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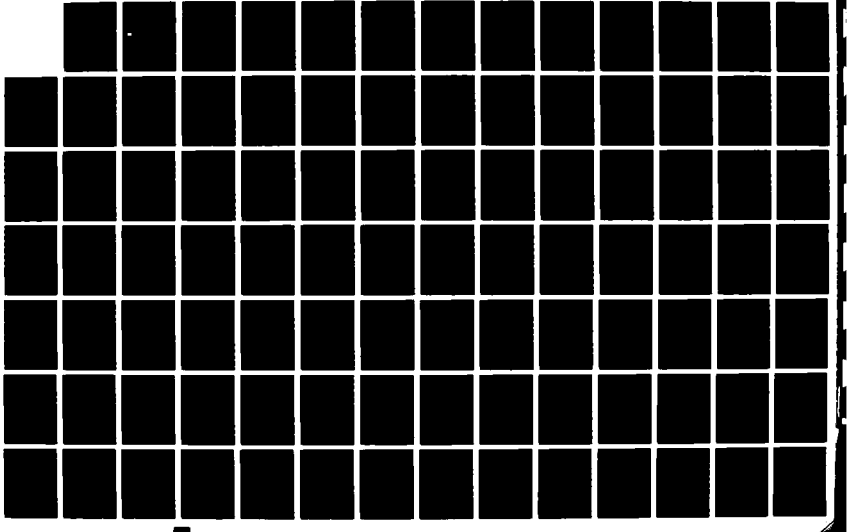

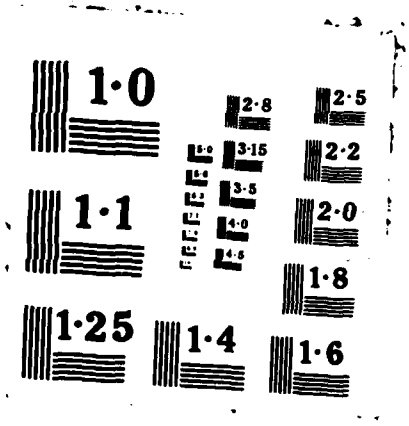
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MANY-BODY TREATMENT OF NAVIER-STROKES FLUIDS

R. J. Becker

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September 1987

Final Report

AIR FORCE WEAPONS LABORATORY
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A Lagrangian has been developed that is equivalent to the full Navier-Stokes (NS) equation for a three-dimensional, subsonic single-component fluid, including viscous pressure gradients and advective terms. Dissipation is incorporated into the Lagrangian by using hypercomplex fields for the velocity potentials. This Lagrangian has been used to derive a field-theory description of fluid flow based on a diagonalized Hamiltonian and the corressponding Poisson-bracket relations. Green's functions for the linearized system and rules for drawing diagrams have been worked out. Perturbation expansions based on the linearized Hamiltonian converge as the Mach number, rather than as the Reynolds number as in earlier attempts to formulate Hamiltonians for the NS equation. This is achieved by expressing the Hamiltonian in terms of the velocity potentials, rather than directly in terms of the velocity fields. The interaction terms in the diagonalized Hamiltonian are of the same form as that for the electron-phonon interaction in quantum field theory.

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PREFACE

This report describes the development of a new approach to modeling turbulent flow. Heat transfer in a fluid is also addressed. This effort is in response to an initiative from Kirtland Air Force Base calling for novel methods of modeling incompressible turbulent flow in convectively cooled systems. The approach taken has been to derive a Hamiltonian which preserves all of the physics contained in the Navier-Stokes equation, and to diagonalize this Hamiltonian (i.e., transform the variables to variables which are preserved with respect to the Hamiltonian). This Hamiltonian can be used as the starting point for a number of different calculations. For example, it can be used to obtain a partition function that will be used to describe the statistical properties of a turbulent system. Indeed, this is an approach favored by the author. The Hamiltonian can also be used as the basis of the newer class of calculations referred to as "Renormalization Group" and "Chaos" descriptions. Hence, it may be used in models which are sometimes referred to as "deterministic" as well as those which are known as "probabilistic." This is natural, since the Hamiltonian itself does contain the same physics as the Navier-Stokes equation.

Rationale

The rationale for the adopted approach is that it has worked well in other areas of physics which share with turbulence the properties of nonlinearity and a large number of degrees of freedom. These are the key features of what are known as many-body systems. Examples of systems in which the field theory or many-body approach has proved extremely useful are nuclear physics, high-energy physics, quantum-electrodynamics, and condensed matter physics. In all of these

fields, the Hamiltonian is separated into the sum of a linear, basis part, and a nonlinear, "perturbing" part. Solutions for the full, exact system are expressed in terms of integrals involving solutions to the linearized system. Usually this involves some form of perturbation theory, which immediately raises the question of how rapidly a perturbation series will converge, or even whether it will converge at all. This is especially true of turbulent flow, in which closure of equations involving moment expansions of the velocity components is a common problem. In the above areas, the representation of the complicated integrals which arise in perturbation series by simple diagrams has been a great aid in summing perturbation series, sometimes to infinite order. This has worked even in cases in which the series converges very slowly, or even diverges. The Hamiltonian described in this report has perturbations which go as the Mach number, and so perturbation expansions are expected to converge well for subsonic flow.

The usefulness of diagrams to represent complicated integrals goes beyond its great simplification of a bookkeeping function. The diagrams have a psychological appeal that aids the physical intuition of the theoretician in his formulation of a particular calculation involving interactions between various degrees of freedom of the system (e.g., fluid elements, or normal modes of the linearized system). In the above fields, the various degrees of freedom of the system are throught of as particles, and interactions are described in terms of scattering events between particles. For example, an oscillation of a sound field at a particular frequency may be regarded as a phonon. This viewpoint gives a deeper meaning to the term many-body system. We see then that a many-body description of a problem is helpful in performing diagram calculations of perturbation expansions and highlights parallels with other systems which have

been extensively analyzed. Furthermore, it provides a means of making a generic analysis, since calculations are made in terms of the normal modes of the linearized system, without actually specifying explicitly what those modes are. The boundary conditions and explicit solutions are included at the end of a generic calculation. This feature may facilitate comparisons of the behavior of systems with different geometries and flow conditions.

Overview

The various aspects of treatment of a system based on field theory are shown in block form in Figure 1. Most calculations begin with the full Hamiltonian, H, as a starting point. The Hamiltonian is expressed in terms of fields which are the solutions to the partial differential equation of interest, in this case the Navier-Stokes and heat budget equations; hence the name field theory. Note that at the outset field theory gives a generic description. The fields are the wavefunctions, or solutions to the partial differential equation; calculations are done in terms of these wavefunctions without explicitly solving for them. Many calculations can be done in terms of a Poisson bracket (P.B.) formulation. Poisson bracket calculations have the advantage of being independent of a particular coordinate system, and are very powerful. The above mentioned many-body systems are all quantum systems, and the analogous equations to P.B. relations in those systems are commutator relations.

It is emphasized that the formulation described in this report is entirely classical. At no point is a quantum description used. Although the formulation closely parallels field theory descriptions of quantum systems, such as the use of Poisson brackets instead of commutators, the integrity of a fully classical description is preserved throughout, and leads to significant variances from the form of quantum calculations.

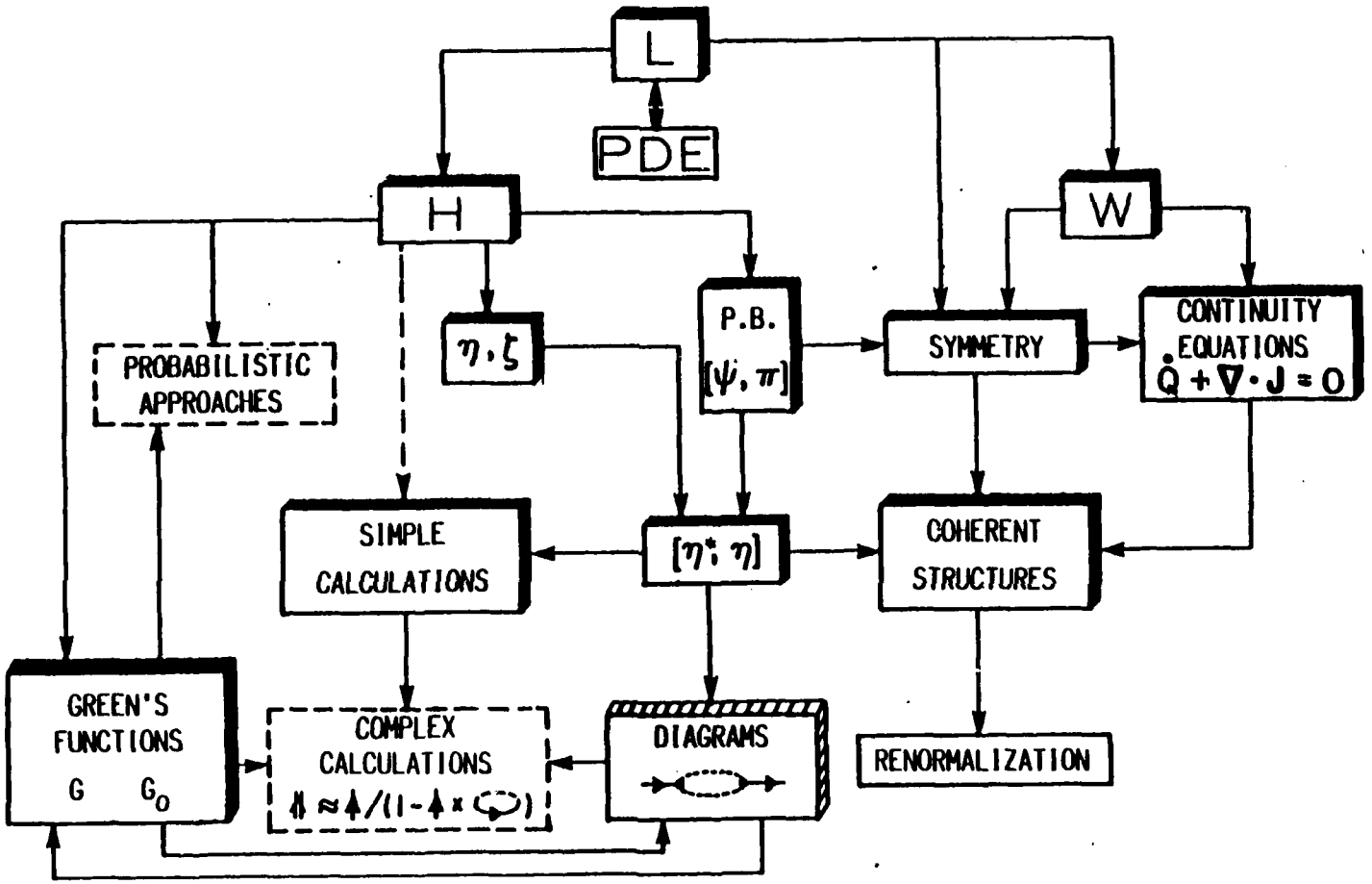

Figure 1. Block Diagram of the Relationships Between Various Aspects of Field Theory.

Often the initial wavefunctions and related fields known as conjugate momenta (denoted by ψ and π in Figure 1) are transformed to new field variables which are said to diagnonalize the Hamiltonian. In quantum systems these new variables become creation and annihilation operators. In the present formulation, they are simply Fourier coefficients, and are represented by η and ζ in Figure 1. Poisson bracket calculations can be made in terms of the diagonalized fields as well as the initial fields. In many cases calculations are easier to perform in terms of the new diagonalized field variables.

Using the Poisson bracket relations, an equation of motion for any field variable can be derived from the Hamiltonian. Many calculations are based on these Hamiltonian equations of motion, which are fully equivalent to the origin partial differential equation (e.g., Navier-Stokes equation) to be solved. An alternative, an extremely powerful procedure, is to find the Green's function, G, corresponding to H. If the initial partial differential equation is thought of as an operator, O, operating on the field variables, ψ, then G plays the role of the inverse operator to O. The Green's function G to the full Hamiltonian, H, can be derived from a perturbation expansion about the Green's function G_o to the linearized problem. G_o is easily derived from the linearized Hamiltonian, H_o, so that a procedure for solving a complex problem can be built up systematically beginning with the solution to a simpler problem. Green's functions are central quantities in field theory. From them the densities of states, densities of excitations, transition probabilities, lifetimes of excitations, dispersion relations, total energy, equation of state, and partition function can be derived.

Benefits

One of the hallmarks of the field theory approach is that the development of the entire theory is completely systematic, once the true starting point, the development of a functional L known as the Lagrangian (from which H can easily be derived) has been attained. Thus theoreticians skilled in field theory will know immediately how to perform relevant calculations similar to those done in quantum systems, given the appropriate forms for L and H. Novices need not learn quantum field theory to perform calculations. They can begin at once with the rules for drawing and summing diagrams. Indeed, there is a modern trend toward teaching quantum field theory beginning with diagrams, without any reference to a Lagrangian, or a deep discussion of the theory. Hence many-body calculations can be made accessible to a wide community. This is in keeping with the goal of the research described in this report: to facilitate accurate calculations of the properties of real turbulent flow fields in practical systems.

Notwithstanding the main goal of a practical calculational tool, it is worth noting that the Lagrangian can be analyzed to obtain a deeper insight into both the properties of a many-body system and the character of the theory. Given the Lagrangian, the stress-energy tensor, W (the stress-energy tensor is usually denoted by T in quantum field theory), can be derived; and the symmetry properties of L and W can be analyzed to obtain all the conservation laws for the system. Examples in fluid mechanics are the conservation of mass, energy, and momentum. Note that these conservation laws are not postulated in addition to the Navier-Stokes equation, but can be systematically derived directly from L, i.e., a collection of initial postulates is not needed to describe the

system, only the initial Lagrangian and a systematic procedure for developing the theory. This feature gives the theory a strong foundation.

Equations of motion for the fields can be derived from the Lagrangian by postulating that the variation in the functional L will be a minimum, and that the fields are fixed at the boundaries. Then integration by parts and the calculus of variations lead to Euler-Lagrange equations of motion (see Appendix A). Hence the true starting point for the development of a field theory should be the correct formulation of the Lagrangian. This task has been accomplished in the program and is the most important result of the effort.

The value of deriving a Lagrangian for fluid motion has long been recognized, and there have been several previous attempts at this task. Those efforts were largely unsuccessful, primarily because of the difficulty involved in accounting for dissipation, but also because of the way in which the pressure gradient term was treated. The present theory, which we now summarize, departs from earlier efforts in both respects.

Summary

The technical description is divided into three parts, which follow. In Part I, the Lagrangian is presented and an analysis is made of the quadratic terms, which correspond to the linear (noninteracting) system. The Lagrangian is expressed in terms of the scalar and solenoidal velocity potentials, Ψ and $\mathcal{A}$, rather than directly in terms of the velocity. This approach highlights the natural symmetries of the system and brings in the pressure gradient term without introducing an extra field. Velocity is accounted for by introducing adjoint fields for the velocity potential. These adjoint fields would grow rather than damp with time if they were complex conjugates of the initial

fields. To avoid this difficulty, the adjoint fields may be considered to be hypercomplex.

It should be noted that the innovation of using hypercomplex adjoint fields to describe terms of odd order in the time and/or spatial derivatives of an equation of motion is applicable to the solution of a wide class of partial differential equations, including equations for passive scalars and many other dissipative systems.

Equations of motion for the velocity potential are derived using Euler-Lagrange equations, and found to correspond to established results for the linearized problem. The continuum mechanics formalism is extended to derive the Hamiltonian and Poisson bracket relations, as well as the stress-energy tensor. At each step equations of motion are generated anew and compared to the initial equations to check for internal consistency. The Hamiltonian is then Fourier transformed and diagonalized. Finally, Green's functions for the system are derived.

Part II explores some of the consequences of including the nonlinear terms (which correspond to the advective term and higher-order parts of the pressure gradient term in the Navier-Stokes equation). One advantage of the development is that the perturbation terms scale as the Mach number, rather than as the Reynolds number, as is the case in earlier works. However, since the advective term is of second order in the velocity, the corresponding terms in the Lagrangian are of third order in the fields. This immediately causes a break-down of phase invariance, a problem which does not arise in quantum field theory. This may mean that the Lagrangian must be transformed to new variables.

The discussion in Part II is based on terms of the Fourier-transformed fields. These fields are further transformed to make contact with various calculations made on such critical phenomena as superfluidity and superconductivity. Modified equations of motion are also derived for the original Fourier-transformed variables, and comparisons are made with the equations for various critical systems.

In Part III procedures for making diagram calculations are indicated and a simple illustration of a diagram calculation is given. An extensive renormalization calculation of the effective viscosity, due to John Erdei, is also presented. Next a procedure for deriving Lagrangians corresponding to equations of motion for passive scalars is discussed, with an extensive example of the heat budget equation. Justifications for several lengthy procedural steps are given in the Appendices.

Parts I, II, and III are each written as self-contained works, complete with their own references for convenience.

MANY-BODY TREATMENT OF NAVIER-STOKES FLUIDS

PART I, BASIC FORMULATION

1. INTRODUCTION

Flowfields of practical importance are usually hard-driven, nonlinear systems of great complexity that are far from their final relaxed state which we term equilibrium. Traditional approaches to describing such systems yield diminished results and are not amenable to application to a variety of systems possessing diverse boundary conditions. By contrast, Many-Body Theory is a description of a complex system which has proved quite useful in a variety of areas that include strongly fluctuating quantities.

Experimental aspects of turbulent flow, such as the slow decay of large-scale structures under appropriate operating conditions,[1] the bifurcation and quasiperiodic behavior of characteristic modes preceding turbulent behavior,[2] and the onset of turbulence at critical Reynolds numbers,[3] are reminiscent of critical phenomena. A comparison between turbulence and critical systems would be greatly augmented by the use of a Many-Body formalism. It would be especially useful for semiquantitative calculations of scaling behavior.

A Lagrangian/Hamiltonian formalism has been developed for a single-component, viscous, subsonic flow in three dimensions. These functions will describe the full Navier-Stokes equation without approximation. Results include the eigenfunctions of the linearized system implicitly. Only the fundamental bracket relations are required. Poisson bracket equations are independent of coordinate systems.[4] Results can be applied to various boundary conditions by explicitly inserting the particular eigenfunctions for those conditions.

Since the Hamiltonian and the Poisson brackets can be shown to be canonically invariant, the initial basis functions may readily be transformed to other variables more suitable to calculation. Finally, an approach based on Many-Body techniques may make contact with the vast literature on field theory. In this way, many useful results may be obtained by a relatively small effort.

Much of the early work on field theoretical calculations concerning fluid mechanics was prior to a maturation in field theory and is expressed in a sophisticated format which cannot readily be disseminated to a broad community. Attempts at applying diagrammatic techniques borrowed from quantum field theory to turbulence in incompressible flow have been made by Wyld,[5] Lee,[6] Edwards,[7] Kawasaki,[8,9] Martin, et al.[10] and Gledzer and Monin.[11] Except for Kawasaki, who used the Lagrangian equation as a model, all of these authors simply made perturbation expansions directly from the Navier-Stokes equations, including the entire pressure gradient term with the advective term. No systematic development from Poisson brackets was considered; rather intuition was used to renormalize perturbation series that began with the Reynolds number as an expansion parameter. It is not surprising that these expansions tended to become intractable after the inclusion of the fourth-order term. While demonstrating that the application of field theory to fluid mechanics is a plausible calculation, and providing useful bases for comparison, this work did not follow a systematic development based on a canonical formulation of the governing partial differential (Navier-Stokes) equation, as is standard in field theory.

A Lagrangian density corresponding to the Navier-Stokes equation[12,13] has been described in earlier work. However, this Lagrangian (expressed in terms of the velocity itself, rather than its potential) is developed from a Lagrangian

corresponding to sound waves in a Lagrangian coordinate system, augmented by continuity equations as constraints. Viscous damping is brought in by "posits" rather than derived directly from a term in the Lagrangian.

There are many ways of constructing the Lagrangian density upon which the Poisson bracket relations are based. Several different Lagrangians for Eulerian flow have been described by various authors.[14] Most of these are expressed directly in terms of the velocity and pressure. The resulting formalisms are awkward. Moreover, they avoid the truculent dissipative term. The Lagrangian density described below is equivalent to the full Navier-Stokes equation for subsonic flow and is expressed in terms of the scalar potential function and the solenoidal stream function. This is a more natural approach, as it implicitly couples the velocity and pressure and nicely separates potential flow from rotational flow. In addition, the symmetries and transformation properties of the system are more apparent using this formalism.

We begin our formalism with a fluid density ρ, a scalar potential Ψ, and a solenoidal potential, $\mathcal{A}$. Since the dissipative term in the Navier-Stokes equation is of even order in the space and time derivatives, while the other terms are of odd order, it is necessary to invoke adjoint fields. This is done by expressing the Lagrangian density in terms of products of new velocity potentials and their adjoint fields, and then expressing Ψ and $\mathcal{A}$ as the sum of these fields. This procedure requires that the "scalar" velocity potentials and their adjoint fields become pseudo-scalars in four spaces and that the "vector" potentials and their adjoint fields become axial vectors. It would be notationally simpler to work with the complex conjugates of the potential fields rather than with the true adjoint fields. This procedure would require that we

express odd-order derivatives of Ψ and $\mathcal{A}$ in terms of the differences of the corresponding derivatives of the new potential fields and their adjoint fields, rather than their sums.

