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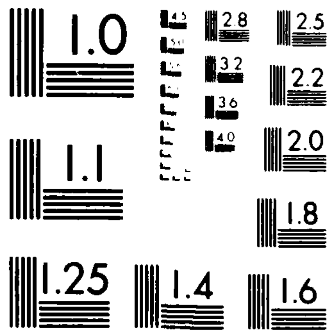
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Technical Report No. 3

EQUILIBRIUM SHOCKS IN PLANE DEFORMATIONS OF INCOMPRESSIBLE ELASTIC MATERIALS

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EQUILIBRIUM SHOCKS IN PLANE DEFORMATIONS OF INCOMPRESSIBLE ELASTIC MATERIALS

by

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ABSTRACT

This paper is concerned with piecewise smooth plane deformations in an isotropic, incompressible elastic material. An explicit necessary and sufficient condition for the existence of piecewise homogeneous equilibrium states is established, and the set of all such states is precisely characterized. A particularly simple expression is derived for the "driving traction" on a surface of discontinuity in the deformation gradient.

1. Introduction

When finitely deformed, certain elastic materials are capable of sustaining deformation fields which involve discontinuous gradients. In such circumstances, the deformation varies smoothly throughout the body except at certain surfaces across which the displacement gradient suffers a jump discontinuity, even though the displacement itself remains continuous. Continuum mechanical treatments of stress-induced phase transformations in solids involve such deformations; see, for example, [1-3].

In this paper we first examine conditions under which a piecewise homogeneous plane deformation can be sustained by an incompressible, isotropic elastic material. The question of the existence of such fields has been investigated previously. For example, the analysis in [4] addresses this issue within the context of <u>compressible</u> elastic materials undergoing plane deformations, while [5] contains a similar study for incompressible materials, and the investigation in [6] approaches the subject from a variational point of view. By restricting attention in this study to materials that are isotropic and incompressible and to plane deformations, we are able to derive a single necessary and sufficient condition for the existence of piecewise homogeneous deformations. Moreover, this condition is expressed in a form that is particularly useful; in addition to providing information on existence, it also allows us to characterize the set of all possible piecewise homogeneous equilibrium states. A similar result for <u>anti</u>-plane deformations was established in [7].

When the theory of elasticity is broadened to allow for equilibrium fields with discontinuous deformation gradients, it turns out that the usual balance between the rate of external work and the rate of storage of elastic energy during a quasi-static motion no longer holds; see [8]. Instead, one finds that mechanical energy may be dissipated at points on the surfaces of discontinuity. This is analogous to the dissipation of energy in an <u>inviscid</u> compressible fluid when the flow involves a shock; the analogy suggests the term "equilibrium shock" for surfaces bearing discontinuities in the displacement gradient in elastostatic fields.

As discussed in [9-12], the altered energetics of fields with equilibrium shocks lead to the notion of a scalar "shock driving traction" f which may be viewed as a normal traction that the body applies to the shock at each of its points. At each point on a moving shock during a quasi-static motion, the product of f with the normal velocity of the shock represents the local rate of dissipation. The presence of dissipation suggests that elastic fields with equilibrium shocks might be used to model certain types of inelastic behavior in solids. Investigations that confirm this possibility are summarized in [10-12], where the nature of the shock traction f plays a major role.

The problems treated in [10-12] are essentially one-dimensional; for these, the structure of the shock traction and its dependence on the deformation are well understood. For problems involving more dimensions, the matter is less fully explored, although Yatomi and Nishimura [13] have obtained some valuable results. The second principal objective of the present paper is the derivation of an especially simple expression for the shock driving traction for isotropic, incompressible elastic materials in the case of plane deformations.

In Section 2 we present some background material on equilibrium shocks in the appropriate setting. The structure of piecewise homogeneous equilibrium states is described in Section 3, and a necessary and sufficient condition for the existence of such states is established and discussed in Section 4. In Section 5, the representation for shock traction mentioned above is derived and discussed.

2. Preliminaries on Finite Plane Strain and Equilibrium Shocks.

Consider an incompressible body which, in a reference configuration, occupies a cylindrical region of space; denote by D the open middle cross-section of this region. Let $X = \{0; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a fixed rectangular cartesian frame whose origin 0 and basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$ are in the plane of D.

A plane deformation of the body is described by the one-to-one mapping $\mathbf{y} = \hat{\mathbf{y}}(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x})$, where $\mathbf{x}$ and $\mathbf{y}$ are the position vectors of a particle in D and $D_* = \hat{\mathbf{y}}(D)$, respectively. For the present, it is assumed that $\mathbf{u} \in C^2(D)$, and we let $\mathbf{F}$ and $\mathbf{G}$ denote the two-dimensional deformation gradient tensor and the left Cauchy-Green deformation tensor, respectively:

$$\mathbf{F} = \nabla \mathbf{y}, \qquad \mathbf{G} = \mathbf{F}\mathbf{F}^T. \tag{2.1}$$

Since the material is incompressible, the deformation invariants I and J obey

$$I = \mathrm{tr}(\mathbf{F}\mathbf{F}^T) = \lambda_1{}^2 + \lambda_2{}^2 \geq 2, \qquad J = \det \mathbf{F} = \lambda_1 \lambda_2 = 1, \tag{2.2}$$

where $\lambda_1(\mathbf{x}) \geq 0$ and $\lambda_2(\mathbf{x}) \geq 0$, are the principal stretches of the deformation at a point in $\mathbf{x}$ in D.

The following kinematical result has been established in [5]: Every two-dimensional tensor $\underset{\sim}{F}$ with unit determinant admits the two decompositions

$$\underset{\sim}{F} = \underset{\sim}{Q}_1\underset{\sim}{K}_1 = \underset{\sim}{K}_2\underset{\sim}{Q}_2, \tag{2.3}$$

where $\underset{\sim}{Q}_1$, $\underset{\sim}{Q}_2$ are proper orthogonal tensors, and $\underset{\sim}{K}_1$, $\underset{\sim}{K}_2$, are tensors whose matrices of components in suitable rectangular cartesian frames X_1 and X_2 have the common value

$$[K_1^{X_1}] = [K_2^{X_2}] = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \tag{2.4}$$

with

$$k = \sqrt{(I-2)} \geq 0, \text{ where } I = \mathrm{tr}\underset{\sim}{F}\underset{\sim}{F}^T. \tag{2.5}$$

<u>It follows that any plane volume-preserving deformation can be decomposed locally into the product of a simple shear in a suitable direction followed, or preceded, by a suitable rotation</u>; k denotes the amount of this shear.

Next, let $\underset{\sim}{\sigma}(\underset{\sim}{x})$ be the two-dimensional nominal stress tensor field on D and $\underset{\sim}{\tau}(\underset{\sim}{y})$ the corresponding Cauchy stress tensor field on D_*:

$$\underset{\sim}{\sigma}(\underset{\sim}{x}) = \underset{\sim}{\tau}(\hat{\underset{\sim}{y}}(\underset{\sim}{x}))\ \underset{\sim}{F}^{-T}(\underset{\sim}{x}), \qquad \underset{\sim}{x} \in D. \tag{2.6}$$

If C is a smooth arc in D, and $C_* = \hat{\underset{\sim}{y}}(C)$ is its image after deformation, the nominal traction $\underset{\sim}{s}$ on C and the true traction $\underset{\sim}{t}$ on C_* are

$$\underset{\sim}{s} = \underset{\sim}{\sigma}\underset{\sim}{N} \text{ on } C, \qquad \underset{\sim}{t} = \underset{\sim}{\tau}\underset{\sim}{n} \text{ on } C_*, \tag{2.7}$$

where $\underset{\sim}{N}$ and $\underset{\sim}{n}$ are corresponding unit normals to C and C_* respectively. Assuming for the present that $\underset{\sim}{\sigma}(\underset{\sim}{x})$ is continuously differentiable on D, equilibrium in the absence of body forces requires

$$\operatorname{div} \underset{\sim}{\sigma} = \underset{\sim}{0}, \qquad \underset{\sim}{\sigma}\underset{\sim}{F}^T = \underset{\sim}{F}\underset{\sim}{\sigma}^T \qquad \text{on } D, \tag{2.8}$$

or equivalently,

$$\operatorname{div} \underset{\sim}{\tau} = \underset{\sim}{0}, \qquad \underset{\sim}{\tau} = \underset{\sim}{\tau}^T \qquad \text{on } D_*. \tag{2.9}$$

Turning to the constitutive law of the incompressible material at hand, we suppose that it is homogeneous, isotropic and hyperelastic. The plane strain elastic potential $\hat{W}(\underset{\sim}{F})$ then has the form

$$\hat{W}(\underset{\sim}{F}) = W(I); \tag{2.10}$$

the value of W is the strain energy per unit reference volume. The associated constitutive law takes the equivalent alternate forms

$$\underset{\sim}{\sigma} = 2W'(I)\underset{\sim}{F} - p\underset{\sim}{F}^{-T}, \qquad \underset{\sim}{\tau} = 2W'(I)\underset{\sim}{F}\underset{\sim}{F}^T - p\underset{\sim}{1}, \tag{2.11}$$

where the scalar field p arises because of the incompressibility constraint.

