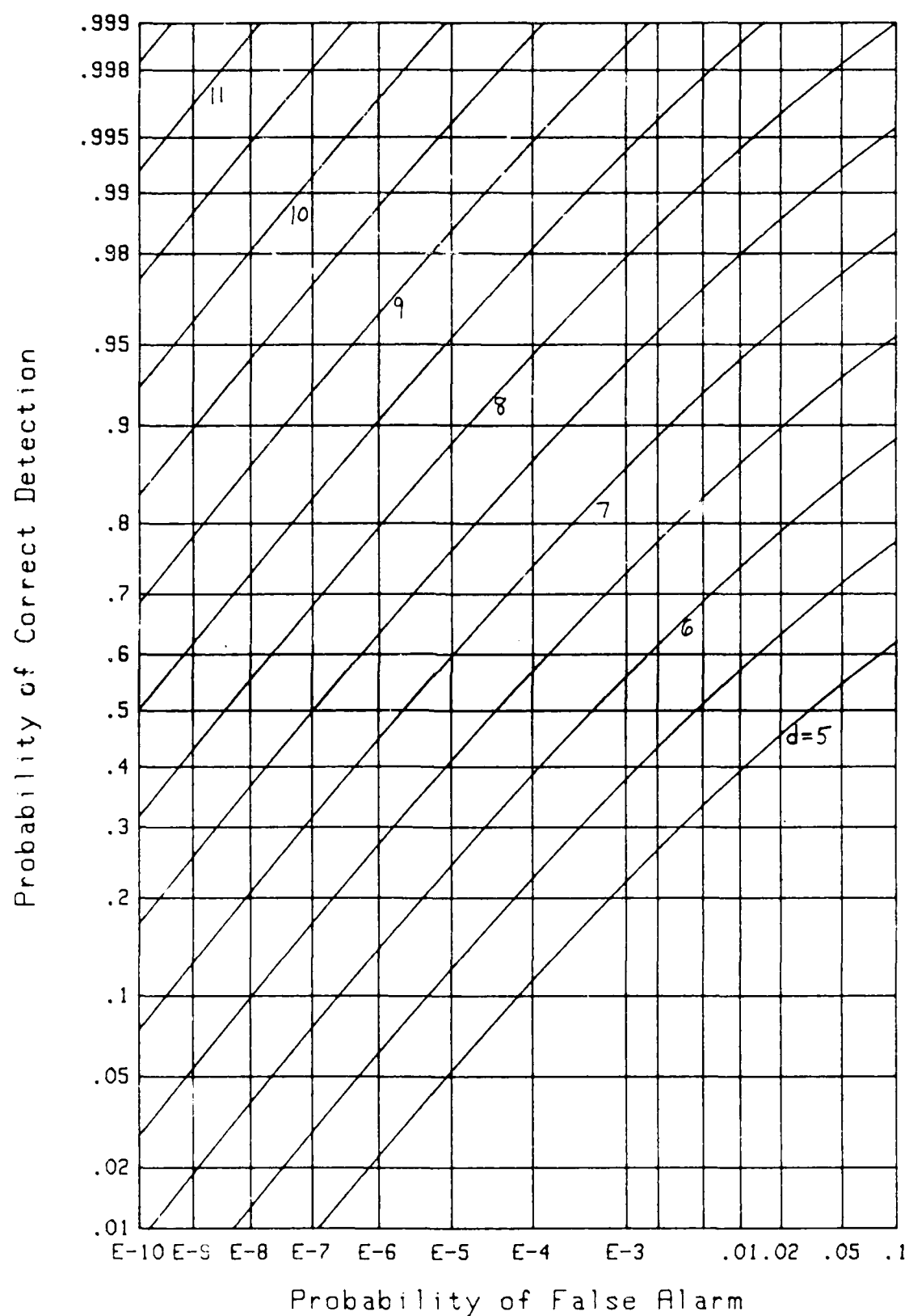
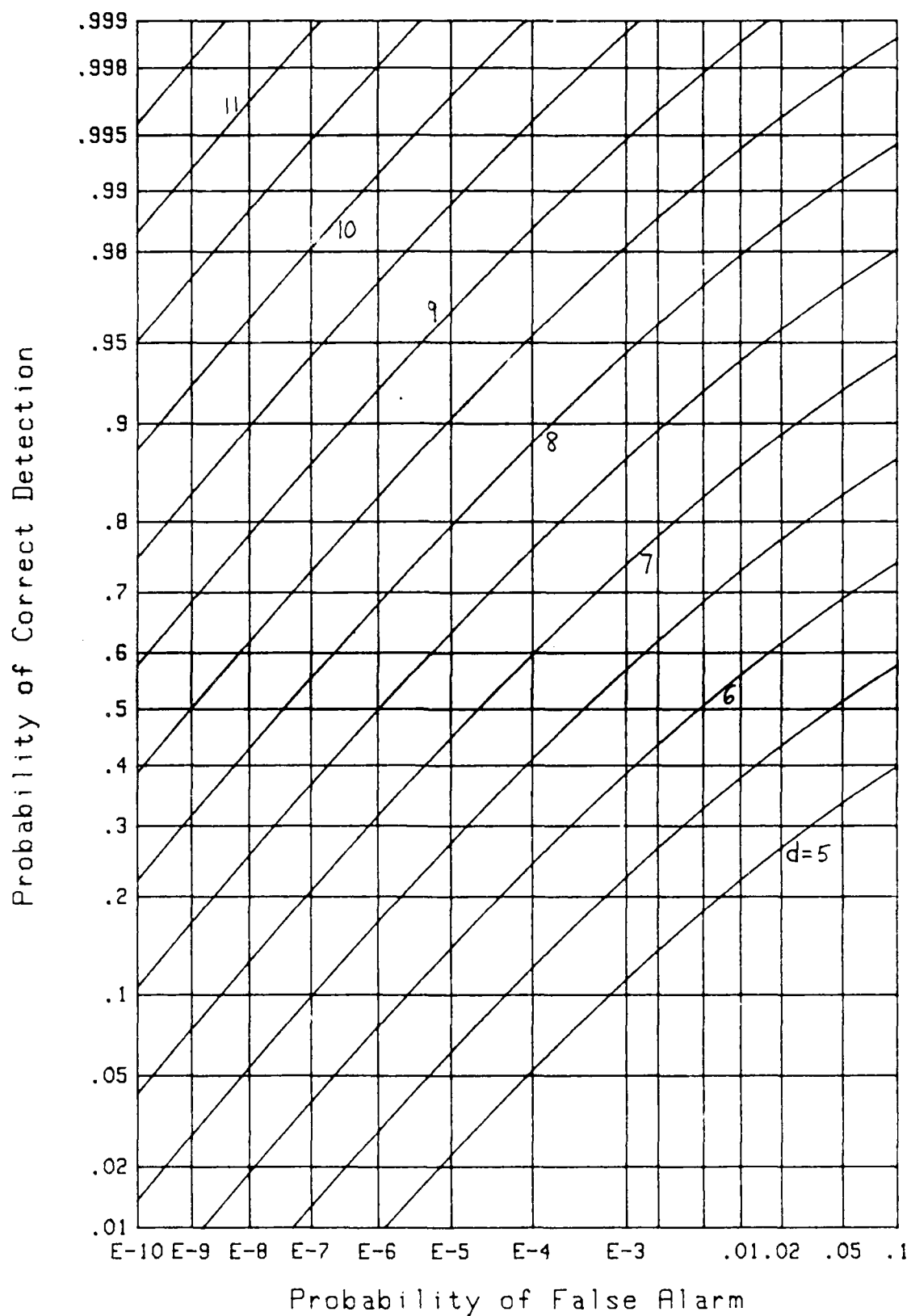