Expressions for new variables are obtained on the basis of the symmetry properties of the Lagrangian. The Poisson bracket relations for the symmetry-generating variables are also examined. It is not necessary to use a matrix formalism to obtain the diagonalization. As a corollary, expressions for the conserved currents of the system are obtained, and the key continuity equations are again generated. This work solidifies the basis for the formalism.

2. NAVIER-STOKES LAGRANGIAN

We wish to solve the Navier-Stokes equation for an isothermal, single-component fluid with constant kinematic viscosity, ν:

$$\dot{\mathbf{v}} + \mathbf{v}\cdot\nabla\mathbf{v} + \nu\nabla\times\nabla\times\mathbf{v} - D\nabla^2\mathbf{v} + \frac{1}{\rho}\nabla P = F/\rho + \tau/\rho \quad , \tag{2.1}$$

where P is the pressure, ρ is the fluid density, and

$$D = \zeta/\rho + \frac{1}{3}\nu \quad . \tag{2.2}$$

To do so, we construct the appropriate Lagrangian, L, in terms of fields for the velocity potential. L is an integral over a Lagrangian density $\mathcal{L}$:

$$L = \int dt d^3r \mathcal{L} \tag{2.3}$$

We express v in terms of a scalar potential field Ψ and a solenoid potential field $\mathcal{A}$:

$$\mathbf{v} = \nabla\Psi + \nabla\times\mathcal{A} \, , \tag{2.4}$$

with

$$\mathcal{A} = A + \bar{A} \tag{2.5}$$

and

$$\Psi = \psi + \bar{\psi} \ , \tag{2.6}$$

where $^-$ denotes the hypercomplex adjoint. We make the approximation

$$P \approx - \rho\dot{\Psi} - \frac{1}{2}\rho \left[|\nabla\Psi|^2 - \frac{1}{c^2}\dot{\Psi}^2 \right] + P_o \ , \tag{2.7}$$

where P_o is a constant. Higher-order approximations for P may be obtained by adding correction terms to Equation (2.7). Denoting the speed of sound by c, the required Lagrangian density is given by

$$\mathcal{L}(\psi, \bar{\psi}, A, \bar{A}) = \mathcal{L}_\psi + \mathcal{L}_A + \mathcal{L}_I + \mathcal{L}_{NL} + \mathcal{L}_P \ , \tag{2.8}$$

where

$$2\frac{c^2}{\rho}\mathcal{L}_\psi = \dot{\bar{\psi}}\dot{\psi} - \frac{1}{2}D(\dot{\bar{\psi}}\gamma_o\nabla^2\psi - \nabla^2\bar{\psi}\gamma_o\dot{\psi}) - c^2\nabla\bar{\psi}\cdot\nabla\psi + \frac{1}{2}(\psi + \bar{\psi})\epsilon, \tag{2.9}$$

$$2\frac{c^2}{\rho}\mathcal{L}_A = \dot{\bar{A}}\cdot\dot{A} + \frac{1}{2}\nu(\dot{\bar{A}}\gamma_o\cdot\nabla x\nabla xA - \nabla x\nabla x\bar{A}\gamma_o\cdot\dot{A}) + \frac{1}{2}(A + \bar{A})B, \tag{2.10}$$

$$8\frac{c^2}{\rho}\mathcal{L}_I = \frac{1}{2}(\gamma_o\dot{\psi} + \dot{\bar{\psi}}\gamma_o)(\nabla xA - \nabla x\bar{A})\cdot(\nabla\psi - \nabla\bar{\psi})$$

$$+ \frac{1}{2}(\gamma_o\dot{\psi} + \dot{\bar{\psi}}\gamma_o)(\nabla xA - \nabla x\bar{A})\cdot(\nabla xA - \nabla x\bar{A}), \tag{2.11}$$

$$8\frac{c^2}{\rho}\mathcal{L}_{NL} = (\gamma_o\dot{\psi} + \dot{\bar{\psi}}\gamma_o)(\nabla\psi - \nabla\bar{\psi})\cdot(\nabla\psi - \nabla\bar{\psi}) \ , \tag{2.12}$$

and

$$2\frac{c^2}{\rho}\mathcal{L}_P = (\gamma_o\dot{\psi} + \dot{\bar{\psi}}\gamma_o)\left[\nabla\psi - \nabla\bar{\psi}|^2\right] - \frac{1}{c^2}(\dot{\psi} - \dot{\bar{\psi}})^2 \ . \tag{2.13}$$

In Equation (2.9), we let

$$F = - \nabla U \tag{2.14}$$

and

$$\varepsilon = - \dot{U}/\rho \ . \tag{2.15}$$

Similarly, we let

$$B = (\nabla x \dot{\tau})/\rho \ . \tag{2.16}$$

$\mathcal{L}_\psi$ describes the potential velocity field, $\mathcal{L}_A$ describes the solenoidal velocity field, and $\mathcal{L}_I$ describes the interaction between these two fields due to the advective term in the Navier-Stokes equation. The nonlinearity in the Navier-Stokes equation is contained in $\mathcal{L}_I$ and $\mathcal{L}_{NL}$ and in correction terms to Equation (2.7).

2.1 Equations of Motion

The Euler-Lagrange equations of motion for ψ, $\bar{\psi}$, A, and $\bar{A}$ are obtained by minimizing L with respect to these fields through the calculus of variations, using integration by parts (see Appendix A). These equations may then be used in conjunction with Equations (2.4) through (2.6) to obtain an equation of motion for v. Since the potential field equations will be of second order in time, the resulting equation must be integrated with respect to time to regain Equation (2.1). Calculations based on $\mathcal{L}$ will be made by treating $\mathcal{L}_I$, $\mathcal{L}_{NL}$ and the nonlinear terms in the series approximation for P as perturbations about the linear term in $\mathcal{L}_\psi$ and $\mathcal{L}_A$. In general the expansion parameters should go roughly as $\frac{1}{2}(v/c)$, so the series should converge rapidly for subsonic flow. As the development of the formalism proceeds, the formalism will take on a structure that closely parallels that of quantum field theory. Nevertheless, the system

remains completely classical, as explained in Appendix B. We begin by investigating the linear Lagrangian density $\mathcal{L}_o$, where

$$\mathcal{L}_o = \mathcal{L}_\psi + \mathcal{L}_A \quad . \tag{2.17}$$

Variation of $\bar{\psi}$ in $\mathcal{L}_0$ gives

$$\ddot{\psi} - D\gamma_o\nabla^2\dot{\psi} - c^2\nabla^2\psi = \frac{1}{2}\epsilon \quad . \tag{2.18}$$

A corresponding variation of ψ in $\mathcal{L}_0$ gives

$$\ddot{\bar{\psi}} + D\nabla^2\dot{\bar{\psi}}\gamma_o - c^2\nabla^2\bar{\psi} = \frac{1}{2}\epsilon \quad . \tag{2.19}$$

Variation of $\bar{A}$ in $\mathcal{L}_0$ yields

$$\ddot{A} + \nu\gamma_o\nabla x\nabla x\dot{A} = \frac{1}{2}B \quad , \tag{2.20}$$

while variation of A in $\mathcal{L}_0$ results in

$$\ddot{\bar{A}} - \nu\nabla x\nabla x\dot{\bar{A}}\gamma_o = \frac{1}{2}B \quad . \tag{2.21}$$

These results are in agreement with those of other authors.[15,16] The momenta conjugate to ψ, $\bar{\psi}$, A, and $\bar{A}$ are respectively:

$$\pi = \partial\mathcal{L}_0/\partial\dot{\psi} = \frac{\rho}{2c^2}(\dot{\bar{\psi}} + \frac{1}{2}D\nabla^2\bar{\psi}\gamma_o), \tag{2.22}$$

$$\bar{\pi} = \partial\mathcal{L}_0/\partial\dot{\bar{\psi}} = \frac{\rho}{2c^2}(\dot{\psi} - \frac{1}{2}D\gamma_o\nabla^2\psi), \tag{2.23}$$

$$p = \partial\mathcal{L}_0/\partial\dot{A} = \frac{\rho}{2c^2}(\dot{\bar{A}} - \frac{1}{2}\nu\nabla x\nabla x\dot{A}\gamma_o), \tag{2.24}$$

and

$$\bar{p} = \partial \mathcal{L}_0/\partial \dot{\bar{A}} = \frac{\rho}{2c^2}(\dot{A} + \frac{1}{2}\nu\gamma_o\nabla x\nabla xA). \tag{2.25}$$

2.2 Hamiltonian

The Hamiltonian density corresponding to $\mathcal{L}_\psi$, $\mathcal{H}_\psi$ is given by

$$\mathcal{H}_\psi = \pi\dot{\psi} + \bar{\pi}\,\dot{\bar{\psi}} - \mathcal{L}_\psi \tag{2.26}$$

$$= \frac{\rho}{2c^2}\left[\dot{\bar{\psi}}\dot{\psi} + c^2\nabla\bar{\psi}\nabla\psi - \frac{1}{2}(\psi + \bar{\psi})\epsilon\right]$$

$$= \frac{\rho}{2c^2}\left[(\frac{2c^2}{\rho}\pi - \frac{1}{2}D\nabla^2\bar{\psi}\gamma_o)(\frac{2c^2}{\rho}\bar{\pi} + \frac{1}{2}D\gamma_o\nabla^2\psi) + c^2\nabla\bar{\psi}\nabla\psi - \frac{1}{2}(\psi + \bar{\psi})\epsilon\right]. \tag{2.27}$$

Similarily, the Hamiltonian density $\mathcal{H}_A$ corresponding to $\mathcal{L}_A$ is given by

$$\mathcal{H}_A = p\cdot\dot{A} + \bar{p}\cdot\dot{\bar{A}} - \mathcal{L}_A \tag{2.28}$$

$$= \frac{\rho}{2c^2}\left[\dot{\bar{A}}\cdot\dot{A} - \frac{1}{2}(A + \bar{A})B\right]$$

$$= \frac{\rho}{2c^2}\left[(\frac{2c^2}{\rho}p + \frac{1}{2}\nu\nabla x\nabla x\bar{A}\gamma_o)\cdot(\frac{2c^2}{\rho}\bar{p} - \frac{1}{2}\nu\gamma_o\nabla x\nabla xA) - \frac{1}{2}(A + \bar{A})B\right]. \tag{2.29}$$

The Hamiltonian equations of motion for ψ, π, $\bar{\psi}$, and $\bar{\pi}$ are

$$\dot{\psi} = \delta H/\delta\pi = \frac{2c^2}{\rho}\bar{\pi} + \frac{1}{2}D\gamma_o\nabla^2\psi, \tag{2.30}$$

$$\dot{\pi} = -\delta H/\delta\psi = \frac{\rho}{2c^2}(-\frac{1}{2}D\nabla^2\dot{\bar{\psi}}\gamma_o + c^2\nabla^2\bar{\psi} + \frac{1}{2}\epsilon), \tag{2.31}$$

$$\dot{\bar{\psi}} = \delta H/\delta\bar{\pi} = \frac{2c^2}{\rho}\pi - \frac{1}{2}D\nabla^2\bar{\psi}\gamma_o \ , \tag{2.32}$$

and

$$\dot{\pi} = -\delta H/\delta\bar{\psi} = \frac{\rho}{2c^2}(\frac{1}{2}\ D\gamma_o\nabla^2\dot{\psi} + c^2\nabla^2\psi + \frac{1}{2}\epsilon). \tag{2.33}$$

For the vector potential fields we find

$$\dot{A} = \delta H/\delta p = \frac{2c^2}{\rho}\bar{p} - \frac{1}{2}\nu\gamma_o\nabla x\nabla xA \ , \tag{2.34}$$

$$\dot{p} = -\ \delta H/\delta A = \frac{\rho}{4c^2}\left[\nu\nabla x\bar{\nabla}x\dot{A}\gamma_o + B\ \right] \tag{2.35}$$

$$\dot{\bar{A}} = \delta H/\delta\bar{p} = \frac{2c^2}{\rho}p + \frac{1}{2}\nu\nabla x\nabla x\bar{A}\gamma_o \ , \tag{2.36}$$

and

$$\dot{\bar{p}} = -\ \delta H/\delta\bar{A} = \frac{\rho}{4c^2}\left[\nu\gamma_o\nabla x\nabla x\dot{A} - B\ \right] . \tag{2.37}$$

Differentiating Equation (2.30) with respect to time and then using Equation (2.33) yields Equation (2.18), as it should. Similarly, differentiating Equation (2.32) with respect to time and then using Equation (2.31) gives Equation (2.19). We now assert that (see Appendix C)

$$\nabla\Psi = \gamma_k\nabla\psi - \nabla\bar{\psi}\gamma_k \ , \tag{2.38}$$

$$\dot{\Psi} = \gamma_o\dot{\psi} - \dot{\bar{\psi}}\gamma_o \ , \tag{2.39}$$

$$A^{\cdot} = \gamma_o\dot{A} - \dot{\bar{A}}\gamma_o \ , \tag{2.40}$$

and

$$\nabla x \mathcal{A} = \gamma_k \nabla x A - \nabla x \bar{A} \gamma_k . \tag{2.41}$$

Then, adding Equations (2.18) and (2.19), we obtain

$$\ddot{\Psi} - D\nabla^2 \dot{\Psi} - c^2 \nabla^2 \Psi = \epsilon \quad . \tag{2.42}$$

Adding Equation (2.20) and (2.21), we find

$$\ddot{\mathcal{A}} + \nu \nabla x \nabla x \dot{\mathcal{A}} = B \quad . \tag{2.43}$$

2.3 Nonlinear Terms

Assuming the gauge equation

$$\nabla \cdot A = 0 \quad , \tag{2.44}$$

the addition of the second expression in $\mathcal{L}_I$ to $\mathcal{L}_0$ does not affect Equations (2.20) and (2.21), but adds the term

$$\frac{1}{2}\gamma_o \frac{\partial}{\partial t}[(\nabla x A - \nabla x \bar{A}) \cdot (\nabla x A - \nabla x \bar{A})]$$

to both Equation (2.18) and Equation (2.19). The corresponding change in Equation (2.40) is

$$\frac{\partial}{\partial t}(\nabla x \mathcal{A} \cdot \nabla x \mathcal{A}) \quad .$$

The addition of L_{NL} to L_ψ adds the terms

$$-(\gamma_o \dot{\psi} + \dot{\bar{\psi}} \gamma_o)\ (\nabla^2 \psi - \nabla^2 \bar{\psi})$$

on the left-hand side of Equation (2.18) and

$$(\nabla \psi - \nabla \bar{\psi}) \cdot (\gamma_o \nabla \dot{\psi} + \nabla \dot{\bar{\psi}} \gamma_o) + \frac{1}{2}(\gamma_o \dot{\psi} + \dot{\bar{\psi}} \gamma_o)(\nabla^2 \psi - \nabla^2 \bar{\psi})$$

on the left-hand side of Equation (2.19). The corresponding change in Equation (2.42) is

$$(\nabla\psi - \nabla\bar{\psi})\cdot(\gamma_o\nabla\dot{\psi} - \nabla\dot{\bar{\psi}}\gamma_o)$$

$$= \frac{1}{2}\partial(\nabla\Psi\cdot\nabla\Psi)/\partial t \quad .$$

We may integrate Equations (2.42) and (2.43) with respect to time, and use Equations (2.4), (2.15), and (2.16) to obtain the linearized counterpart to Equation (2.1).

The addition of the first expression in $\mathcal{L}_I$ to $\mathcal{L}_o$ does not affect Equations (2.20) and (2.21), but adds the terms

$$\gamma_o\frac{\partial}{\partial t}\left[(\nabla\psi - \nabla\bar{\psi})\cdot(\nabla xA - \nabla x\bar{A})\right] + (\gamma_o\nabla\dot{\psi} + \nabla\dot{\bar{\psi}}\gamma_o)\cdot(\nabla xA - \nabla x\bar{A})$$

to the left side of Equation (2.18) and the terms

$$\gamma_o\frac{\partial}{\partial t}\left[(\nabla\psi - \nabla\bar{\psi})\cdot(\nabla xA - \nabla x\bar{A})\right] - (\gamma_o\nabla\dot{\psi} + \nabla\dot{\bar{\psi}}\gamma_o)\cdot(\nabla xA - \nabla x\bar{A})$$

to the left side of Equation (2.19). The corresponding additions to Equation (2.42) are terms of the form

$$\frac{1}{2}\nabla\dot{\Psi}\cdot(\nabla x\mathcal{A}) + \frac{1}{2}\nabla\Psi\cdot(\nabla x\dot{\mathcal{A}}),$$

provided again that Equations (2.36) and (2.37) hold. As a check, we differentiate Equation (2.1) with respect to time, and substitute in for v using the Equation (2.4). We make use of the linear approximation for the pressure gradient term,

$$\kappa\dot{P} + \nabla\cdot v \simeq 0, \qquad (2.45)$$

to obtain the relations

$$\nabla\times[\ddot{\mathcal{A}} + \nu\nabla\times\nabla\times\dot{\mathcal{A}}] - B = 0 \tag{2.46}$$

and

$$\nabla[\ddot{\Psi} - D\nabla^2\dot{\Psi} - \frac{1}{c^2}\nabla^2\Psi + \frac{1}{2}\frac{\partial}{\partial t}(\nabla\Psi\cdot\nabla\Psi) + \frac{\partial}{\partial t}(\nabla\times\mathcal{A}\cdot\nabla\Psi) + \frac{\partial}{\partial t}(\nabla\times\mathcal{A}\cdot\nabla\times\mathcal{A}) - \epsilon] = 0. \tag{2.47}$$

2.4 Poisson Bracket Relations

We will define Poisson brackets in terms of a variational derivative:

$$[X,Y] = \sum_i \left[\frac{\delta X}{\delta\Phi_i}\frac{\delta Y}{\delta\Pi_i} - \frac{\delta X}{\delta\Pi_i}\frac{\delta Y}{\delta\Phi_i} \right] , \tag{2.48}$$

where the fields Φ_i are given by ψ, $\bar{\psi}$, A, and $\bar{A}$ and the Π_i fields are their conjugate momenta π, $\bar{\pi}$, p, and $\bar{p}$.

We have the usual relations

$$[\Phi_i(r), \Phi_j(r')] = [\Pi_i(r), \Pi_j(r')] = 0. \tag{2.49}$$

For the conjugate fields we write

$$[\Phi_i(r), \Pi_j(r')] = C\,\delta_{ij}\,\delta(r-r'). \tag{2.50}$$

For the scalar fields we have C = 1, but for the vector fields we have C = ½, as we shall show in Section 3.