The particular deformation described by

$$y_1 = x_1 + kx_2, \qquad y_2 = x_2, \tag{2.12}$$

is a simple shear, where the constant k denotes the amount of shear. The corresponding shear stress component τ_{12} is found from (2.12), (2.11), (2.1), (2.2) to be $\tau_{12} = \hat{\tau}(k)$, where

$$\hat{\tau}(k) = 2kW'(2+k^2), \qquad -\infty < k < \infty. \tag{2.13}$$

The odd function $\hat{\tau}(k)$ is the <u>shear stress response function</u> of the material in simple shear. Throughout thi paper it will be assumed that the <u>secant modulus of shear</u> $M(k)$ is positive:

$$M(k) = 2\, W'(2+k^2) > 0 \quad \text{for } -\infty < k < \infty\ ; \tag{2.14}$$

this is consistent with the Baker-Ericksen inequality. While materials for which $M(k)$ is not always positive are of interest (see [11]), this case will not be pursued here. Observe from (2.13) that

$$W(I) = \int_0^{\sqrt{(I-2)}} \hat{\tau}(k)\, dk \quad \text{for} \quad I \geq 2, \tag{2.15}$$

which implies that <u>the in-plane response, in all plane deformations, of an isotropic, incompressible elastic material is completely characterized by specifying the shear stress response function</u> $\hat{\tau}(k)$.

The displacement equations of equilibrium for plane strain in a homogeneous, isotropic, incompressible material are, by (2.11), (2.8), (2.2),

$$\left.\begin{aligned} c_{\alpha\beta\gamma\delta}(\underset{\sim}{F}) u_{\gamma,\beta\delta} - p_{,\beta}\, F_{\beta\alpha}{}^{-1} &= 0 \\ \det \underset{\sim}{F} &= 1 \end{aligned}\right\} \quad \text{on } D, \tag{2.16}$$

where $F_{\alpha\beta} = \delta_{\alpha\beta} + u_{\alpha,\beta}$ and

$$c_{\alpha\beta\gamma\delta}(\underset{\sim}{F}) = 2W'(I)\delta_{\alpha\gamma}\delta_{\beta\delta} + 4W''(I)F_{\alpha\beta}F_{\gamma\delta}. \tag{2.17}$$

Here, a comma followed by a subscript indicates partial differentiation with respect to the corresponding x-coordinate, while repeated subscripts are summed over 1,2.

It has been shown in [5] that, in the presence of (2.14), the system of partial differential equations (2.16) is <u>elliptic</u> at a solution $\underset{\sim}{u}$, p and at a point $\underset{\sim}{x}$, if and only if

$$\tau'(k(\underset{\sim}{x})) > 0, \tag{2.18}$$

where $k(\underset{\sim}{x}) = \surd(I(\underset{\sim}{x})-2) = \surd(\mathrm{tr}\underset{\sim}{F}(\underset{\sim}{x})\underset{\sim}{F}^T(\underset{\sim}{x})-2)$ is the amount of shear associated with the deformation at that point. From (2.13), (2.14) one has $\hat{\tau}'(0) > 0$, so that ellipticity necessarily prevails at infinitesimal deformations. If, however, $\hat{\tau}$ fails to be monotonically increasing on $-\infty < k < \infty$, ellipticity will be lost if the deformation is severe enough. If $\hat{\tau}'(k) > 0$ for all k, we say that the <u>material</u> is elliptic for plane deformations.

One consequence of a loss of ellipticity of the governing partial differential equations is the possible occurrence of elastostatic fields which are less than classically smooth. In order to account for such weak solutions, the smoothness assumptions made previously must be relaxed. Of particular interest is the case wherein the field quantities possess the aforementioned degree of smoothness everywhere expect on one or more arcs in D. Accordingly, we now allow for the possibility that although $\underset{\sim}{u}$ is continuous in D, there is a single smooth curve $C \subset D$ such that $\underset{\sim}{\sigma}$ and $\underset{\sim}{u}$ are respectively once and twice continuously differentiable in D-C, and $\underset{\sim}{\sigma}$ and $\nabla\underset{\sim}{u}$ suffer finite jump discontinuities across C. In these new circumstances, the field equations discussed previously continue to hold in D-C. In addition, equilibrium considerations require that the nominal traction $\underset{\sim}{s}$ be continuous across C; this in turn implies that the true traction $\underset{\sim}{t}$ is continuous across C_*, the deformation-image of C. Thus

$$[[\underset{\sim}{\sigma}]]\underset{\sim}{N} = \underset{\sim}{0} \text{ on } C, \qquad [[\underset{\sim}{\tau}]]\underset{\sim}{n} = \underset{\sim}{0} \text{ on } C_*, \tag{2.19}$$

where $\underset{\sim}{N}$ and $\underset{\sim}{n}$ are unit normals on C and C_* respectively, and $[[\cdot]]$ indicates the jump across the appropriate curve. Note that displacement continuity across C requires

$$[[\underset{\sim}{F}]]\underset{\sim}{L} = \underset{\sim}{0} \text{ on } C, \tag{2.20}$$

where $\underset{\sim}{L}$ is a unit tangent vector on C. A curve C carrying jump discontinuities in $\underset{\sim}{F}$, p and $\underset{\sim}{\sigma}$ while preserving continuity of displacement and traction is called an <u>equilibrium shock</u>.

3. <u>Piecewise homogeneous equilibrium states</u>.

In order to investigate many of the local issues related to equilibrium shocks, it is sufficient to consider the case in which D coincides with the entire (x_1,x_2)-plane, C is a straight line through the origin, and the deformation gradient $\underset{\sim}{F}$ as well as the hydrostatic pressure p are constant on either side of C. Under these conditions, the field equations (2.8), (2.11) will be trivially satisfied in D - C, and the requirements remaining to be fulfilled are (2.20) and either of (2.19).

Let $\underset{\sim}{L}$ be a unit vector along the straight line C through the origin, and let $\underset{\sim}{N} = \underset{\sim}{e}_3 \times \underset{\sim}{L}$ be the unit normal to C associated with $\underset{\sim}{L}$, so that $(\underset{\sim}{L},\underset{\sim}{N},\underset{\sim}{e}_3)$ is a right-handed orthonormal triplet. Let $\overset{+}{D} = \{\underset{\sim}{x} \mid \underset{\sim}{x}\cdot\underset{\sim}{N} > 0\}$, $\overset{-}{D} = \{\underset{\sim}{x} \mid \underset{\sim}{x}\cdot\underset{\sim}{N} < 0\}$ be the two open half-planes into which C divides D; see Figure 1. Points in $\overset{+}{D}$ lie on the <u>positive</u> side of the shock.