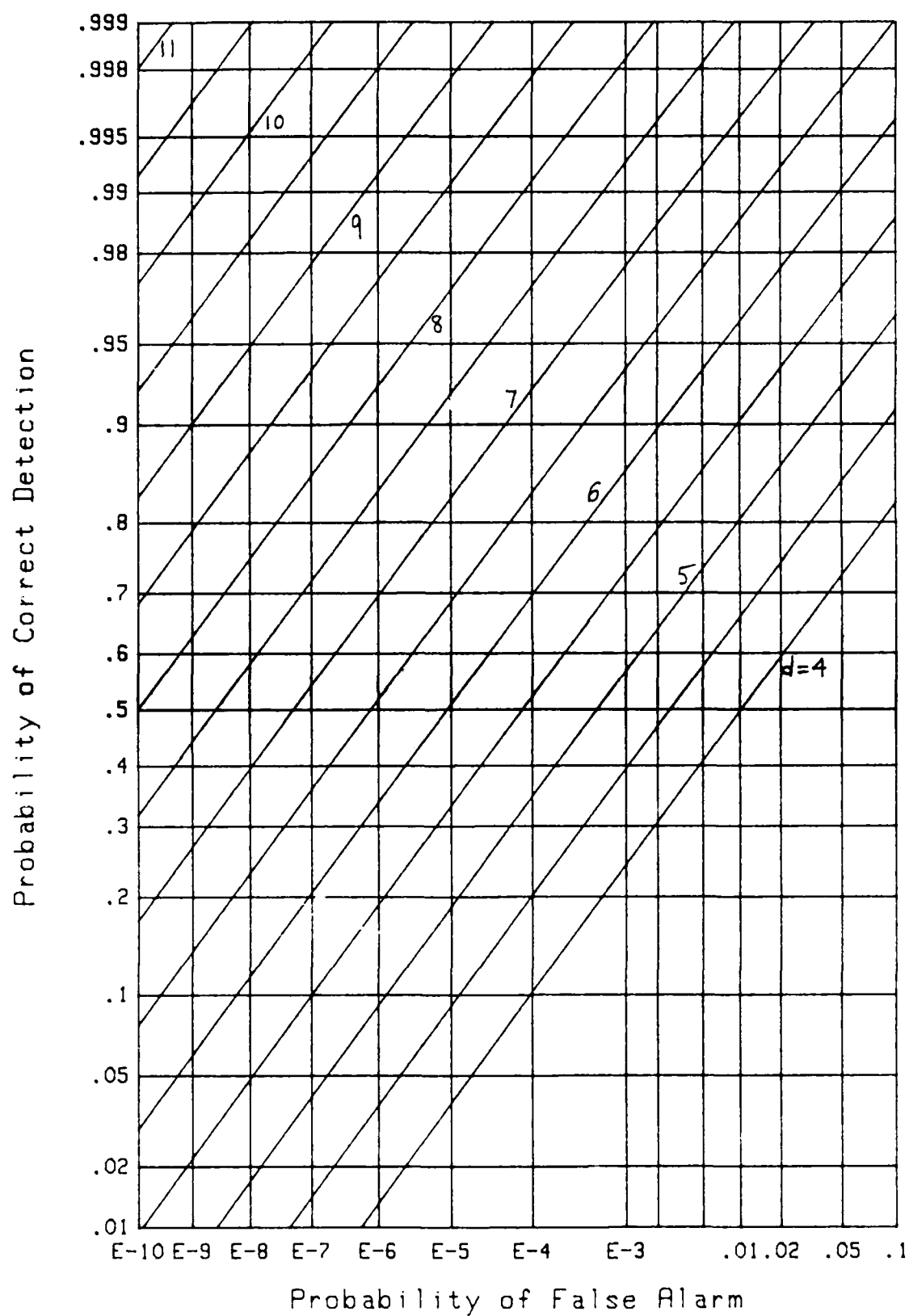
Figure 22. ROC for $M=5$, $N=1000$

Figure 23. ROC for $M=6$, $N=1$

Figure 24. ROC for $M=6$, $N=10$

Figure 25. ROC for $M=6$, $N=100$

Figure 26. ROC for $M=6$, $N=1000$

Figure 27. ROC for $M=7$, $N=1$

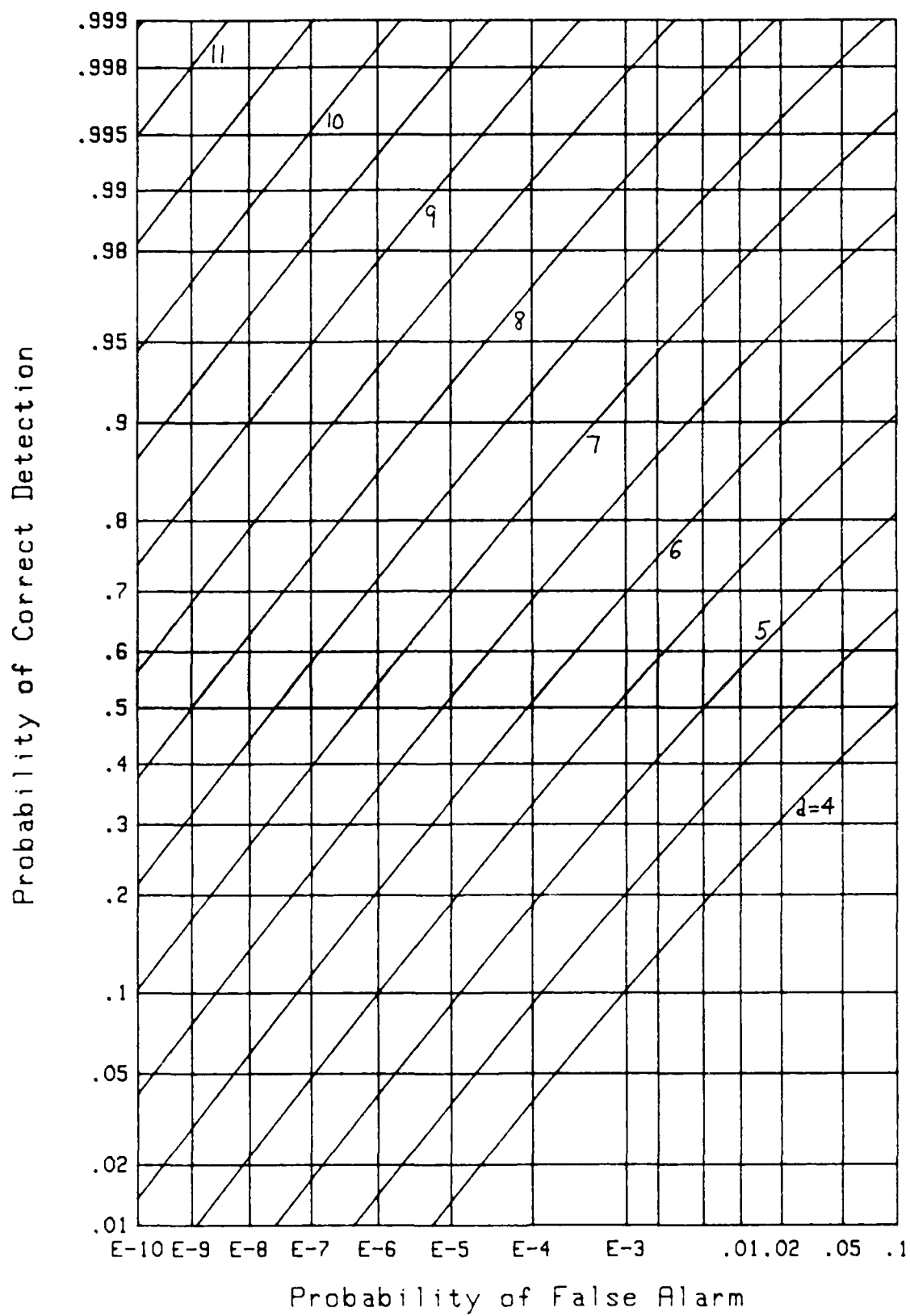
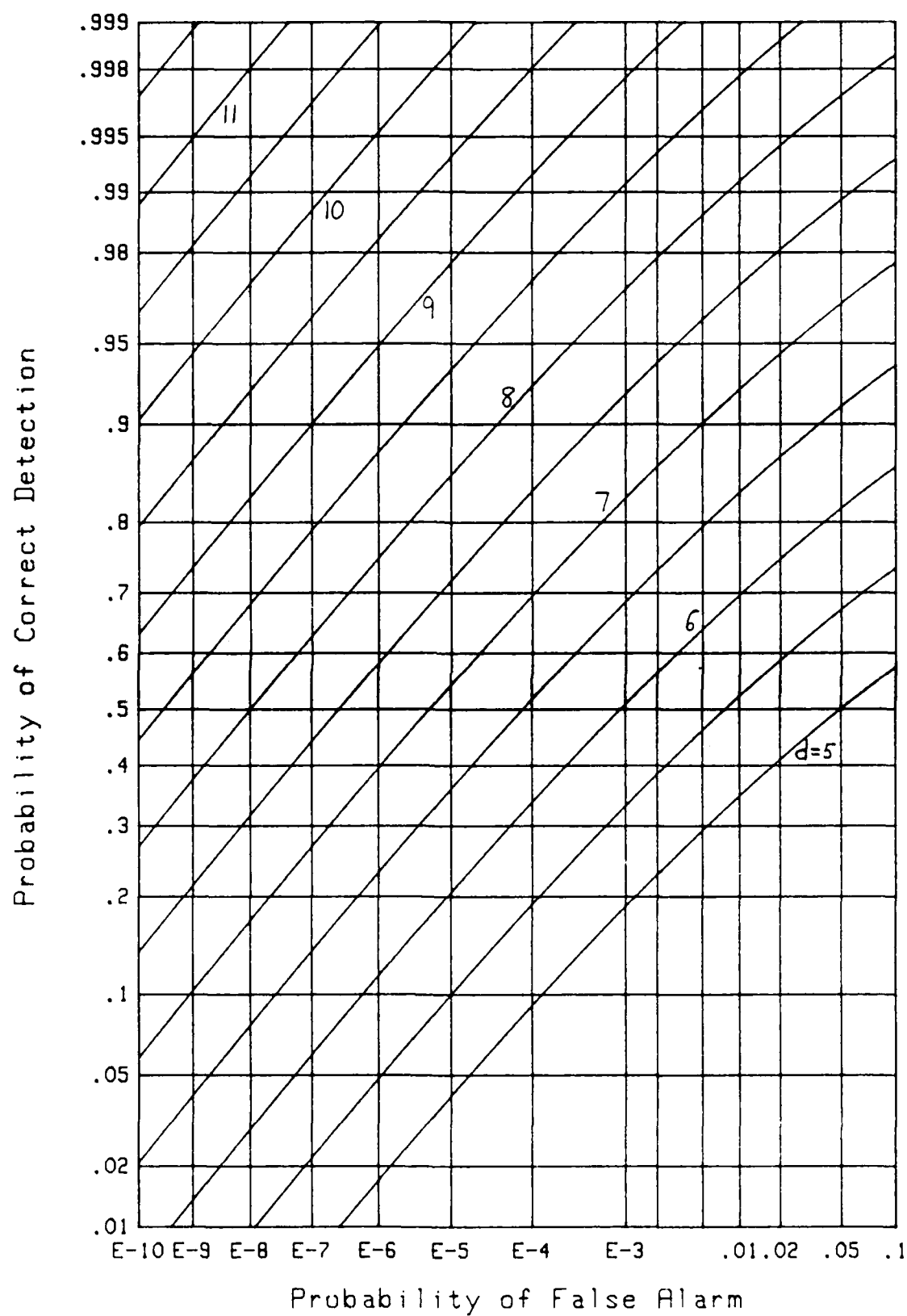
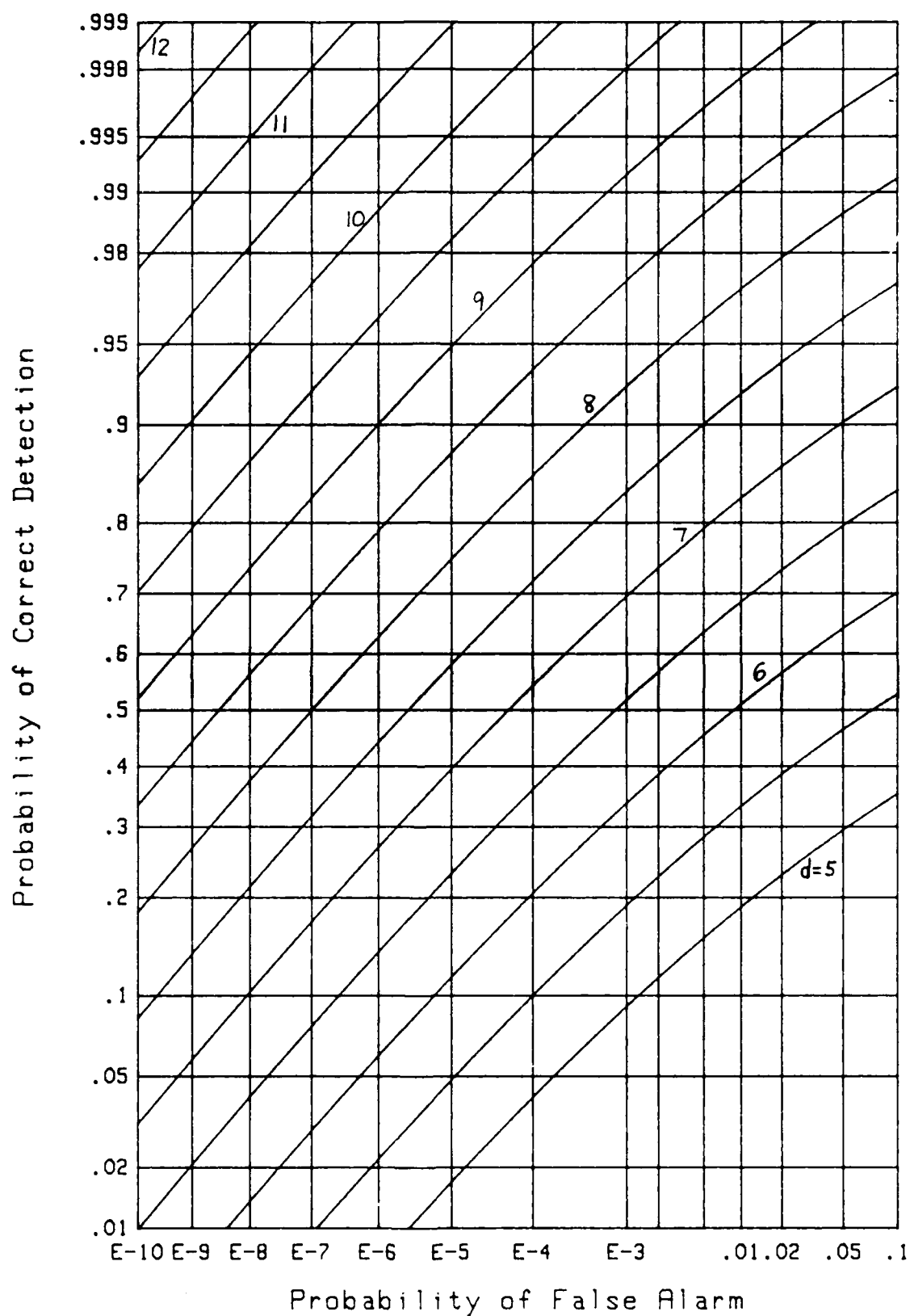
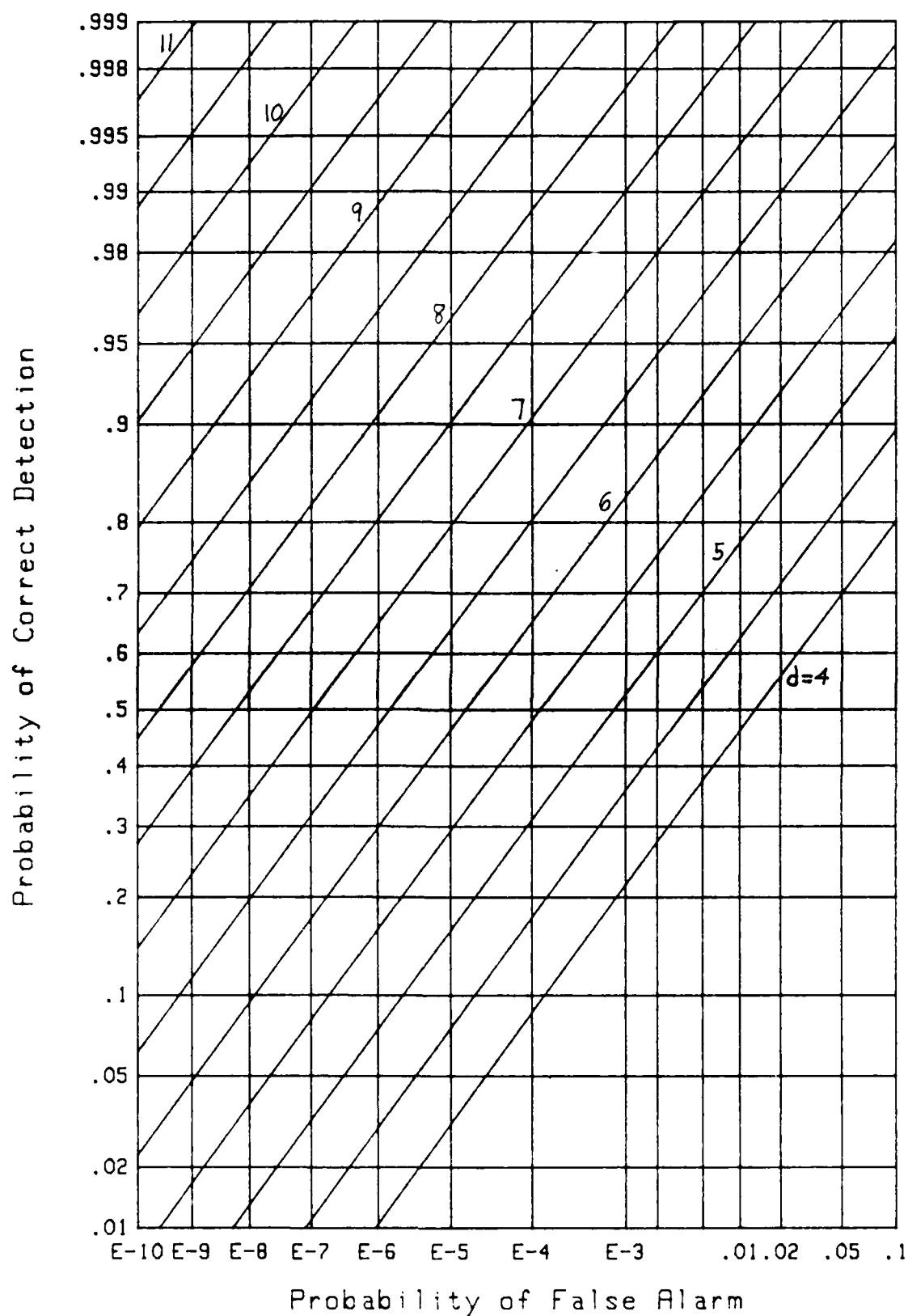
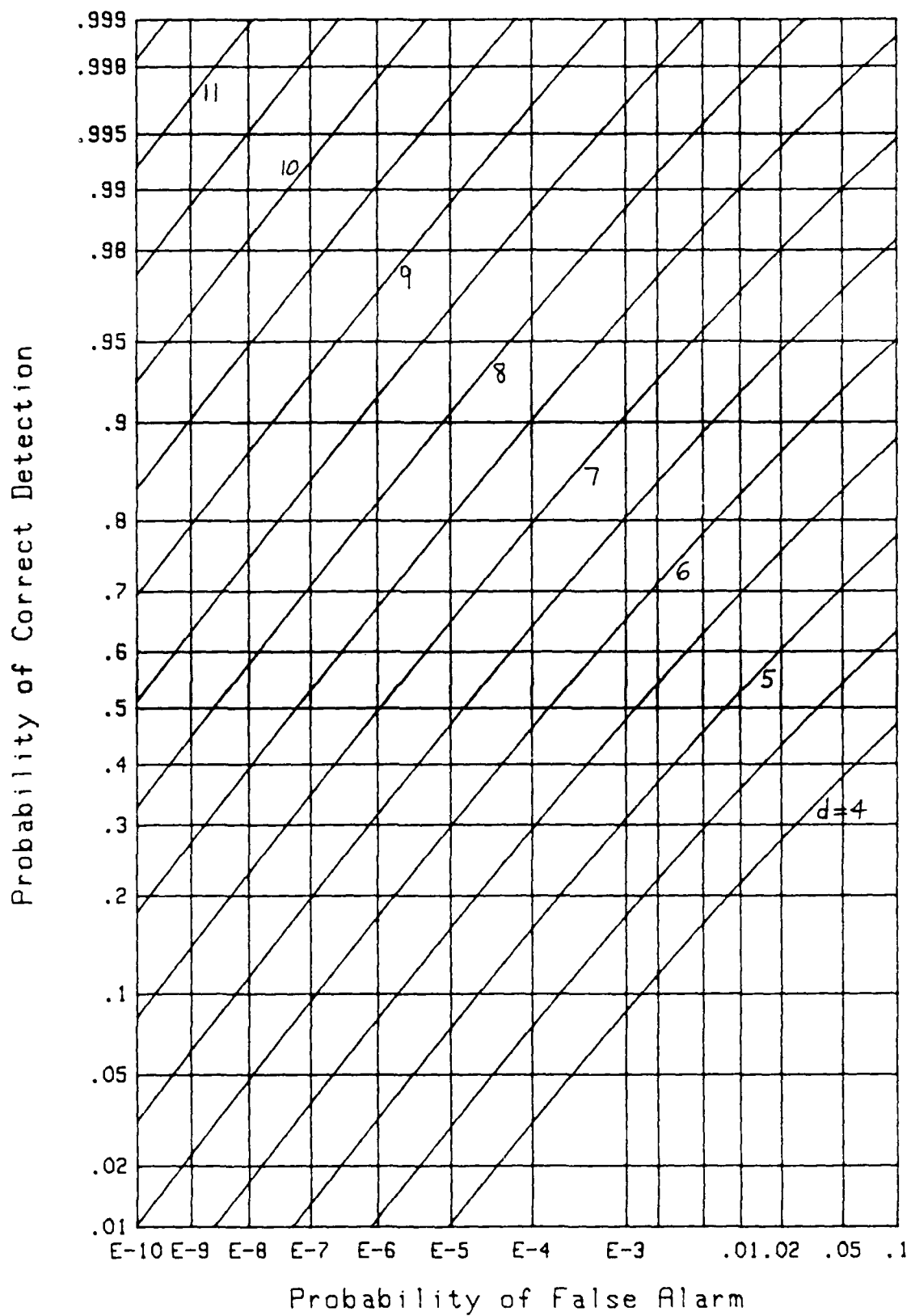