Keeping in mind that our Poisson brackets are defined in terms of variational derivatives, the equations of motion expressed in terms of Poisson brackets are in standard form:[17]

$$\partial F/\partial\Phi_i = [F, \Pi_i] , \tag{2.51}$$

$$-\partial F/\partial\Pi_i = [F, \Phi_i] , \tag{2.52}$$

and

$$\dot{F} = [F, H] + \frac{\partial F}{\partial t} \quad , \tag{2.53}$$

for any variable F. We can check our formalism for consistency by using the relation (2.53) to retrieve our Equations of motion. With H_A given by:

$$H_A = \frac{\rho}{2c^2}\left(\frac{2c^2}{\rho}p + \frac{1}{2}\nu\nabla x\nabla x\bar{A}\gamma_o\right) \cdot \left(\frac{2c^2}{\rho}\bar{p} - \frac{1}{2}\nu\gamma_o\nabla x\nabla xA\right) \tag{2.54}$$

we find

$$\dot{A} = [A, H_A]$$

$$= 2[A, p]\left(\frac{2c^2}{\rho}\bar{p} - \frac{1}{2}\nu\gamma_o\nabla x\nabla xA\right) \tag{2.55}$$

$$= \frac{2c^2}{\rho}\bar{p} - \frac{1}{2}\nu\gamma_o\nabla x\nabla xA \quad . \tag{2.56}$$

Similarly, recalling that our brackets are defined in terms of variational derivatives,

$$\dot{p} = [p, H_A]$$

$$= \frac{\rho}{2c^2}\left(\frac{2c^2}{\rho}p + \frac{1}{2}\nu\nabla x\nabla x\bar{A}\gamma_o\right) \quad \frac{1}{2}\nu\gamma_o[\nabla x\nabla xA, p]$$

$$= \frac{\rho}{4c^2}\nu\gamma_o\nabla x\nabla x\left(\frac{2c^2}{\rho}p + \frac{1}{2}\nu\nabla x\nabla x\bar{A}\gamma_o\right) \tag{2.57}$$

$$= \frac{\rho}{4c^2}\nu\nabla x\nabla x\dot{\bar{A}} \quad . \tag{2.58}$$

With H_ψ given by

$$H_\psi = \frac{\rho}{2c^2}\left[\left(\frac{2c^2}{\rho}\pi - \frac{1}{2}D\gamma_o\nabla^2\bar{\psi}\right)\left(\frac{2c^2}{\rho}\bar{\pi} + \frac{1}{2}D\gamma_o\nabla^2\psi\right) + c^2\nabla\bar{\psi}\cdot\nabla\psi\right] \tag{2.59}$$

we have

$$\dot{\psi} = [\psi, H_\psi]$$

$$= \frac{\rho}{2c^2}\left[\left(\frac{2c^2}{\rho}\bar{\pi} + \frac{1}{2}D\gamma_o\nabla^2\psi\right)\frac{2c^2}{\rho}[\psi,\pi]\right] \tag{2.60}$$

$$= \frac{2c^2}{\rho}\bar{\pi} + \frac{1}{2}D\gamma_o\nabla^2\psi \quad . \tag{2.61}$$

Similarly,

$$\dot{\bar{\pi}} = [\bar{\pi}, H_\psi]$$

$$= \frac{\rho}{2c^2}\left[\left(\frac{2c^2}{\rho}\pi - \frac{1}{2}D\nabla^2\bar{\psi}\gamma_o\right)\cdot\frac{1}{2}D[\bar{\pi},\nabla^2\psi] + c^2\nabla\bar{\psi}\cdot[\bar{\pi},\nabla\psi]\right]$$

$$= \frac{\rho}{2c^2}\left[-\frac{1}{2}D\nabla^2\left(\frac{2c^2}{\rho}\pi - \frac{1}{2}D\nabla^2\bar{\psi}\gamma_o\right) + c^2\nabla^2\bar{\psi}\right] \tag{2.62}$$

$$= \frac{\rho}{2c^2}\left(-\frac{1}{2}D\nabla^2\dot{\bar{\psi}}\gamma_o + c^2\nabla^2\bar{\psi}\right) \tag{2.63}$$

in the absence of external forces. In the same manner we can reproduce the equations of motion for

$$\bar{A},\ \bar{p},\ \bar{\psi},\ \text{and}\ \bar{\pi}.$$

3. CONSERVED CURRENTS

The prediction of conserved currents on the basis of the symmetry of the Lagrangian density $\mathcal{L}$ is discussed in detail by Roman[18] and by Mandel and Shaw.[19] Although their discussion is directed towards the quantum systems of particle physics, all the arguments can be applied to a classical system. Nothier's theorem states that the invariance of the Lagrangian density with respect to a symmetry transformation implies the existence of a conserved variable. Consider a four-vector, f^{μ}, such that

$$\partial_{\mu} \cdot f^{\mu} = 0. \tag{3.1}$$

Integrating Equation (3.1) over coordinate space, and making the identification

$$F = \int d^3 r f_o \quad , \tag{3.2}$$

yields

$$\dot{F} = - \int d^3 r \sum_{i=1}^{3} \partial f_i / \partial x_i \quad , \tag{3.3}$$

$$= - \int dA f \to 0 \quad . \tag{3.4}$$

Consequently, if the variation of $\mathcal{L}$ with respect to some symmetry transformation yields an equation of the form of Equation (3.1) for some four vector f_{μ}, we are led to a conserved variable F. As shown in Mandel and Shaw,[19] gauge invariance, expressed by

$$\begin{aligned} \psi &\to e^{i\epsilon} \psi \quad , \\ \bar{\psi} &\to e^{-i\epsilon} \bar{\psi} \quad , \end{aligned} \tag{3.5}$$

leads to the conservation of the quantity

$$\int d^3 r [\pi \psi - \bar{\pi} \bar{\psi}] = m/\rho \quad . \tag{3.6}$$

The continuity equation for m is

$$\dot{m} + \nabla\cdot\dot{j} = 0 \quad , \tag{3.7}$$

with

$$\dot{j} = \bar{\psi}\nabla\psi - \psi\nabla\bar{\psi} \quad . \tag{3.8}$$

Invariance with respect to the four-translation

$$\begin{aligned} x_\mu &\to x_\mu + \delta x_\mu \equiv x'_\mu \quad , \\ \psi(x_\mu) &\to \psi'(x'_\mu) \quad , \\ A(x_\mu) &\to A'(x'_\mu) \quad , \end{aligned} \tag{3.9}$$

leads to the conservation of the stress-energy tensors T^{ψ} and T^{A} (see Appendix D).

The general form of the elements of T^{ψ} is given by:

$$T^{\psi}_{\nu\sigma} = \psi_\nu \frac{\partial \ell}{\partial \psi_\sigma} + 2\sum_\rho \left[\psi_{\nu\rho} \frac{\partial \ell}{\partial \psi_{\sigma\rho}} \right] + \bar{\psi}_\nu \frac{\partial \ell}{\partial \bar{\psi}_\sigma} + 2\sum_\rho \left[\bar{\psi}_{\nu\rho} \frac{\partial \ell}{\partial \bar{\psi}_{\sigma\rho}} \right] - \ell_{\nu\sigma} \quad . \tag{3.10}$$

This definition of W^{ψ} is a necessary extension of the stress tensors described by Morse and Feshback[16] and by Morse and Ingard[15] if dissipation is to be included in L. A similar definition will hold for W^{A}. Explicit forms for the elements of the stress tensor are

$$T^{\psi}_{11} = \frac{\rho}{2c^2}\left[\dot{\bar{\psi}}\dot{\psi} + c^2 \nabla\bar{\psi}\nabla\psi - \frac{1}{2}(\bar{\psi} + \psi)\epsilon \right] = H \quad ; \tag{3.11}$$

$$T^{\psi}_{1r} = \frac{\rho}{2c^2}\left[D(\nabla\dot{\bar{\psi}}\gamma_o\psi - \bar{\psi}\gamma_o\nabla\dot{\psi}) - c^2(\dot{\psi}\nabla\bar{\psi} - \bar{\psi}\nabla\psi) \right] \equiv I \quad ; \tag{3.12}$$

$$T_{rl} = \frac{\rho}{2c^2}(\pi\nabla\psi + \bar{\pi}\nabla\bar{\psi}) = \frac{\rho}{2c^2}P \tag{3.13}$$

$$T_{xy} = -\frac{1}{2}\rho\psi_x\psi_y \quad , \quad \text{etc.} \quad ; \tag{3.14}$$

and

$$T_{xx} = \frac{-\rho}{2c^2}\left[\dot{\bar{\psi}}\dot{\psi} + D\gamma_o(\dot{\psi}\bar{\psi}_{xx} - \dot{\bar{\psi}}\psi_{xx}) - c^2(\bar{\psi}_y\psi_y + \bar{\psi}_z\psi_z - \psi_x\psi_x) \ , \ \text{etc.}\right]. \tag{3.15}$$

The components of T^{ψ} satisfy the continuity relations

$$\dot{H} + \nabla\cdot I = 0 \tag{3.16}$$

and

$$P + \nabla\cdot W = 0 \quad , \tag{3.17}$$

where W is a 3x3 tensor containing the spatial parts of T^{ψ}, known as the wave-stress tensor.

We see that the continuity equation for the stress-energy tensor follows directly from the form of our Lagrangian density, i.e., we can <u>derive</u> it from our Lagrangian; it does not represent a subsidiary condition on our formalism.

Although $\mathcal{L}_o$ is invariant with respect to gauge transformations, the nonlinear terms in our full Lagrangian density break gauge symmetry. Neither the A nor the ψ fields can be made to satisfy local gauge invariance in any simple or reasonable manner. However, the full Lagrangian density is preserved with respect to four-translations.

The symmetries of the Lagrangian/Hamiltonian density are important not only for generating additional equations of motion, but because they provide valuable clues and, in fact, a systematic procedure for examining the behavior of the system when it undergoes a change of state, e.g., from laminar to turbulent flow.

Since the conserved variables may be derived from the symmetry properties of the Lagrangian, their Poisson bracket relations with other variables offer clues to the behavior of the system with respect to various symmetries. In particular, if F is a conserved variable, and if there exist (possibly the same) variables such that

$$[F(\theta), X] = Y \tag{3.18}$$

with

$$\langle Y \rangle \neq 0, Y \gtrless \theta_c \tag{3.19}$$

for some parameter θ and a critical value of that parameter, θ_c, then Equations (3.18) and (3.19) signal a breaking of symmetry, usually associated with a change in state, at θ_c. We seek such relations in an attempt to develop a theory of coherent structures at the onset of turbulence. Consequently, it is of interest to examine various Poisson bracket relations pertaining to conserved variables. We have:

$$[m, \psi] = -\rho\psi \quad , \tag{3.20}$$

$$[m, \bar{\psi}] = \rho\bar{\psi} \quad , \tag{3.21}$$

$$[m, \pi] = \rho\pi \quad , \tag{3.22}$$

$$[m, \bar{\pi}] = -\rho\bar{\pi} \quad . \tag{3.23}$$

Equations (3.1) and (3.10) to (3.13) lead to the conservation of P in the form[18,19]

$$P = \int d^3r [\pi\nabla\psi + \bar{\pi}\nabla\bar{\psi}] \quad . \tag{3.24}$$

The function P has the property that:

$$[F, P] = \nabla F \quad . \tag{3.25}$$

From Equations (3.18) and (3.7) we see that if:

$$\langle v \rangle = \langle \nabla\psi - \nabla\bar{\psi} \rangle \neq 0 \quad , \tag{3.26}$$

then translation symmetry, generated by P, will be broken.

4. DIAGONALIZATION

We will now make linear transformations of the scalar potentials which will diagonalize H_ψ in the absence of external driving forces. It will be helpful in performing the required algebra to define a frequency ω_k, given by

$$\omega_k^2 = c^2k^2 + \frac{1}{4}D^2k^4, \tag{4.1}$$

for a mode with wavenumber k. We will ignore external driving forces in the following discussion.

We first expand ψ and $\bar{\psi}$ in terms of the normal modes of system. We express the normal modes as functions of their wavevectors k through the argument ik•r. We will assume a stationary system, and express $\psi_k(t)$ in terms of the phases

$$\theta^+_{\eta k} = e^{i\omega_{\eta k}t}, \tag{4.2}$$

$$\theta^-_{\eta k} = e^{-i\omega_{\eta k}t}, \tag{4.3}$$

$$\theta^+_{\zeta k} = e^{i\omega_{\zeta k}t}, \tag{4.4}$$

and

$$\theta^-_{\zeta k} = e^{-i\omega_{\zeta k}t}. \tag{4.5}$$

The scalar fields are expanded as

$$\psi = \sum_k \frac{c}{\sqrt{\rho\omega_k}} \left(\zeta^*_k \theta^+_{\zeta k} \bar{u}_k + \eta_k \theta^-_{\eta k} u_k \right), \tag{4.6}$$

$$\bar{\psi} = \sum_k \frac{c}{\sqrt{\rho\omega_k}} \left(\zeta_k \theta^-_{\zeta k} u_k + \eta^*_k \theta^+_{\eta k} \bar{u}_k \right), \tag{4.7}$$

where we use the notation

$$Q_k = u_k^* \gamma_A \quad . \tag{4.8}$$

The constant factor $c/\sqrt{\rho\omega_k}$ is inserted into the definitions given by Equations (4.6) and (4.7) to produce expressions for quantities of interest in terms of the Fourier transformed fields that are compact and that highlight the analogy with work in other areas using second quantization.[18-23] The system is "quantized" only to the extent that we are using expansions in Equations (4.6) and (4.7) in series rather than continuous transforms. Our fields are classical fields, and we have not as yet introduced eigenvectors or operators.

Two species of functions, η_k and ζ_k in the transformations given by Equations (4.6) and (4.7) are needed because the equation of motion for ψ contains terms that are of zero, first, and second order in time, as well as terms that are of third order in the field strength. The presence of terms of odd order in either the time or space derivatives or in the field strength dictates the use of complex fields. If the equation of motion is genuinely of second order with respect to time and the Lagrangian is expressed in terms of complex fields, then two species are required for diagonalization.[20] This occurs for example, in the description of a charged meson field.[18-20,24] If the equation of motion is of second order and only a real field is required in the Lagrangian, then only one species of function is required for diagonalization. Examples are phonon fields and the neutral π meson fields.[19,24-26] A second-order equation is equivalent to two first-order equations, so complex fields with first-order equations of motion can be expressed in terms of a real field

obeying a second-order equation of motion.[20] This occurs, for example, with the Schrodinger equation[20,23] and the diffusion equation.[16] The equation of motion for the A fields is similar in form to the latter two cases, and we will need only one species of function to diagonalize H_A, as we shall show below.

We emphasize that Equations (4.6) to (4.8) show how the development may be applied to any geometry. In the remainder of the discussion we will concentrate on the functions $\psi_k(t)$ and $\bar{\psi}_k(t)$ and their derivatives. This discussion will be completely general. To apply the results to a given geometry, we need only substitute the appropriate functions $u_k(ik\cdot r)$ and $\bar{u}_k(-ik\cdot r)$. In the discussion, whenever we refer to these functions explicitly, we will use the plane wave solutions $u_k e^{ik\cdot r}$ and $u_k \gamma_A e^{-ik\cdot r} = \tilde{u}_k e^{-ik\cdot r}$.

4.1 Scalar Potential

Since

$$\ddot{\psi} - D\gamma_o \nabla^2 \dot{\psi} - c^2 \nabla^2 \psi = 0 \quad , \tag{4.9}$$

and

$$\ddot{\tilde{\psi}} + D\nabla^2 \dot{\tilde{\psi}} \gamma_o - c^2 \nabla^2 \tilde{\psi} = 0 \quad , \tag{4.10}$$

we have

$$\omega_{\eta k}^2 + iD\gamma_o \omega_{\eta k} k^2 - c^2 k^2 = 0 \tag{4.11}$$

and

$$\omega_{\zeta k}^2 - iD\gamma_o \omega_{\zeta k} k^2 - c^2 k^2 = 0 \quad . \tag{4.12}$$

Equations (4.11) and (4.12) imply that

$$\omega_{\eta k} = \omega_{\zeta k} - iD\gamma_o k^2 \tag{4.13}$$

and

$$\omega_{\eta k} + \tfrac{1}{2}iD\gamma_o k^2 = \omega_k = \omega_{\zeta k} - \tfrac{1}{2}iD\gamma_o k^2 \quad . \tag{4.14}$$

Note that

$$\omega_{\eta k}\,\omega_{\zeta k} = c^2k^2 \quad , \tag{4.15}$$

so that substituting Equations (4.6) and (4.7) into the relation

$$\mathcal{H}_\psi = \frac{\rho}{2c^2}\left[\dot{\bar{\psi}}\dot{\psi} + c^2\nabla\bar{\psi}\cdot\nabla\psi\right] \tag{4.16}$$

yields

$$H_\psi = \sum_k \left[\left(\omega_{\eta k}^2 + c^2k^2\right)\eta_k^*\eta_k + \left(\omega_{\zeta k}^2 + c^2k^2\right)\zeta_k^*\zeta_k\right] \Big/ 2\omega_k \quad , \tag{4.17}$$

assuming that

$$\int d^3r\; u_k(r)\,\bar{u}_{k'}(r) = \delta_{k,k'} \quad , \tag{4.18}$$

i.e., that the u_k are orthonormal.

We now rewrite Equation (4.11) in the form

$$\omega_{\eta k}^2 + c^2k^2 = 2(\omega_{\eta k} + \tfrac{1}{2}iD\gamma_o k^2)\omega_{\eta k} \quad , \tag{4.19}$$

or

$$\omega_{\eta k}^2 + c^2k^2 = 2\omega_k\omega_{\eta k} \quad . \tag{4.20}$$

Similarly, we may rewrite Equation (4.12) as

$$\omega_{\zeta k}^2 + c^2k^2 = 2(\omega_{\zeta k} - \tfrac{1}{2}iD\gamma_o k^2)\omega_{\zeta k} \quad , \tag{4.21}$$

or

$$\omega_{\zeta k}^2 + c^2k^2 = 2\omega_k\omega_{\zeta k} \quad . \tag{4.22}$$

Substituting Equations (4.20) and (4.22) into Equation (4.17), we obtain at last

$$H_\psi = \sum_k \left(\omega_{\eta k} \eta_k^* \eta_k + \omega_{\zeta k} \zeta_k^* \zeta_k \right) , \tag{4.23}$$

in standard form, [18,22,27] albeit with complex eigenfrequencies. It will often be convenient to use the compact notation

$$\psi(r,t) = \sum_k \psi_k(t)\, u_k(ik\cdot r) , \tag{4.24}$$

$$\bar{\psi}(r,t) = \sum_k \bar{\psi}_k(t)\, \bar{u}_k(-ik\cdot r) , \tag{4.25}$$

where

$$\psi_k = \frac{c}{(\rho\omega_k)^{1/2}} \left(\zeta_{\underline{k}}^* \theta_{\zeta k}^+ + \eta_k \theta_{\eta k}^- \right) , \tag{4.26}$$

and

$$\bar{\psi}_k = \frac{c}{(\rho\omega_k)^{1/2}} \left(\zeta_{\underline{k}} \theta_{\zeta k}^- + \eta_k^* \theta_{\eta k}^+ \right) , \tag{4.27}$$

The fields $\bar{\pi}$ and π may also be expanded in a manner similar to $\bar{\psi}$ and ψ:

$$\pi = \sum_k \pi_k(t) \bar{u}_k(-ik\cdot r) , \tag{4.28}$$

and

$$\bar{\pi} = \sum_k \bar{\pi}_k(t) u_k(ik\cdot r) . \tag{4.29}$$

From the relations

$$\pi = \frac{\rho}{2c^2} \left(\dot{\bar{\psi}} + \tfrac{1}{2} D \nabla^2 \bar{\psi} \gamma_o \right) \tag{4.30}$$

and

$$\bar{\pi} = \frac{\rho}{2c^2} \left(\dot{\psi} - \tfrac{1}{2} D \gamma_o \nabla^2 \psi \right) , \tag{4.31}$$

and Equations (4.6) and (4.7) we find

$$\pi_k(t) = \frac{1}{2c}(\rho/\omega_k)^{\frac{1}{2}} \left[(i\omega_{\eta k} - \tfrac{1}{2}D\gamma_o k^2)\eta_k^* \theta_{\eta k}^+ - (i\omega_{\zeta k} + \tfrac{1}{2}D\gamma_o k^2)\zeta_{\underline{k}}\theta_{\zeta k}^-\right] \quad (4.32)$$

and

$$\bar{\pi}_k(t) = \frac{1}{2c}(\rho/\omega_k)^{\frac{1}{2}} \left[(-i\omega_{\eta k} + \tfrac{1}{2}D\gamma_o k^2)\eta_k \theta_{\eta k}^- + (i\omega_{\zeta k} + \tfrac{1}{2}D\gamma_o k^2)\zeta_{\underline{k}}^*\theta_{\zeta k}^+\right]. \quad (4.33)$$

Now from Equation (4.14) note that

$$i\omega_{\eta k} - \tfrac{1}{2}D\gamma_o k^2 = i\omega_k = i\omega_{\zeta k} + \tfrac{1}{2}D\gamma_o k^2 \quad ; \quad (4.34)$$

consequently Equations (4.31) and (4.32) may be simplified to

$$\pi_k(t) = i(\rho\omega_k)^{\frac{1}{2}}/2c \ [\eta_k^*\theta_{\eta k}^+ - \zeta_{\underline{k}}\theta_{\zeta k}^-] \quad (4.35)$$

and

$$\bar{\pi}_k(t) = i(\rho\omega_k)^{\frac{1}{2}}/2c \ [\zeta_{\underline{k}}^*\theta_{\zeta k}^+ - \eta_k\theta_{\eta k}^-] \quad . \quad (4.36)$$

Equations (4.6), (4.7), (4.35), and (4.36) are the building block expressions for the remaining development. We now have all the information we need to proceed. First expressions for the quantities η_k^*, η_k, ζ_k^*, and ζ_k (which are independent of r and t) are obtained in terms of the original fields:

$$\eta_k^* = \tfrac{1}{2}\left[\sqrt{\rho\omega_k}/c \ \bar{\psi}_k + 2c/i\sqrt{\rho\omega_k}\pi_k\right] \theta_{\eta k}^- \quad , \quad (4.37)$$

$$\eta_k = \tfrac{1}{2}\left[\sqrt{\rho\omega_k}/c \ \psi_k - 2c/i\sqrt{\rho\omega_k}\bar{\pi}_k \right] \theta_{\eta k}^+ \quad , \quad (4.38)$$

$$\zeta_{\underline{k}}^* = \tfrac{1}{2}\left[\sqrt{\rho\omega_k}/c \ \psi_k + 2c/i\sqrt{\rho\omega_k}\bar{\pi}_k \right] \theta_{\zeta k}^- \quad . \quad (4.39)$$

and

$$\zeta_{\bar{k}} = \tfrac{1}{2}\left[\sqrt{\rho\omega_k}/c\ \bar{\psi}_k - 2c/i\sqrt{\rho\omega_k}\pi_k\right]\ \theta^+_{\zeta k} \quad . \tag{4.40}$$