Consider the piecewise homogeneous deformation

$$\underset{\sim}{y} = \begin{cases} \overset{+}{\underset{\sim}{F}}\,\underset{\sim}{x} & \text{for } \underset{\sim}{x}\in \overset{+}{D}\ , \\ \overset{-}{\underset{\sim}{F}}\,\underset{\sim}{x} & \text{for } \underset{\sim}{x}\in \overset{-}{D}\ , \end{cases} \tag{3.1}$$

where $\overset{+}{\underset{\sim}{F}}$ and $\overset{-}{\underset{\sim}{F}}$ are distinct, constant unimodular tensors:

$$\overset{+}{\underset{\sim}{F}} \neq \overset{-}{\underset{\sim}{F}}\ , \qquad \det \overset{+}{\underset{\sim}{F}} = \det \overset{-}{\underset{\sim}{F}} = 1. \tag{3.2}$$

Define $\overset{+}{\underset{\sim}{G}}$, $\overset{-}{\underset{\sim}{G}}$, $\overset{+}{I}$, $\overset{-}{I}$, k_+ and k_- by

$$\overset{+}{\underset{\sim}{G}} = \overset{+}{\underset{\sim}{F}}\,\overset{+}{\underset{\sim}{F}}{}^T\ , \qquad \overset{-}{\underset{\sim}{G}} = \overset{-}{\underset{\sim}{F}}\,\overset{-}{\underset{\sim}{F}}{}^T\ , \tag{3.3}$$

$$\overset{+}{I} = \operatorname{tr} \overset{+}{\underset{\sim}{G}}\ , \qquad \overset{-}{I} = \operatorname{tr} \overset{-}{\underset{\sim}{G}}\ , \tag{3.4}$$

$$k_+ = (\overset{+}{I} - 2)^{1/2} \geq 0\ , \qquad k_- = (\overset{-}{I} - 2)^{1/2} \geq 0\ . \tag{3.5}$$

In view of the kinematical result (2.3)-(2.5), k_+ and k_- represent the respective amounts of shear associated with the deformations on $\overset{+}{D}$ and $\overset{-}{D}$.

The displacement corresponding to the deformation (3.1) will be continuous across C if and only if

$$\overset{+}{\underset{\sim}{F}}\,\underset{\sim}{L} = \overset{-}{\underset{\sim}{F}}\,\underset{\sim}{L}\ . \tag{3.6}$$

Assume that (3.6) holds, and let

$$\underset{\sim}{\ell} = g\,\overset{+}{\underset{\sim}{F}}\underset{\sim}{L} = g\,\overset{-}{\underset{\sim}{F}}\underset{\sim}{L}\ , \tag{3.7}$$

where

$$g = 1/|\overset{+}{\underset{\sim}{F}}\underset{\sim}{L}| = 1/|\overset{-}{\underset{\sim}{F}}\underset{\sim}{L}| > 0; \tag{3.8}$$

the unit vector $\underset{\sim}{\ell}$ is tangent to the image C_* of C under the deformation (3.1). The vector $\underset{\sim}{n}$ defined by

$$\underset{\sim}{n} = \underset{\sim}{e}_3 \times \underset{\sim}{\ell} \tag{3.9}$$

is the unit normal to C_* associated with $\underset{\sim}{\ell}$; $(\underset{\sim}{\ell}, \underset{\sim}{n}, \underset{\sim}{e}_3)$ is a right-handed orthonormal triplet. Using the definitions of $\underset{\sim}{N}$, $\underset{\sim}{\ell}$ and $\underset{\sim}{n}$ as well as the facts that $\det \overset{+}{\underset{\sim}{F}} > 0$, $\det \overset{-}{\underset{\sim}{F}} > 0$, one can show that $\overset{+}{D}_* = \{\underset{\sim}{y} \mid \underset{\sim}{y} \cdot \underset{\sim}{n} > 0\}$ and $\overset{-}{D}_* = \{\underset{\sim}{y} \mid \underset{\sim}{y} \cdot \underset{\sim}{n} < 0\}$ are the respective images of $\overset{+}{D}$ and $\overset{-}{D}$ under the deformation (3.1). Because $\det \overset{+}{\underset{\sim}{F}} = \det \overset{-}{\underset{\sim}{F}} = 1$, (3.7) and (3.9) imply that

$$\overset{+}{\underset{\sim}{F}}{}^T \underset{\sim}{n} = \overset{-}{\underset{\sim}{F}}{}^T \underset{\sim}{n} = g\, \underset{\sim}{N}, \tag{3.10}$$

from which it further follows that

$$g = (\underset{\sim}{n} \cdot \overset{+}{\underset{\sim}{G}}\underset{\sim}{n})^{1/2} = (\underset{\sim}{n} \cdot \overset{-}{\underset{\sim}{G}}\underset{\sim}{n})^{1/2}. \tag{3.11}$$

Note from (3.11) that

$$[[\underset{\sim}{n} \cdot \underset{\sim}{G}\underset{\sim}{n}]]_-^+ = 0, \tag{3.12}$$

where $[[\cdot]]$ indicates the jump across the shock.

A fiber whose orientation in the reference configuration is parallel to the shock C suffers a relative extension $(1-g)/g$; when $g = 1$, such a fiber is unextended by the deformation (3.1). When this is the case, we say the shock is <u>normal</u>; otherwise, it is <u>oblique</u>.

Let a unit vector $\underset{\sim}{L}$ and a unimodular tensor $\overset{+}{\underset{\sim}{F}}$ be given. Define g and $\underset{\sim}{\ell}$ through the first equalities in (3.8) and (3.7), respectively, and define $\underset{\sim}{n}$ by (3.9). A unimodular tensor $\overset{-}{\underset{\sim}{F}}$ will satisfy (3.6) if and only if there is a number κ such that

$$\overset{-}{\mathbf{F}} = (\mathbf{1} + \kappa\, \boldsymbol{\ell} \otimes \mathbf{n})\overset{+}{\mathbf{F}} , \tag{3.13}$$

where $\otimes$ denotes tensor multiplication of two vectors. To prove this, suppose first that F satisfies (3.6). It follows that $\overset{-}{\mathbf{F}} = \overset{+}{\mathbf{F}} + \mathbf{h} \otimes \mathbf{N}$, where $\mathbf{N} = \mathbf{e}_3 \times \mathbf{L}$ and $\mathbf{h} = \overset{-}{\mathbf{F}}\mathbf{N} - \overset{+}{\mathbf{F}}\mathbf{N}$. Alternatively, we may write $\overset{-}{\mathbf{F}} = (\mathbf{1} + (\mathbf{h}\otimes\mathbf{N})\overset{+}{\mathbf{F}}{}^{-1})\overset{+}{\mathbf{F}} = (\mathbf{1} + \mathbf{h}\otimes(\overset{+}{\mathbf{F}}{}^{-1})^T\mathbf{N})\overset{+}{\mathbf{F}}$ so that by the first equality in (3.10), $\overset{-}{\mathbf{F}} = (\mathbf{1} + (1/g)\mathbf{h} \otimes\mathbf{n})\overset{+}{\mathbf{F}}$. Taking the determinant of both sides of this equality and recalling $(3.2)_2$ implies that $\mathbf{h} \cdot \mathbf{n} = 0$. Consequently $\mathbf{h}$ is parallel to $\boldsymbol{\ell}$, and we may write $\mathbf{h} = (\kappa g)\, \boldsymbol{\ell}$, which establishes (3.13). The definition of $\mathbf{h}$ together with (3.10) and $\kappa = (1/g)\mathbf{h}\cdot\boldsymbol{\ell}$ gives

$$\kappa = - (1/g^2)[[\boldsymbol{\ell} \cdot \mathbf{G}\mathbf{n}]]^{+}_{-} . \tag{3.14}$$

Conversely, if $\overset{-}{\mathbf{F}}$ is given by (3.13), then it is unimodular and satisfies (3.6).

Geometrically, (3.13) asserts that the deformation on $\overset{-}{D}$ may be viewed as the composition of two deformations, the first of which is the extension to $\overset{-}{D}$ of the defomation on $\overset{+}{D}$, while the second is a simple shear parallel to $C_\star$ in which the additional amount of shear is κ.