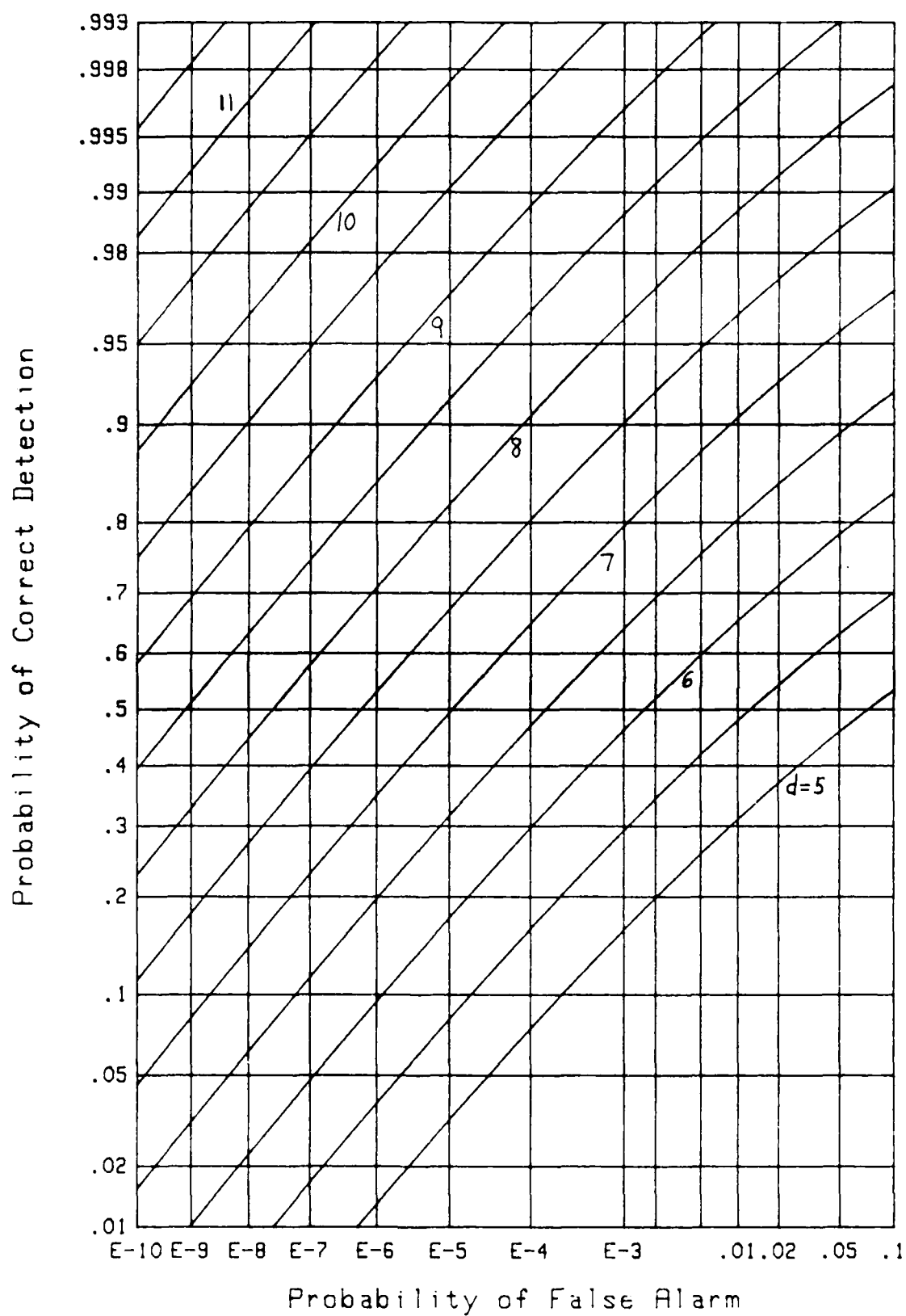
Figure 28. ROC for $M=7$, $N=10$

Figure 29. ROC for $M=7$, $N=100$

Figure 30. ROC for $M=7$, $N=1000$

Figure 31. ROC for $M=8$, $N=1$

Figure 32. ROC for $M=8$, $N=10$

Figure 33. ROC for $M=8$, $N=100$