Of course, we may also write

$$\eta^*_k = \frac{1}{2}\int d^3r \left[\ \frac{\sqrt{\rho\omega_k}}{c}\ \bar{\psi} + \frac{2c}{i\sqrt{\rho\omega_k}}\ \pi\ \right] \theta^-_{\eta k} u_k \quad , \tag{4.41}$$

$$\eta_k = \frac{1}{2}\int d^3r \left[\ \frac{\sqrt{\rho\omega_k}}{c}\ \psi - \frac{2c}{i\sqrt{\rho\omega_k}}\ \bar{\pi}\ \right] \theta^+_{\eta k} \bar{u}_k \quad , \tag{4.42}$$

$$\zeta^*_k = \frac{1}{2}\int d^3r \left[\ \frac{\sqrt{\rho\omega_k}}{c}\ \bar{\psi} + \frac{2c}{i\sqrt{\rho\omega_k}}\ \bar{\pi}\ \right] \theta^-_{\zeta k} u_k \quad , \tag{4.43}$$

and

$$\zeta_k = \frac{1}{2}\int d^3r \left[\ \frac{\sqrt{\rho\omega_k}}{c}\ \psi - \frac{2c}{i\sqrt{\rho\omega_k}}\ \pi\ \right] \theta^+_{\zeta k} \bar{u}_k \quad . \tag{4.44}$$

4.2 Vector Potential

The Hamiltonian for the A fields, H_A, is already diagonalized, as can be seen from the equation

$$\mathcal{H}_A = \frac{\rho}{2c^2}\ \dot{\bar{A}}\cdot\dot{A} \tag{4.45}$$

in the absence of external forces. The A fields are expanded as follows:

$$A = \sum_k A_k(t)u_k(r) = \sum_k c\sqrt{2/\rho\omega_{ak}}\ a_k\theta^*_{ak}u_k \tag{4.46}$$

and

$$\bar{A} = \sum_k \bar{A}_k(t)\bar{u}_k(r) = \sum_k c\sqrt{2/\rho\omega_{ak}}\ a^*_k\theta_{ak}\bar{u}_k \quad , \tag{4.47}$$

where

$$\theta_{ak} = e^{i\omega_a t} \quad , \tag{4.48}$$

and

$$\theta^*_{ak} = e^{-i\omega_a t} \quad . \tag{4.49}$$

From the equations of motion for A and $\bar{A}$ in the absence of external forces,

$$\ddot{A} + \nu\gamma_o \nabla x \nabla x \dot{A} = 0 \quad . \tag{4.50}$$

and

$$\ddot{\bar{A}} - \nu \nabla x \nabla x \dot{\bar{A}} \gamma_o = 0 \tag{4.51}$$

we obtain the dispersion relation

$$i\omega_{ak} = -\nu\gamma_o k^2 \tag{4.52}$$

or

$$\omega_{ak} = i\nu\gamma_o k^2 \quad , \tag{4.53}$$

in the agreement with the well-known results.[16]

Substituting Equations (4.46), (4.47) in Equation (4.45), we find

$$H_A = \sum_k \omega_{ak} a^*_k a_k \quad . \tag{4.54}$$

We may also expand the momenta conjugate to the A fields:

$$p = \sum_k p_k(t) u_k(r) \quad , \tag{4.55}$$

$$\bar{p} = \sum_k \bar{p}_k(t) \bar{u}_k(r) \quad . \tag{4.56}$$

Then, from the relations

$$p = \frac{\rho}{2c^2} (\dot{\bar{A}} - \tfrac{1}{2}\nu \nabla x \nabla x \bar{A} \gamma_o) \tag{4.57}$$

and

$$\bar{p} = \frac{\rho}{2c^2}\,(\dot{A} + \tfrac{1}{2}\nu\gamma_o\nabla x\nabla x A) \tag{4.58}$$

we find, using Equations (4.46), (4.47), (4.52), and (4.53),

$$p_k = i\sqrt{\rho\omega_{ak}/8c^2}\; a_k^*\,\theta_{ak} \tag{4.59}$$

and

$$\bar{p}_k = -\,i\sqrt{\rho\omega_{ak}/8c^2}\; a_k\,\theta_{ak}^* \quad . \tag{4.60}$$

The inverse transformations corresponding to H_A are

$$a_k = \sqrt{\rho\omega_{ak}/2c^2}\; A_k\,\theta_{ak} \quad , \tag{4.61}$$

$$a_k^* = \sqrt{\rho\omega_{ak}/2c^2}\; \bar{A}_k\,\theta_{ak}^* \quad , \tag{4.62}$$

$$a_k = i\sqrt{8c^2/\rho\omega_{ak}}\; \bar{p}_k\,\theta_{ak} \quad , \tag{4.63}$$

and

$$a_k^* = -i\sqrt{8c^2/\rho\omega_{ak}}\,p_k\,\theta_{ak}^* \quad . \tag{4.64}$$

We preserve the expression for H_A in terms of the functions $a_k^* a_k$ in standard form, while maintaining Poisson bracket relations in the diagonal representation similar to the ones commonly found in field theory. We recall that the complex first-order equations for the A fields (these fields are not explicitly of first order, but clearly are equivalent to first-order equations) may be related to a real field that obeys a truly second-order equation of motion. Let this field

be defined by the equations

$$A_k = \alpha_k + \dot{\alpha}_k / i\omega_{ak} \tag{4.65}$$

and

$$\bar{A}_k = \alpha_k - \dot{\alpha}_k / i\omega_{ak} \quad , \tag{4.66}$$

where

$$\ddot{\alpha}_k + \omega_{ak}^2 \alpha_k = 0 \quad . \tag{4.67}$$

Then we have

$$\dot{A}_k = i\omega_{ak} A_k \quad . \tag{4.68}$$

and

$$\dot{\bar{A}}_k = -i\omega_{ak} \bar{A}_k \quad . \tag{4.69}$$

Using our dispersion relation for the vector fields,

$$\omega_{ak} = - i\gamma_o \nu k^2 \quad , \tag{4.70}$$

and the relation

$$\nabla x \nabla x A_k = - k^2 A_k \quad , \tag{4.71}$$

we see that Equations (4.68) and (4.69) are equivalent to our equations of motion for the vector fields. At the same time, Equation (4.67) is the equation of motion for a harmonic oscillator with coordinate α_k, frequency ω_{ak}, and conjugate momentum p_{ak} given by

$$p_{\alpha k} = \dot{\alpha}_k \quad . \tag{4.72}$$

Consequently, we must have

$$[\alpha_k, \dot{\alpha}_{k'}] = [\alpha_k, p_{k'}] = \delta_{k,k'} \quad , \tag{4.73}$$

and

$$[\alpha_k, \alpha_{k'}] = [\dot{\alpha}_k, \dot{\alpha}_{k'}] = 0 \quad . \tag{4.74}$$

Equations (4.75) together with Equations (4.71) to (4.74) imply that

$$[A_k, p_{k'}] = [\bar{A}_k, \bar{p}_{k'}] = \frac{1}{2}\delta_{k,k'} \quad , \tag{4.75}$$

and

$$[a_k^*, a_{k'}] = i\delta_{k,k'} \quad , \tag{4.76}$$

as asserted following Equations (2.47).

From Equations (3.48) to (3.51), (3.68) to (3.71), (2.45), and (2.46) we obtain the relations

$$[x_i, x_j] = [x_i^*, x_j^*] = 0 \quad . \tag{4.77}$$

and

$$[x_i^*, x_j] = i\delta_{ij} \quad , \tag{4.78}$$

where the fields x_i are given by η_k and ζ_k. We may show this explicitly as follows:

$$[\eta_k^*, \eta_{k'}] = \frac{1}{4}(2i[\bar{\psi}_k, \bar{\pi}_{k'}] - 2i[\pi_k, \psi_k]) = i\delta_{k,k'} \tag{4.79}$$

and

$$[\zeta_k^*, \zeta_{k'}] = \frac{1}{4}(2i[\psi_k, \pi_{k'}] - 2i[\bar{\pi}_k, \bar{\psi}_{k'}]) = i\delta_{k,k'} \quad . \tag{4.80}$$

The factor i in Equation (4.78) is due to our use of Poisson brackets to describe classical fields rather than commutators to describe quantum fields.

4.3 Conserved Currents

We can define variables analogous to the number operators in quantum mechanics. Because we have a classic system described by Poisson brackets, we must introduce a factor of i in our definitions;

$$N_\eta = \sum_k n_{\eta k} \equiv \sum_k i\eta_k^* \eta_k \quad , \tag{4.81}$$

$$N_\zeta = \sum_k n_{\zeta k} \equiv \sum_k i\zeta_k^* \zeta_k \quad , \tag{4.82}$$

and

$$N_a = \sum_k n_{ak} \equiv \sum_k i a_k^* a_k \quad . \tag{4.83}$$

With these definitions we obtain the standard relations

$$[n_{\eta k}, \eta_{k'}] = -\eta_k \delta_{kk'} \quad , \tag{4.84}$$

$$[n_{\eta k}, \eta_{k'}^*] = \eta_k^* \delta_{kk'} \quad , \tag{4.85}$$

$$[n_{\zeta k}, \zeta_{k'}] = -\zeta_k \delta_{kk'} \quad , \tag{4.86}$$

$$[n_{\zeta k}, \zeta_{k'}^*] = \zeta_k^* \delta_{kk'} \quad , \tag{4.87}$$

$$[n_{ak}, a_{k'}] = -a_k \delta_{kk'} \quad , \tag{4.88}$$

and

$$[n_{ak}, a_{k'}^*] = a_k^* \delta_{kk'} \quad . \tag{4.89}$$

With

$$H_o = H_\psi + H_A \quad , \tag{4.90}$$

where

$$H_\psi = \sum_k (\omega_{\eta k} \eta_k^* \eta_k + \omega_{\zeta k} \zeta_k^* \zeta_k) \quad , \tag{4.91}$$

and

$$H_A = \sum_k \omega_{ak} a_k^* a_k \quad , \tag{4.92}$$

we can immediately obtain the time derivative of the diagonalized variables from relations of the form

$$\dot{\chi}_k = [\chi_k, H_o] \quad . \tag{4.93}$$

These relations are of a familiar form:

$$\dot{\eta}_k = -i\omega_{\eta k}\eta_k \quad , \quad \dot{\eta}_k^* = i\omega_{\eta k}\eta_k^* \quad , \tag{4.94}$$

$$\dot{\zeta}_k = -i\omega_{\zeta k}\zeta_k \quad , \quad \dot{\zeta}_k^* = i\omega_{\zeta k}\zeta_k^* \quad ,$$

and

$$\dot{a}_k = -i\omega_{ak}a_k \quad , \quad \dot{a}_k^* = i\omega_{ak}\dot{a}_k^* \quad . \tag{4.95}$$

We now look for variables that are conserved with respect to H_o. Some immediate examples are N_η, N_ζ, N_a and

$$m = i\rho\sum_k(\eta_k^*\eta_k - \zeta_k^*\zeta_k) \quad , \tag{4.96}$$

and

$$P = \sum_k k(\eta_k^*\eta_k + \zeta_k^*\zeta_k) \quad . \tag{4.97}$$

5. GREEN'S FUNCTIONS

The Green's function $G_\omega(r|r')$ is the solution to the equation

$$(H_A - \omega)G_\omega(r|r') = -\delta(r - r') \quad . \tag{5.1}$$

In Equation (5.1) we will assume that H_A is a scalar, as we have previously. All of the basis Hamiltonians which we have discussed are given in terms of a quadratic product of the field amplitudes, which are vectors. Thus far, we have

assumed that this product is a dot product, i.e.,

$$H_j \sim \langle \chi^* | \chi \rangle \tag{5.2}$$

for any Hamiltonian H_j in terms of the appropriate fields χ and χ^*.

5.1 Derivations

$G_\omega(r|r')$ is first expanded in terms of the eigenmodes of H_A, as is commonly done.[16,23,28-31] Usually a linear expansion is made, but in anticipation that since G_ω is the inverse of H_A it will be bilinear in the fields, we write

$$G_\omega(r|r') = \sum_{kk'} G_{kk'} a^*_{k'} e^{-ik'\cdot r'} a_k e^{ik\cdot r} \quad . \tag{5.3}$$

Substituting Equation (5.3) into Equation (5.1) we find[29]

$$\delta(r - r') = \sum_{kk'} G_{kk'}(\omega - \omega_{ak}) a^*_{k'} e^{-ik'\cdot r'} a_k e^{ik\cdot r} \quad . \tag{5.4}$$

We will continue with this formulation. We could also express the Hamiltonians in terms of the vector product, $|\chi\rangle\langle\chi|$, (See Appendix C), in which case we would have an operator formulation. In that case the discussion would parallel the dot product case, and all of the following treatment would apply. Most of the earlier discussion would also be valid, but it would be necessary to replace the Poisson brackets with commutators.

Since

$$\sum_k a_k a^*_k e^{ik\cdot(r' - r)} = \delta(r - r') \tag{5.5}$$

we must have

$$G_{kk'} = \frac{\delta_{kk'}}{\omega - \omega_{ak}} \quad . \tag{5.6}$$

Consequently,

$$G_\omega(r|r') = \sum_k \frac{a_k a_k^* e^{ik\cdot(r-r')}}{\omega - \omega_{ak}} \quad . \tag{5.7}$$

Our expression for G_ω is time independent. We may bring in the time dependence of the Green's function through the relations

$$a_k \rightarrow a_k e^{-i\omega t} \tag{5.8}$$

and

$$a_k^* \rightarrow a_k^* e^{i\omega t'} \quad . \tag{5.9}$$

Then we define the Fourier transform of G_ω as

$$G(r,t|r',t') = \frac{1}{2\pi}\int \sum_k \frac{a_k a_k^* e^{ik\cdot(r-r')-i\omega(t-t')}}{\omega - \omega_{ak}} d\omega \quad . \tag{5.10}$$

From Equation (5.5) we see that

$$G(r,t|r',t') = \frac{\delta(r - r')}{2\pi} \int d\omega \frac{e^{-i\omega(t-t')}}{\omega - \omega_{ak}} \quad . \tag{5.11}$$

Now from Equation (4.54)

$$\omega_{ak} = i\gamma_o \nu k^2 \tag{5.12}$$

is purely imaginary, so the pole in Equation (5.11) is on the imaginary axis. The contour integration gives, upon interchanging t and t',

$$G(r,t|r',t') = i\delta(r - r')e^{-\nu k^2(t-t')} \quad , \; t > t' \tag{5.13}$$

$$= 0 \quad , \; t < t' \quad .$$

Performing the integration over ω and integrating over

$$r = r - r' \tag{5.14}$$

in Equation (5.10) rather than summing over k gives

$$G(k,t-t') = G(k,\tau)$$

$$= i\sum_{k} a_k a_k^* e^{-\nu k^2 \tau} \tag{5.15}$$

$$= \int dr\, G(r,t|r',t') \quad . \tag{5.16}$$

Proceeding similarly we may find the Green's function $S_\omega(r|r')$ corresponding to H_ψ. S_ω must satisfy the equation.

$$(H_\psi - \omega)S_\omega(r|r') = -\delta(r - r') \quad . \tag{5.17}$$

Assume[18,19] that S_ω is of the form

$$S_\omega(r|r') = \sum_{kk}\left[S^{\eta}_{kk',\eta_k,\eta_k^*} e^{i(k\cdot r - k'\cdot r')} - S^{\zeta}_{kk',\zeta_{k'k}\zeta^*} e^{i(k\cdot r - k'\cdot r')}\right] . \tag{5.18}$$

Then, substituting Equation (5.18) into Equation (5.17), we find, from Equation (4.23) and an argument parallel to that above, that

$$S^{\eta}_{kk'} = \frac{\delta_{kk'}}{\omega - \omega_{\eta k}} \tag{5.19}$$

and

$$S^{\zeta}_{kk'} = \frac{\delta_{kk'}}{\omega - \omega_{\zeta k}} \quad . \tag{5.20}$$

Therefore we may write

$$S_\omega(r|r') = S^{\eta}_{\omega}(r - r') - S^{\zeta}_{\omega}(r - r') \tag{5.21}$$

$$= \sum_{k}\left[\frac{\eta_k \eta_k^*}{\omega - \omega_{\eta k}} - \frac{\zeta_k \zeta_k^*}{\omega - \omega_{\zeta k}}\right] e^{ik\cdot(r-r')} \quad . \tag{5.22}$$

The Fourier transform of S_ω, $S(r,t|r',t')$, is then given by

$$S(r,t|r',t') = \theta(\tau)S^{\eta}(r,t|r',t') - \theta(-\tau)S^{\zeta}(r,t|r',t') \tag{5.23}$$

$$= \int \frac{1}{2\pi} \sum_k \left[\frac{\eta_k \eta_k^*}{\omega - \omega_{\eta k}} - \frac{\zeta_k \zeta_k^*}{\omega + \omega_{\zeta k}} \right] d\omega \; e^{ik\cdot(r'-r) - i\omega(t-t')} \tag{5.24}$$

$$= \frac{\delta(r - r')}{2\pi} \int d\omega \left[\frac{1}{\omega - \omega_{\eta k}} - \frac{1}{\omega + \omega_{\zeta k}} \right] e^{-i\omega(t-t')} \; . \tag{5.25}$$

With

$$\omega_{\eta k} = \omega_k - iD\gamma_o k^2 \tag{5.26}$$

and

$$\omega_{\zeta k} = \omega_k + iD\gamma_o k^2 \tag{5.27}$$

the contour integral gives

$$S^{\eta}(r,t|r',t') = i\delta(r - r')e^{-i\omega_k \tau - Dk^2 \tau} \quad , \quad t > t' \tag{5.28}$$

$$= 0 \; , \; t < t' ,$$

$$S^{\zeta}(r,t|r',t') = -i\delta(r - r')e^{i\omega_k \tau - Dk^2 \tau} \quad , \quad t < t' \tag{5.29}$$

$$= 0 \; , \; t > t' \; .$$

From Equations (5.25) to (5.27) and (4.15) we find

$$S(r,t|r',t') = \frac{\delta(r - r')}{2\pi} \int d\omega \; \frac{2\omega_k}{\omega^2 - \omega_k^2 + 2i\omega\gamma_o Dk^2} \; e^{-i\omega(t-t')} \; . \tag{5.30}$$

Finally, we find

$$S(k,\tau) = \theta(\tau)S^{\eta}(k,\tau) - \theta(-\tau)S^{\zeta}(k,\tau) \tag{5.31}$$

$$= i \sum_k \eta_k \eta_k^* e^{-i\omega_k \tau - Dk^2\tau} \quad , \; t > t',$$

$$= -i \sum_k \zeta_k \zeta_k^* e^{i\omega_k \tau - Dk^2\tau} \quad , \; t < t' \; . \tag{5.32}$$

5.2 Density of States

The phase space volume Ω corresponding to a variable ϕ is given by

$$\Omega = \int d\phi \; d\pi = \int d\Gamma \quad . \tag{5.33}$$

From Equations (4.6) to (4.8), (4.32), (4.39), (4.46), (4.57), (4.58), (4.64) and (4.66) we see that

$$\int d\Gamma \rightarrow \sum_k \quad . \tag{5.34}$$

By way of illustration, let us assume an isotropic system. Then we have

$$\sum_k \rightarrow 4\pi \int k^2 dk \quad . \tag{5.35}$$

We wish to convert the integral in k space to a frequency integral:

$$4\pi \int k^2 dk \rightarrow \int \mathcal{D}(\omega) \; d\omega \quad . \tag{5.36}$$

This implies that the density of states, $\mathcal{D}(\omega)$ is given by

$$\mathcal{D}(\omega) = 4\pi \left| k^2(\omega) \right| / \left| d\omega/dk \right| \quad . \tag{5.37}$$

Thus we can find $\mathcal{D}(\omega)$, once we have the appropriate dispersion relation. The required relations are given in Equations (4.11), (4.12), (4.63), and (4.64). An immediate complication arises from the fact that $\omega_{\eta k}$ and $\omega_{\zeta k}$ are either complex or pure imaginary for $k \neq 0$, and ω_{ak} is pure imaginary. This means that the population of the corresponding states changes in time. Each value of k

corresponds to a unique value of $\omega_{\eta k}$, $\omega_{\zeta k}$, and ω_{ak}; so the dispersion relations define curves in the complex ω plane. Let the real part of a frequency ω be denoted by ω' and the imaginary part by ω''. Then we have[32]:

$$\frac{d\omega}{dk} = \left[\frac{d\omega'}{dk}^2 + \frac{d\omega''}{dk}^2 \right]^{1/2} . \tag{5.38}$$

For the A fields we have the simple relation

$$\omega'' = \omega_{ak} = \nu k^2 . \tag{5.39}$$

Consequently

$$| k^2 | dk = \frac{1}{2}\nu^{-1.5} \omega_a^{0.5} d\omega_a . \tag{5.40}$$

If we assume the existence of a cutoff frequency ω_m, we can normalize $\mathcal{D}_a(\omega)$ to obtain

$$\mathcal{D}_a(\omega) = \frac{3}{2} \frac{\omega_a^{\frac{1}{2}}}{\omega_m^{1.5}} . \tag{5.41}$$

For the scalar fields we have

$$\omega'_k = \sqrt{c^2k^2 - \tfrac{1}{4}Dk^4} \tag{5.42}$$

and

$$\omega''_k = \pm iDk^2 . \tag{5.43}$$

In Equation (5.43) the + sign refers to the ζ modes and the - sign to the η modes. From Equations (5.42) and (5.43) we find

$$| \omega^2_{\eta k} | = | \omega^2_{\zeta k} | = c^2k^2 . \tag{5.44}$$

Using Equations (5.38), (5.42), and (5.44) we find, with

$$w \equiv D\omega/2c^2 \tag{5.45}$$

and ω representing either ω_η or ω_ζ:

$$\mathcal{D}(\omega) = \tfrac{1}{2}w^2\sqrt{\frac{1 - w^2}{1 - 3w^2 + \frac{3}{2}w^4}} \quad . \tag{5.46}$$

In the limit

$$w << 1 \tag{5.47}$$

we find

$$\mathcal{D}(\omega) \to \tfrac{1}{2}\omega^2 \quad . \tag{5.48}$$

We note from Equation (5.42) that $\mathcal{D}(\omega')$ has a maximum value ω'_m given by

$$\mathcal{D}\left(\omega'_m\right) = \sqrt{2c^2/D} \quad . \tag{5.49}$$

This is because the contribution to ω'_k from diffusion causes the $\omega'(k)$ curve to bend over at large k, and in fact to reach zero at a critical value of k given by

$$k_c^2 = 2c/D \quad . \tag{5.50}$$

If D is very small, we can make the rough approximation

$$\omega \simeq \omega' \quad , \tag{5.51}$$

or

$$\omega'' \to 0 \quad . \tag{5.52}$$

6. DISCUSSION

The approach taken in Section 2 to handle the difficulties of dissipation is related to the work of Morse, Feshback, and others,[15,16,33,34] and in fact, is inspired by the earlier work of these authors. However, the treatment given here varies from earlier work not only in that it treats a more complicated system, but in the diagonalization of H. Earlier authors separated H into a conservative part that they diagonalized and a dissipative part that required special rules for incorporation into the theory. In Section 4, the entire Hamiltonian is completely diagonalized, and dissipation is incorporated directly into a formalism that is similar to that of more familiar systems. This approach is especially convenient for the derivation of Green's functions. A review of the papers cited above and other attempts to treat dissipation in a Lagrangian theory is given by Dekker.[35]

The diagonalized Hamiltonian for the scalar fields involves two species of functions in the diagonal or N representation. In field theory, this corresponds to the presence of a hidden variable that is not explicitly present in the equations of motion.[18-21] Usually this is a charge. Identifying this variable for the scalar fields thus becomes a central problem.