We turn next to the continuity of traction across the shock. By $(2.19)_2$, $(2.11)_2$, (2.14) and (3.3), this requires

$$[[M(k)\mathbf{G}]]^{+}_{-}\mathbf{n} = [[p]]^{+}_{-}\mathbf{n} , \tag{3.15}$$

where M(k) is the secant shear modulus. Taking the dot product of (3.15) with $\mathbf{n}$ and using (3.11) gives the jump in pressure as

$$[[p]]^{+}_{-} = [[M(k)]]^{+}_{-}g^2 , \tag{3.16}$$

while the dot product of (3.15) with $\underset{\sim}{\ell}$ gives

$$\underset{\sim}{\ell} \cdot [[M(k)\underset{\sim}{G}]]_{-}^{+}\underset{\sim}{n} = 0 \, . \tag{3.17}$$

Conversely, if (3.16) and (3.17) hold, then so does (3.15). Finally, using (3.14) to eliminate $\underset{\sim}{\ell}\cdot\bar{\underset{\sim}{G}}\underset{\sim}{n}$ in (3.17) yields

$$\kappa = [M(k_+) - M(k_-)] \, (\underset{\sim}{\ell} \cdot \overset{+}{\underset{\sim}{G}}\underset{\sim}{n}) / \, [g^2 M(k_-)] . \tag{3.18}$$

Several useful facts follow from (3.18). First, in a piecewise homogeneous equilibrium state, necessarily $k_+ \neq k_-$; otherwise, by (3.18), $\kappa = 0$, so that by (3.13), $\bar{\underset{\sim}{F}} = \overset{+}{\underset{\sim}{F}}$, contradicting the first of (3.2). Second, one must have $k_+ \neq 0$, $k_- \neq 0$. If, for example, $k_+ = 0$, then (3.3) and (2.3), (2.4) applied to $\overset{+}{\underset{\sim}{F}}$ show that $\overset{+}{\underset{\sim}{G}}$ is the identity. From (3.18) it again follows that $\kappa = 0$, which is impossible. Similarly, $k_- \neq 0$. Thus the deformations on both sides of the shock must involve nontrivial, distinct amounts of shear k_+, k_- .

Let a unimodular tensor $\overset{+}{\underset{\sim}{F}}$ be given. The <u>shock problem</u> requires the determination of a unimodular tensor $\bar{\underset{\sim}{F}} \neq \overset{+}{\underset{\sim}{F}}$ and a unit vector $\underset{\sim}{L}$ such that (3.6) and (3.17) hold; in (3.17), $\overset{+}{\underset{\sim}{G}}$ and $\bar{\underset{\sim}{G}}$ are to be given by (3.3), k_+ and k_- by (3.5), $g = 1/|\overset{+}{\underset{\sim}{F}}\underset{\sim}{L}|$, $\underset{\sim}{\ell} = g\, \overset{+}{\underset{\sim}{F}}\underset{\sim}{L}$, and $\underset{\sim}{n}$ is as in (3.9). For a given $\overset{+}{\underset{\sim}{F}}$, let $\bar{\underset{\sim}{F}}$, $\underset{\sim}{L}$ be a solution of the shock problem, and let $\overset{+}{p}$ be a given constant. Define $\bar{p}$ by means of (3.16); then $\overset{+}{\underset{\sim}{F}}$, $\bar{\underset{\sim}{F}}$, $\overset{+}{p}$, and $\bar{p}$ will generate a piecewise homogeneous equilibrium state through (3.1) and the constitutive law $(2.11)_1$. The shock C is the straight line through the origin determined by $\underset{\sim}{L}$.

4. Solutions of the Shock Problem.

Here we establish a necessary and sufficient condition for the existence of solutions to the shock problem, and we discuss its interpretation and some of its implications.

Proposition: Let $\overset{+}{\underset{\sim}{F}}$ be a given unimodular tensor, and let k_+ be given by

$$k_+ = \sqrt{\{\mathrm{tr}(\overset{+}{\underset{\sim}{F}}\,\overset{+}{\underset{\sim}{F}}{}^T) - 2\}}. \tag{4.1}$$

A solution to the shock problem corresponding to $\overset{+}{\underset{\sim}{F}}$ exists if and only if there is a number $k_- \neq k_+$ such that

$$[\hat{\tau}(k_+) - \hat{\tau}(k_-)](k_+ - k_-) \leq 0 , \tag{4.2}$$

where $\hat{\tau}(k)$ is the shear stress response function characteristic of the material.

We first assume that the shock problem has a solution $\overset{-}{\underset{\sim}{F}}$, $\underset{\sim}{L}$ for the given $\overset{+}{\underset{\sim}{F}}$ and show that the result stated above is necessary. We begin by recording the identity

$$(\underset{\sim}{n}\cdot\underset{\sim}{G}\underset{\sim}{n})^2 + (\underset{\sim}{\ell}\cdot\underset{\sim}{G}\underset{\sim}{n})^2 = |\underset{\sim}{G}\underset{\sim}{n}|^2 = (\underset{\sim}{n}\cdot\underset{\sim}{G}^2\underset{\sim}{n}). \tag{4.3}$$

The Cayley-Hamilton theorem applied to the unimodular tensors $\overset{+}{\underset{\sim}{G}}$ and $\overset{-}{\underset{\sim}{G}}$, together with (4.3) and (3.11), yields

$$(\underset{\sim}{\ell}\cdot\overset{+}{\underset{\sim}{G}}\underset{\sim}{n})^2 = -g^4 + \overset{+}{I}g^2 - 1 , \quad (\underset{\sim}{\ell}\cdot\overset{-}{\underset{\sim}{G}}\underset{\sim}{n})^2 = -g^4 + \overset{-}{I}g^2 - 1 . \tag{4.4}$$

Using (4.4) and (3.5) in (3.17) gives the following equation for g in terms of k_+ and k_- :

$$[[M^2(k)]]^{+}_{-}g^4 - [[M^2(k)(2+k^2)]]^{+}_{-}g^2 + [[M^2(k)]]^{+}_{-} = 0 \tag{4.5}$$

Because of $(3.11)_1$, we may write

$$g^2 = \overset{+}{\lambda}_1{}^2 \cos^2\theta + \overset{+}{\lambda}_2{}^2 \sin^2\theta \ , \tag{4.6}$$

where θ is the angle in $[0,\pi]$ between $\underset{\sim}{n}$ and the principal direction of $\overset{+}{\underset{\sim}{G}}$ that corresponds to the principal stretch $\overset{+}{\lambda}_1$. Without loss of generality, we assume that $\overset{+}{\lambda}_1 \leq \overset{+}{\lambda}_2$, so that it then follows from (4.6) that

$$\overset{+}{\lambda}_1 \leq g \leq \overset{+}{\lambda}_2 . \tag{4.7}$$

Returning to (4.5), we note that if $[[M^2(k)]]_-^+ = 0$, we would have $g = 0$, which is impossible. Thus (4.5) may be rewritten as

$$[[M^2(k)k^2]]_-^+ / \ [[M^2(k)]]_-^+ = g^2 + g^{-2} - 2 \geq 0 . \tag{4.8}$$

Since $M(k) = \hat{\tau}(k)/k$, (4.8) is equivalent to

$$[[\hat{\tau}^2(k)]]_-^+ / \ [[\hat{\tau}^2(k)/k^2]]_-^+ = g^2 + g^{-2} - 2 \geq 0 . \tag{4.9}$$

From (4.6) and the fact that $\overset{+}{\lambda}_1 = 1/\overset{+}{\lambda}_2$, one shows readily that $g^2 + g^{-2} \leq \overset{+}{\lambda}_1{}^2 + \overset{+}{\lambda}_2{}^2 = \overset{+}{I} = 2 + k_+{}^2$. It then follows from (4.9) that necessarily

$$0 \leq [[\hat{\tau}^2(k)]]_-^+ \ / \ [[\hat{\tau}^2(k)/k^2]]_-^+ \leq k_+{}^2 \ . \tag{4.10}$$

Finally, one can show that (4.10) is equivalent to

$$[\hat{\tau}(k_+) - \hat{\tau}(k_-)](k_+ - k_-) \leq 0 \ , \tag{4.11}$$

thus establishing the necessity of (4.2).