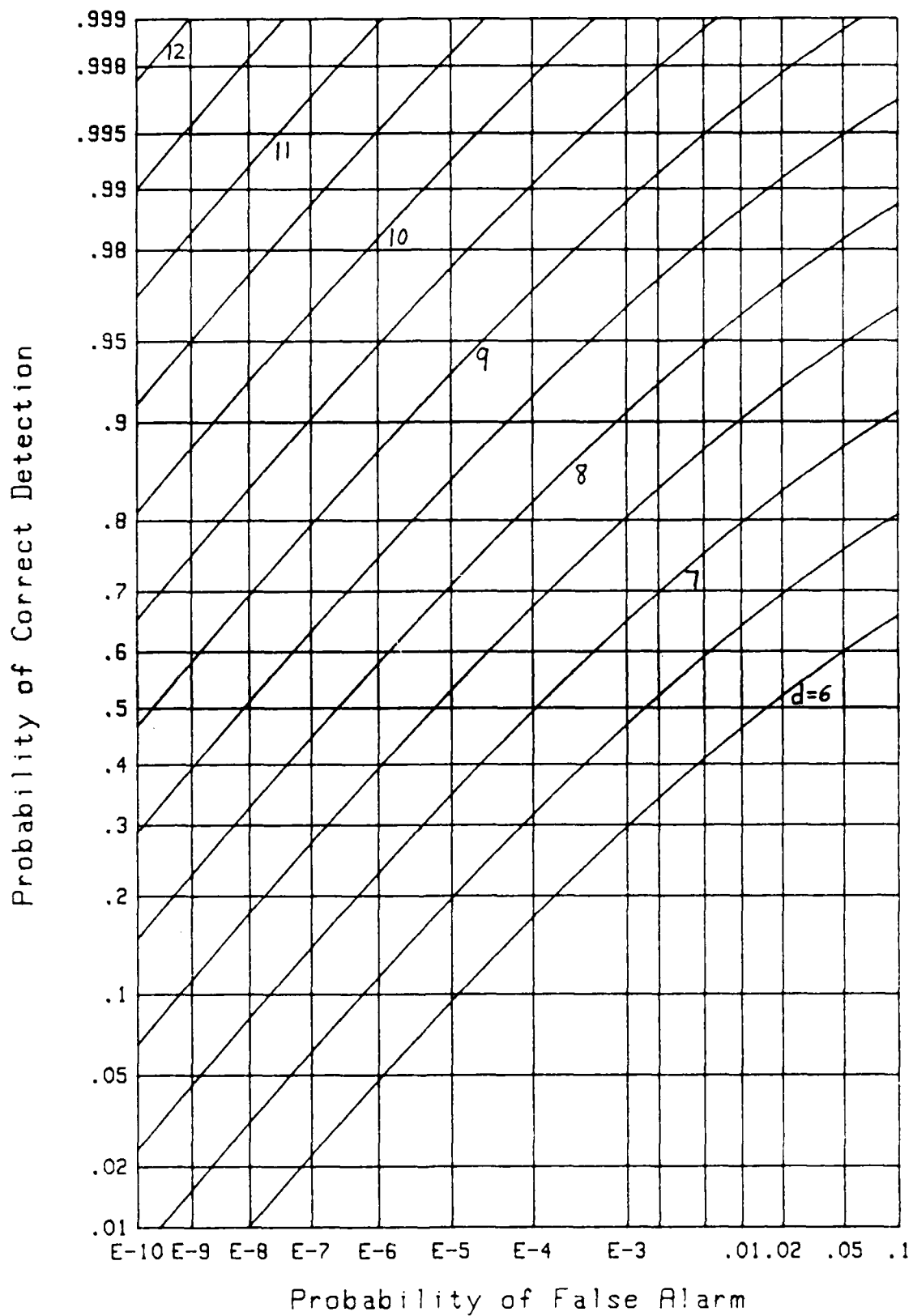


Figure 34. ROC for M=8, N=1000

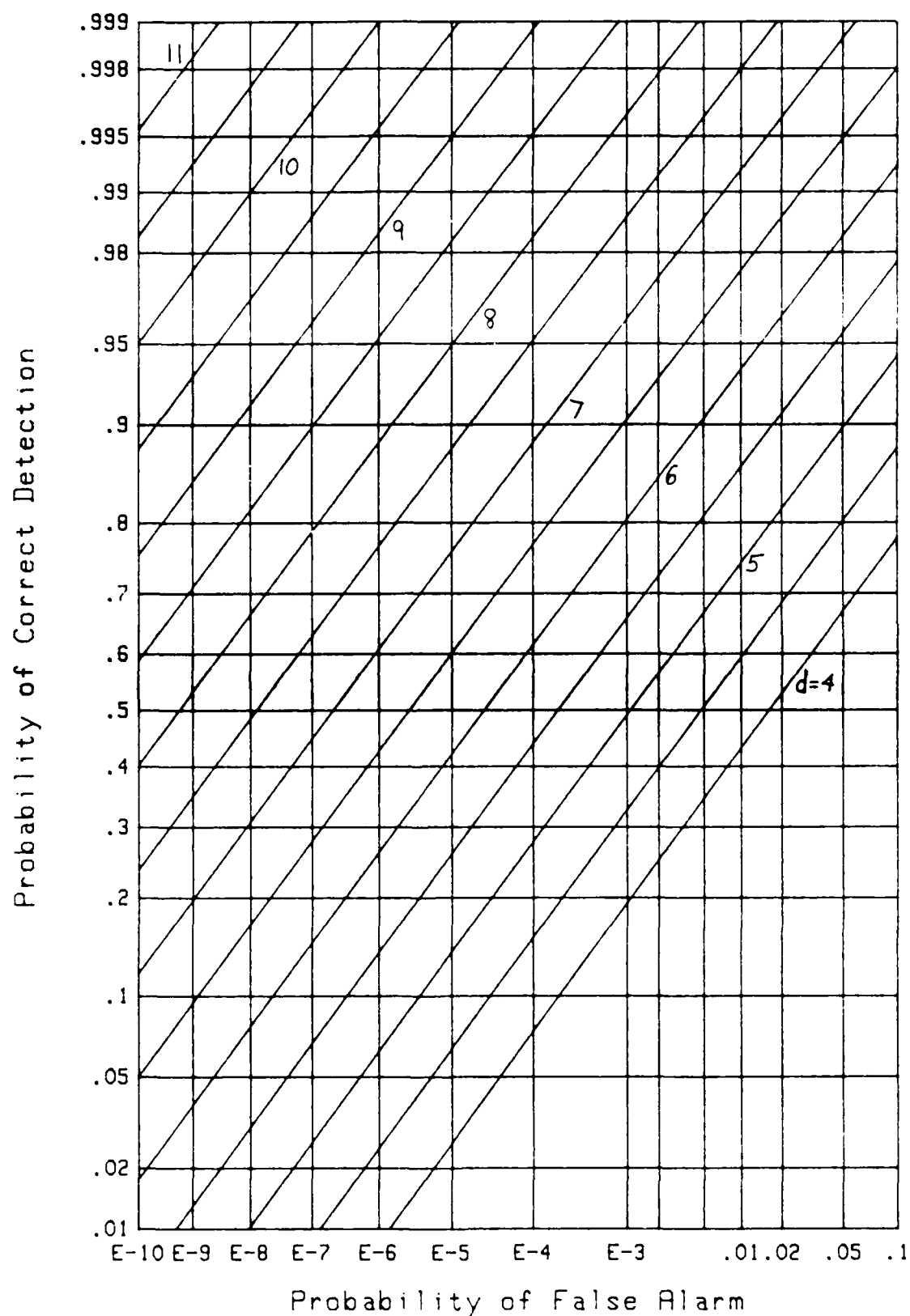
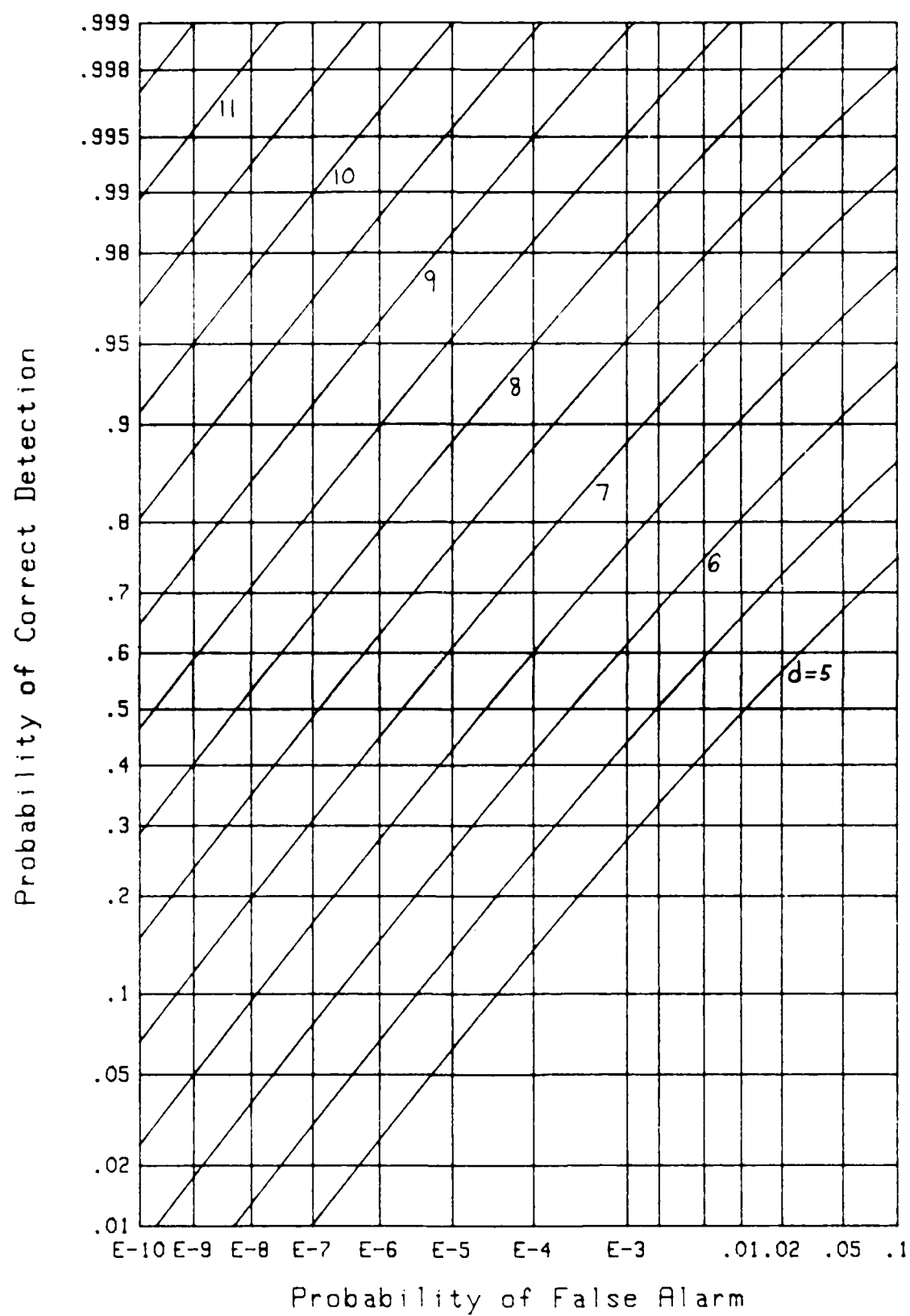
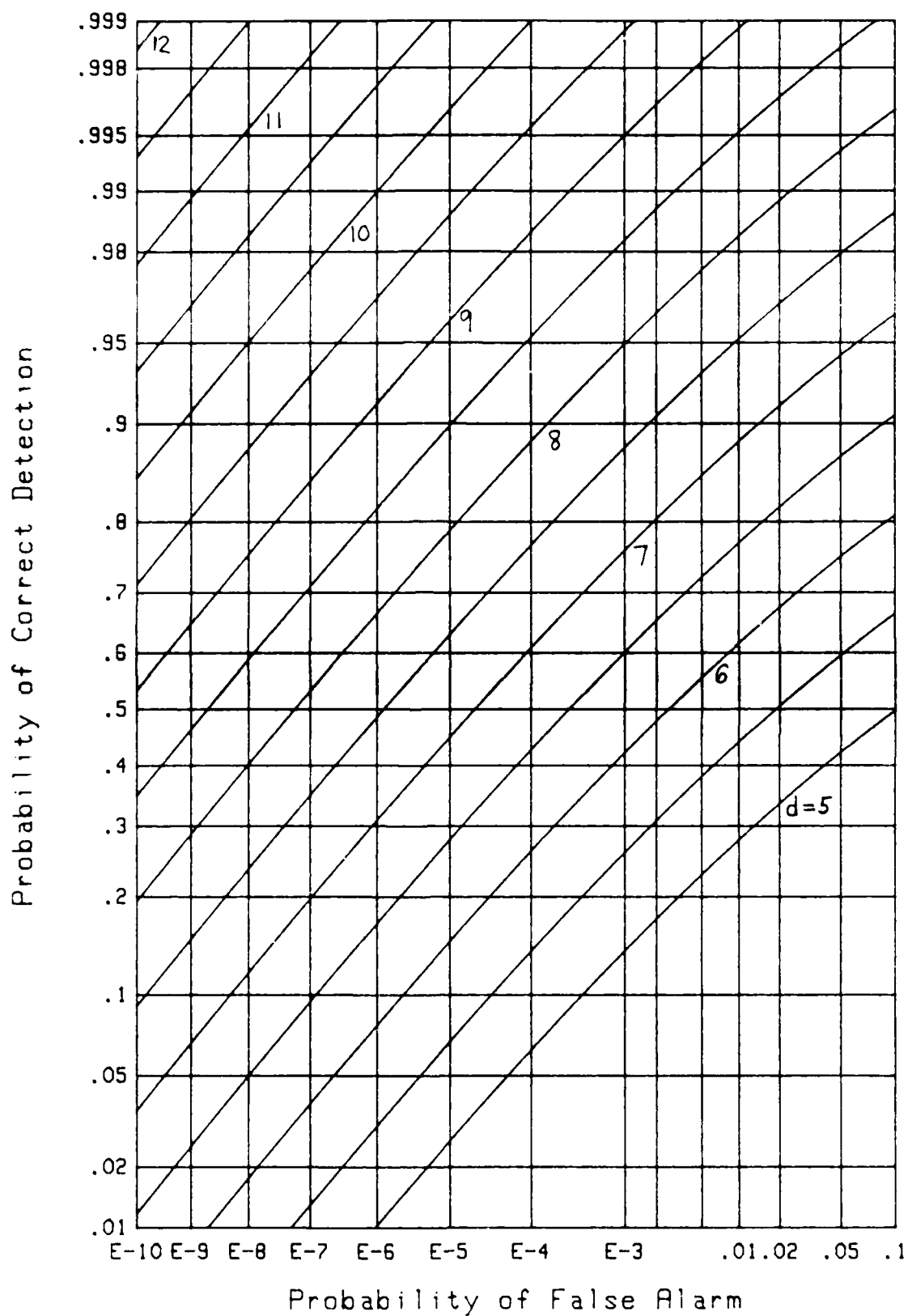
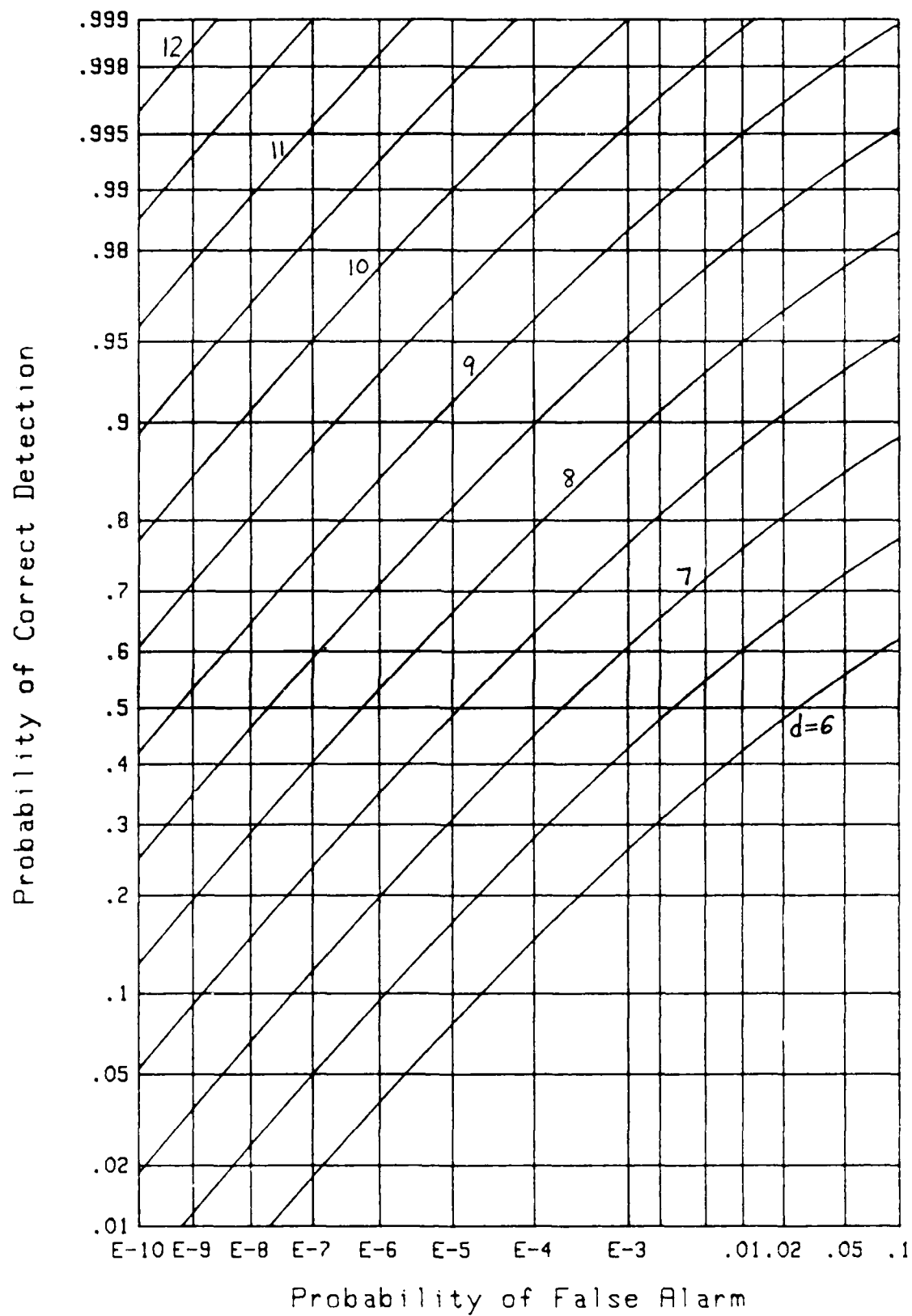
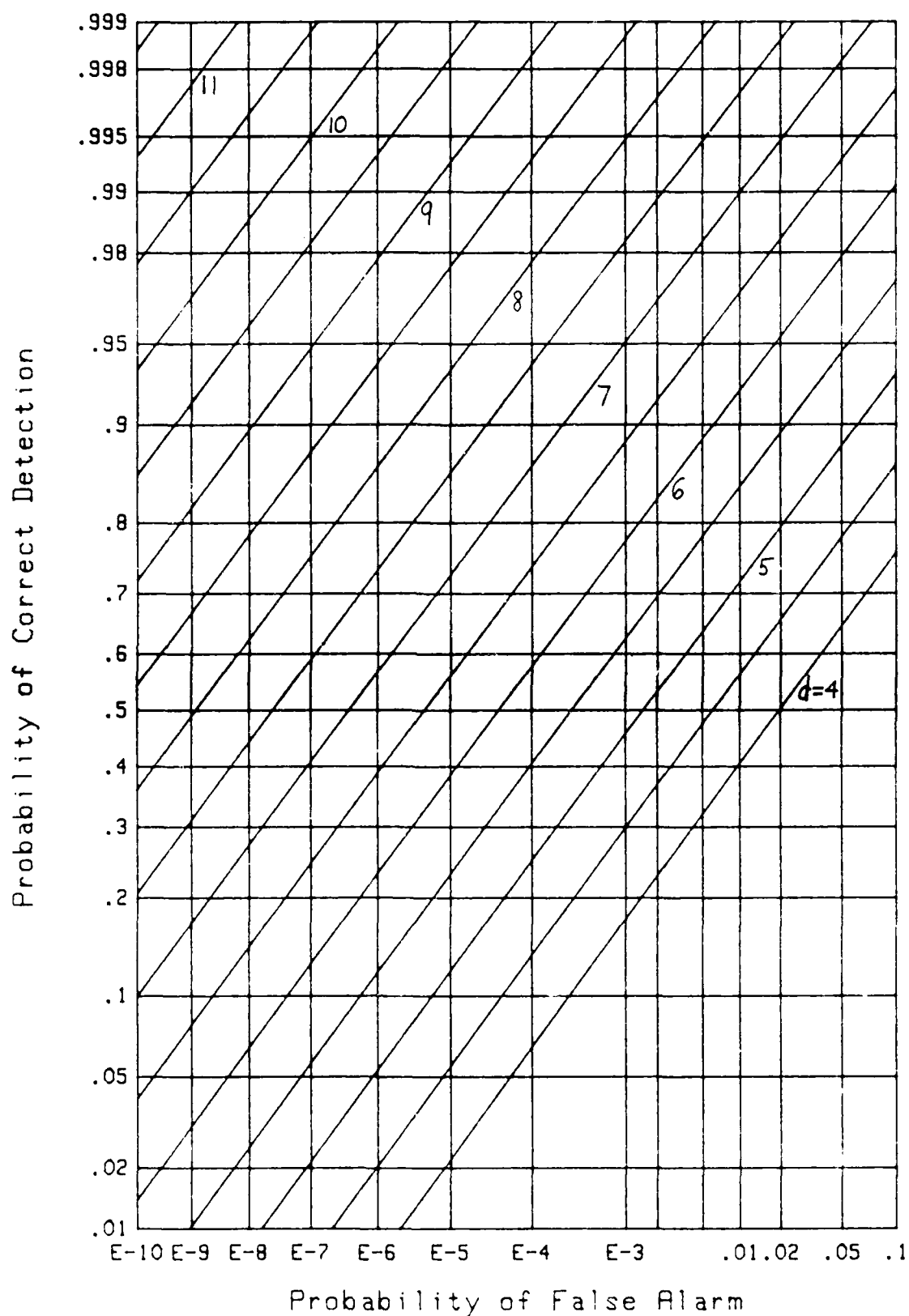