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MANY-BODY TREATMENT OF NAVIER-STOKES FLUIDS

PART II, EFFECTS OF NONLINEAR TERMS

1. INTRODUCTION

The primary use of a field theory treatment for fluid flow will stem from the nature of the perturbation generated. The character of such perturbation series is determined by the relationship between the interaction terms and the basis Hamiltonian, H_o. In Part I, the bulk of the formalism for a many-body treatment of subsonic Navier-Stokes fluids was developed, although the discussion was confined to H_o. In the following text, the nonlinear terms in the full Hamiltonian, H, are considered. The discussion will be directed towards flows with low compressibility, so that the nonlinear terms arising from pressure gradients will be ignored.

We have succeeded in describing coupling of the scalar and vector potential modes. This is done by expanding H_I in terms of the diagonalized fields, and then constructing a linear transformation of those fields that yield a new rediagonalization, given certain approximations. The new dispersion relations have a non-zero threshold level if the vector and scalar fields are both excited. Also, the analysis of mode coupling due to the nonlinear terms in the Hamiltonian has been extended to examine several similarities with the literature on critical behavior and suggest a connection with "deterministic chaos."

2. MODE COUPLING

The perturbation Hamiltonians, H_I and H_{NL}, are now included along with H_o to form the total Hamiltonian H:

$$H = H_o + H_I + H_{NL} \quad . \tag{2.1}$$

The objective is to diagonalize H. The Hamiltonian density, $\mathcal{H}_I$ is easily obtained from $\mathcal{L}_I$, given in Equations (2.8) to (2.12) of Part I. We have

$$\mathcal{H}_I = \mathcal{H}_1 + \mathcal{H}_2 \quad , \tag{2.2}$$

with

$$\mathcal{H}_1 = \frac{\rho}{8c^2}(\gamma_o\dot{\psi} + \dot{\bar{\psi}}\gamma_o)(\nabla\psi - \nabla\bar{\psi})\cdot(\nabla x A - \nabla x \bar{A}) \tag{2.3}$$

and

$$\mathcal{H}_2 = \frac{\rho}{8c^2}(\gamma_o\dot{\psi} + \dot{\bar{\psi}}\gamma_o)(\nabla x A - \nabla x \bar{A})\cdot(\nabla x A - \nabla x \bar{A}) \quad . \tag{2.4}$$

Equations (2.3) and (2.4) are expressed in terms of the time derivative of the generalized coordinates rather than conjugate momenta for convenience. An alternative form for $\mathcal{H}_1$ and $\mathcal{H}_2$ valid in the limit of constant vorticity is discussed in Appendix E.

It will prove very convenient to use a compact notation in the remaining discussion. The Fourier-transformed variables defined in Section 4 of Part I are now expressed in the shorthand form

$$\eta_q \theta^-_{\eta q} e^{iq\cdot r} \rightarrow \eta_q \quad , \tag{2.5}$$

$$\eta^*_q \theta^+_{\eta q} e^{-iq\cdot r}\gamma_A \rightarrow \eta^\dagger_q \quad , \tag{2.6}$$

$$a_k \theta^*_{ak} e^{ik\cdot r} \rightarrow a_k \quad , \tag{2.7}$$

$$a^*_k \theta_{ak} e^{-ik\cdot r}\gamma_A \rightarrow a^\dagger_k \quad , \text{ etc.} \tag{2.8}$$

The use of the symbol † is not meant to convey a change from field variables to operators, as in quantum mechanics; rather it is intended to emphasize that the compressed notation includes γ_A. Using the transformed version of the field

variables given in Part I, Equations (4.2) through (4.7) and Equations (4.47) through (4.50), the following expressions are obtained:

$$\nabla x\bar{A} - \nabla xA = i\sum_k c\sqrt{2/\rho\omega_{ak}}\, kx(a_k^\dagger + a_k) \quad , \tag{2.9}$$

$$\gamma_o\dot{\psi} + \dot{\bar{\psi}}\gamma_o = i\gamma_o\sum_q \frac{c}{\sqrt{\rho\omega_q}}\left[\omega_{\eta q}(\eta_q^\dagger - \eta_q) + \omega_{\zeta q}(\zeta_q^\dagger - \zeta_q)\right] \quad , \tag{2.10}$$

$$\nabla\psi - \nabla\bar{\psi} = i\gamma_o\sum_q \frac{c}{\sqrt{\rho\omega_q}}\, q(\zeta_q^\dagger + \zeta_q - \eta_q^\dagger - \eta_q) \quad , \tag{2.11}$$

$$\nabla x\dot{\bar{A}}\gamma_o + \gamma_o\nabla x\dot{A} = c\gamma_o\sum_k \frac{\sqrt{2\omega_{ak}}}{\rho}\, kx(a_k^\dagger + a_k) \quad , \tag{2.12}$$

Throughout the discussion, frequent use will be made of the identity

$$\int_{-\infty}^{\infty} d^3r\; e^{i(p\pm q\pm k)\cdot r} = \delta(p \pm q \pm k) \quad . \tag{2.13}$$

We will also take liberties with our summations, noting that they go from -N to N with N large. It will be assumed that

$$\sum_k = \sum_{\bar{k}} = \sum_{k+q} \quad , \qquad N >> q \quad , \tag{2.14}$$

i.e., we assume that the dominant interactions occur at long wavelengths. We begin with H_1:

$$H_1 = \int d^3r\, \mathcal{H}_1$$

$$= \frac{-i\gamma_o c}{8\sqrt{\rho}} \sum_{qk} kx(a_k - a_{\bar{k}}^\dagger)\cdot q\left[\omega_{\eta q+k}^\dagger(\eta_{q+k}^\dagger - \eta_{q+k}) + \omega_{\zeta q+k}(\zeta_{q+k}^\dagger - \zeta_{q+k})\right]$$

$$(\zeta_q^\dagger + \zeta_q - \eta_q^\dagger - \eta_q)/\omega_{ak} \quad . \tag{2.15}$$

We also have

$$H_2 = \int d^3r \mathcal{H}_2$$

$$= \frac{\rho\gamma_o}{16c^2} \sum_{kq} \frac{2c^3}{\rho\omega_{ak}\sqrt{\rho\omega_q}} [\omega_{\eta q}(\eta_{\underline{q}}^\dagger - \eta_q) + \omega_{\zeta q}(\zeta_{\underline{q}}^\dagger - \zeta_q)]$$

$$\sum_j [kxa_k \cdot jxa_j \delta(k + j + q) + kxa_k^\dagger \cdot jxa_j \delta(j - k + q)$$

$$+ kxa_k \cdot jxa_j^\dagger \delta(k - j + q) + kxa_k^\dagger \cdot jxa_j^\dagger \delta(j + k - q)] \quad . \qquad (2.16)$$

H_2 reduces to

$$H_2 = \frac{c\gamma_o}{8\sqrt{\rho}} \sum_{kq} \frac{1}{\omega_{ak}\sqrt{\omega_q}} [\omega_{\eta q}(\eta_q - \eta_{\underline{q}}^\dagger) + \omega_{\zeta q}(\zeta_q - \zeta_{\underline{q}}^\dagger)]$$

$$\left\{ \left[k\cdot(k + q) a_k a_{\underline{k+q}} - 2k\cdot(k + q) a_{k+q}^\dagger a_k + k\cdot(k + q) a_{\underline{k}}^\dagger a_{k+q}^\dagger \right] \right.$$

$$\left. - \left[k\cdot a_{\underline{k+q}} (k + q)\cdot a_k - 2k\cdot a_{k+q}^\dagger (k + q)\cdot a_k + k\cdot a_{k+q}^\dagger (k + q)\cdot a_{\underline{k}}^\dagger \right] \right\} \quad . \qquad (2.17)$$

Finally, consider H_{NL}, which is given by

$$H_{NL} = \int d^3r \mathcal{H}_{NL}$$

$$= \frac{i\rho\gamma_o}{16c^2} \sum_{pq} \left(\frac{c}{\sqrt{\rho}} \right)^3 \frac{p\cdot(p + q)}{\sqrt{\omega_q \omega_p \omega_{p+q}}} [\omega_{\eta q}(\eta_q^\dagger - \eta_q) + \omega_{\zeta q}(\zeta_q^\dagger - \zeta_q)]$$

$$[\eta^{\dagger}_{p+q}\eta_p + \zeta^{\dagger}_{p+q}\zeta_p - \eta^{\dagger}_{p+q}\zeta_p - \zeta^{\dagger}_{p+q}\eta_p - \eta_{\underline{\quad}p+q}\eta_p - \zeta_{\underline{\quad}p+q}\zeta_p$$

$$+ \eta_{\underline{\quad}p+q}\zeta_p + \zeta_{\underline{\quad}p+q}\eta_p - \eta^{\dagger}_{\underline{\quad}p+q\,p}\eta^{\dagger} - \zeta^{\dagger}_{\underline{\quad}p+q\,p}\zeta^{\dagger} + \eta^{\dagger}_{\underline{\quad}p+q\,p}\zeta^{\dagger} + \zeta^{\dagger}_{\underline{\quad}p+q\,p}\eta^{\dagger} + \eta^{\dagger}\eta_{p\,p+q}$$

$$+ \zeta^{\dagger}_p\zeta_{p+q} - \eta^{\dagger}_p\zeta_{p+q} - \zeta^{\dagger}_p\eta_{p+q}] \quad . \tag{2.18}$$

2.1 Canonical Transformations

H_{NL} can be transformed to obtain more manageable expressions. The needed transformations are

$$\mathcal{B}_k = u_k a_k - v_{\underline{k}}\,a^{\dagger}_{\underline{k}} \quad , \tag{2.19}$$

$$\mathcal{B}^{\dagger}_k = u^{\dagger}_k a^{\dagger}_k - v_{\underline{k}}\,a_{\underline{k}} \quad , \tag{2.20}$$

$$\mathcal{B}^{\dagger}_{k+q} = u^{\dagger}_{k+q} a^{\dagger}_{k+q} - v_{\underline{\quad}k+q}\,a_{\underline{\quad}k+q} \quad , \tag{2.21}$$

$$\mathcal{A}_k = k\cdot(u_k a_k - v_{\underline{k}}\,a^{\dagger}_{\underline{k}}) \quad , \tag{2.22}$$

$$\mathcal{A}^{\dagger}_k = k\cdot(u^{\dagger}_k a^{\dagger}_k - v_{\underline{k}}\,a_{\underline{k}}) \quad , \tag{2.23}$$

$$\mathcal{A}^{\dagger}_{k+q} = (k+q)\cdot(u^{\dagger}_{k+q} a^{\dagger}_{k+q} - v_{\underline{\quad}k+q}\,a_{\underline{\quad}k+q}) \quad , \tag{2.24}$$

$$c_q = \zeta_q - \eta^{\dagger}_q + \eta_q - \zeta^{\dagger}_{\underline{q}} \quad , \tag{2.25}$$

$$c^{\dagger}_q = \zeta^{\dagger}_q - \eta_q + \eta^{\dagger}_{\underline{q}} - \zeta_{\underline{q}} \quad , \tag{2.26}$$

$$d_q = \zeta_q + \eta^{\dagger}_{\underline{q}} - \eta_q - \zeta^{\dagger}_{\underline{q}} \quad , \tag{2.27}$$

$$d_q^\dagger = \zeta_q^\dagger + \eta_{\underline{q}} - \eta_q^\dagger - \zeta_{\underline{q}} \quad , \tag{2.28}$$

$$\phi_p = u_p(\eta_p - \zeta_p) - v_{\underline{p}}^\dagger(\eta_{\underline{p}}^\dagger - \zeta_{\underline{p}}^\dagger) \quad , \tag{2.29}$$

$$\phi_p^\dagger = u_p^\dagger(\eta_p^\dagger - \zeta_p^\dagger) - v_{\underline{p}}(\eta_{\underline{p}} - \zeta_{\underline{p}}) \quad , \tag{2.30}$$

and $$\phi_{p+q}^\dagger = u_{p+q}^\dagger(\eta_{p+q}^\dagger - \zeta_{p+q}^\dagger) - v_{\underline{\underline{p+q}}}(\eta_{\underline{\underline{p+q}}} - \zeta_{\underline{\underline{p+q}}}) \quad . \tag{2.31}$$

In Equations (2.19) to (2.21) and (2.29) to (2.31), as well as in the following material, the transformed Hamiltonians will only be equivalent to the original Hamiltonians if the respective u's and v's are equal to one. Unfortunately, if the transformations are to be canonical, the transformation functions must be chosen to satisfy

$$u_k^2 - v_{\underline{k}}^2 = 1 \quad , \tag{2.32}$$

$$u_q^2 - v_{\underline{q}}^2 = 1 \quad , \tag{2.33}$$

and

$$u_p^2 - v_{\underline{p}}^2 = 1 \quad . \tag{2.34}$$

These requirements overdetermine the u's and v's. A compromise may be found in which (letting j represent k, p, or q)

$$u_j = N_j + \epsilon_j \tag{2.35}$$

and

$$v_j = N_j - \epsilon_j \tag{2.36}$$

where

$$N_j >> 1 \tag{2.37}$$

and

$$\epsilon_j << 1 \tag{2.38}$$

in such a way that

$$N_j \epsilon_j = 1 \quad . \tag{2.39}$$

The error for finite N is equal to $1/16N_j^2$ if the eigenvalues of the diagonalized terms in the respective Hamiltonians are reduced by N_j^2 for each j. In the remainder of this treatment, difficulties with proper specifications for u_j and v_j will be ignored.

Using Equations (2.29) and (2.31) H_{NL} may be rewritten as

$$H_{NL} = \frac{-ic\gamma_o}{16\sqrt{\rho}} \sum\sum_{pq} [\omega_{\eta q}(\eta_q - \eta_{\underline{q}}^\dagger) + \omega_{\zeta q}(\zeta_q - \zeta_{\underline{q}}^\dagger)] \phi_{p+q}^\dagger \phi_p \sqrt{\omega_p \omega_q \omega_{p+q}}\, p\cdot(p+q)$$

$$= H_3 + H_4 \quad . \tag{2.40}$$

Similarly, H_2 may be expressed as

$$H_2 = \frac{c\gamma_o}{8\sqrt{\rho}} \sum_{kq} \frac{1}{\omega_{ak}\sqrt{\omega_q}} [\omega_{\eta q}(\eta_q - \eta_{\underline{q}}^\dagger) + \omega_{\zeta q}(\zeta_q - \zeta_{\underline{q}}^\dagger)]\, b_{k+q}^\dagger b_k\, k\cdot(k+q)$$

$$= H_5 + H_6 \quad , \tag{2.41}$$

where the shorthand notation

$$k\cdot(k+q) b_{k+q}^\dagger b_k = k\cdot(k+q) \beta_{k+q}^\dagger \beta_k - \ell_{k+q}^\dagger \ell_k \tag{2.42}$$

is used.

H_1 is simplified through the change

$$H_1 = \frac{-i\gamma_o c}{8\sqrt{\rho}} \sum_{qk} kx(a_k - a_{\underline{k}}^\dagger)\cdot q \left[\omega_{q+k} c_{q+k}^\dagger c_q - i\nu_{q+k} d_{q+k}^\dagger d_q\right]/\omega_{ak} \tag{2.43}$$

where

$$\nu_p = D\gamma_o p^2 \quad . \tag{2.44}$$

Depending on the mean amplitude of the A and ψ fields it will be convenient to group the various terms in H as either

$$H = (H_\eta + H_3 + H_5) + (H_\zeta + H_4 + H_6) + (H_A + H_1) \tag{2.45}$$

or as

$$H = (f_1 H_\psi + H_1) + (f_2 H_\psi + H_{NL}) + (H_A + H_2) \quad , \tag{2.46}$$

where f_1 and f_2 are fractions such that

$$f_1 + f_2 = 1 \tag{2.47}$$

and we define

$$H_\eta = \sum_q \omega_{\eta q} \eta_q^\dagger \eta_q \tag{2.48}$$

$$H_\zeta = \sum_q \omega_{\zeta q} \zeta_q^\dagger \zeta_q \quad . \tag{2.49}$$

Consider the grouping given by Equation (2.45). We write this as

$$H = H_\beta + H_r + H_s \quad . \tag{2.50}$$

We have

$$H_\beta = \sum_k [\omega_{ak} a_k^\dagger a_k - \frac{ic\gamma_o}{4\sqrt{2\rho\omega_{ak}}} kx(a_k - a_{\underline{k}}^\dagger) \cdot \sum_q q(\omega_{q+k} c_{q+k}^\dagger c_q - i\nu_{q+k} d_{q+k}^\dagger d_q)/\omega_q] \tag{2.51}$$

$$= \sum_k [\omega_{ak} a_k^\dagger a_k - if\gamma_o(a_k - a_{\underline{k}}^\dagger) \cdot \sum_q q(\omega_{q+k} n_{cqk} - i\nu_{q+k} n_{dqk})/\omega_q] \quad , \tag{2.52}$$

$$H_r = \sum_q [\omega_{\eta q} \eta_q^\dagger \eta_q + \frac{c\gamma_o}{8\sqrt{\rho\omega_q}} \omega_{\eta q} (\eta_q - \eta_{\underline{q}}^\dagger) \sum_k k(k+q) b_{k+q}^\dagger b_k / \omega_{ak}$$

$$- \frac{ic\gamma_o}{16\sqrt{\rho\omega_q}} \omega_{\eta q} (\eta_q - \eta_{\underline{q}}^\dagger) \sum_p p(p+q) \phi_{p+q}^\dagger \phi_p / \sqrt{\omega_p \omega_{p+q}}] \qquad (2.53)$$

$$= \sum_q [\omega_{\eta q} \eta_q^\dagger \eta_q + g_q \gamma_o (\eta_q - \eta_{\underline{q}}^\dagger) \sum_k k(k+q) n_{bkq} / \omega_{ak}$$

$$- \frac{i}{2} g_q \gamma_o (\eta_q - \eta_{\underline{q}}^\dagger) \sum_p p(p+q) n_{\phi pq} \sqrt{\omega_p \omega_{p+q}}] \quad , \qquad (2.54)$$

and

$$H_s = \sum_q \Big[\omega_{\zeta q} \zeta_q^\dagger \zeta_q + \frac{c\gamma_o}{8\sqrt{\rho\omega_q}} \omega_{\zeta q} \Big(\zeta_q - \zeta_{\underline{q}}^\dagger \Big) \sum_k k(k+q) b_{k+q}^\dagger b_k / \omega_{ak}$$

$$- \frac{ic\gamma_o}{16\sqrt{\rho\omega_q}} \omega_{\zeta q} \Big(\zeta_q - \zeta_{\underline{q}}^\dagger \Big) \sum_p p(p+q) \phi_{p+q}^\dagger \phi_p \sqrt{\omega_p \omega_{p+q}} \Big] \qquad (2.55)$$

$$= \sum_q \Big[\omega_{\zeta q} \zeta_q^\dagger \zeta_q + g_q^\dagger \gamma_o \Big(\zeta_q - \zeta_{\underline{q}}^\dagger \Big) \sum_k k(k+q) n_{bkq} / \omega_{ak}$$

$$- \frac{i}{2} g_q^\dagger \sum_k p(p+q) n_{\phi pq} \sqrt{\omega_p \omega_{p+q}} \Big] \quad . \qquad (2.56)$$

2.2 Emery Transformations

Decoupled fields are now obtained by applying canonical transformations to Equations (2.48), (2.50), and (2.52) of a type originally due to Emery.[1,2]

H_β is transformed by means of

$$J_k = \sum_q (\omega_{q+k} n_{cqk} - i\nu_{q+k} n_{dqk}) / \omega_q \quad , \qquad (2.57)$$

$$\omega_{ak} = i\nu k^2 = -\omega_{ak}^* \quad , \tag{2.58}$$

$$a_k = \beta_k + if_k\gamma_o J_{\underline{k}}/\omega_{ak} \quad , \tag{2.59}$$

$$a_k^\dagger = \beta_k^\dagger + if_k\gamma_o J_k/\omega_{ak} \quad , \tag{2.60}$$

$$a_{\underline{k}}^\dagger = \beta_{\underline{k}}^\dagger + if_k\gamma_o J_{\underline{k}}/\omega_{ak} \quad . \tag{2.61}$$

In Equation (2.54) we make use of Equation (2.52) and the fact that

$$J_k^\dagger = -J_{\underline{k}} \quad , \tag{2.62}$$

as can be seen from Equations (2.47) and (2.48). Using Equations (2.55) through (2.57) in the relation

$$H_\beta = \sum_k \left[\omega_{ak} a_k^\dagger a_k + if_k\gamma_o(a_k - a_{\underline{k}}^\dagger)J_k\right] \tag{2.63}$$

we find, remembering that

$$\sum_k \beta_k J_{\underline{k}} = \sum_k \beta_{\underline{k}} J_k \tag{2.64}$$

(Since the sums go from -N to N), and that $\gamma_o^2 = -1$:

$$H_\beta = \sum_k \left[\omega_{ak}\beta_k^\dagger\beta_k - f_k^2 J_k^2/\omega_{ak}\right] \quad . \tag{2.65}$$

This shows that the normal modes corresponding to the vector potential in the excited medium are dressed.