We now show that the conditions stated in the proposition are sufficient for the existence of an equilibrium shock. Let $\overset{+}{\underset{\sim}{F}}$ be a given unimodular tensor, and let k_+ be defined by (4.1). Suppose that there is a number $k_- \neq k_+$ satisfying (4.2); note that there may be many such numbers. There is then a root g of (4.9) satisfying (4.7) for this k_- ; in fact, there are two such roots. For each such g, there is an angle θ satisfying (4.6) and hence determining a unit vector $\underset{\sim}{n}$. Corresponding to this pair $g, \underset{\sim}{n}$, one defines $\underset{\sim}{N} = (1/g)\, \overset{+}{\underset{\sim}{F}}{}^{T}\underset{\sim}{n}$; see (3.10). Let $\underset{\sim}{L} = \underset{\sim}{N} \times \underset{\sim}{e}_3$, $\underset{\sim}{\ell} = \underset{\sim}{n} \times \underset{\sim}{e}_3$, and define κ by (3.18), using $\overset{+}{\underset{\sim}{G}} = \overset{+}{\underset{\sim}{F}}\,\overset{+}{\underset{\sim}{F}}{}^{T}$. Finally, define $\overset{-}{\underset{\sim}{F}}$ by (3.13). With $\underset{\sim}{L}$ and $\overset{-}{\underset{\sim}{F}}$ constructed in this way, (3.6) is automatically satisfied. It is readily verified that (3.17) holds as well. Thus $\overset{-}{\underset{\sim}{F}}$ and $\underset{\sim}{L}$ furnish a solution of the shock problem corresponding to the given $\overset{+}{\underset{\sim}{F}}$, and the proof is complete.

The proposition just proved may be interpreted geometrically with the help of a set Σ in the (k_1, k_2)-plane defined as follows:

$$\Sigma = \{(k_1,k_2) \mid [\hat{\tau}(k_1) - \hat{\tau}(k_2)](k_1 - k_2) \le 0,\ k_1>0,\ k_2>0,\ k_1 \ne k_2\}. \tag{4.12}$$

Given a unimodular tensor $\overset{+}{\underset{\sim}{F}}$, the proposition asserts that the corresponding shock problem has a solution if and only if there is a number k_- such that $(k_+, k_-) \in \Sigma$, where $k_+ = \sqrt{[\mathrm{tr}(\overset{+}{\underset{\sim}{F}}\overset{+}{\underset{\sim}{F}}{}^{T}) - 2]}$ is the amount of shear associated with $\overset{+}{\underset{\sim}{F}}$. For illustration, consider a material whose shear stress response function $\hat{\tau}(k)$ is as shown in Figure 2. Let Γ be the closed curve defined by

$$\Gamma = \{(k_1,k_2) \mid \hat{\tau}(k_1) = \hat{\tau}(k_2),\ k_1 > 0,\ k_2 > 0\}; \tag{4.13}$$

the set Σ for this material consists of all points on and within Γ <u>except</u> for the points on the diagonal line $k_1 = k_2$. Σ is symmetric about this diagonal. Figure 3 shows the curve Γ and the region Σ for a material of the kind described above whose shear stress response is given by

$$\hat{\tau}(k) = \mu(k - \gamma k^3 + k^5), \quad \gamma = 1.875\ , \ \mu > 0\ ; \tag{4.14}$$

the qualitative nature of Figure 3, however, does not depend on this special choice, as long as $\hat{\tau}(k)$ has the form shown in Figure 2. The shaded region in Figure 3 consists of points (k_1,k_2) at which $\hat{\tau}'(k_1)$ and $\hat{\tau}'(k_2)$ are both positive; the corresponding piecewise homogeneous equilibrium states always involve deformations that are elliptic on <u>both</u> sides of the shock. This is not the case for points (k_1,k_2) in the unshaded part of Σ.

For the special case of a <u>normal</u> shock, $g = 1$, and (4.9) immediately yields

$$\hat{\tau}(k_+) = \hat{\tau}(k_-) \ ; \tag{4.15}$$

thus the set of points in Σ that correspond to normal shocks are precisely the points on the boundary Γ of Σ except, of course, for the points with $k_+ = k_-$.

If $\hat{\tau}'(k) > 0$ for all k, so that the shear stress in simple shear is a monotone increasing function of the amount of shear, the material under consideration is elliptic for plane deformations. It is an immediate consequence of the proposition that Σ is empty in this case, and no piecewise homogeneous equilibrium states of plane strain exist for such a material. On the other hand, if the material is such that $\hat{\tau}'(k)$ is non-positive throughout some interval, then - since $\hat{\tau}'(0) > 0$ by (2.13), (2.14) - piecewise homogeneous plane equilibrium deformations will certainly exist for suitably chosen values of $\overset{+}{\underset{\sim}{F}}$.

The observations of the preceding paragraph were made in [5] on the basis of a different analysis. In addition to providing information on the existence of shocks, the approach taken here also characterizes all possible piecewise homogeneous equilibrium states corresponding to a given $\underset{\sim}{F}$. For each $k > 0$, let Σ_k be the set in the (k_1,k_2)-plane defined by

$$\Sigma_k = \{\kappa \mid (k,\kappa) \in \Sigma\} = \{\kappa \mid (\kappa,k) \in \Sigma\} . \tag{4.16}$$

Σ_k is the cross-section of Σ at either $k_1 = k$ or $k_2 = k$. If $\underset{\sim}{F}$ is a unimodular tensor, let $k = \sqrt{[\mathrm{tr}(\underset{\sim}{F}\underset{\sim}{F}^T) - 2]}$; if Σ_k is non-empty, then there exists an equilibrium shock corresponding to every k_- in Σ_k .

We turn now to a discussion of the structure of the driving traction associated with equilibrium shocks.

5. Shock Traction

For the purposes of this section, it is necessary to consider quasi-static motions of the body. Let $\{\underset{\sim}{u}(\cdot,t),\ p(\cdot,t)\}$, $t_0 \le t \le t_1$, be a one-parameter family of weak solutions of the displacement equations of equilibrium (2.16) of the type described in Section 2; let $C_t \subset D$ be the associated family of shocks. Assume that the particle velocity $\underset{\sim}{v}(\underset{\sim}{x},t) = \partial\underset{\sim}{u}(\underset{\sim}{x},t)/\partial t$ exists and is continuous in $(\underset{\sim}{x},t)$ for $\underset{\sim}{x} \in D\text{-}C_t$, $t_0 \le t \le t_1$, and that $\underset{\sim}{v}$ is piecewise continuous on $D \times [t_0,t_1]$.

Let $\underset{\sim}{V}(\underset{\sim}{x},t)$ be the velocity of a point on the moving curve C_t which at time t is located at $\underset{\sim}{x}$, and denote by $\underset{\sim}{N}(\underset{\sim}{x},t)$ the unit normal to C_t at this same point. Note that, even though the deformations here need not be piecewise homogeneous, equations (3.2)-(3.18) and (4.1)-(4.12) continue to hold at each point on C_t and each instant t, provided that one interprets $\overset{+}{\underset{\sim}{F}}(\underset{\sim}{x},t)$ or $\overset{-}{\underset{\sim}{F}}(\underset{\sim}{x},t)$ as the limiting values of $\underset{\sim}{F}$ as the point $\underset{\sim}{x}$ on C_t is approached from its positive or negative side, respectively. The positive side of C_t is the one into which $\underset{\sim}{N}$ points.

As shown in [8], the presence of the shock affects the balance of mechanical energy during the motion in an essential way. For any fixed regular sub-region Π of D, let d(t) represent the difference between the rate of external work on Π and the rate at which elastic energy is being stored:

$$d(t) = \int_{\partial\Pi} \underset{\sim}{s}\cdot\underset{\sim}{v}\, ds - \frac{d}{dt}\int_{\Pi} W(\underset{\sim}{F})\, dA, \qquad t_0 \le t \le t_1; \tag{5.1}$$

d(t) is the rate of dissipation of mechanical energy associated with the particles which occupy the region Π in the reference state. By adapting to plane deformations an argument given in [8], one can show that d(t) may be written as

$$d(t) = \int_{C_t\cap\Pi} [[\underset{\sim}{P}]]_{-}^{+}\underset{\sim}{N} \cdot \underset{\sim}{V}\, ds, \tag{5.2}$$

where $\underset{\sim}{P}(\underset{\sim}{x},t)$ is the two-dimensional energy-momentum tensor,

$$\underset{\sim}{P} = W(\underset{\sim}{F})\underset{\sim}{1} - \underset{\sim}{F}^T \underset{\sim}{\sigma}, \qquad \underset{\sim}{x} \in D\text{-}C_t,\ t_0 \le t \le t_1. \tag{5.3}$$

If $\underset{\sim}{P}(\cdot,t)$ happens to be continuous across C_t at an instant t, (5.2) gives d(t) = 0. In general, however, if Π intersects C_t, d(t) ≠ 0 .

