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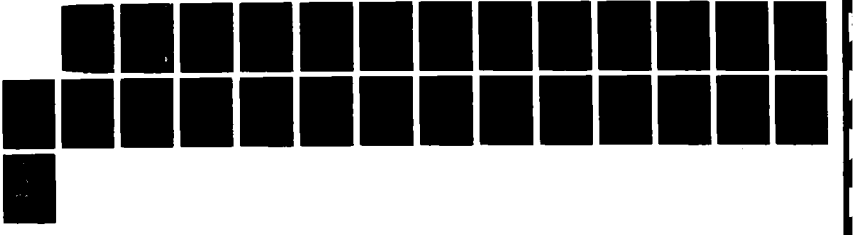

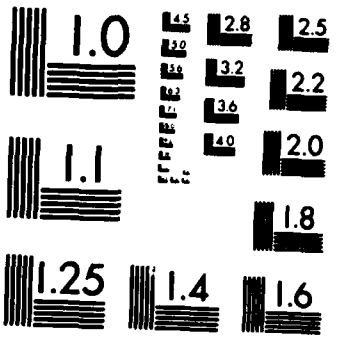

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A Quadratic-Form Analysis of the Collisionless Tearing Mode

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A quadratic-form method is developed to study the collisionless tearing mode properties in the neutral sheet geometry in cases where complicated orbit integrals must be evaluated even at marginal stability. The method is based on a quadratic form of a self-adjoint integro-differential operator describing the linear tearing mode. The equilibrium orbits are included exactly and analytically. The objective of the paper is to provide a rigorous theoretical basis for the method. Application to physical systems is illustrated using an example. This method is particularly suited for problems where the repeated numerical evaluation of the orbit integrals is time-consuming. In such cases, the present method can yield stability criteria with minimal computation.

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CONTENTS

A QUADRATIC-FORM ANALYSIS OF THE COLLISIONLESS TEARING MODE

I. INTRODUCTION

The properties of the collisionless tearing instability in the neutral sheet geometry have been studied extensively.[1-16] A physical system in which the instability may be particularly relevant is the earth's magnetotail.[7,10,13,14] In its classic form [1-3], the inertia of the current carriers leads to the instability. The equations to describe the linear behavior of the instability can be derived using the standard method of characteristics. This yields integro-differential equations for the perturbed quantities. In principle, the solution to these equations gives a complete description of the linear collisionless tearing instability. In practice, however, the equations are difficult to solve because evaluation of complicated orbit integrals is required. Thus, the majority of previous theoretical works have attempted to extract useful information without evaluating the orbit integrals exactly. These methods include energy principles, [3,6,8] the use of approximate orbits, [3,5,7,9,13,14] and expansion of orbit integrals in velocity moments.[10]

However, as the physical system of interest becomes more complex, the approximate methods of evaluating the orbit integrals may become untenable. For example, if ions have an anisotropic bi-Maxwellian distribution with an isotropic electron distribution, then the large ion orbit contribution to the instability is significant.[13] In this case, the neutral sheet has to be treated by a 3-region (as opposed to the conventional 2-region) matching

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technique. The straight-line orbit and constant-ψ approximations limit the validity to relatively small ion anisotropy $T_\perp/T_{||} \lesssim 1.5$. If the distribution function is highly non-Maxwellian, then these approximations completely break down and the full integro-differential equations must be solved. In the problem treated by Holdren[17], the equilibrium orbits and orbit integrals for a relativistic neutral sheet were numerically calculated using discretized variables. The integro-differential equation for the perturbed vector potential was then solved using an iterative scheme and the required numerics were substantial. More recently, Chen and Lee[15] provided an integro-differential treatment of collisionless tearing instability in a highly non-Maxwellian neutral sheet. In this work, the exact unperturbed particle orbits were used analytically in the orbit integrals. Using the Galerkin method,[18] the integro-differential equation for the perturbed vector potential was converted into a finite dimensional matrix and solved to obtain the dispersion relation and the eigenmode structure. It was found that the eigenmode is strongly localized to a region of the order of $(\rho_e x_p)^{1/2}$ at the null plane and that the eigenmode is highly structured in this region, where ρ_e is the electron Larmor radius in the asymptotic magnetic field and x_p is the neutral sheet half-thickness. For $\underline{k} || \underline{B}_o$, the dispersion relation was found to have the general form $\gamma/kv_{e||} = (\rho_e/x_p)D$ where D is a function of temperatures only and $v_{e||}$ is the electron thermal velocity along $\underline{B}_o$, the equilibrium magnetic field. For this problem, the usual straight-line orbit and constant-ψ approximations would indeed have been invalid. Another point of interest to note is that the contribution to the perturbed current density $J_{1y}(x)$ at a given point x primarily comes from those axis-crossing orbits with turning points at x. It is easier to calculate such contribution accurately using analytic orbits than using discretized variables.