H_r is transformed in a similar manner. We rewrite Equation (2.53) as

$$H_r = \sum_q [\omega_{\eta q}\eta_q^\dagger\eta_q + g_q\gamma_o(\eta_q - \eta_{\underline{q}}^\dagger)[k(k+q)J_{bq} - ip(p+q)J_{\phi q}]] \tag{2.66}$$

and make the transformation

$$\eta_q = r_q + g_q\gamma_o[k(k+q)J_{b\overline{q}} - ip(p+q)J_{\phi\overline{q}}]/\omega_{\eta q} \quad , \tag{2.67}$$

$$\eta_q^\dagger = r_q^\dagger - g^*\gamma_o[k(k+q)J_{bq} - ip(p+q)J_{\phi q}]\ /\omega_{\eta q}^* \quad , \tag{2.68}$$

and

$$\eta_{\underline{q}}^\dagger = r_{\underline{q}}^\dagger + g^*\gamma_o[k(k+q)J_{b\underline{q}} - ip(p+q)J_{\phi\underline{q}}]\ /\omega_{\eta q}^* \quad . \tag{2.69}$$

From Equations (2.49), (2.50), and (2.61), it follows that

$$J_{bq}^\dagger = J_{b\underline{q}} \tag{2.70}$$

and

$$J_{\phi q}^\dagger = J_{d\underline{q}} \quad . \tag{2.71}$$

Furthermore, since $g_q \sim \omega_{\eta q}$,

$$g_q^*/\omega_{\eta q}^* = g_q/\omega_{\eta q} \quad . \tag{2.72}$$

Using Equations (2.61) through (2.67) we find

$$H_r = \sum_q \Big\{ \omega_{\eta q} r_q^\dagger r_q + g_q^2[k(k+q)J_{b\underline{q}} - ip(p+q)J_{\phi\underline{q}}]$$

$$[k(k+q)J_{bq} - ip(p+q)J_{\phi q}]/\omega_{\eta q}^* \Big\} \quad . \tag{2.73}$$

Clearly, we may make a like transformation for H_s. To wit:

$$H_s = \sum_q [\omega_{\zeta q}\zeta_q^\dagger\zeta_q + g_q^*\gamma_o(\zeta_q - \zeta_{\underline{q}}^\dagger)][k(k+q)J_{bq} - ip(p+q)J_{\phi q})]$$

$$\zeta_q = s_q + g_q^*\gamma_o[k(k+q)J_{b\underline{q}} - ip(p+q)J_{\phi q}]/\omega_{\zeta q} \quad , \tag{2.74}$$

$$\zeta_q^\dagger = s_q^\dagger - g_q\gamma_o[k(k+q)J_{bq}^\dagger - ip(p+q)J_{\phi q}]\ /\omega_{\zeta q}^* \quad , \tag{2.75}$$

$$\zeta_q^\dagger = s_q^\dagger + g_q\gamma_o[k(k+q)J_{bq}^\dagger - ip(p+q)J_{\phi q}]\ /\omega_{\zeta q}^* \quad ; \tag{2.76}$$

leading to the result

$$H_s = \left\{ \sum_q \omega_{\zeta q} s_q^\dagger s_q - g_q^{*2} [k(k+q) J_{bq} - ip(p+q) J_{\phi q}] \right.$$

$$\left. [k(k+q) J_{bq} - ip(p+q) J_{\phi q}] / \omega_{\zeta q}^* \right\} \quad . \tag{2.77}$$

3. COLLECTIVE EXCITATIONS

The form of the Hamiltonians H_β, H_r, and H_s is very similar to that of electron-phonon coupling.[3-9] This suggests that canonical transformations can be made to new Hamiltonians H'_β, H'_r, and H'_s, which contain terms that are only quadratic and fourth order in the fields, with no terms containing both vector and scalar fields, and that the new Hamiltonians can be analyzed along the lines of the BCS theory for superconductivity.[8-18] We now pursue each of the desired transformations in turn. H_β is given by Equation (2.49):

$$H_\beta = \sum_k [\omega_{ak} a_k^\dagger a_k - i\gamma_o f_k kx(a_k - a_{\underline{k}}^\dagger) \cdot \sum_q q(\omega_{q+k} c_{q+k}^\dagger c_q - i\nu_{q+k} d_{q+k}^\dagger d_q)/\omega_q] \quad , \tag{3.1}$$

where

$$f_k = c/4\sqrt{2\rho\omega_{ak}} \quad . \tag{3.2}$$

Equation (3.1) may be written as

$$H_\beta = H_{\beta 0} + H_{\beta 1} \quad . \tag{3.3}$$

The transformed Hamiltonian, H'_β, may be written as

$$H'_\beta = H_{\beta_o} + H'_{\beta_1} \quad , \tag{3.4}$$

with

$$H_{\beta o} = \sum_k \omega_{ak} a_k^* a_k \tag{3.5}$$

and, (see Appendix F) again recalling that $\gamma_o^2 = -1$,

$$H'_{\beta 1} = -\sum_k \omega_{ak} k^2 f_k^2$$

$$\sum_{qq'} \frac{\omega_{q+k}\,\omega_{q'-k} c^\dagger_{q+k} c_q c^\dagger_{q'-k} c_{q'}}{[(\omega_{cq} - \omega_{cq-k})^2 + \nu^2 k^4]\omega_q^2} + \frac{\nu_{q+k}\,\nu_{q'-k} d^\dagger_{q+k} d_q d^\dagger_{q'-k} d_{q'}}{[(\omega_{dq} - \omega_{dq-k})^2 + \nu^2 k^4]\omega_q^2} \quad . \tag{3.6}$$

The meaning of the frequencies ω_{cq}, $\omega_{cq-k'}$, ω_{dq}, and $\omega_{dq-k'}$ is discussed in Appendix F, as are the frequencies ω_{bq}, $\omega_{bq-k'}$, ω_{dq}, and $\omega_{\phi q-k'}$ to be found below.

H_r is given by Equations (2.50) and (2.51):

$$H_r = \sum_q \left\{ \omega_{\eta q} \eta^\dagger_q \eta_q + \gamma_o g_q (\eta_q - \eta^\dagger_{\underline{q}}) \right.$$

$$\left. \left[\sum_k k(k+q) b^\dagger_{k+q} b_k / \omega_{ak} - \frac{i}{2} p(p+q) \phi^\dagger_{p+q} \phi_p / \sqrt{\omega_p \omega_{p+q}} \right] \right\} \quad . \tag{3.7}$$

with

$$g_q = c\omega_{\eta q} / 8\sqrt{\rho \omega_q} \quad . \tag{3.8}$$

We write Equation (3.7) as

$$H_r = H_{r0} + H_{r1} \quad , \tag{3.9}$$

and obtain a transformed Hamiltonian H'_r given by (see Appendix F)

$$H'_r = H_{r0} + H'_{r1} \quad , \tag{3.10}$$

where

$$H_{r0} = \sum_q \omega_{\eta q} \eta^\dagger_q \eta_q \tag{3.11}$$

and

$$H_r' = \frac{1}{2}\sum_q g_q^2\omega_q \left\{ \sum_{pp'} p(p+q)p'(p'+q)\phi^\dagger_{p+q}\phi_p\phi^\dagger_{p'-q}\phi_{p'}/[(\omega_{\phi p} - \omega_{\phi p-q})^2 - \omega_{\eta q}^2]\omega_p\omega_{p+q} \right.$$

$$\left. - \sum_{kk'} k(k+q)k'(k'+q)b^\dagger_{k+q}b_k b^\dagger_{k'-q}b_{k'}/[(\omega_{bk} - \omega_{bk-q})^2 + \omega_{\eta q}^2]\omega_{ak}^2 \right\}. \quad (3.12)$$

We have used a - sign before the second term in Equation (3.12) and in front of ω_q^2 within that term as a reminder that the frequencies ω_{bk} and ω_{bk-q} are purely imaginary.

The expressions for H_s and H_s' are similar to those for H_r and H_r':

$$H_s = H_{s0} + H_{s1} = \sum_q \left\{ \omega_{\zeta q}\zeta^\dagger_q\zeta_q + \gamma_o g_q(\zeta_q - \zeta^\dagger_q) \right.$$

$$\left. \left[\sum_k k(k+q)b^\dagger_{k+q}b_k/\omega_{ak} - \frac{i}{2}p(p+q)\phi^\dagger_{p+q}\phi_p/\sqrt{\omega_p\omega_{p+q}} \right] \right\} , \quad (3.13)$$

where

$$g_q^* = c\omega_{\zeta q}/8\sqrt{\rho\omega_q} \quad ; \quad (3.14)$$

$$H_s' = H_{s0} + H_{s1}' \quad , \quad (3.15)$$

where

$$H_{s0} = \sum_q \omega_{\zeta q}\zeta^\dagger_q\zeta_q \quad ; \quad (3.16)$$

and

$$H'_{s1} = -\frac{1}{2}g_q^{*2}\sum\omega_q$$

$$\left\{\sum_{pp'} p(p+q)p'(p'+q)\phi^{\dagger}_{p+q}\phi_p\phi^{\dagger}_{p'-q}\phi_{p'}[(\omega_{\phi p}-\omega_{\phi p-q})^2-\omega_{\zeta q}^2]\omega_p\omega_{p+q}\right.$$

$$\left.+\sum_{kk'} k(k+q)k'(k'+q)b^{\dagger}_{k+q}b_k b^{\dagger}_{k'-q}b_{k'}[(\omega_{bk}-\omega_{bk-q})^2+\omega_{\zeta q}^2]\omega_{ak}^2\right\} . \quad (3.17)$$

We now make the definitions:

$$\Delta'_{bq} = \frac{1}{2}g_q^2\omega_q\sum_k \frac{k(k+q)b^{\dagger}_{k+q}b_k}{[(\omega_{bk}-\omega_{bk-q})^2+\omega_{nq}^2]\omega_{ak}^2} , \quad (3.18)$$

$$\Delta''_{bq} = \frac{1}{2}g_q^{*2}\omega_q\sum_k \frac{k(k+q)b^{\dagger}_{k+q}b_k}{[(\omega_{bk}-\omega_{bk-q})^2+\omega_{\zeta q}^2]\omega_{ak}^2} , \quad (3.19)$$

$$\Delta_{ck} = \frac{1}{2}\omega_{ak}k^2f_k^2\sum_q \frac{\omega_{q+k}c^{\dagger}_{q+k}c_q}{[(\omega_{cq}-\omega_{cq-k})^2+\nu^2k^4]\omega_q^2} , \quad (3.20)$$

$$\Delta_{dk} = \frac{i}{2}\omega_{ak}k^2f_k^2\sum_q \frac{\nu_{q+k}d^{\dagger}_{q+k}d_q}{[(\omega_{dq}-\omega_{dq-k})^2+\nu^2k^4]\omega_q^2} , \quad (3.21)$$

$$\Delta'_{\phi q} = \frac{1}{2}g_q^2\omega_q\sum_p \frac{p(p+q)\phi^{\dagger}_{p+q}\phi_p}{[(\omega_{\phi p}-\omega_{\phi p-q})^2-\omega_{\zeta q}^2]\omega_p\omega_{p+q}} , \quad (3.22)$$

and

$$\Delta''_{\phi q} = \frac{1}{2} g^{*2}_q \omega_q \sum_p \frac{p(p+q)\phi^\dagger_{p+q}\phi_p}{[(\omega_{\phi p} - \omega_{\phi p-q})^2 - \omega^2_{\zeta q}]\omega_p \omega_{p+q}} \quad . \tag{3.23}$$

We now interchange the primed and unprimed variables in the double summations of Equations (3.6), (3.12), and (3.17), sum over the primed variables, and use the definitions (3.18) to (3.23) to write

$$H = H_\alpha + H_\chi \quad , \tag{3.24}$$

where

$$H_\alpha = \sum_k \left[\omega_{ak} a^\dagger_k a_k - \sum_q b^\dagger_{k-q} b_k (\Delta'^\dagger_{bq} + \Delta''_{bq}) \right] \quad , \tag{3.25}$$

and

$$H_\chi = \sum_q \Big\{ \left[\omega_{\eta q} \eta^\dagger_q \eta_q + \omega_{\zeta q} \zeta^\dagger_q \zeta_q \right] + \sum_k \left[\omega_{q-k} c^\dagger_{q-k} c_q \Delta_{ck} - i\nu_{q-k} \sum_k d^\dagger_{q-k} d_q \Delta_{dk} \right]$$

$$- \sum_p \phi^\dagger_{p-q} \phi_p (\Delta'_{\phi q} + \Delta''_{\phi q}) \Big\} \quad . \tag{3.26}$$

Equations (3.25) and (3.26) are now used to find the normal modes of the coupled system, given the averages assumed by the definitions (3.18) to (3.23). To simplify the problem, some sort of approximation is needed. Let us choose to replace the inner summations in Equations (3.25) and (3.26) with typical values, which we will designate with capital letters. For example, we let

$$\sum_p \rightarrow P \quad .$$

Then new canonical transformations can be made to obtain the dominant modes of the system. We first note from the definitions (3.18) to (3.23) that the various Δs have resonant denominators at small wavenumbers, so their main

contribution should occur at long wavelengths. Let us begin with H_α. With the transformations

$$B_{+k} = b_k + b_{k-Q} \tag{3.27}$$

and

$$B_{-k} = b_k - b_{k-Q} \tag{3.28}$$

the average value of the second term in H_α may be expressed as

$$(\Delta'_{bQ} + \Delta''_{bQ})(B^\dagger_{+k}B_{+k} - B^\dagger_{-k}B_{-k}) \quad .$$

In the limit as $Q \to 0$ and the Δs remain finite, we find

$$H_\alpha = \sum_k \left[\omega_{ak} a_k^\dagger a_k - 2(\Delta'_{bo} + \Delta''_{bo})(a_k - a_{\bar{k}}^\dagger)(a_k^\dagger - a_{\bar{k}})\right] \quad . \tag{3.29}$$

Again using a standard procedure[3], H_α is diagonalized with the transformations

$$\alpha_k = u_k a_k - v_{\bar{k}}^\dagger a_{\bar{k}}^\dagger \quad , \tag{3.30}$$

$$\alpha_k^\dagger = u_k^\dagger a_k^\dagger - v_{\bar{k}} a_{\bar{k}} \quad , \tag{3.31}$$

where

$$u_k^2 - v_{\bar{k}}^2 \to 1$$

and

$$u_k, v_k \to 1 \quad . \tag{3.32}$$

The eigenfrequency for the diagonalized Hamiltonian, $\omega_{\alpha k}$, is given by

$$\omega_{\alpha k} = \omega_{ok} + \omega_{1k} \quad , \tag{3.33}$$

where

$$\omega_{ok}^2 = \omega_{ak}^2 + \omega_{1k}^2 \quad , \tag{3.34}$$

and

$$\omega_{1k} = 2(\Delta'_{bo} + \Delta''_{bo}) \quad . \tag{3.35}$$

$\omega_{\alpha k}$ now is complex and is no longer a simple quadratic in k.

The expression for the scalar Hamiltonian is considerably more complicated than for the vector Hamiltonian. Consequently, it is helpful to make a number of simplifications. First we interchange the p and q indices in the last term in H_χ. We then make the following approximations:

- let $p,k \to 0$.
- let $\lim_{q\to 0} \Delta_{cq} = \Delta_{dq} = \Delta_o$,
- Keep only terms with the same value of q in the last (ϕ) term.
- Drop Dq^2 compared to cq for small q.

We then make the definitions

$$\Delta_\phi = \lim_{q\to 0}(\Delta'_{dq} + \Delta''_{\phi q}) \quad , \tag{3.36}$$

$$\bar{\omega}_q = \lim_{q,k\to 0} \frac{2q(\omega_{ak} + \omega_q)}{\sqrt{v_q w_q}} \Delta_o \quad , \tag{3.37}$$

$$\omega_{1q} = 2(\bar{\omega}_q - \Delta_\phi) \quad , \tag{3.38}$$

and

$$\omega_{oq} = \omega_q + 2\bar{\omega}_q + 2\Delta_\phi \quad ; \tag{3.39}$$

and find

$$H_\chi \approx \sum_q [\omega_{oq}(\eta_q^\dagger \eta_q + \zeta_q^\dagger \zeta_q) + \omega_{1q}(\eta_q^\dagger \zeta_q^\dagger + \eta_q \zeta_q)]. \tag{3.40}$$

Proceeding as for ω_α, we find the eigenfrequencies of H_χ to be given by

$$\omega_\chi^2 = \omega_o^2 - \omega_1^2 \quad . \tag{3.41}$$

The development given above has strong parallels with the Bogoliubov theory of superfluidity[8,9,15-19] and the BCS theory of superconductivity. However, the similarity in form of Equations (3.6), (3.12), (3.17), (3.25), and (3.26) to the Hamiltonians found in those theories is not sufficient to indicate similar critical behavior. In the BCS theory, the Poisson brackets are replaced with (Fermion) anticommutators, leading to a dispersion relation of the form

$$\omega^2 = \omega_o^2 + \omega_1^2 \quad , \tag{3.42}$$

where ω_1 is a constant. This produces a gap in the dispersion relation at low wavevectors. In the Bogoliubov theory, ω_o scales as k^2, with ω_1 again a constant. This leads to a different k dependence for the eigenfrequency as k increases. A dip in the dispersion curve can result for appropriate interaction potentials, creating an equivalent gap in the dispersion curve.

In the treatment given above, we have no gap or dip in the dispersion curve and hence no similar critical behavior. A gap will occur at small k or q if the system is bounded, but this boundary condition effect is different from an intrinsic bulk effect due to many-body interactions. Consequently, no clear conclusion can be reached regarding critical behavior, as the inclusion of higher order terms or an examination of the various Δs may yet indicate non-monotonic dispersion curves.

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MANY-BODY TREATMENT OF NAVIER-STOKES FLUIDS
PART III, FURTHER TOPICS

1. RULES FOR DIAGRAMS

Rules for drawing diagrams, summing them, and calculating the quantities they represent can be derived by examining the form of the Green's function given in Section 5 of Part I and following the diagram procedures used in quantum field theory.[1-8] The vector fields obey a diffusion equation, which is very similar in form to the Shroedinger equation for a nonrelativistic electron. One problem that arises is that the isolated disturbances of the solenoidal fields do not propagate, but simply diffuse. The Green's functions for the scalar fields are of the same form as those for phonons, which is not surprising, since the scalar oscillations are essentially sound waves. Therefore, the diagrams for the scalar fields should obey rules similar to those for phonon diagrams.