It has been shown in [8] that, because of displacement and traction continuity across C_t , the vector $[[\underset{\sim}{P}]]_{-}^{+}\underset{\sim}{N}$ is normal to C_t at each of its points. Thus we may write (5.2) in the alternate form

$$d(t) = \int_{C_t\cap\Pi} f\, \underset{\sim}{N}\cdot\underset{\sim}{V}\, ds, \tag{5.4}$$

where $f(\underset{\sim}{x},t)$ is defined by

$$f = [[\underset{\sim}{P}]]\overset{+}{\underset{\sim}{N}} \cdot \underset{\sim}{N}, \qquad \underset{\sim}{x} \in C_t, \quad t_0 \le t \le t_1. \tag{5.5}$$

Combining (5.1) with (5.4) yields

$$\int_{\partial\Pi} \underset{\sim}{s}\cdot\underset{\sim}{v}\, ds + \int_{C_t\cap\Pi} (-f\underset{\sim}{N})\cdot\underset{\sim}{V}\, ds = \frac{d}{dt}\int_{\Pi} W(\underset{\sim}{F})\, dA, \qquad t_0\le t\le t_1, \tag{5.6}$$

suggesting that $+f\underset{\sim}{N}$ be viewed as a fictitious nominal "shock driving traction" exerted on the shock by the surrounding material; the scalar f determines the magnitude of this traction. This concept is related to the notion of the "force on a defect" introduced by Eshelby [9]. The expressions (5.5), (5.3) together comprise a special case of a result given in [14]; see also Rice [15].

When treated from a thermomechanical point of view, the dissipation rate d(t) can be shown to be identical to the product of temperature and entropy production rate, provided that the temperature is spatially uniform and constant in time. The Clausius-Duhem inequality then requires the dissipation rate d(t) to be non-negative for all sub-regions Π and all instants t; see [8]. Equivalently, one must have

$$f\, V_n \ge 0, \quad \underset{\sim}{x}\in C_t, \quad t_0 \le t \le t_1, \tag{5.7}$$

where V_n is the normal velocity of a point on the shock C_t :

$$V_n = \underset{\sim}{V} \cdot \underset{\sim}{N}. \tag{5.8}$$

The dissipation inequality (5.7) is trivially satisfied in the special case when f happens to be zero. In general, however, this is not the case, and given an equilibrium state, (5.7) specifies the direction in which the the shock may move in a quasi-static motion commencing from this state.

Following Yatomi and Nishimura [13], one can eliminate the shock orientation $\underset{\sim}{N}$ from (5.5), (5.3) as follows: From the discussion following (3.13),

$$[[\underset{\sim}{F}]]_{-}^{+} = [[\underset{\sim}{F}\underset{\sim}{N}]]_{-}^{+} \otimes \underset{\sim}{N}. \tag{5.9}$$

Using this and $(2.19)_1$ in (5.5), (5.3) yields the desired result:

$$f = [[W]]_{-}^{+} - [[\underset{\sim}{F}]]_{-}^{+} \cdot \overset{+}{\underset{\sim}{\sigma}}, \tag{5.10}$$

where the dot product of two tensors $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is defined to be $\underset{\sim}{A} \cdot \underset{\sim}{B}$ = trace$(\underset{\sim}{A}\underset{\sim}{B}^T)$.

A more illuminating expression for f may be obtained as follows. First, from (5.5), (5.3), (2.19) and (3.13), one finds

$$f = [[W]]_{-}^{+} + \kappa\, (\overset{+}{\underset{\sim}{\sigma}}\underset{\sim}{N} \cdot \underset{\sim}{\ell})\, (\overset{+}{\underset{\sim}{F}}{}^T\underset{\sim}{n} \cdot \underset{\sim}{N}). \tag{5.11}$$

Using (3.10), (2.6) in (5.11) provides

$$f = [[W]]_{-}^{+} + (\overset{+}{\underset{\sim}{\tau}}\underset{\sim}{n} \cdot \underset{\sim}{\ell})\kappa; \tag{5.12}$$

$\overset{+}{\underset{\sim}{\tau}}\underset{\sim}{n} \cdot \underset{\sim}{\ell}$ is the shear traction parallel to the shock, while κ is the amount of the additional shear introduced in the representation (3.13).

Finally, from (5.12) we derive an expression for f that is appropriate for plane deformations of isotropic, incompressible, elastic materials and that does not involve the shock orientation. By $(2.11)_2$, (2.14) and (3.3) we may write (5.12) as

$$f = [[W]]_{-}^{+} + \kappa\, M(k_+)\, \underset{\sim}{\ell}\cdot\overset{+}{\underset{\sim\sim}{Gn}}. \tag{5.13}$$

Using (3.18) to express κ in terms of $\underset{\sim}{\ell}\cdot\overset{+}{\underset{\sim\sim}{Gn}}$ and then eliminating the latter from (5.13) with the help of (4.4), we find

$$f = [[W]]_{-}^{+} + (\overset{+}{I} - g^2 - g^{-2})(M(k_+)/M(k_-))[[M(k)]]_{-}^{+}. \tag{5.14}$$

Writing $\overset{+}{I} = 2 + k_+^2$ and using (4.8) to eliminate g from (5.14) yields

$$f = [[W(2+k^2)]]_{-}^{+} - \frac{M(k_+)\, M(k_-)}{M(k_+) + M(k_-)}(k_+^2 - k_-^2). \tag{5.15}$$

Equivalently, we may write

$$f = H(k_+, k_-) \quad \text{on } C_t\ , \quad t_0 \le t \le t_1\ , \tag{5.16}$$

where H is the function defined on Σ by

$$H(k_1,k_2) = \int_{k_2}^{k_1} \tau(k)\, dk - \frac{\tau(k_1)\tau(k_2)}{\tau(k_1)k_2 + \tau(k_2)k_1}(k_1^2 - k_2^2), \quad (k_1,k_2)\in\Sigma. \tag{5.17}$$

By (5.16), the shock traction f at a point on the shock C_t depends only on the local amounts of shear k_+, k_- associated with the deformations on the two sides of C_t ; in particular, f does not depend on the the orientation of the shock.

Although derived here for plane deformations, the expression (5.16), (5.17) for the shock traction is formally identical to the corresponding result derived by Yatomi and Nishimura for anti-plane deformations (see equation (3.3) of [13]). This is not surprising, in view of the representations (2.3) for plane, area-preserving deformations.

In the special case of a normal shock, we have $\hat{\tau}(k_+) = \hat{\tau}(k_-)$ by (4.15), so that (5.16), (5.17) reduce to

$$f = \int_{k_-}^{k_+} \hat{\tau}(k)\, dk - \hat{\tau}(k_+)\,(k_+ - k_-). \tag{5.18}$$

From (5.18), the value of f in the case of a normal shock may be interpreted geometrically as the difference $A_1 - A_2$, where A_1 is the area under the stress-strain curve between k_- and k_+, and A_2 is the area of the rectangle on the same base with height $\hat{\tau}(k_+)$.

In general, $H(k_1,k_2)=0$ defines a curve lying in the region Σ of the (k_1,k_2)-plane. The shock traction corresponding to points on this curve is zero, and therefore so is the local rate of dissipation. If at some instant, the limiting deformation gradient tensors $\overset{+}{\underset{\sim}{F}}$ and $\overset{-}{\underset{\sim}{F}}$ at a particle on C_t are distinct but such that $H(k_+,k_-) = 0$, we say that this particle is in a "Maxwell state". For a particle which happens to involve a normal shock, the Maxwell states as characterized through (5.18) with $f = 0$ satisfy the well-known "equal-area rule".