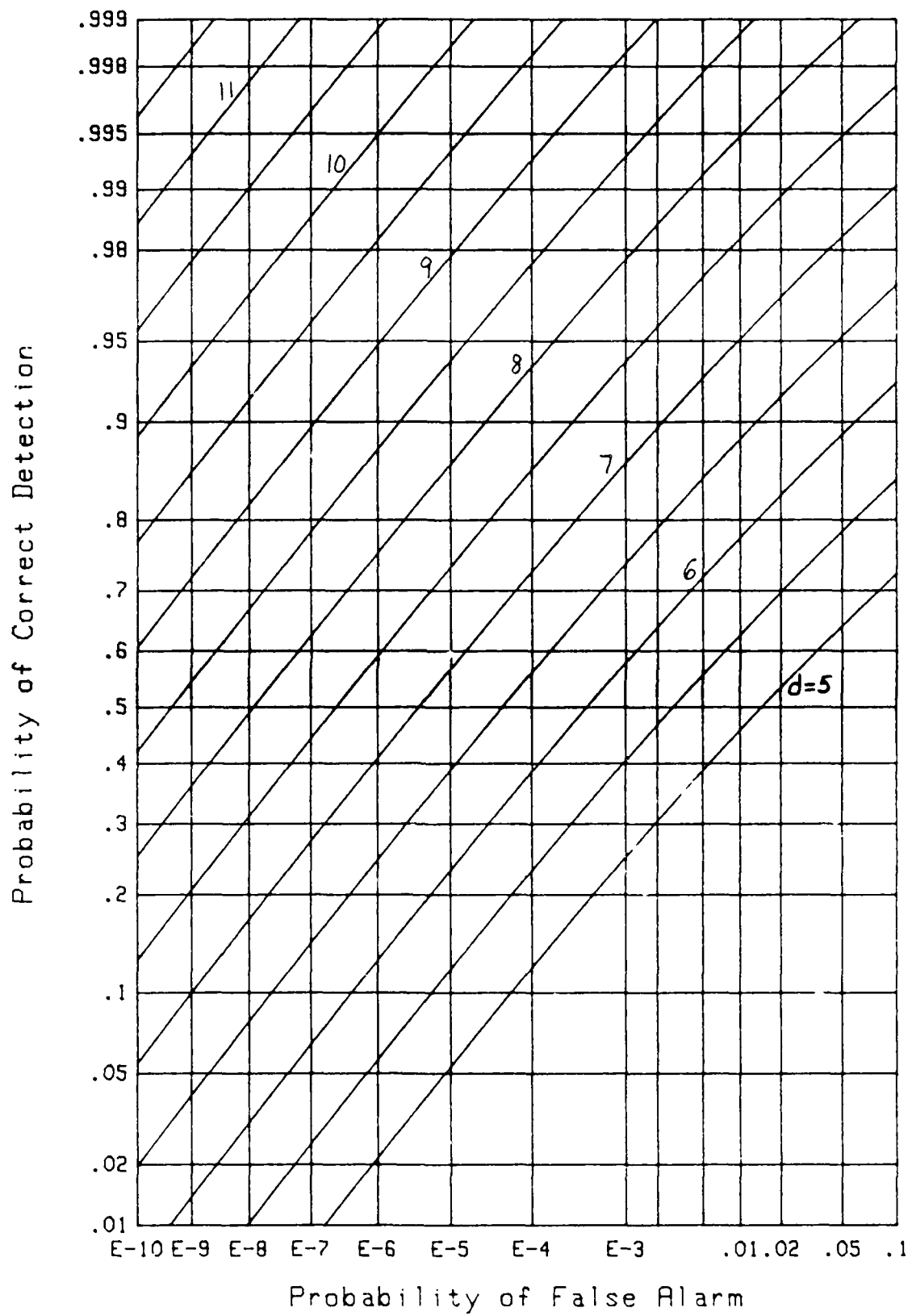
Figure 35. ROC for $M=9$, $N=1$

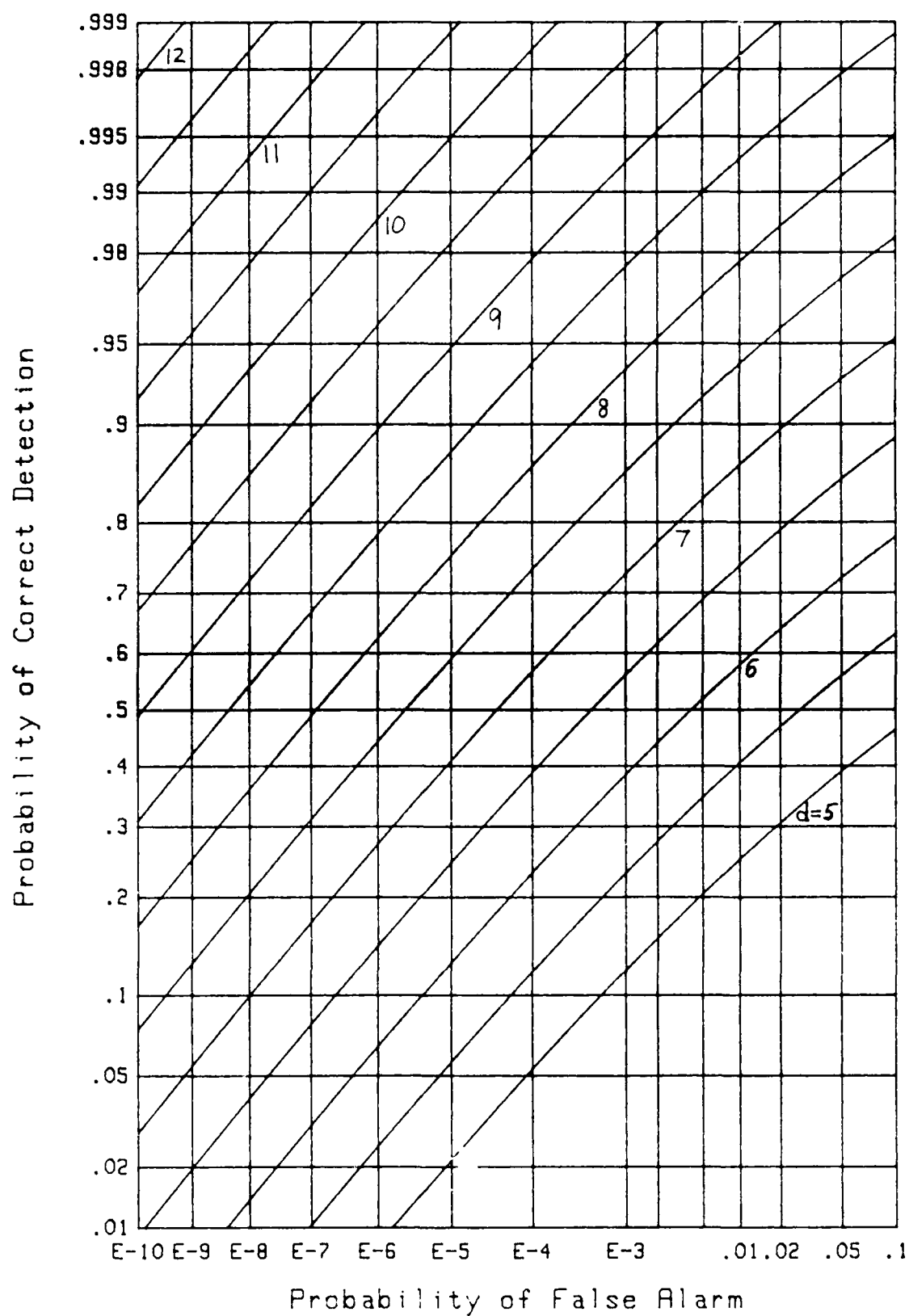
Figure 36. ROC for $M=9$, $N=10$

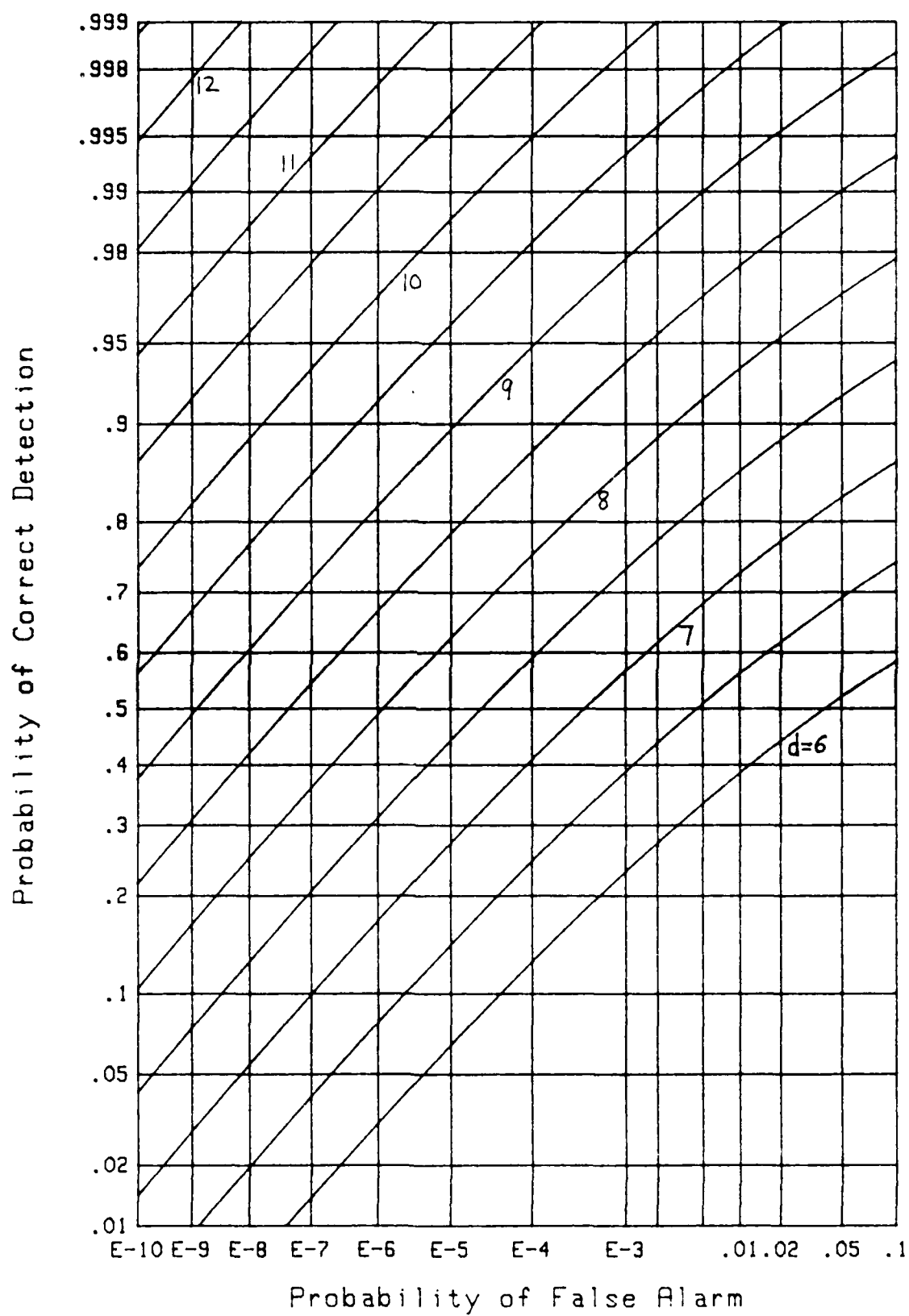
Figure 37. ROC for $M=9$, $N=100$

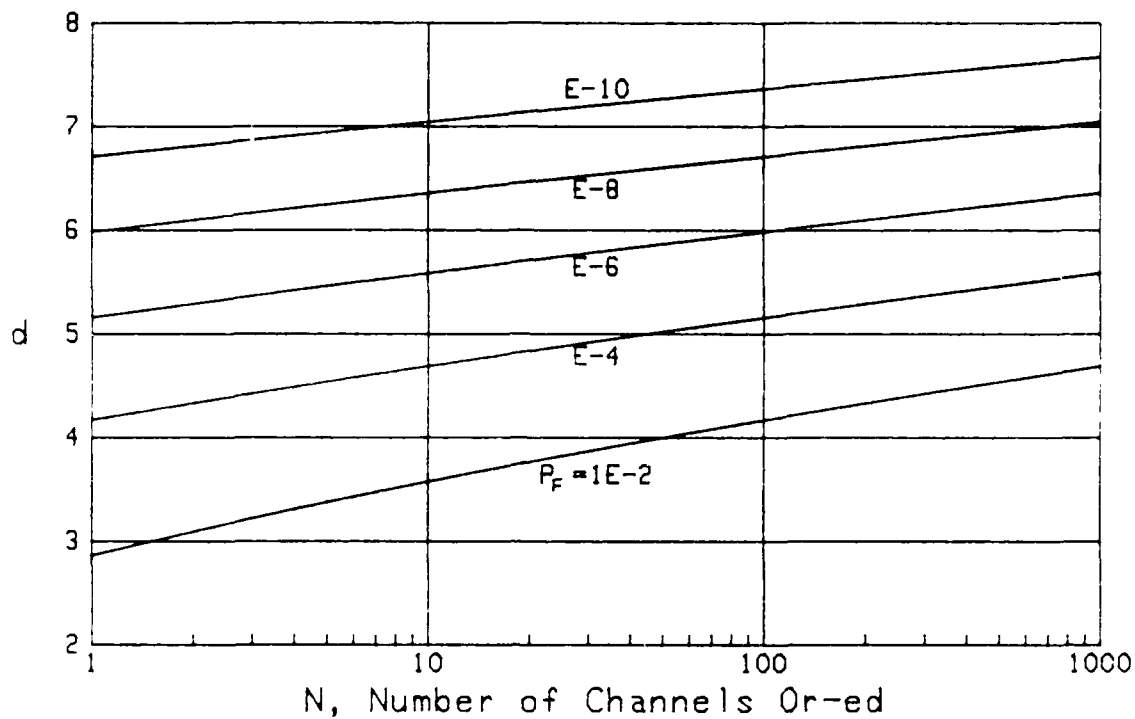
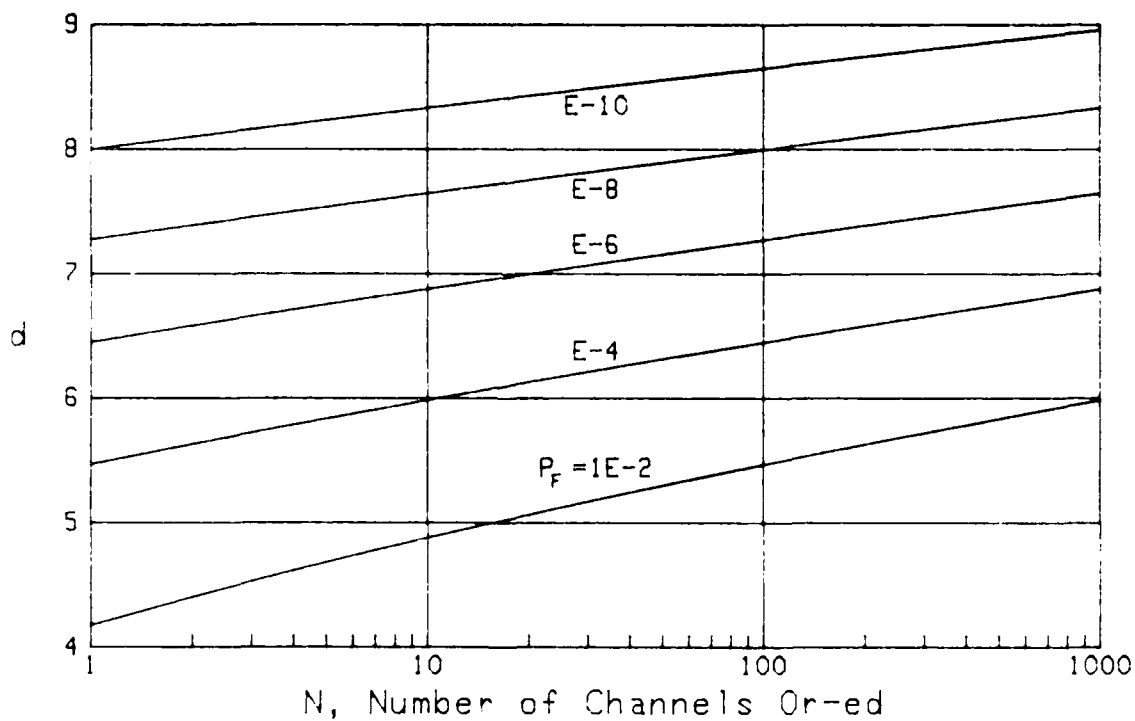
Figure 38. ROC for $M=9$, $N=1000$