The following rules are very close to those given by Shultz:[1]

(1) Draw all connected, topologically nonequivalent diagrams with 2n vertices and two external points, where three lines meet at each vertex. At least one and as many as three of these lines may be dotted lines with the remainder (zero to two) drawn as solid lines.

(2) Write down the contribution from each diagram.

(3) Sum these contributions.

The contributions to a calculated quantity corresponding to various diagrams are as follows:

(1) For the contributions to the numerator of the two-particle Green function draw two points labelling them r_1' and r_2' (at which lines will start) and two points labelling them r_1 and r_2 (at which lines will end), anywhere on the paper. Every diagram will ultimately have a line with an arrow on it leaving each point labelled with an r' and a line

line arriving at each point labelled with an r. For the vacuum diagrams, there are no points labelled with r's or r"s.

(2) To get a contribution in which interaction occur n_2 times, draw n_2 points labelling them $y_1, y_2, \ldots . y_{n_2}$

(3) Add directed solid lines and dotted lines between points so that each "internal point" (i.e., a y or y') has three lines either going into or out of it, and each "external point" (i.e., an r or r') has one line out of it (if it is an r') or into it (if it is an r).

(4) For every element y_i •——— y_j, write $G_o(y_i|y_j)$.

(5) For every element y_i •---------• y_j, write $S(y_i|y_j)$. Dashed (phonon) lines need not be directed, because (S) = S(-k). It is usually convenient to direct phonon lines just to keep everything consistent in a diagram.

(6) For every element y_i, write $\rho_o(y_i) = \lim_{t \to +o} G(r_1 - r_2, -\tau)$.

(7) Integrate over all values from $-\infty$ to ∞ of the time and three spatial coordinates of every internal point.

In practical flow fields, it will be necessary to operate in coordinate space. However, since it is usually much easier to perform calculations in momentum space, the rules for summing diagrams in momentum space are also given:

(1) Draw all distinct structure, as in r-space. All vortex lines must be directed. Direct phonon lines for convenience.

(2) Assign p's and q's to all lines so that the sum of the energy and momenta entering a vertex equals the sum of the energy and momenta leaving a vertex.

(3) For every element $\xleftarrow{p}$ include a factor $G_o(p)$.

(4) For every element $\xleftarrow{q}$ include a factor $S(q)$.

An inspection of the nonlinear terms in the Hamiltonian, discussed in Part II, shows that there will be three types of vertices, depending on whether the number of solid lines is 0, 1, or 2 as shown in Figure 2. If there are two solid lines (Figure 2a), then a vertex factor of

$$U = \frac{\gamma_o c}{8\omega_{ak}} \sqrt{\frac{\omega_q}{\rho}}\, k(k + q) \tag{1.1}$$

should be included. If there is one solid line (Figure 2b), the vertex factor is

$$V = \frac{-i\gamma_o c}{8\sqrt{2\rho\omega_{ak}}} \frac{\omega_{k+q}}{\omega_k} kq \quad . \tag{1.2}$$

If there are only dotted lines (Figure 2c), the vertex factor is

$$W = \frac{i\gamma_o c}{16\sqrt{\rho\omega_{p+q}}} p(p + q) \quad . \tag{1.3}$$

Since the vertices each correspond to an odd number of fields, diagrams will contain an even number of vertices. Assume that the Linked Cluster Theorem holds

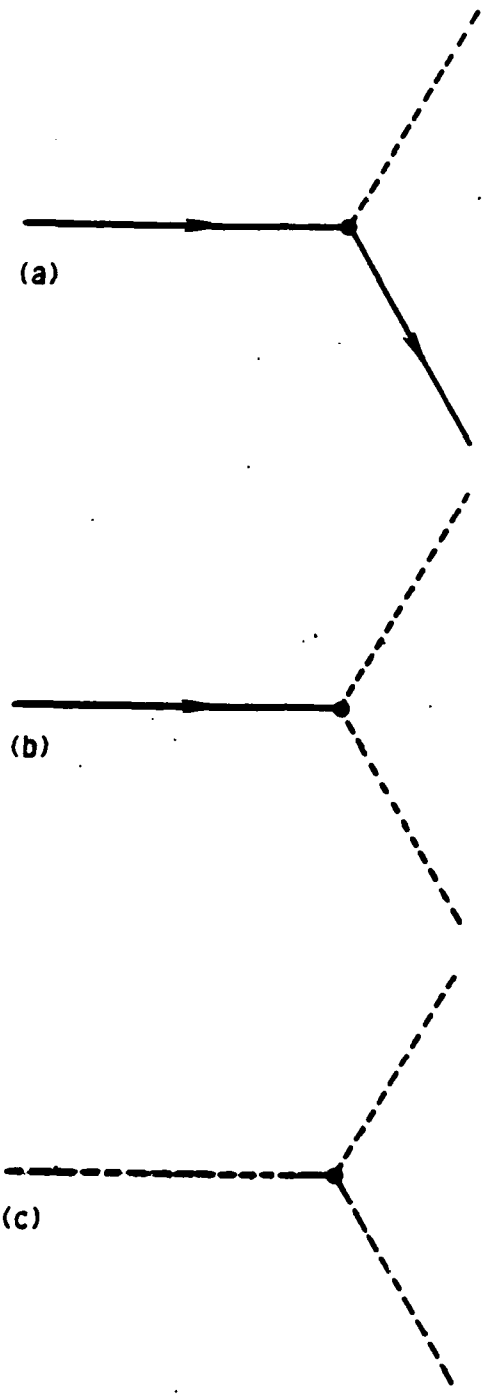


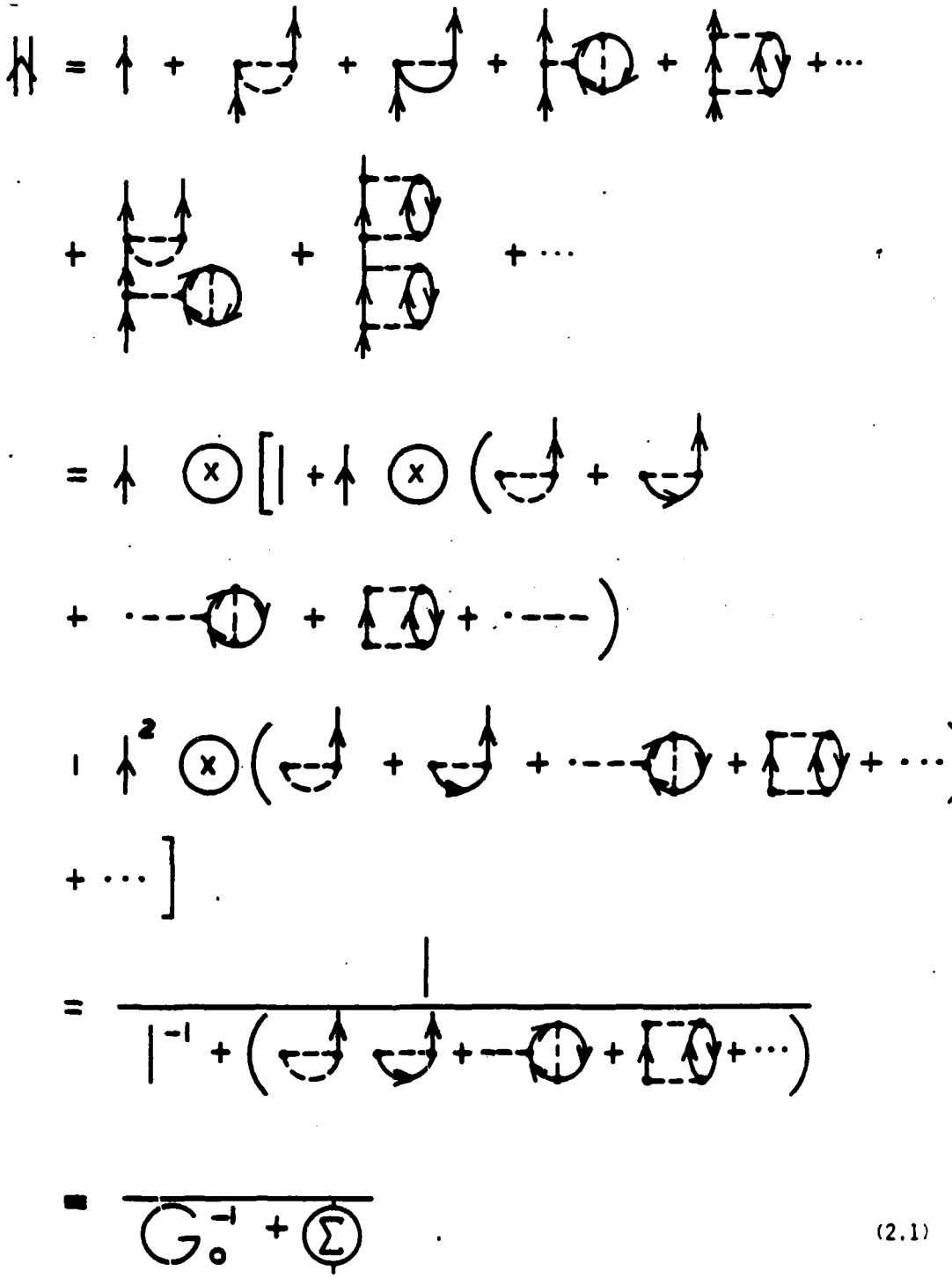
Figure 2. Basic Diagram Building Blocks.

and consider only linked graphs. The simplest two-vertex graphs are shown in Figures 3 and 4. Apparently, the "bubble graphs" shown in Figure 3 are not allowed except when the line joining the bubble to the rest of the diagram carries zero momentum. Some 4-vertex diagrams are shown in Figures 5 and 6. Obviously, the diagrams rapidly become complicated as the number of vertices increases.

Propagators are represented by uninterrupted lines; those without interactions are given by single lines, while those with interactions are given by double lines. Thus G is given by , G by , S by , and S by  .

2. SAMPLE DIAGRAM CALCULATIONS

Standard results can be borrowed from the literature on diagram calculations[3,9,10] in quantum many-body theory, albeit some of the diagrams for the Navier-Stokes problem will have a slightly different form than the corresponding condensed matter diagrams. First, some definitions[3] may be adopted without alteration. A self-energy part is defined as any diagram without external legs that can be inserted into a given line. A self-energy part which cannot be broken into two unconnected self-energy parts by removing one line is defined as an "irreducible" or "proper" self-energy part. Let us begin with Dyson's equation, taking the vortex propagator as an example. The procedure is to show that the sum of all proper diagrams can be arranged in a geometric series:



(2.1)

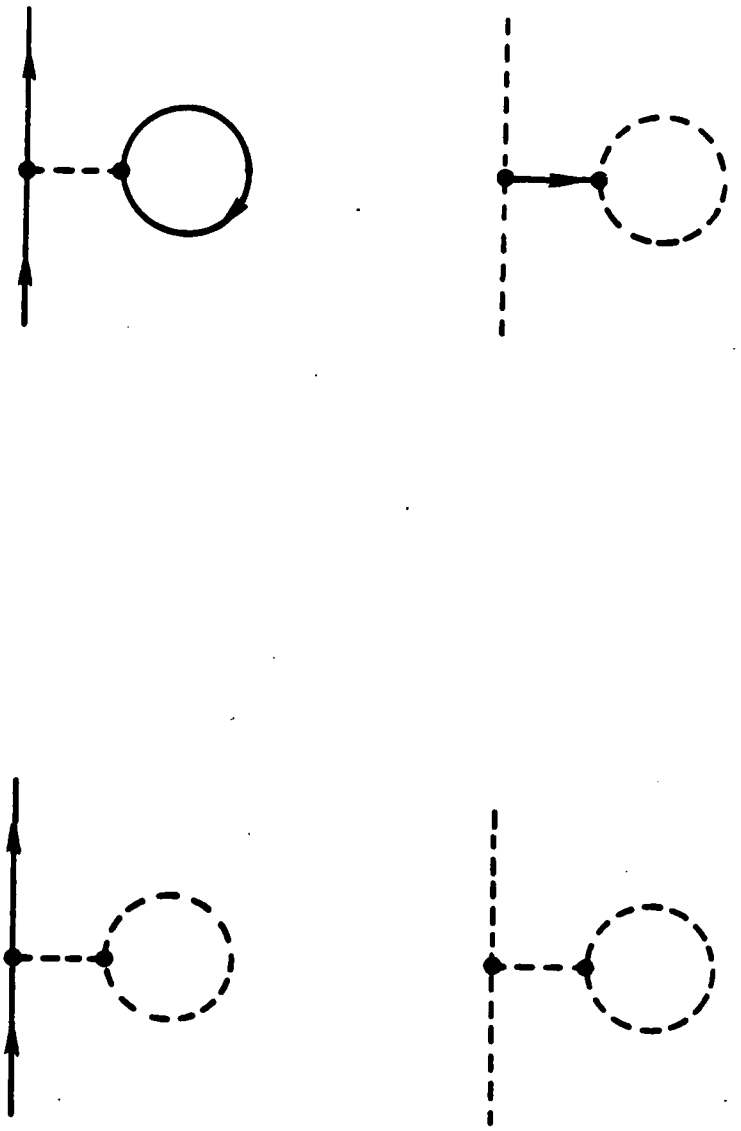


Figure 3. Two-Vertex Bubble Diagrams.

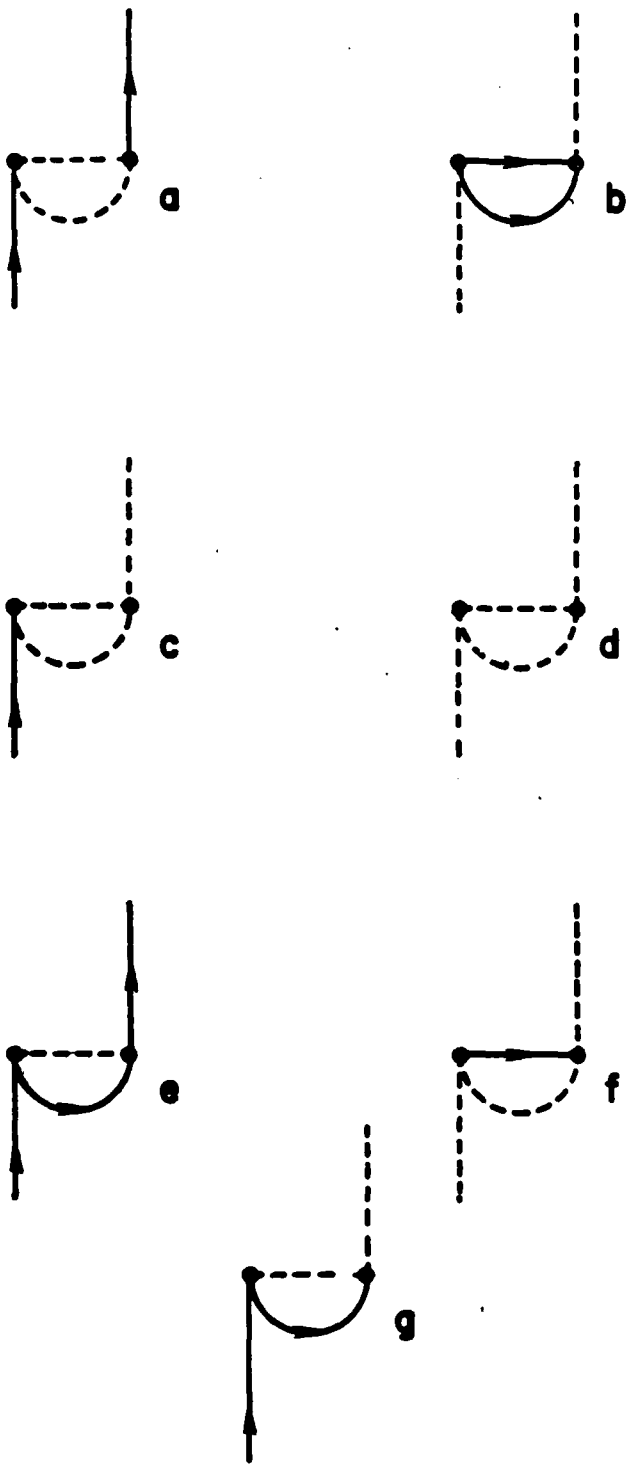


Figure 4. Oyster Diagrams.

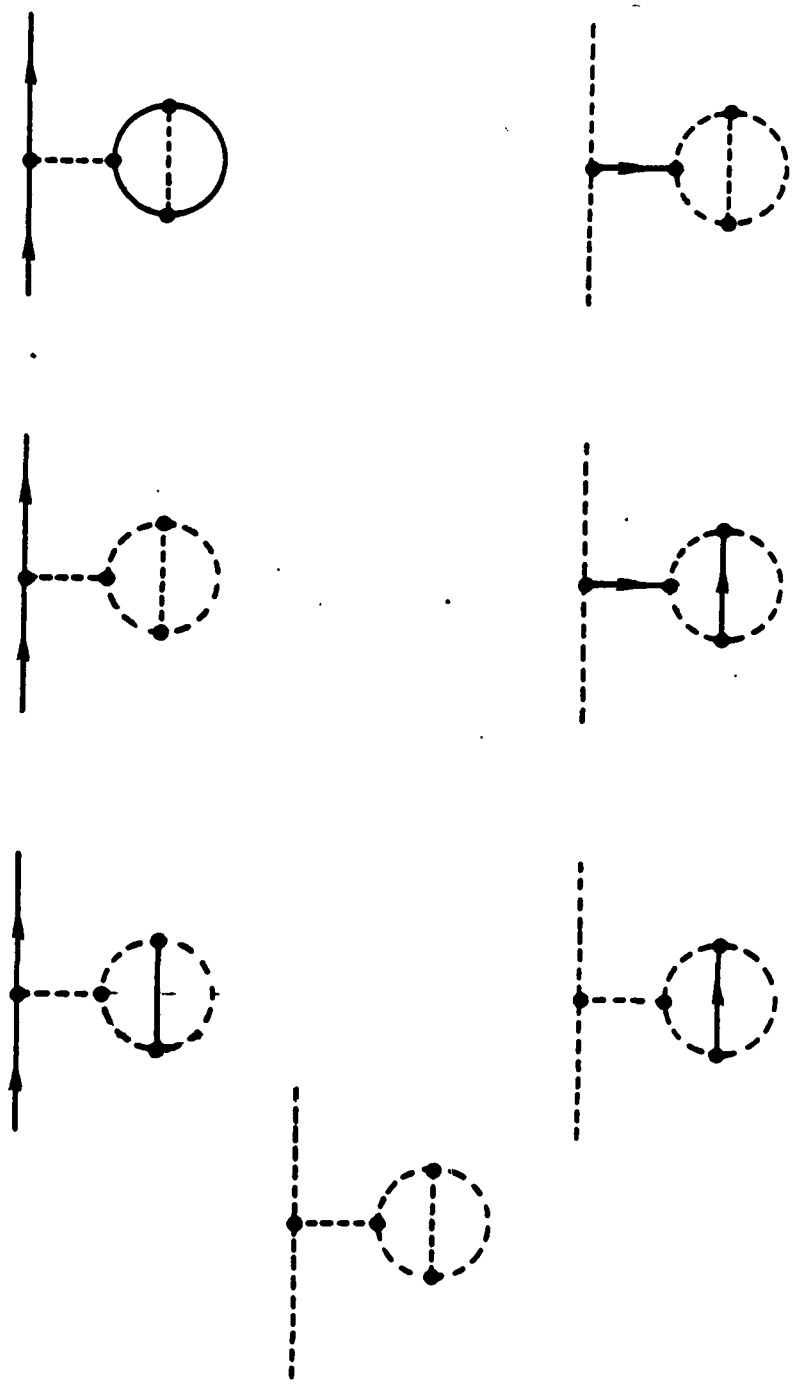


Figure 5. Four-Vertex Bubble Diagrams.