Figure 5 shows the curve $H(k_1,k_2)=0$ for a material characterized by a shear stress response function $\hat{\tau}(k)$ of the form shown in Figure 2. For the purpose of drawing the figure, we have used the particular response function given in (4.14). The shaded area indicates points at which H is positive; the unshaded area corresponds to negative values of H. When $\hat{\tau}(k)$ has the form shown in Figure 2, one can show that the maximum and minimum values of H occur on the boundary of the set Σ, i.e. at certain normal shocks.

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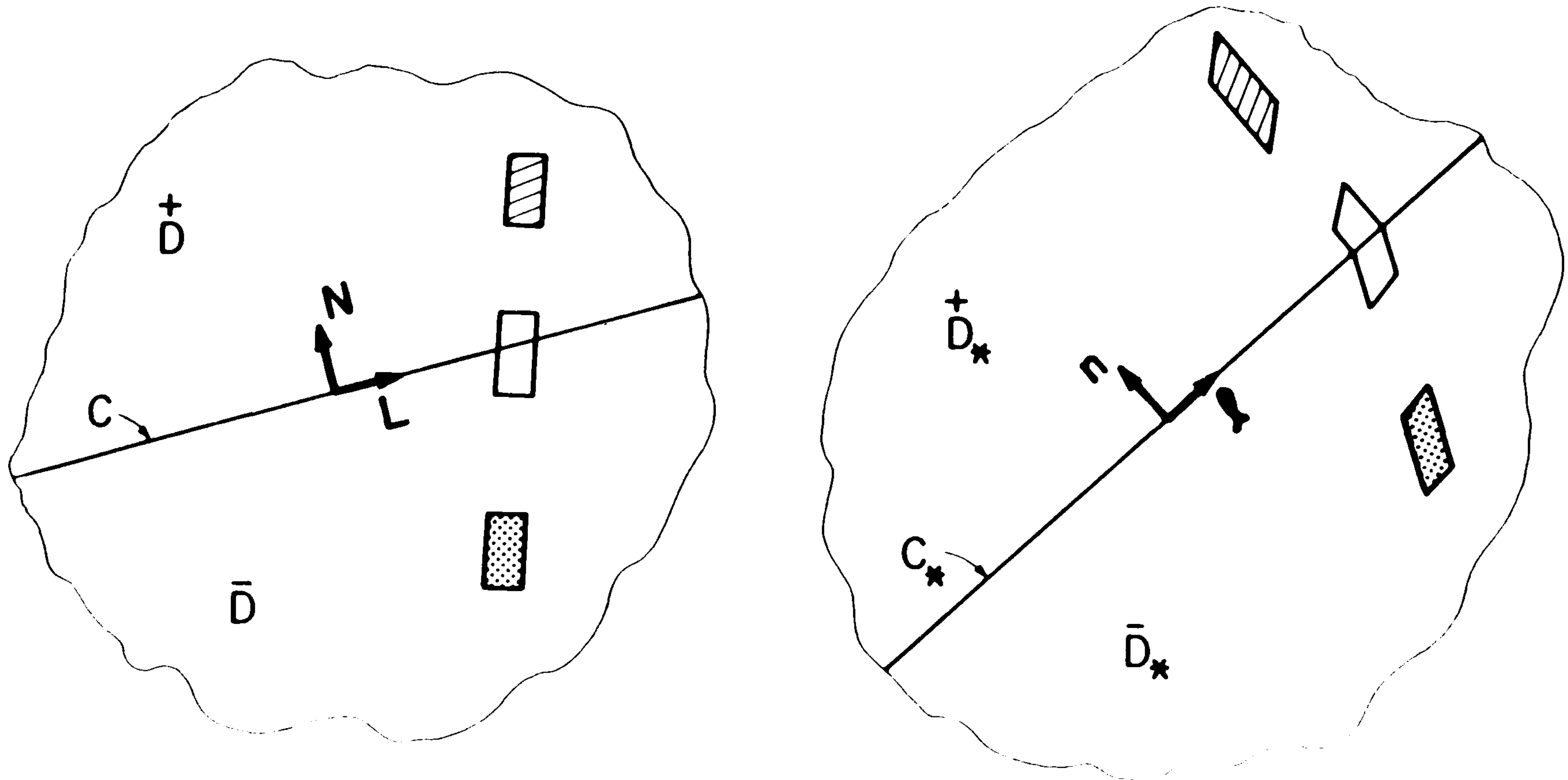