In Ref. 15, due to the relative simplicity of the geometry, the orbit integrals and matrix elements could be calculated independently of k and ω. Thus, only minimal numerical calculations were required. However, in more complex systems, the required computation may be more substantial, i.e., the matrix elements may have to be evaluated for each iteration for different values of ω/kv to determine the dispersion relation. In such cases, it would be desirable to determine some overall stability properties without extensive computation. In addition, if unstable, it would be desirable to determine the lower and upper bounds for the growth rate. Such considerations have led to formulation of various energy principle methods. For example, for ideal MHD, the energy principle method[19] seeks to determine the stability of a magnetized plasma by considering the potential energy associated with perturbations of an equilibrium. For the collisionless tearing mode, a number of energy principle analyses have been performed.[3,6,8] These methods generally provide the sufficient condition for stability. In these works, approximate orbit integrals are used, or in certain simple cases, no orbit integral is needed.[8] If, however, no valid approximations are available for the necessary orbit integrals, then these methods do not yield accurate results or estimates of growth rates.

In this paper, we consider another method which is analogous to energy principle methods but which is more versatile. In the present method, we evaluate the orbit integrals using the exact equilibrium orbits and no trial functions are used. It generally yields a sufficient condition with a lower and upper bounds for the growth rate for unstable systems. However, the method can also be iterated to give accurate growth rates for the unstable modes and the eigenfunctions.

The method depends on certain general features of the eigenmodes. One important feature is that the eigenmodes are generally well localized. The technique is potentially applicable to a wide variety of systems. The

basic ideas are simple and we first describe the method. The objective of this paper is to provide a rigorous theoretical basis for the method. For this purpose, we prove a number of general properties of the quadratic form based on a self-adjoint integro-differential operator. We then illustrate the technique by applying it to the collisionless tearing mode in a highly non-Maxwellian neutral sheet that has been solved previously.

II. A Quadratic Form Formulation

In this section, we present the formalism for a quadratic-form method for treating the collisionless tearing instabilities. We first describe the general framework and then provide a specific application. For the sake of concreteness, we will use a neutral sheet geometry. The method can be applied to other configurations.

Figure 1 shows the magnetic field and the coordinate system. The equilibrium magnetic field is given by $\underline{B}_o(x) = B_z(x)\,\hat{\underline{z}}$ with $B_z(-x) = -B(x)$ and $B_z = 0$ at $x = 0$. The current density is $\underline{J}_o(x) = J_o(x)\hat{\underline{y}}$ with $J_o(-x) = J_o(x)$. Because of pressure balance, the particle density $n_o(x)$ and current density $J_o(x)$ are peaked about $x = 0$ with a characteristic half-thickness of x_p. For $|x| > x_p$, B_z is nearly constant and the current density is much smaller than for $|x| < x_p$. The tearing instability in a neutral sheet can be described by perturbations of the form $\psi(x,z,t) = \psi(x)\exp(ikz - i\omega t)$, where the wave vector $\underline{k} = k\,\hat{\underline{z}}$ is taken to be parallel to the equilibrium magnetic field. We assume that all perturbations vanish as $t \to -\infty$. In this paper, we will neglect the scalar potential. The inclusion of the scalar potential gives rise to another coupled integro-differential equation whose basic character is similar to that of the equation to be treated here. Thus, the present method can be generalized to include the scalar potential in a straightforward manner. In reality, the neglect of the scalar potential implies that high perturbation frequencies ($|\omega| \gtrsim \omega_{ci}$)

cannot be treated adequately, where ω_{ci} is the ion cyclotron frequency in the asymptotic magnetic field.