Figure 39. ROC for $M=10$, $N=1$

Figure 40. ROC for $M=10$, $N=10$

Figure 41. ROC for $M=10$, $N=100$

Figure 42. ROC for $M=10$, $N=1000$


 Figure 43. Required d Values for $M=1$, $P_{cd}=0.5$

 Figure 44. Required d Values for $M=1$, $P_c=0.9$

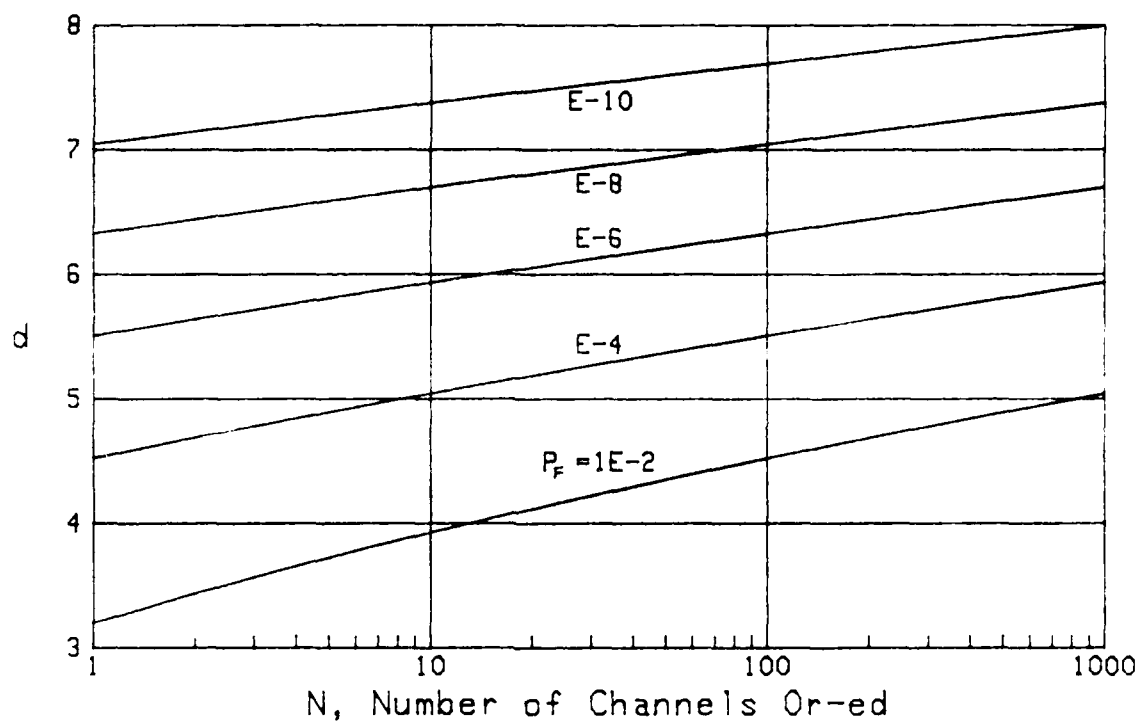


Figure 45. Required d Values for $M=2$, $P_{cd}=.5$

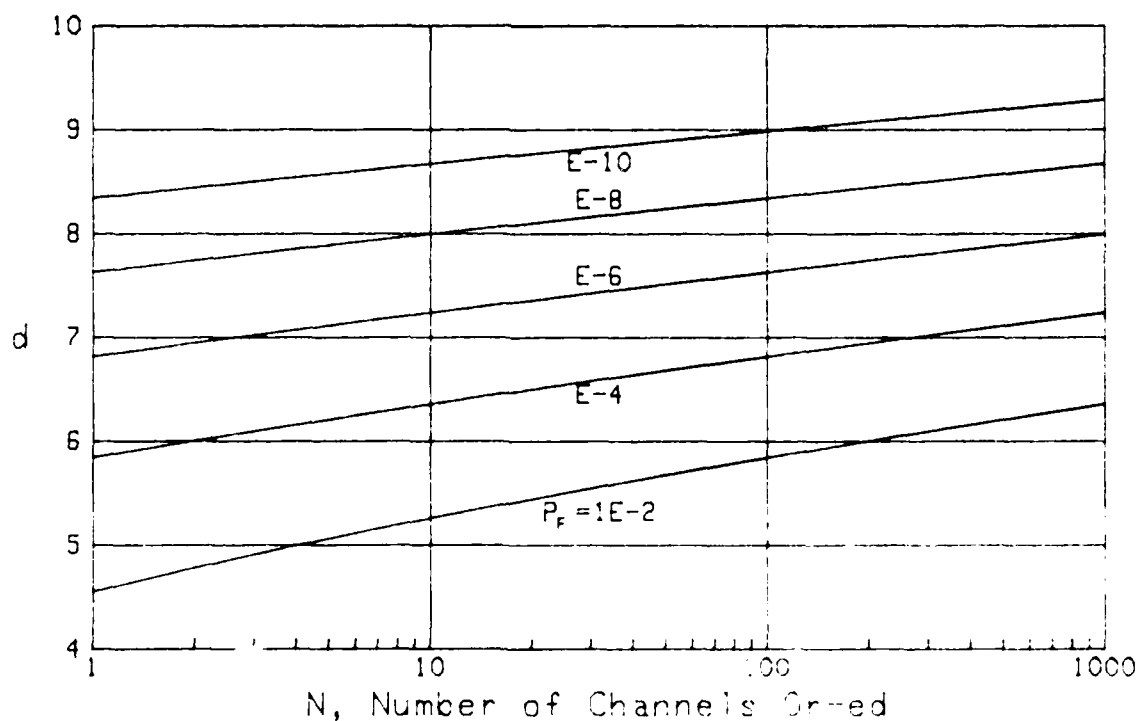
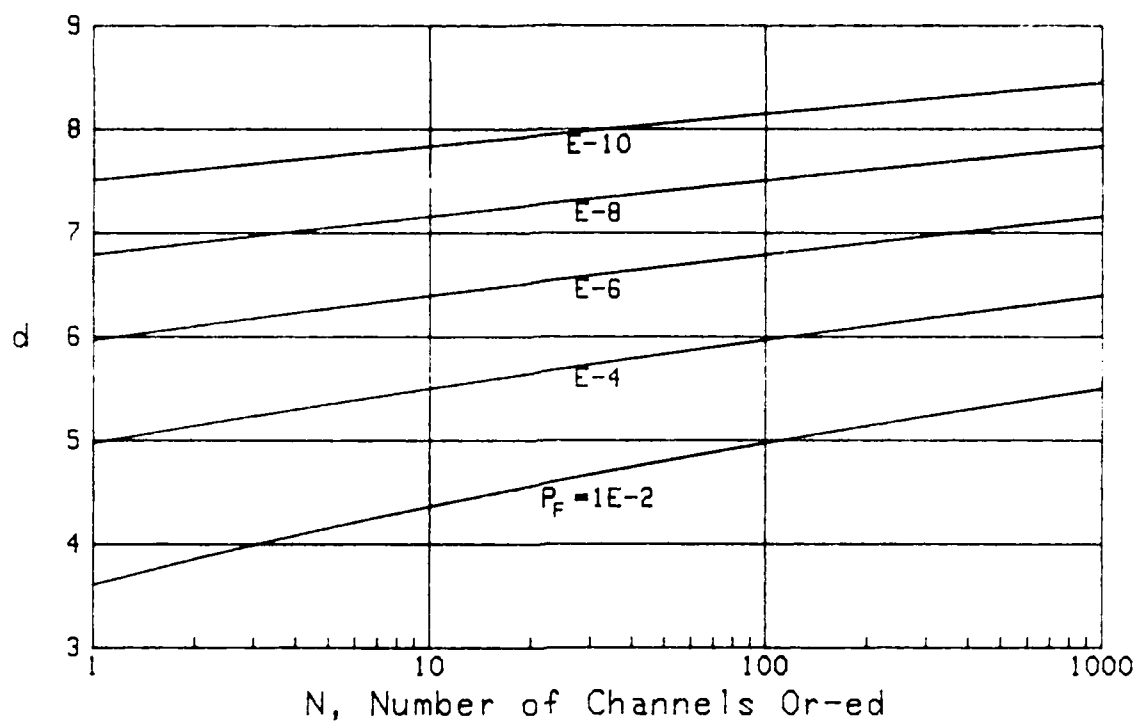
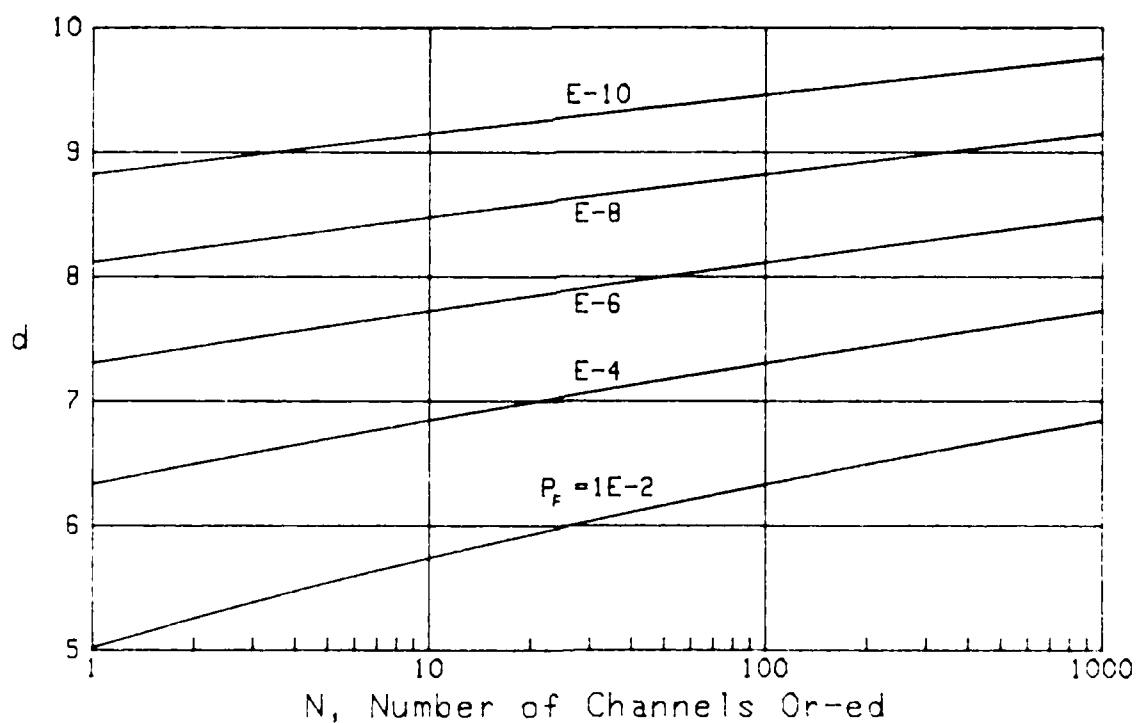


Figure 46. Required d Values for $M=2$, $P_c=.9$

Figure 47. Required d Values for $M=4$, $P_c=.5$ Figure 48. Required d Values for $M=4$, $P_c=.9$

APPENDIX A. Q_M -FUNCTION RELATIONSHIPS

Let $\{x_m\}_1^M$ and $\{y_m\}_1^M$ be independent identically distributed Gaussian random variables, each with zero mean and common variance σ^2 , and let $\{a_m\}_1^M$ and $\{b_m\}_1^M$ be arbitrary fixed constants. Define "total" parameter

$$d^2 = \frac{1}{\sigma^2} \sum_{m=1}^M (a_m^2 + b_m^2) . \quad (A-1)$$

Chi-Squared Variate

We are interested in the statistical description of the noncentral chi-squared random variable of $2M$ degrees of freedom,

$$v = \sum_{m=1}^M [(x_m + a_m)^2 + (y_m + b_m)^2] . \quad (A-2)$$

We will only list results here, and will not give detailed derivations.