(a) UNDEFORMED BODY (b) DEFORMED BODY

FIGURE 1. PIECEWISE HOMOGENEOUS DEFORMATION

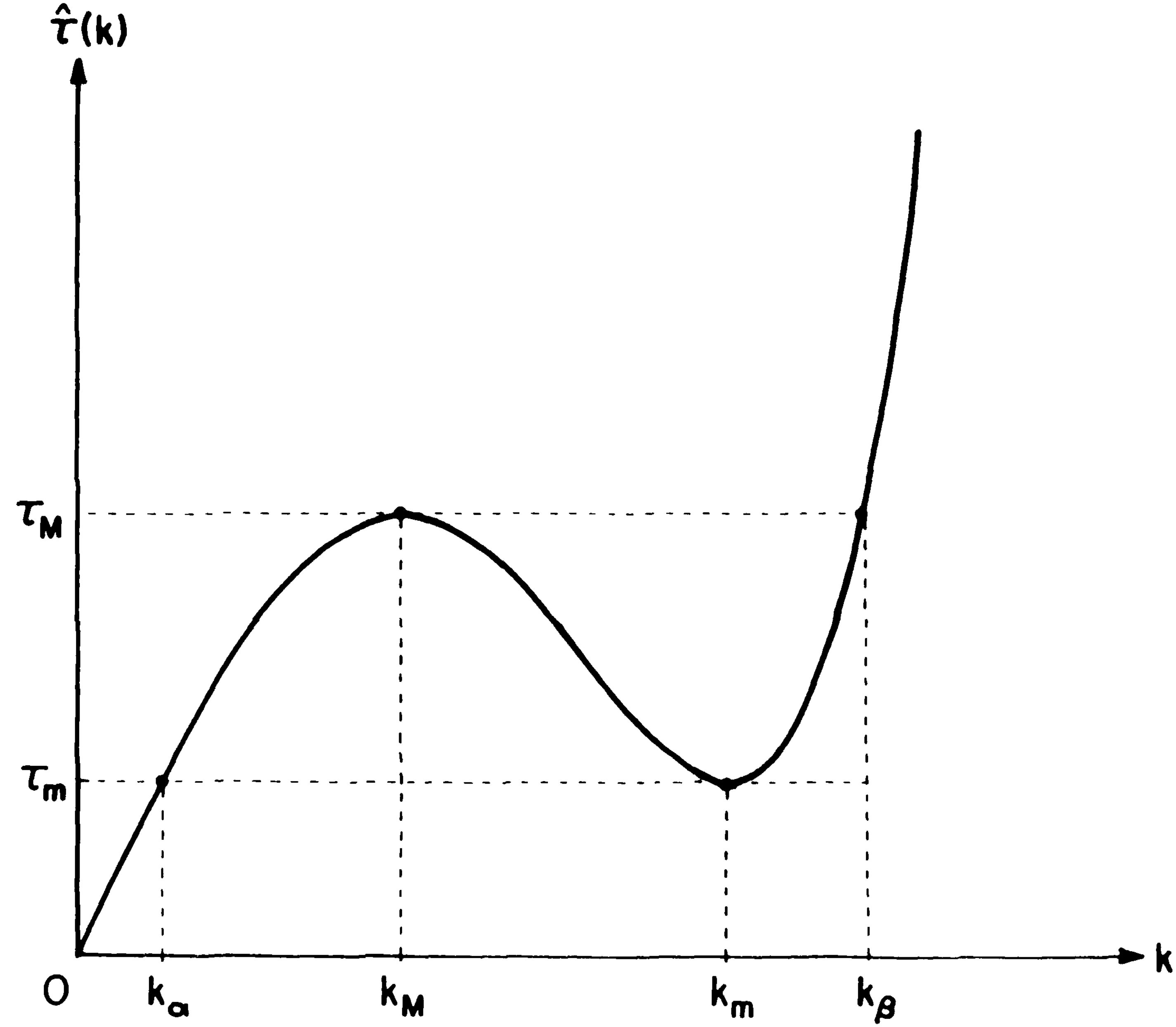

FIGURE 2. RESPONSE CURVE IN SHEAR

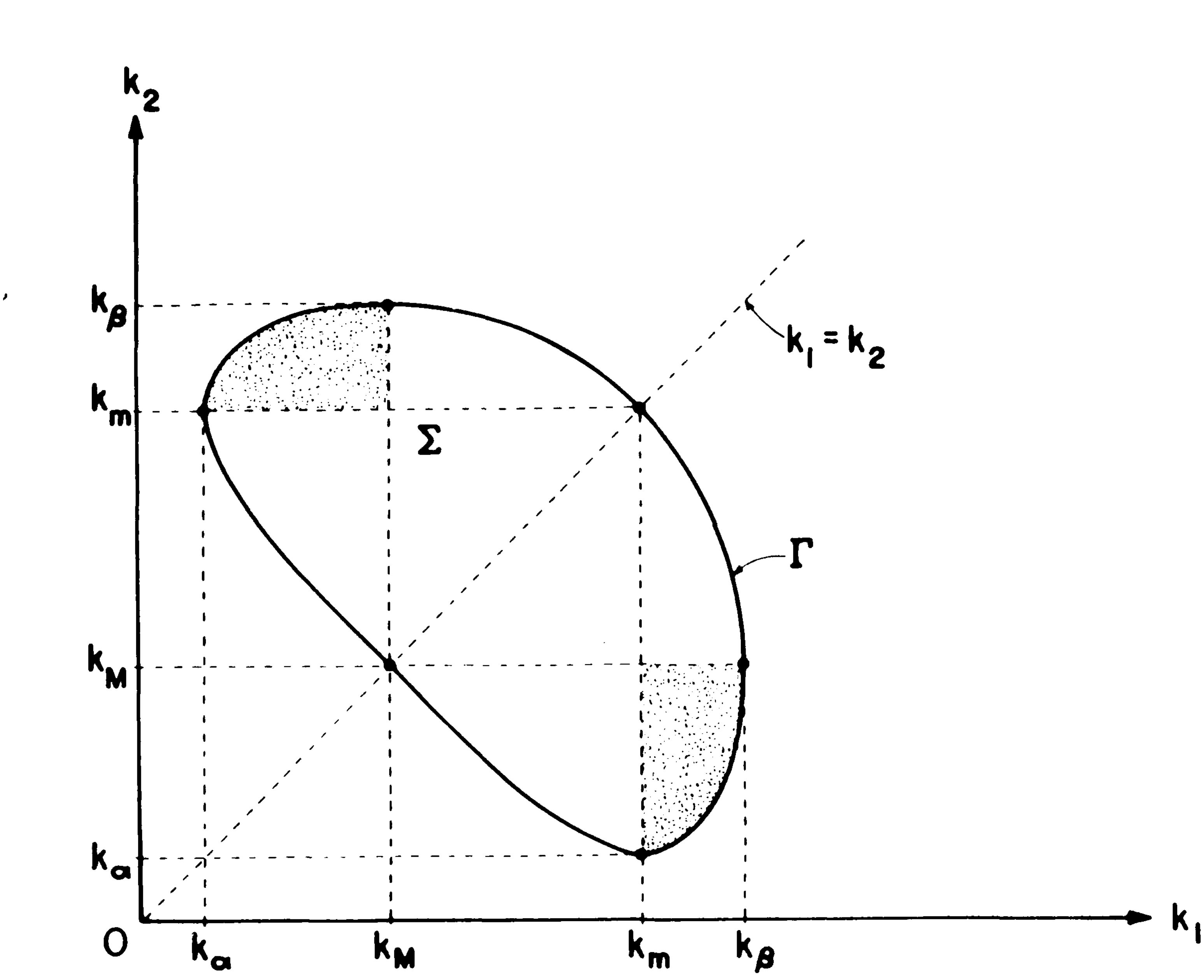

FIGURE 3. THE SET Σ CHARACTERIZING ALL POSSIBLE SHOCK STATES

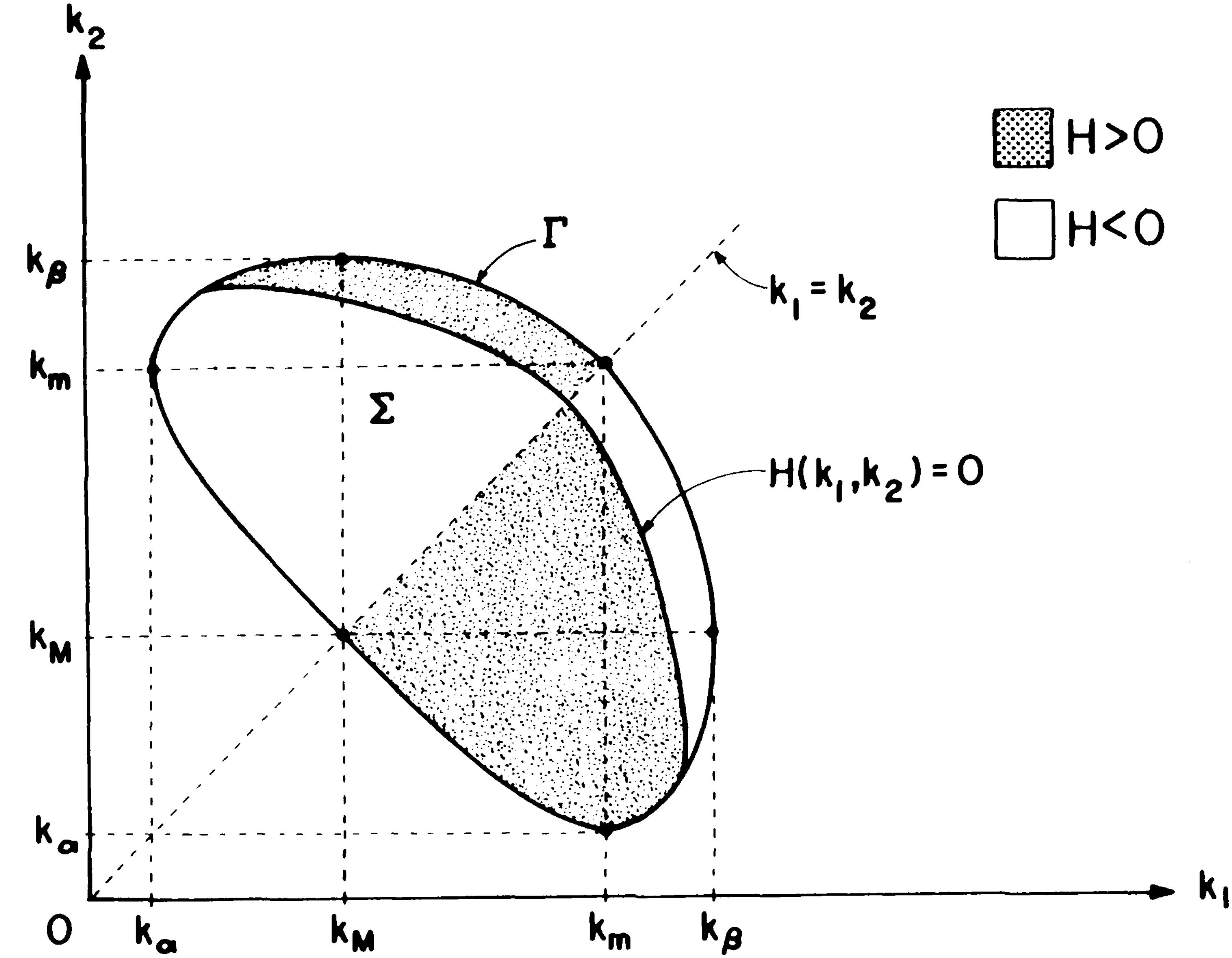

FIGURE 4. REGIONS OF POSITIVE, NEGATIVE AND VANISHING SHOCK TRACTION

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This paper is concerned with piecewise smooth plane deformations in an isotropic, incompressible elastic material. An explicit necessary and sufficient condition for the existence of piecewise homogeneous equilibrium states is established, and the set of all such states is precisely characterized. A particularly simple expression is derived for the "driving traction" on a surface of discontinuity in the deformation gradient.

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