The Ampere's law gives

$$\frac{d^2\psi}{dx^2} - k^2\psi(x) + \frac{4\pi}{c}J_{1y}(x) = 0 \ , \tag{1}$$

where the perturbed vector potential is $\underline{A}_1 \equiv \psi(x,z,t)\hat{\underline{y}}$ and J_{1y} is the perturbed current density. It is in evaluating J_{1y} that one must evaluate orbit-integrals. These integrals may be regarded as linear operators acting on ψ. Thus we can re-write Eq. (1) as

$$L\psi = 0, \tag{2}$$

where L is an integro-differential operator to be determined for each specific problem. We can also define a quadratic form I by

$$I(\phi|\psi) \equiv \int_{-\infty}^{\infty} dx \phi^*(x) L\psi(x) \ , \tag{3}$$

where ϕ^* is the complex conjugate of ϕ. It is clear that Eq. (2) can be recovered by the variational principle requirement $\delta I(\psi|\psi) = 0$, which automatically implies $I(\psi|\psi) = 0$. If we let ϕ_m be an eigenfunction of L corresponding to the eigenvalue λ_m, then we can write

$$\lambda_m = I(\phi_m|\phi_m)$$

$$= -\int_{-\infty}^{\infty} dx \left|\frac{d\phi_m}{dx}\right|^2 - k^2 + \int_{-\infty}^{\infty} dx \phi_m^* L \phi_m, \tag{4}$$

where $\tilde{L}$ is the linear operator describing the orbit integrals. Here, we have used $\phi_m \to 0$ as $|x| \to \infty$ and assumed ϕ_m to be normalized to unity. Note that the eignevalues depend on k and ω and that they are primarily determined by the first and third terms of the above equation. However, Eq. (2) can also be thought of as an eigenvalue equation $L\psi(x) = \lambda\psi(x)$, where $\lambda = \lambda(k, \omega)$. Equation (2) is then equivalent to setting $\lambda = 0$. Thus, the dispersion relation can be obtained from

$$\lambda(k, \omega) = 0. \tag{5}$$

In this paper, we will prove a number of mathematical properties of the eigenvalues λ_m and the integro-differential equation. These properties will be used in a method to determine the stability condition for a given system with respect to the tearing mode. It is an objective of this paper to provide a sound theoretical basis for the method and to illustrate the application.

Following the treatment of Ref. 15, we use an equilibrium distribution function of the form $F_\perp = F_{\perp j}(H_{\perp j} - V_j P_{yj})F_{\|j}(H_{\|j})$, where j = e, i and $(H_{\perp j}, P_{yj}, H_{\|j})$ are the single-particle constants of motion. Here, $H_{\perp j} = (m_j/2)(v_x^2 + v_y^2)$, $P_{yj} = m_j v_y + (q_j/c)A_y^o(x)$ where $A_y^o(x)$ is the equilibrium vector and $H_{\|j} = (m_j/2)\, v_z^2$. For simplicity, we will choose $F_\|$ to be Maxwellian with

$$F_{\|j}(H_\|) = (2\pi T_\|/m_j)^{-1/2}\exp(-H_\|/T_\|). \tag{6}$$

For the species j, m_j is the mass, q_j is the charge and V_j is the mean drift velocity in the y direction. In the remainder of the paper, we will omit the species index j when no confusion results. Then, it can be

shown[15] that the left-hand side of Eq. (1) can be written as an integro-differential operator L given by

$$L\psi = \frac{d^2\psi}{dx^2} - k^2\psi + \frac{4\pi e^2}{c}\sum_j \beta_j \psi(x) \int d^3\underline{v} v_y \frac{\partial F_\perp}{\partial H_\perp}$$

$$+ \frac{4\pi e^2}{c}\sum_j \frac{1}{m_j^2 c} \int dx' \left(\frac{1}{T_{\|j}} K_1(x,x') + K_2(x,x')\right) \psi(x') \ . \qquad (7)$$