The characteristic function of v is [6; page 11]

$$f_v(\xi) = \overline{\exp(i\xi v)} = (1 - i\xi 2\sigma^2)^{-M} \exp \left[\frac{i\xi d^2 \sigma^2}{1 - i\xi 2\sigma^2} \right] , \quad (A-3)$$

which is seen to depend on the arbitrary constants $\{a_m\}$ and $\{b_m\}$ only through the sum d^2 in (A-1). The probability density function of random variable v is [7; 6.631 4]

$$p_v(u) = \frac{1}{2\sigma^2} \left(\frac{\sqrt{u}}{d\sigma} \right)^{M-1} I_{M-1} \left(\frac{d\sqrt{u}}{\sigma} \right) \exp \left(-\frac{u}{2\sigma^2} - \frac{d^2}{2} \right) \quad \text{for } u > 0. \quad (\text{A-4})$$

The cumulative distribution function of random variable v is

$$\text{Prob}(v < u) = P_v(u) = \int_0^u dt p_v(t), \quad (\text{A-5})$$

and the exceedance distribution function is

$$1 - P_v(u) = Q_M(d, \sqrt{u}/\sigma) \quad \text{for } u > 0, \quad (\text{A-6})$$

where the Q_M -function is

$$Q_M(\alpha, \beta) = \int_\beta^\infty dx x \left(\frac{x}{\alpha} \right)^{M-1} I_{M-1}(\alpha x) \exp \left(-\frac{x^2 + \alpha^2}{2} \right). \quad (\text{A-7})$$

As special cases of (A-4) and (A-6), for $d = 0$, we have probability density function

$$p_v^{(0)}(u) = \frac{u^{M-1}}{(M-1)! (2\sigma^2)^M} \exp \left(-\frac{u}{2\sigma^2} \right) \quad \text{for } u > 0 \quad (\text{A-8})$$

and exceedance distribution function

$$1 - P_v^{(0)}(u) = \exp \left(-\frac{u}{2\sigma^2} \right) e_{M-1} \left(\frac{u}{2\sigma^2} \right) = E_{M-1} \left(\frac{u}{2\sigma^2} \right) \quad \text{for } u > 0, \quad (\text{A-9})$$

where [5; 6.5.11]

$$e_n(x) = \sum_{k=0}^n x^k / k! \quad (\text{A-10})$$

is the partial exponential and where we define

$$E_n(x) = \exp(-x) e_n(x) . \quad (A-11)$$

Returning to the general case of $d > 0$ for random variable v again, the cumulants of v are

$$\lambda_v(n) = (2\sigma^2)^n n! \left(\frac{M}{n} + \frac{d^2}{2} \right) \text{ for } n \geq 1 , \quad (A-12)$$

the v -th moments are

$$\overline{v^v} = (2\sigma^2)^v \frac{\Gamma(M+v)}{\Gamma(M)} {}_1F_1(-v; M; -d^2/2) \text{ for } v > -M, \quad (A-13)$$

and the n -th moments are

$$\overline{v^n} = (2\sigma^2)^n n! L_n^{(M-1)}(-d^2/2). \quad (A-14)$$

Chi Variate

The noncentral chi variate of $2M$ degrees of freedom is

$$z = v^{1/2} = \left\{ \sum_{m=1}^M [(x_m + a_m)^2 + (y_m + b_m)^2] \right\}^{1/2} . \quad (A-15)$$

Its probability density function is

$$p_z(u) = \frac{u}{\sigma^2} \left(\frac{u}{d\sigma} \right)^{M-1} I_{M-1} \left(\frac{du}{\sigma} \right) \exp \left(-\frac{u^2}{2\sigma^2} - \frac{d^2}{2} \right) \text{ for } u > 0, \quad (A-16)$$

and its exceedance distribution function is

$$1 - P_Z(u) = Q_M(d, u/\sigma) \quad \text{for } u > 0. \quad (\text{A-17})$$

As special cases of (A-16) and (A-17), for $d = 0$, we have probability density function

$$p_z^{(0)}(u) = \frac{2 u^{2M-1}}{(M-1)! (2\sigma^2)^M} \exp\left(\frac{-u^2}{2\sigma^2}\right) \quad \text{for } u > 0 \quad (\text{A-18})$$

and exceedance distribution function

$$1 - p_z^{(0)}(u) = E_{M-1}\left(\frac{u^2}{2\sigma^2}\right) \quad \text{for } u > 0, \quad (\text{A-19})$$

in terms of the functions defined in (A-10) and (A-11).

In general, for $d > 0$, the v -th moment of random variable z is

$$\overline{z^v} = \sigma^v 2^{v/2} \frac{\Gamma(M + v/2)}{\Gamma(M)} {}_1F_1(-v/2; M; -d^2/2) \quad \text{for } v > -2M. \quad (\text{A-20})$$

The characteristic function and cumulants of z are not available in any compact form.

Special Case

If the constants in random variable v in (A-2), and in random variable z in (A-15), satisfy

$$a_m = A \cos \theta_m, \quad b_m = A \sin \theta_m, \quad (\text{A-21})$$

where $\{\theta_m\}$ are arbitrary, then (A-1) reduces to

$$d^2 = M A^2 / \sigma^2, \quad (\text{A-22})$$

independent of the particular values of $\{\theta_m\}$. So if $\{\theta_m\}$ were random variables instead of constants, the statistics of v and z in (A-2) and (A-15), respectively, would be unaffected. This conclusion follows immediately from (A-3).

In this latter case of random $\{\theta_m\}$, if they are also uniformly distributed over 2π , it is sometimes useful to define an individual (common) signal-to-noise ratio

$$R = \frac{\overline{a_m^2}}{\overline{x_m^2}} = \frac{\overline{b_m^2}}{\overline{y_m^2}} = \frac{A^2/2}{\sigma^2} \quad \text{for all } m. \quad (\text{A-23})$$

Then the parameter d^2 in (A-22) can be expressed as

$$d^2 = 2 M R. \quad (\text{A-24})$$

More generally, if

$$a_m = A_m \cos \theta_m, \quad b_m = A_m \sin \theta_m, \quad (\text{A-25})$$

where $\{A_m\}$ are arbitrary constants, then (A-1) reduces to

$$d^2 = \frac{1}{\sigma^2} \sum_{m=1}^M A_m^2. \quad (\text{A-26})$$

Again, presuming $\{\theta_m\}$ to be uniformly distributed random variables over 2π , if we define the individual component signal-to-noise ratios as

$$R_m = \frac{\overline{a_m^2}}{\overline{x_m^2}} = \frac{\overline{b_m^2}}{\overline{y_m^2}} = \frac{A_m^2/2}{\sigma^2} \quad , \quad (\text{A-27})$$

then (A-26) can be expressed as

$$d^2 = 2 \sum_{m=1}^M R_m \quad . \quad (\text{A-28})$$

These relations, (A-24) and (A-28), afford an alternative interpretation of the "total" parameter d^2 in terms of component signal-to-noise ratios.

APPENDIX B. TABULATION OF P_{CD} AND $Q_M(d,T)$

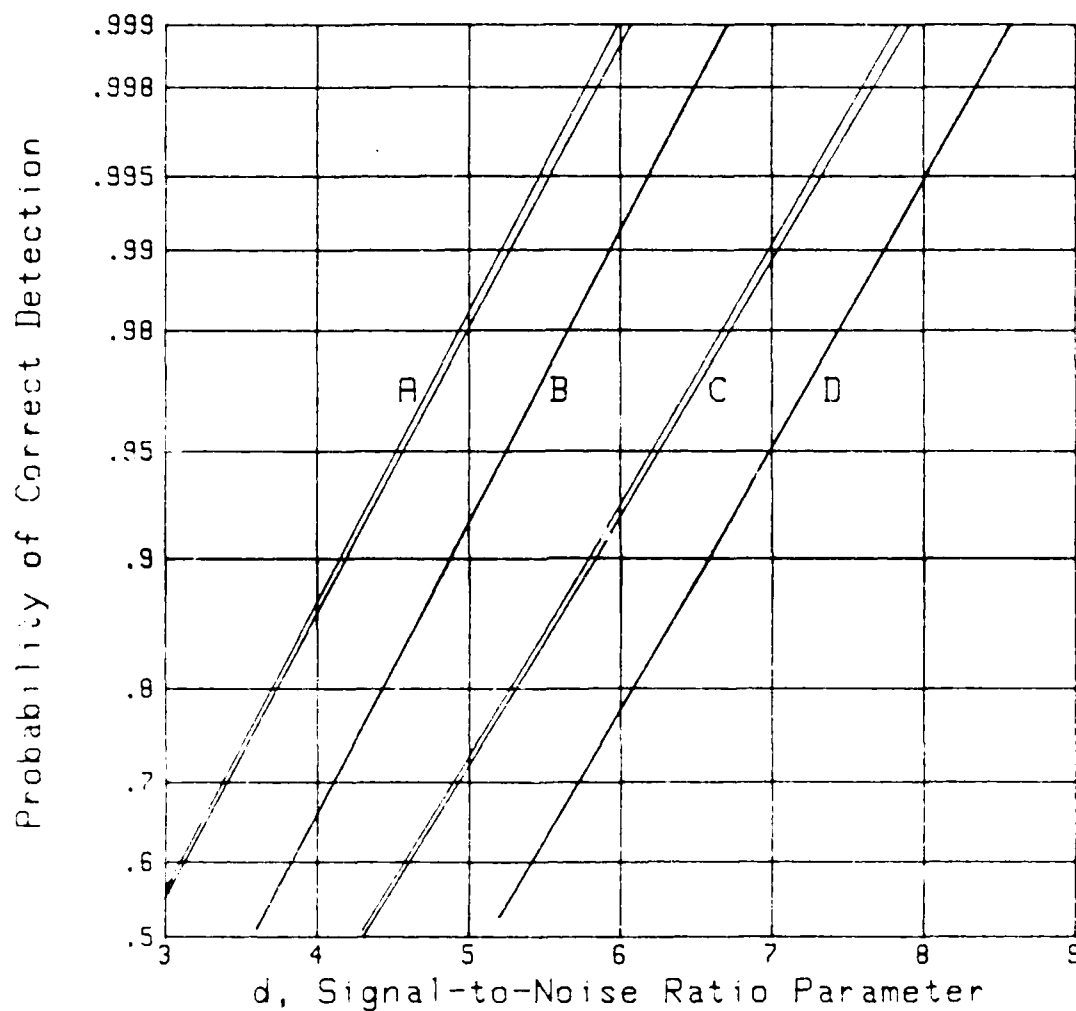
For the eight possible combinations of $M=1,10$ with $N=2,10,100,1000$, values of the exact value of P_{CD} and the approximation afforded by $Q_M(d,T)$ are tabulated here. An explanation of table B-1, which pertains to $M=1, N=2$, follows:

For threshold $T = 2.40$, the false alarm probability $P_F = .10912$. Holding these values fixed, then as d is varied from 2.2 to 5.4, the detection probabilities vary over the values .5, .9, .99, .999 (approximately). This case is covered by the top four lines in table B-1.

When the threshold T is changed to 3.25, the new false alarm probability is $P_F = .01015$, and the second group of four lines in table B-1 pertains. This procedure is continued for all the M,N combinations, while P_F ranges over the values .1, .01, .001 (approximately). The comparisons for smaller P_F values are not conducted because the discrepancies are very small, as may be seen by inspection of the tables.

The greatest discrepancies between probabilities P_{CD} and $Q_M(d,T)$ occur in tables B-3 and B-4, where $N=10$. These particular cases are plotted in figure B-1, for false alarm probabilities in the .1 and .01 regime. For example, the two curves labelled by A, which pertain to $M=1, N=10$, $P_F \approx .1$, show a very slight difference between the two probabilities over the range (.5, .999). The label B actually pertains to two overlapping curves for $M=1, N=10, P_F \approx .01$; that is, the plotted values for P_{CD} and

$Q_M(d,T)$ are indistinguishable at this level of false alarm probability
 The situation for C and D is exactly similar, except that in these latter cases, we have $M=10$, $N=10$.



A: Table B-3, $P_f = .10272$
 B: Table B-3, $P_f = .01021$
 C: Table B-4, $P_f = .10175$
 D: Table B-4, $P_f = .01002$

Figure B-1. Comparison of Probabilities

	d	P _{CD}	Q _M (d,T)
T = 2.40 P _F = .10912	2.2	.50220	.51005
	3.6	.91017	.91506
	4.6	.98935	.99062
	5.4	.99890	.99915
T = 3.25 P _F = .01015	3.1	.50353	.50409
	4.4	.89997	.90032
	5.5	.99099	.99106
	6.3	.99920	.99921
T = 3.89 P _F = .00104	3.8	.51653	.51658
	5.0	.88951	.88954
	6.2	.99205	.99206
	6.9	.99905	.99905

Table B-1. Probability Comparison for M=1, N=2

	d	P _{CD}	Q _M (d,T)
T = 5.59 P _F = .10129	3.5	.49914	.50646
	5.1	.89637	.90120
	6.4	.99083	.99185
	7.3	.99913	.99931
T = 6.32 P _F = .01013	4.6	.51111	.51166
	6.1	.90493	.90526
	7.3	.99132	.99138
	8.1	.99905	.99906
T = 6.89 P _F = .00101	5.3	.49032	.49036
	6.8	.90322	.90325
	8.0	.99162	.99162
	8.8	.99913	.99913

Table B-2. Probability Comparison for M=10, N=2

	d	P _{CD}	Q _M (d,T)
T = 3.01 P _F = .10272	2.9	.51382	.52498
	4.2	.90180	.90857
	5.3	.99074	.99213
	6.1	.99909	.99933
T = 3.71 P _F = .01021	3.6	.51050	.51140
	4.9	.90404	.90457
	6.0	.99161	.99171
	6.8	.99927	.99929
T = 4.29 P _F = .00101	4.2	.51145	.51153
	5.5	.90544	.90548
	6.6	.99188	.99189
	7.3	.99903	.99903

Table B-3. Probability Comparison for M=1, N=10

	d	P _{CD}	Q _M (d,T)
T = 6.11 P _F = .10175	4.3	.49943	.51032
	5.9	.90911	.91532
	7.1	.99133	.99257
	7.9	.99898	.99921
T = 6.73 P _F = .01002	5.2	.52682	.52769
	6.6	.90232	.90282
	7.8	.99133	.99143
	8.6	.99908	.99910
T = 7.23 P _F = .00104	5.8	.51336	.51344
	7.2	.90126	.90131
	8.4	.99160	.99160
	9.2	.99914	.99914

Table B-4. Probability Comparison for M=10, N=10

	d	P _{CD}	Q _M (d,T)
T = 3.70 P _F = .10105	3.6	.50535	.51547
	4.9	.90037	.90628
	6.0	.99080	.99193
	6.8	.99914	.99931
T = 4.29 P _F = .01003	4.2	.51067	.51153
	5.5	.90499	.90548
	6.6	.99180	.99189
	7.3	.99901	.99903
T = 4.79 P _F = .00104	4.7	.50637	.50645
	6.0	.90378	.90382
	7.1	.99170	.99170
	7.8	.99900	.99900