Here, the kernels are given by

$$K_1(x,x') = \sum_n \int \frac{dP_y dH_\perp}{T(H_\perp, P_y)} W(b_n) F_\perp \frac{v_y(x) v_y'(x')}{|v_x(x) v_x'(x')|} \exp\left[-in\Omega(t-t')\right] \qquad (8)$$

and

$$K_2(x,x') = \sum_n \int \frac{dP_y dH_\perp}{T(H_\perp, P_y)} \left(1 + \frac{n\Omega}{kv_{j\|}} Z(b_n)\right) \frac{\partial F_\perp}{\partial H_\perp} \frac{v_y(x) v_y'(x')}{|v_x(x) v_x'(x')|} \exp\left[-in\Omega(t-t')\right] \qquad (9)$$

where the orbital period is given by

$$T(H_\perp, P_y) \equiv \oint \frac{dx}{|v_x(x)|} \ , \qquad (10)$$

and

$$b_n \equiv (\omega + n\Omega)/kv_{j\|} \ .$$

Here, the integration is carried out over one complete cycle. The coordinates $(\underline{x}', \underline{v}')$ represent the particle orbits in the equilibrium field with the conditions $\underline{x}'(t' = t) = \underline{x}$ and $\underline{v}'(t' = t) = \underline{v}$. The dispersion functions are given by

$$Z(\xi) = \pi^{-1/2} \int_{-\infty}^{\infty} dt \ \frac{\exp(-t^2)}{t - \xi}$$

and

$$W(\xi) = 1 + \xi Z(\xi).$$

In the above expressions, $v_{j\parallel}$ is the thermal velocity of the species j associated with the motion parallel to the equilibrium magnetic field. In deriving Eqs. (7), (8) and (9), we have used

$$v_x(x')\psi(x') = \sum_{n=0}^{\infty} \Phi_n(x)\exp\,[in\Omega(t - t')] \,. \tag{11}$$

with

$$\Phi_n(x) \equiv \frac{1}{T}\int_0^T d\tau[v_x(x')\psi(x')]\exp[-in\Omega(t - \tau)] \tag{12}$$

where $T = T(H_\perp, P_y)$ and

$$\Omega = 2\pi/T(H_\perp, P_y),$$

with the orbital period T given by Eq. (10).

In writing Eq. (11), use has been made of the fact that an equilibrium orbit can be thought of as a function of the present coordinates at $t' = t$. The kernels K_1 and K_2 generally must be evaluated numerically. However, there is no need to follow equilibrium orbits. The velocity components can be evaluated analytically as functions of $H_\perp$ and P_y. In doing the velocity space integration, all the equilibrium orbits are included exactly.

It is easy to see that the third term on the right hand side of Eq. (7) is the so-called adiabatic term. The third and fourth terms combined give the perturbed current density. In deriving Eqs. (8) and (9), we have

included all harmonics of the orbital periods. However, the conventional tearing mode corresponds to the n = 0 case. In this paper, we will also set n = 0 henceforth for simplicity and restrict ourselves to the purely growing mode, $\omega = i\gamma$.

III. A Non-Maxwellian Neutral Sheet

If we set n = 0 and $\omega = i\gamma$, the operator L is self-adjoint since it is real and symmetric in x and x'. Then, its eigenvalues are real and eigenfunctions can be chosen to be real and orthonormal without loss of generality. Let $\phi_m(x)$ be such an eigenfunction with the eigenvalue λ_m. Then,

$$I(\phi_m|\phi_m) = \lambda_m$$

$$= -\int dx \left(\frac{d\phi_m}{dx}\right)^2 - k^2 + \frac{4\pi e^2}{c}\sum_j \beta_j \int dx \phi_m^2(x) \int d^3\underline{v} v_y \frac{\partial F_\perp}{\partial H_\perp}$$

$$+ \frac{4\pi e^2}{c}\sum_j \frac{1}{m_j^2 c} \int dx dx' \phi_m(x)\left[\left(\frac{1}{T_{\|j}}\right)K_1(x,x') + K_2(x,x')\right]\phi_m(x'), \qquad (13)$$

where we have used $\phi_m \to 0$ as $|x| \to \infty$. Note that the dependence on m occurs only through the eigenfunctions ϕ_m.

We will demonstrate the following properties. We will first show that the eigenvalue λ_m increases without bounds as γ approaches $-\infty$ for all m and k. Then, we will show that λ_m approaches an asumptotic value for each k and m as γ approaches $+\infty$ and that the asymptotic values are bounded from above but not from below. We then prove that, for any finite value of γ, there exist only a finite number of m values for which λ_m is positive.

We now prove that $\lambda_m \to +\infty$ for all m as $\gamma \to -\infty$. First, we consider an analytic continuation of Eq. (13) to $\gamma < 0$. The first three terms of Eq. (13) have no explicit dependence on γ. Equation (9) shows that the K_2 term also has no γ dependence if n = 0. Equation (8) shows that K_1 depends on γ through the dispersion function W. For large negative γ, we have the asymptotic expression

$$W\left(\frac{i\gamma}{kv_{j\|}}\right) \sim -2\pi^{1/2}\gamma \exp(\gamma^2).$$

Therefore, as $\gamma \to -\infty$, we have $W \to +\infty$. In addition, K_1 can be re-written as

$$K_1(x,x') = W(b_o) \int \frac{dP_y dH_\perp}{T} \left[\int dx \phi_m(x) \frac{v_y(x)}{|v_x(x)|}\right]^2$$

where $b_o = \omega/kv_{j\|}$. Since T is positive definite, the integral is positive definite. Thus, we see $K_1(x,x') \to +\infty$ in the limit $\gamma \to -\infty$. Because all other terms in Eq. (13) are independent of γ, we conclude that

$$\lim_{\gamma \to -\infty} \lambda_m = +\infty \tag{14}$$

for all m. In the other limit $\gamma \to +\infty$, we see that $Z(b_o) \to 0$ and $W(b_o) \to +1$ so that, for a given k, λ_m reaches an asymptotic value for each m. In practice, the dependence of γ/kv becomes unimportant for $\gamma/kv > 4$ where $Z(4i) \tilde{=} 0.1$. The large-γ asymptotic expression for λ_m then becomes

$$\lambda_m \sim -\int dx \left(\frac{d\phi_m}{dx}\right)^2 - k^2 + \frac{4\pi e^2}{c} \sum_j \beta_j \int dx \phi_m^2 \int d^3\underline{v} v_y \frac{\partial F}{\partial H_\perp}$$

$$+ \frac{4\pi e^2}{c} \sum_j \frac{1}{m_j^2 c T_{j\parallel}} \int \frac{dP_y dH_\perp}{T} F_\perp \left[\int dx \frac{v_y \phi_m(x)}{|v_x|} \right]^2$$

$$+ \frac{4\pi e^2}{c} \sum_j \frac{1}{m_j^2 c} \int \frac{dP_y dH_\perp}{T} \frac{\partial F_\perp}{\partial H_\perp} \left[\int dx \frac{v_y \phi_m(x)}{|v_x|} \right]^2 \qquad (15)$$

It is clear that the asymptotic values of λ_m may be positive or negative. It is of interest to note that if $F_\perp$ and $\partial F_\perp / \partial H_\perp$ are integrable at least in the δ-function sense, then the asymptotic values of λ_m are bounded from above. However, they need not be bounded from below since it is possible to make the integral of $(d\phi_m/dx)^2$ arbitrarily large by introducing "wiggles". Since $\lim_{\gamma \to -\infty} \lambda_m \to +\infty$ for all m, and λ_m can be negative for some m as $\gamma \to +\infty$, it is clear that some eigenvalues must cross zero. If the zero-crossings occur for $\gamma > 0$, then the eigenfunctions corresponding to $\lambda = 0$ provide the solutions to the tearing mode equation (2). The values of γ where the crossings occur are the corresponding growth rates.

We now demonstrate an important property that for any finite γ, negative or positive, there are only finitely many positive eigenvalues λ_m. For any given k and a finite γ, it is easy to see that only the first term of Eq. (13) can be infinite. Thus, it is sufficient to show that there are only finitely many m's for which the quantity $\int dx (d\phi_m/dx)^2$ is finite.