Table B-5. Probability Comparison for M=1, N=100

	d	P _{CD}	Q _M (d,T)
T = 6.72 P _F = .09974	5.2	.52240	.53217
	6.6	.89913	.90471
	7.8	.99064	.99169
	8.6	.99895	.99913
T = 7.23 P _F = .01033	5.8	.51258	.51344
	7.2	.90082	.90131
	8.4	.99152	.99160
	9.2	.99913	.99914
T = 7.68 P _F = .00102	6.4	.52928	.52936
	7.8	.91140	.91144
	8.9	.99112	.99113
	9.7	.99910	.99910

Table B-6. Probability Comparison for M=10, N=100

	d	P _{CD}	Q _M (d,T)
T = 4.27 P _F = .10403	4.2	.51031	.51962
	5.5	.90369	.90884
	6.6	.99141	.99233
	7.3	.99892	.99909
T = 4.79 P _F = .01036	4.7	.50564	.50645
	6.0	.90337	.90382
	7.1	.99162	.99170
	7.8	.99899	.99900
T = 5.25 P _F = .00103	5.2	.51839	.51846
	6.5	.90916	.90920
	7.5	.99008	.99009
	8.3	.99911	.99911

Table B-7. Probability Comparison for M=1, N=1000

	d	P _{CD}	Q _M (d,T)
T = 7.22 P _F = .10332	5.8	.50897	.51785
	7.2	.89822	.90320
	8.4	.99098	.99185
	9.2	.99903	.99917
T = 7.68 P _F = .01017	6.3	.49278	.49354
	7.8	.91105	.91144
	8.9	.99105	.99113
	9.7	.99909	.99910
T = 8.08 P _F = .00105	6.8	.50022	.50029
	8.2	.90129	.90133
	9.3	.98975	.98976
	10.1	.99894	.99894

Table B-8. Probability Comparison for M=10, N=1000

APPENDIX C. PROGRAM LISTING

```

10  M=10          ! NUMBER OF FILTER OUTPUTS SUMMED
20  N=1000        ! NUMBER OF CHANNELS OR-ED
30  DIM U(100), Do(1:10,0:3) ! THRESHOLD VALUES
40  COM Pf(100), Pd1(100), Pd2(100), Pd3(100), Pd4(100), Pd5(100)
50  COM Pd6(100), Pd7(100), Pd8(100), Pd9(100), Pd10(100)
60  COM Pd11(100), Pd12(100), Pd13(100), Pd14(100), Pd15(100)
70  COM Pd16(100), Pd17(100)
80  DOUBLE M, N, I, J ! INTEGERS
90  DATA 2,3,4,4,3,3,4,5,3,4,4,5,3,4,4,5,3,4,5,5
100 DATA 3,4,5,5,4,4,5,5,4,4,5,6,4,5,5,6,4,5,5,6
110 READ Do(*) ! STARTING VALUES FOR d
120 U=0.
130 U=U+.01
140 Pf=FNPf(U,M,N)
150 IF Pf>.1 THEN 130 ! UPPER LIMIT ON Pf
160 U1=MAX(U-.01,.01)
170 U=U+.01
180 Pf=FNPf(U,M,N)
190 IF Pf>1E-10 THEN 170 ! LOWER LIMIT ON Pf
200 U2=U
210 Delu=(U2-U1)*100.
220 FOR I=0 TO 100
230 U=U1+Delu*I
240 U(I)=U
250 Pf(I)=FNPf(U,M,N)
260 NEXT I
270 I=LG(N)
280 Do=Do(M,I)
290 PRINTER IS PRT
300 PRINT M,N,Do
310 PRINTER IS CRT
320 FOR J=1 TO 17
330 Ds=Do+(J-1)*.5 ! TOTAL DEFLECTION PARAMETER d
340 FOR I=0 TO 100
350 U=U(I) ! THRESHOLD
360 Pd=FNPd(Ds,U,M)
370 Pd=MIN(Pd,.99999)
380 IF J=1 THEN Pd1(I)=Pd
390 IF J=2 THEN Pd2(I)=Pd
400 IF J=3 THEN Pd3(I)=Pd
410 IF J=4 THEN Pd4(I)=Pd
420 IF J=5 THEN Pd5(I)=Pd
430 IF J=6 THEN Pd6(I)=Pd
440 IF J=7 THEN Pd7(I)=Pd
450 IF J=8 THEN Pd8(I)=Pd
460 IF J=9 THEN Pd9(I)=Pd
470 IF J=10 THEN Pd10(I)=Pd
480 IF J=11 THEN Pd11(I)=Pd
490 IF J=12 THEN Pd12(I)=Pd
500 IF J=13 THEN Pd13(I)=Pd
510 IF J=14 THEN Pd14(I)=Pd
520 IF J=15 THEN Pd15(I)=Pd
530 IF J=16 THEN Pd16(I)=Pd
540 IF J=17 THEN Pd17(I)=Pd
550 NEXT I
560 NEXT J

```

```

570   FOR I=0 TO 100
580   Pfi(I)=FNIinophi(Pfi(I))
590   Pd1(I)=FNIinophi(Pd1(I))
600   Pd2(I)=FNIinophi(Pd2(I))
610   Pd3(I)=FNIinophi(Pd3(I))
620   Pd4(I)=FNIinophi(Pd4(I))
630   Pd5(I)=FNIinophi(Pd5(I))
640   Pd6(I)=FNIinophi(Pd6(I))
650   Pd7(I)=FNIinophi(Pd7(I))
660   Pd8(I)=FNIinophi(Pd8(I))
670   Pd9(I)=FNIinophi(Pd9(I))
680   Pd10(I)=FNIinophi(Pd10(I))
690   Pd11(I)=FNIinophi(Pd11(I))
700   Pd12(I)=FNIinophi(Pd12(I))
710   Pd13(I)=FNIinophi(Pd13(I))
720   Pd14(I)=FNIinophi(Pd14(I))
730   Pd15(I)=FNIinophi(Pd15(I))
740   Pd16(I)=FNIinophi(Pd16(I))
750   Pd17(I)=FNIinophi(Pd17(I))
760   NEXT I
770   CALL A
780   END
790
800   DEF FNIinophi(X)
810   IF X=.5 THEN RETURN 0.
820   P=MIN(X,1.-X)
830   T=-LOG(P)
840   T=SQR(T+T)
850   P=1.+T*(1.432788+T*(.189269+T*(.001308)))
860   P=T-(2.515517+T*(.802853+T*(.010328)))/P
870   IF X<.5 THEN P=-P
880   RETURN P
890   FNIEND
900
910   DEF FNPf(U,DOUBLE M,N)
920   T=FNE(.5*U+U,M-1)
930   Pf=1.-(1.-T)/N
940   RETURN Pf
950   FNIEND
960
970   DEF FNPd(Ds,U,DOUBLE M)
980   Pd=FNDm(M,Ds,U)
990   RETURN Pd
1000  FNIEND
1010
1020  DEF FNE(X,DOUBLE N)
1030  DOUBLE K
1040  T=S=EXP(-X)
1050  FOR K=1 TO N
1060  T=T*X/K
1070  S=S+T
1080  NEXT K
1090  RETURN S
1100  FNIEND
1110

```

AMS 55, 26.2.23

FALSE ALARM PROBABILITY

DETECTION PROBABILITY

UPPER BOUND ON Pd

$\exp(-x) = e^{-x}$

INTEGER

```

1120 DEF FNQM(DOUBLE M,REAL A,B)      ! QM(A,B)
1130 Error=1.E-17
1140 DOUBLE M1,J                      ! INTEGERS
1150 Q3=.5*A*A
1160 Q4=.5*B*B
1170 Q5=EXP(-.5*(Q3+Q4))
1180 Q6=Q7=Q5
1190 M1=M-1
1200 FOR J=1 TO M1
1210 Q7=Q7+Q4/J
1220 Q6=Q6+Q7
1230 NEXT J
1240 Qm=Q5*Q6
1250 FOR J=1 TO 300
1260 Q5=Q5+Q3/J
1270 Q7=Q7+Q4/(J+M1)
1280 Q6=Q6+Q7
1290 Q9=Q5*Q6
1300 Qm=Qm+Q9
1310 IF Q9<=Error*Qm THEN 1340
1320 NEXT J
1330 PRINT "300 TERMS IN FNQM(M,A,B) AT ";M;A;B
1340 RETURN MIN(Qm,1.)
1350 FNEND
1360 !
1370 SUB A      ! PLOT PD VS PF ON NORMAL PROBABILITY PAPER
1380 COM Pf(*),Pd1(*),Pd2(*),Pd3(*),Pd4(*),Pd5(*)
1390 COM Pd6(*),Pd7(*),Pd8(*),Pd9(*),Pd10(*)
1400 COM Pd11(*),Pd12(*),Pd13(*),Pd14(*),Pd15(*)
1410 COM Pd16(*),Pd17(*)
1420 DIM A$(30),B$(32)
1430 DIM Xlabel$(1:30),Ylabel$(1:30)
1440 DIM Xcoord(1:30),Ycoord(1:30)
1450 DIM Xgrid(1:30),Ygrid(1:30)
1460 DOUBLE N,Lx,Ly,Nx,Ny,I          ! INTEGERS
1470 !
1480 A$="Probability of False Alarm"
1490 B$="Probability of Correct Detection"
1500 !
1510 Lx=12
1520 REDIM Xlabel$(1:Lx),Xcoord(1:Lx)
1530 DATA E-10,E-9,E-8,E-7,E-6,E-5,E-4,E-3,.01,.02,.05,.1
1540 READ Xlabel$(*)
1550 DATA 1E-10,1E-9,1E-8,1E-7,1E-6,1E-5,1E-4,1E-3,.01,.02,.05,.1
1560 READ Xcoord(*)
1570 !
1580 Ly=18
1590 REDIM Ylabel$(1:Ly),Ycoord(1:Ly)
1600 DATA .01,.02,.05,.1,.2,.3,.4,.5,.6,.7,.8,.9
1610 DATA .95,.98,.99,.995,.998,.999
1620 READ Ylabel$(*)
1630 DATA .01,.02,.05,.1,.2,.3,.4,.5,.6,.7,.8,.9
1640 DATA .95,.98,.99,.995,.998,.999
1650 READ Ycoord(*)
1660 !
1670 Nx=14
1680 REDIM Xgrid(1:Nx)
1690 DATA 1E-10,1E-9,1E-8,1E-7,1E-6,1E-5,1E-4,1E-3,.002,.005,.01,.02,.05,.1
1700 READ Xgrid(*)
1710 !

```

```

1720      Np=18
1730      REDIM Ygrid(1:Np)
1740      DATA .01,.02,.05,.1,.2,.3,.4,.5,.6,.7,.8,.9
1750      DATA .95,.98,.99,.995,.998,.999
1760      READ Ygrid(*)
1770      I
1780      FOR I=1 TO Lx
1790      Xcoord(I)=FNInophi(Xcoord(I))
1800      NEXT I
1810      FOR I=1 TO Ly
1820      Ycoord(I)=FNInophi(Ycoord(I))
1830      NEXT I
1840      FOR I=1 TO Nx
1850      Xgrid(I)=FNInophi(Xgrid(I))
1860      NEXT I
1870      FOR I=1 TO Ny
1880      Ygrid(I)=FNInophi(Ygrid(I))
1890      NEXT I
1900      X1=Xgrid(1)
1910      X2=Xgrid(Nx)
1920      Y1=Ygrid(1)
1930      Y2=Ygrid(Ny)
1940      Scale=(Y2-Y1)/(X2-X1)
1950      GINIT 200./260.
1960      PLOTTER IS 505,"HPGL"
1970      PRINTER IS 505
1980      PRINT "VS2"
1990      LIMIT PLOTTER 505,0.,200.,0.,260.
2000      VIEWPORT 20.,20.,103./Scale,19.,122.
2010      VIEWPORT 20.,85.,19.,122.
2020      VIEWPORT 22.,85.,59.,122.
2030      VIEWPORT 22.,85.,19.,62.
2040      WINDOW X1,X2,Y1,Y2
2050      FOR I=1 TO Nx
2060      MOVE Xgrid(I),Y1
2070      DRAW Xgrid(I),Y2
2080      NEXT I
2090      FOR I=1 TO Ny
2100      MOVE X1,Ygrid(I)
2110      DRAW X2,Ygrid(I)
2120      NEXT I
2130      PENUP
2140      CSIZE 2.3,.5
2150      LORG 5
2160      Y=Y1-(Y2-Y1)*.02
2170      FOR I=1 TO Lx
2180      MOVE Xcoord(I),Y
2190      LABEL Xlabel$(I)
2200      NEXT I
2210      CSIZE 3.,.5
2220      MOVE .5*(X1+X2),Y1-.06*(Y2-Y1)
2230      LABEL A$
2240      MOVE .5*(X1+X2),Y1-.1*(Y2-Y1)
2250      LABEL "Figure 42. ROC for M=10, N=1000"

```

! VERTICAL PAPER

! 1 GDU = 2 mm

! TOP OF PAPER

! BOTTOM OF PAPER

```
2260 CSIZE 2.3,.5
2270 LORG 8
2280 X=X1-(X2-X1)*.01
2290 FOR I=1 TO Ly
2300 MOVE X,Ycoord(I)
2310 LABEL Ylabel$(I)
2320 NEXT I
2330 LDIR PI*2.
2340 CSIZE 3.,.5
2350 LORG 5
2360 MOVE X1-.15*(X2-X1),.5*(Y1+Y2)
2370 LABEL B$
2380 PENUP
2390 PLOT Pf(*),Pd1(*)
2400 PENUP
2410 PLOT Pf(*),Pd2(*)
2420 PENUP
2430 PLOT Pf(*),Pd3(*)
2440 PENUP
2450 PLOT Pf(*),Pd4(*)
2460 PENUP
2470 PLOT Pf(*),Pd5(*)
2480 PENUP
2490 PLOT Pf(*),Pd6(*)
2500 PENUP
2510 PLOT Pf(*),Pd7(*)
2520 PENUP
2530 PLOT Pf(*),Pd8(*)
2540 PENUP
2550 PLOT Pf(*),Pd9(*)
2560 PENUP
2570 PLOT Pf(*),Pd10(*)
2580 PENUP
2590 PLOT Pf(*),Pd11(*)
2600 PENUP
2610 PLOT Pf(*),Pd12(*)
2620 PENUP
2630 PLOT Pf(*),Pd13(*)
2640 PENUP
2650 PLOT Pf(*),Pd14(*)
2660 PENUP
2670 PLOT Pf(*),Pd15(*)
2680 PENUP
2690 PLOT Pf(*),Pd16(*)
2700 PENUP
2710 PLOT Pf(*),Pd17(*)
2720 PENUP
2730 BEEP 500,2
2740 PRINTER IS CRT
2750 PLOTTER 505 IS TERMINATED
2760 SUBEND
```

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