Let $\{u_k(x)\}$ be a complete set of basis functions. Then, we can write

$$\phi_m(x) = \sum_{k=1}^{\infty} a_{mk} u_k(x). \qquad (16)$$

It is important to consider the nature of the eigenfunctions. We note that Eq. (1) can be cast in the form of a Schroedinger equation with $-k^2$ playing the role of the energy of a particle in a potential well $-(4\pi/c)J_{1y}$. For

the tearing mode problem at hand, the perturbed current is sharply peaked at the magnetic null plane ($x = 0$). Thus, we seek bound state solutions of Eq. (1) localized around $x = 0$. As a general remark, we note that the equilibrium distributions are such that the current and particle densities are localized near $x = 0$ with a characteristic half-thickness x_p as discussed earlier. Then, in the operator L, only the first two terms are important [Eq. (7)]. For $|x| \gg x_p$, the current density is essentially zero and the perturbed eigenfunction vanishes exponentially. As a simple model with these features, we use[15]

$$F_{\perp j} = (\hat{mn}_0/2\pi)\ (H - V_j P_y - T_{j\perp})$$

where V_j and T_j are constants and $n_0 = n_0(0)$. This distribution is such that $n_0(x)$ and $J_0(x)$ are constant for $|x| < x_p$ and zero for $|x| > x_p$ for each species, where

$$x_p^2 = \frac{mV^2 + 2T_\perp}{4\pi\hat{en}_o(V_i - V_e)(qV_j)/c^2}$$

Here, q is the charge of the species. For this distribution function, Eq. (1) can be solved exactly outside the plasma sheet ($|x| > x_p$) to give

$$\psi = \psi(x_p)\exp[-k(|x|-x_p)].$$

Thus, we can write

$$\int_{-\infty}^{\infty} dx \left(\frac{d\phi_m}{dx}\right)^2 = \int_{-x_p}^{x_p} dx \left(\frac{d\phi_m}{dx}\right)^2 + k\phi_m^2(x_p).$$

We define

$$R_m \equiv \int_{-x_p}^{x_p} dx \left(\frac{d\phi_m}{dx}\right)^2 \qquad (17)$$

Then, λ_m is finite if and only if R_m is finite. For simplicity, we treat ϕ_m as if it were entirely confined in the region $(-x_p, x_p)$. Then, the Fourier components for even ϕ_m are simply

$$u_k(x) = d^{-1/2}\cos\left(\frac{k\pi}{d}x\right). \tag{18}$$

Substituting Eq. (18) into Eq. (17), we obtain

$$R_m = \left(\frac{\pi}{d}\right)^2 \sum_{k=1}^{\infty} k^2 a_{mk}^2. \tag{19}$$

The orthonormality of the eigenfunctions ϕ_m requires

$$\sum_{k=1}^{\infty} a_{mk}^2 = 1, \tag{20}$$

for all m. In addition, we have

$$\sum_{m=1}^{\infty} a_{mk}^2 = 1, \tag{21}$$

for all k. This can be seen by noting that the matrix $A = \{a_{ij}\}$ is such that $AA^T = I$ from the orthonormality of eigenfunctions ϕ_m. Since A is a square matrix, A^T is the inverse of A, i.e., A is an orthogonal matrix. Then, Eq. (21) follows. From Eqs. (19) and (20), we see $R_m \geq (\pi/d)^2$ for all m with $R_m = (\pi/d)^2$ if and only if $a_{m1} = 1$ and $a_{mk} = 0$ for all $k \neq 1$.

We now prove that there exist only finitely many m for which R_m is finite. Consider a set S of all finite R_m. Define the following quantities

$$R_m^N \equiv \sum_{k=1}^{N} k^2 a_{mk}^2$$

and

$$Q_N \equiv \sum_{k=N+1}^{\infty} k^2 a_{mk}^2,$$

so that $R_m = R_m^N + Q_N$. Because the sequence R_m^N converges to R_m as $N \to \infty$, we see that for any $R_m \in S$ and any arbitrary $\varepsilon > 0$, there exists a finite integer N_m such that $Q_N < \varepsilon$ for all $N > N_m$. Now, consider a finite subset S_M of S such that S_M has M elements. Let $\langle R_m \rangle_M$ be the average value of the elements of S_M. Then, by denoting the elements of S_M by m = 1,... M, we have

$$\langle R_m \rangle_M = \frac{1}{M} \sum_{m=1}^{M} R_m$$

$$= \frac{1}{M} \sum_{m=1}^{M} \sum_{k=1}^{N} k^2 a_{mk}^2 + \frac{1}{M} \sum_{m=1}^{M} Q_N.$$

Here, for any $\varepsilon > 0$, N is chosen such that Q_N is less than ε for all R_m in S_M. Interchanging m- and k- summations and summing over m to infinity, we obtain

$$\langle R_m \rangle_M < \frac{1}{M} \sum_{k=1}^{N} k^2 \sum_{m=1}^{\infty} a_{mk}^2 + \langle Q_N \rangle_M$$

where $\langle Q_N \rangle_M$ is the average over S_M. Then, using Eq. (21) and carrying out the k-summation, we have

$$\langle R_m \rangle_M < \frac{1}{M} \frac{N(N-2)(2N-1)}{6} + \langle Q_N \rangle_M \qquad (22)$$

for all R_m in S_M. Clearly for any ε, it is easy to show $\langle Q_N \rangle_M < \varepsilon$ by induction. We can choose $\varepsilon < (\pi/d)^2$. Now, we suppose S has infinitely many elements. Then, we can take the limit $M \to \infty$. Recall that $R_m \geq (\pi/d)^2$ for all m.

Therefore, $\lim_{M\to\infty} \langle R_m \rangle_M \geq (\pi/d)^2$. However, the inequality (22) shows that

$$\lim_{M \to \infty} \langle R_m \rangle_M < \langle Q_N \rangle_M < \varepsilon < (\pi/d)^2$$

This is a contradiction due to the supposition that M is infinite. Therefore, S cannot have infinitely many elements.

The properties of λ_m described above suggest a practical method for assessing stability of a neutral sheet when detailed computation is time-consuming. First, compute the positive eigenvalue spectrum at $\gamma = 0$. Suppose there are n_1 positive eigenvalues. Next, compute the positive eigenvalue spectrum at a large $\gamma = \gamma_2$ with n_2 positive eignevalues. If n_1 and n_2 are not equal, then there are at least $|n_2 - n_1|$ values of $\gamma > 0$ for which Eq. (2) is satisfied. That is, there are at least $|n_2 - n_1|$ unstable modes. This method would be particularly useful when evaluation of kernels for each value of k and ω is numerically prohibitive. In cases where some numerical computation is still practical, the above procedure can be iterated by considering successively smaller intervals $\Delta\gamma = (\gamma_2 - \gamma_1)$. From Eqs. (3) and (17), we see that there can be only a finite number of m such that λ_m is finite for finite γ. In particular, there can be only finitely many positive λ_m for $\gamma = 0$. There are infinitely many m such that λ_m's cross zero for $\gamma < 0$. That is, there are infinitely many stable modes and only a finite number of stable modes. It is also clear that if the dimensionality of the matrix is large enough, the values of the positive eigenvalues are not significantly affected by the dimensionality of the matrix. That is, we only need to solve a finite dimensional eigenvalue

equation. Thus, Eq. (2) can be reduced to a finite dimensional matrix equation to find all the unstable modes, if the system is unstable. If $\Delta\gamma$ can be made sufficiently small, then we obtain the growth rate and the eigenmode structure accurately.

IV. Summary and Discussion

We have described a quadratic-form method which may be useful for studying the linear collisionless tearing mode stability of a neutral sheet in cases where the complicated orbit integrals make it impractical to calculate the eigenmode structures. We have provided the necessary mathematical justification. This method may be iterated, where practical, to provide a necessary and sufficient condition regarding the stability of a complicated system and, if unstable, the growth rates.

In this paper, we have used a mathematically tractable distribution function. However, the basic conclusions of the paper are not limited to the idealized model. In the example given above, the kernels K_1 and K_2 depend on the standard dispersion functions Z and W because of the Maxwellian distribution in $H_{||}$. As a result, the asymptotic behavior of K_1 and K_2 for $\gamma \to \pm \infty$ can be determined readily. Similar asymptotic behavior occurs in other systems in which the distribution functions are not Maxwellian in $H_{||}$. It has been shown that the behavior of the eigenvalues λ_m is determined by the quantity R_m [Eq. (17)]. We have shown that R_m is finite only for a finite number of m. This behavior is due to the fact that eigenfunctions ϕ_m are localizable to a finite region, which in this case, is the magnetic null region. Thus, the properties of $I(\phi_m|\phi_m)$ which are necessary for the method can be generalized to a wide variety of systems. We expect that the present quadratic-form technique can be used to analyze more realistic physical systems. In this paper, we have shown how the necessary properties must be assessed for application to a given system.

Acknowledgments

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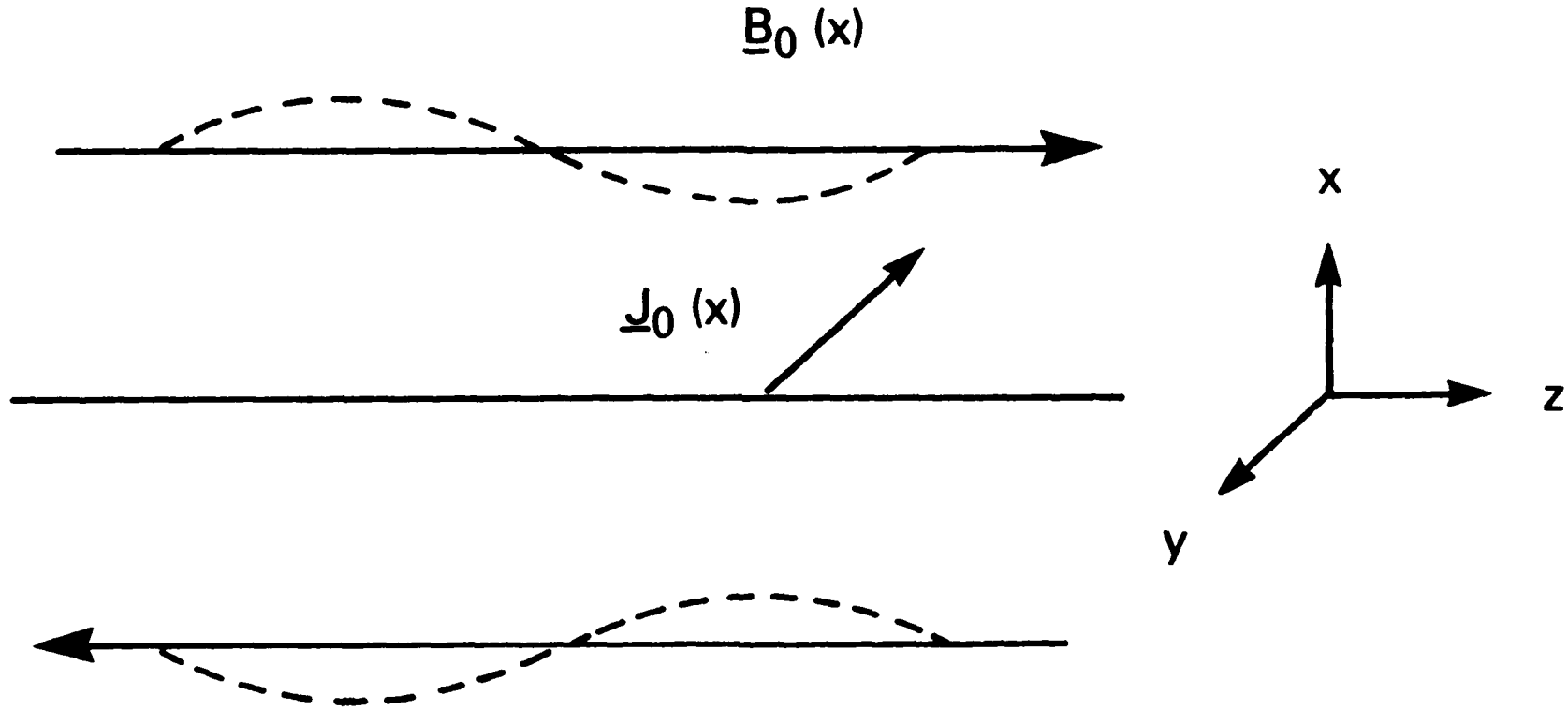

Fig. 1 Schematic drawing of a neutral sheet and the coordinate system. The dashed lines describe the magnetic field including the perturbation.

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