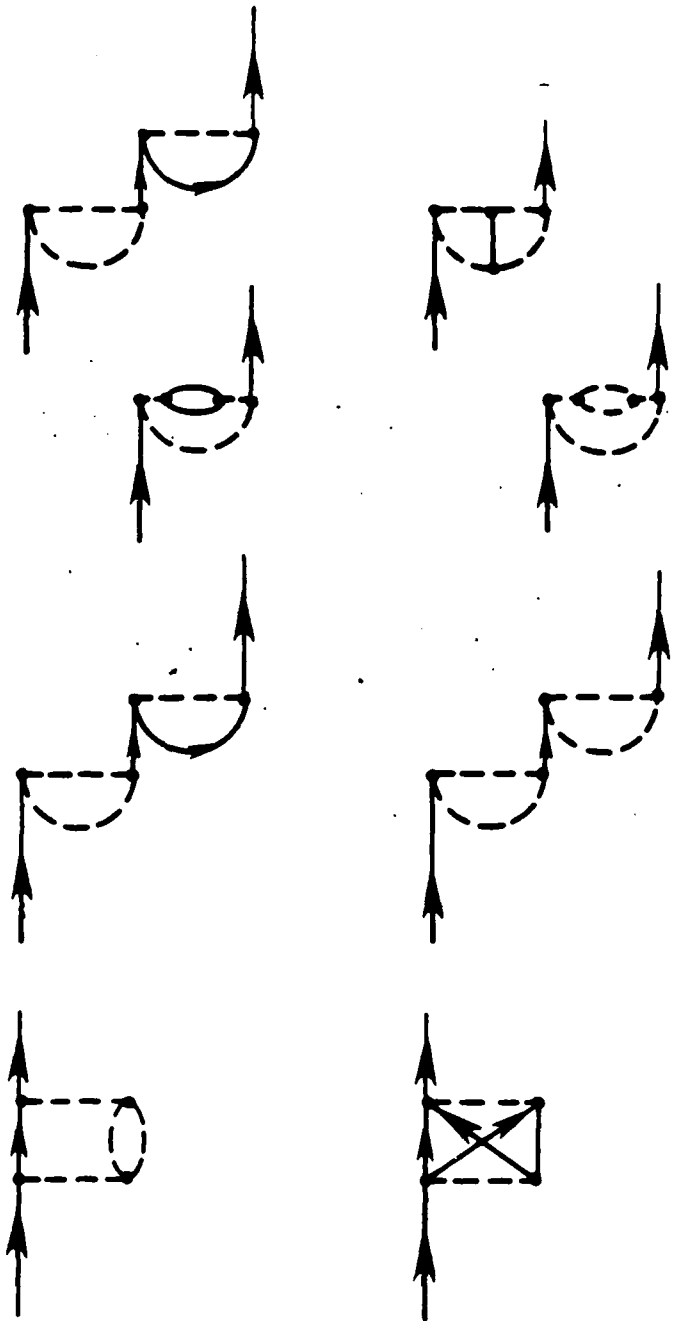


Figure 6. Examples of Four-Vertex Diagrams.

Substituting Equation (5.7) of Part I into-Equation (2.1) yields

$$G = (\omega - \omega_{ak} + \Sigma)^{-1} \quad . \tag{2.2}$$

A similar equation will hold for S. In practice only a few dominant irreducible diagrams are summed to obtain Σ. Fortunately, a good estimate for G can be obtained using Equation (2.2) with only a moderately accurate expression for Σ. Effective interactions due to sums over large numbers of diagrams can be expressed in terms of new types of diagrams. Define any diagram without external legs which may be appended to a vertex as a susceptance part, and any susceptance part which cannot be reduced to two simpler disconnected susceptance parts by breaking a single line as a proper or irreducible susceptance part. Examples are given in Figure 7. Let the sum over all proper parts be denoted by

$$\chi \equiv \text{(X)} \quad . \tag{2.3}$$

Then in a manner similar to that used to obtain Equation (2.1) we find, if bubble graphs can be ignored,

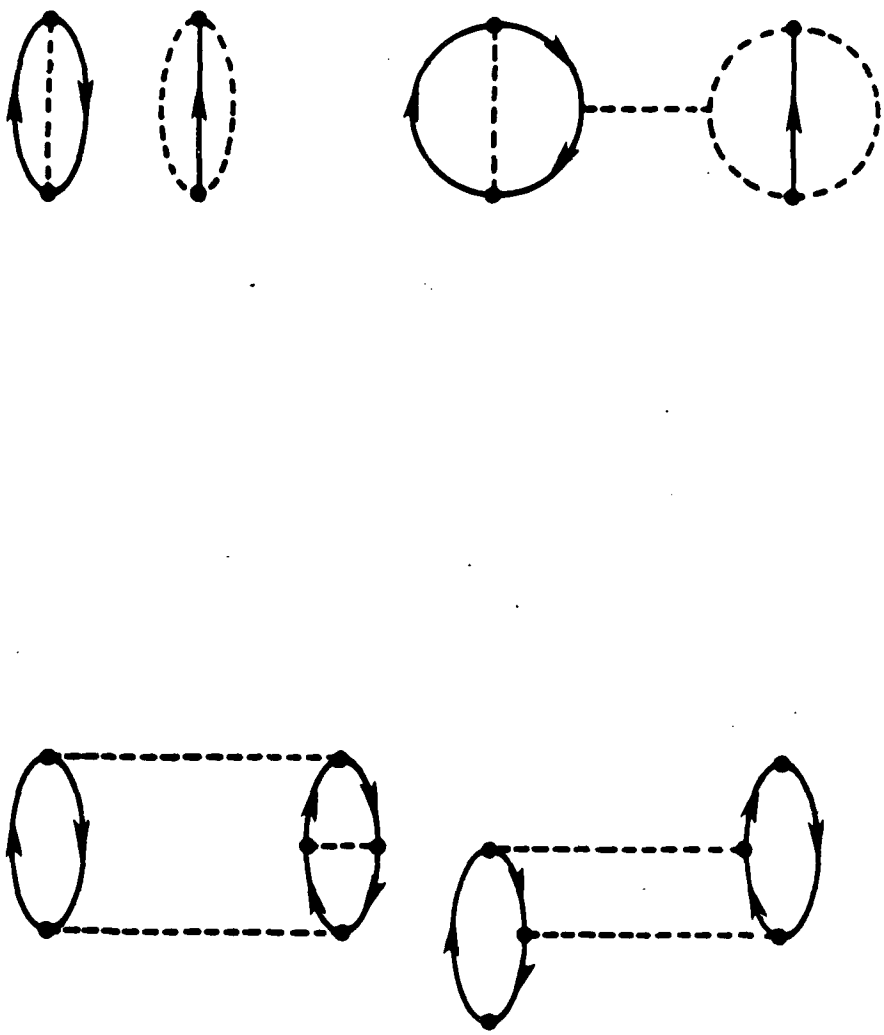
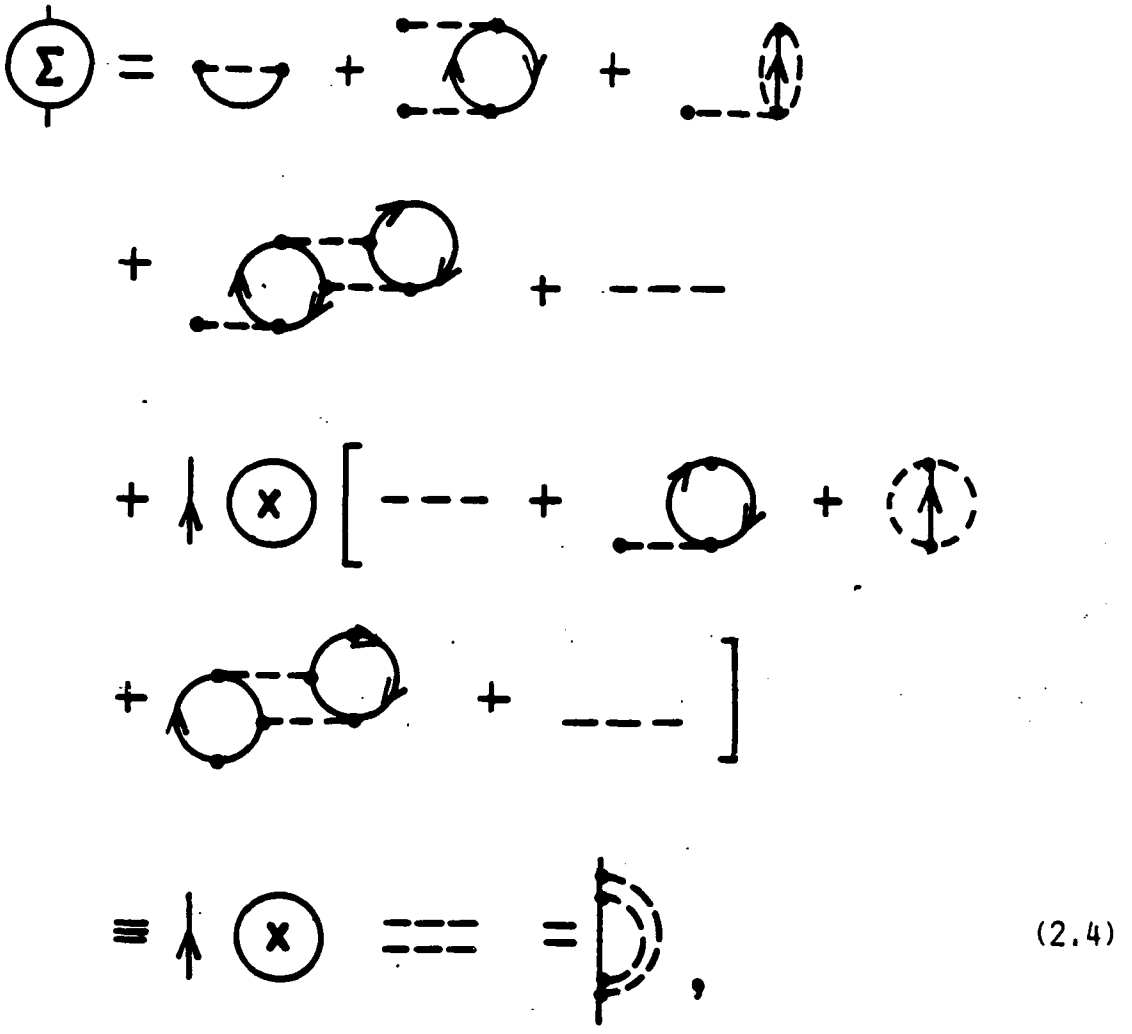
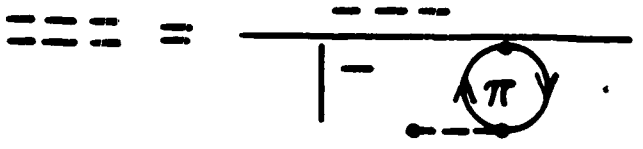


Figure 7. Susceptance (Unlinked) Diagrams.



(2.4)

where



(2.5)

3. HEAT EQUATION

We describe heat transport by the equation[11-16]

$$\dot{\mathcal{T}} + v\cdot\nabla\mathcal{T} + D_T\,\nabla^2\mathcal{T} = \beta'\mathcal{T}\,(\dot{P} + v\cdot\nabla P) + \rho e \quad , \tag{3.1}$$

where

$$\beta' = \beta/\rho c_p \tag{3.2}$$

β is the coefficient of thermal expansion:

$$\beta = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_P \tag{3.3}$$

c_p is the specific heat at constant pressure, and D_T, the coefficient of thermal diffusivity, is given by

$$D_T = \kappa/\rho C_p \quad . \tag{3.4}$$

For gases $\beta \to 1/\mathcal{T}$, while for liquids β becomes very small. D_T is usually described by the symbol χ, which we are reserving for other purposes. The symbol

$$e = \frac{1}{2}\nu \sum_{\alpha,\beta}\left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right)^2 \quad , \tag{3.5}$$

where e describes the generation of heat due to friction. It is small, and we usually ignore it.

3.1 Thermal Lagrangian

A Lagrangian density which describes Equation (3.1), $\mathcal{L}_T$ is given by

$$\mathcal{L}_T = \frac{1}{2}(\bar{T}\gamma_o\dot{T} - \dot{\bar{T}}\gamma_o T) + \frac{1}{2}(\nabla\psi - \nabla\bar{\psi})\cdot(\bar{T}\nabla T - T\nabla\bar{T}) + \frac{1}{2}(\bar{T} - T)^2\cdot(\nabla^2\psi - \nabla^2\bar{\psi})$$

$$- D_T\nabla\bar{T}\cdot\nabla T - \beta'\bar{T}T[\ddot{\psi} + \ddot{\bar{\psi}} + (\nabla\psi - \nabla\bar{\psi})\cdot(\gamma_o\nabla\dot{\psi} + \nabla\dot{\bar{\psi}}\gamma_o)]$$

$$- \frac{1}{2}\nu\rho(T + \bar{T})\sum_{\alpha,\beta}\left(\frac{\partial^2\psi}{\partial x_\alpha\partial x_\beta} - \frac{\partial^2\bar{\psi}}{\partial x_\alpha\partial x_\beta}\right)^2$$

$$+ \alpha_T[(\gamma\dot{T} - \dot{\bar{T}}\gamma_o) - (\gamma\dot{T}_o - \dot{\bar{T}}_o\gamma_o)]g(\psi + \bar{\psi}) \quad , \tag{3.6}$$

with

$$\mathcal{T} = T + \bar{T} \quad . \tag{3.7}$$

The third term in Equation (3.6) has been inserted to avoid adding an extra term to the Navier-Stokes equation. The β' term in Equation (3.6) will add a term to the Navier-Stokes equation of order $\beta' D_T\nabla^2\mathcal{T}$. We will assume that this can be dropped. The last term will not contribute to Equation (3.1), but will add a term of the form[16]

$$\alpha_T(\mathcal{T} - \mathcal{T}_o)g \tag{3.8}$$

to the Navier-Stokes equation. As in the case of the velocity potential, the simultaneous occurrence in the equation of motion of terms that are of both even and odd order in the time and spatial derivatives requires the use of complex fields in the Langrangian. We will again have relations for odd derivatives that will require hypercomplex fields coefficients to satisfy:

$$\dot{\mathcal{T}} = \gamma_o\dot{T} - \dot{\bar{T}}\gamma_o \tag{3.9}$$

and

$$\nabla\not{T} = \gamma_k \nabla T - \nabla \bar{T} \gamma_k \ . \tag{3.10}$$

If we write

$$v = \gamma_k \nabla \psi - \nabla \bar{\psi} \gamma_k \tag{3.11}$$

and

$$P = \rho(\gamma_o \dot{\psi} + \dot{\bar{\psi}} \gamma_o) \tag{3.12}$$

and disregard terms that cancel upon variation of ψ and $\bar{\psi}$ or are included with the velocity equations, then the Lagrangian L_T leads to the equations of motion

$$\gamma_o \dot{T} + v \cdot \nabla T + D_T \nabla^2 T - \beta' T(\gamma_o \dot{P} + v \cdot \nabla P) + \frac{1}{2}\rho e = 0 \tag{3.13}$$

and

$$-\gamma_o \dot{\bar{T}} - v \cdot \nabla \bar{T} + D_T \nabla^2 \bar{T} - \beta' \bar{T}(\gamma_o \dot{P} + v \cdot \nabla P) + \frac{1}{2}\rho e = 0 \ . \tag{3.14}$$

Using Equations (3.9) through (3.12), the addition of Equations (3.13) and (3.14) leads to Equation (3.1), as desired.

The momenta conjugate to T and $\bar{T}$ are given respectively by

$$\pi_T = \frac{1}{2}\bar{T} \tag{3.15}$$

and

$$\bar{\pi}_T = -\frac{1}{2}T \ . \tag{3.16}$$

The "Hamiltonian" density $\mathcal{H}_T$ corresponding to $\mathcal{L}_T$ is given by

$$\mathcal{H}_T = \frac{1}{2}(\nabla\bar{\psi} - \nabla\psi)\cdot(\bar{T}\nabla T - T\nabla\bar{T}) - \frac{1}{2}(\bar{T} - T)^2(\nabla^2\psi - \nabla^2\bar{\psi}) + D_T\nabla\bar{T}\nabla T$$

$$+ \beta'\bar{T}T[\ddot{\psi} + \ddot{\bar{\psi}} + (\nabla\psi - \nabla\bar{\psi})\cdot(\gamma_o\nabla\dot{\psi} + \nabla\dot{\bar{\psi}}\gamma_o)] + \frac{1}{2}\nu_\rho(T + \bar{T})e \quad . \tag{3.17}$$

The Poisson bracket relations are similar to those for the A fields:

$$[T_i(r),\ T_j(r')] = [\bar{T}_i(r),\ \bar{T}_j(r')] = 0 \tag{3.18}$$

and

$$[T_i(r),\ \bar{T}_j(r')] = \frac{1}{2}\delta(r - r'). \tag{3.19}$$

We may expand the fields T and $\bar{T}$ in a familiar way:

$$T = \sum_k T_k e^{-i\omega_{Tk}t} u_k(r) \tag{3.20}$$

and

$$\bar{T} = \sum_k \bar{T}_k e^{i\omega_{Tk}t} \bar{u}_k(r) \quad , \tag{3.21}$$

where we again take

$$u_k(r) = e^{ik\cdot r} \quad . \tag{3.22}$$

From Equations (3.15) and (3.16) we also have

$$\pi_T = \frac{1}{2}\sum_k T_k e^{i\omega_{Tk}t - ik\cdot r}$$

and

$$\bar{\pi}_T = -\frac{1}{2}\sum_k T_k e^{-i\omega_{Tk}t + ik\cdot r}$$

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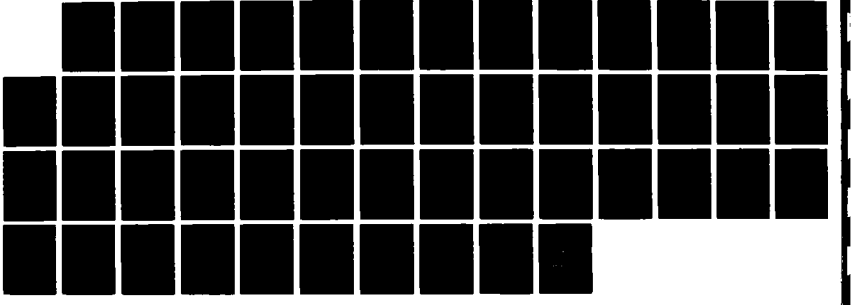

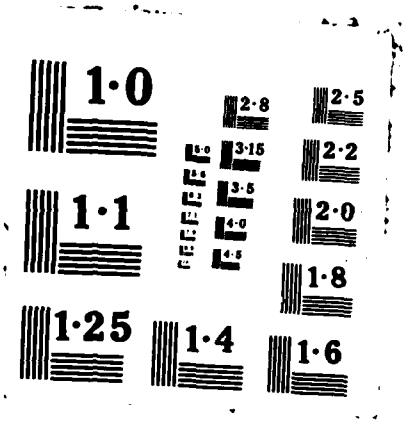
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The dispersion relation obtained from Equations (3.13), (3.14), (3.20) and (3.21) is

$$-i\omega_{Tk} + iv\cdot k - D_T k^2 = \beta'(\gamma_o \dot{P} + v\cdot\nabla P) + \rho e/T \ . \tag{3.25}$$

If we drop the term on the right side of Equation (3.25), we obtain simply

$$\omega_{Tk} = v\cdot k + iD_T k^2 \ . \tag{3.26}$$

We see that temperature disturbances are damped by diffusion processes and are carried by the mean flow; they do not actually propagate, as pressure perturbations do.

The relations

$$\dot{T} = [T, H] \tag{3.27}$$

and

$$\dot{\bar{T}} = [\bar{T}\ , H] \quad , \tag{3.28}$$

together with Equations (3.17), (3.18), and (3.19) yield

$$\gamma_o \dot{T} + v\cdot\nabla T + D_T\nabla^2 T - \beta' T(\gamma_o \dot{P} + v\cdot\nabla P) = \frac{1}{2}\nu\rho e + (T - \bar{T})\nabla v \tag{3.29}$$

and

$$\dot{\bar{T}}\gamma_o + v\cdot\nabla\bar{T} - D_T\nabla^2\bar{T} + \beta'\bar{T}(\gamma_o \dot{P} + v\cdot\nabla P) = \frac{1}{2}\nu\rho e + (T - \bar{T})\nabla v \ . \tag{3.30}$$

Subtracting Equation (3.30) from Equation (3.29), we again obtain Equation (3.1), as we should.

3.2 General Treatment of Passive Scalars

In multicomponent fluids, we encounter equations of the form

$$\dot{\psi} + v\cdot\nabla^2\psi + D_\gamma\nabla^2\psi = f(\psi). \tag{3.31}$$

Equations of this type may be treated by Lagrangian densities of the form

$$2\mathcal{L}_Y = \bar{Y}\gamma_o\dot{Y} - \dot{\bar{Y}}\gamma_o Y + (\nabla\psi - \nabla\bar{\psi})\cdot(\bar{Y}\nabla Y - Y\nabla\bar{Y}) + (\bar{Y} - Y)^2(\nabla^2\psi - \nabla^2\bar{\psi})$$

$$- 2D_Y\nabla\bar{Y}\nabla Y - [\bar{Y}f_o(Y) - Yf_o(\bar{Y})] \quad , \tag{3.32}$$

where

$$\not{\psi} = Y + \bar{Y} \quad , \tag{3.33}$$

$$\dot{\not{\psi}} = \gamma_o\dot{Y} - \dot{\bar{Y}}\gamma \quad , \tag{3.34}$$

$$\nabla\not{\psi} = \gamma_k\nabla Y - \nabla\bar{Y}\gamma_k \quad , \tag{3.35}$$

and f_o is the linearized part of f. The conjugate momenta corresponding to Y and $\bar{Y}$ are respectively

$$\pi_Y = \frac{1}{2}\bar{Y} \tag{3.36}$$

and

$$\bar{\pi}_Y = -\frac{1}{2}Y \quad . \tag{3.37}$$

The Hamiltonian density corresponding to Equations (3.31) and (3.32) is

$$2\mathcal{H}_Y = (\nabla\bar{\psi} - \nabla\bar{\psi})\cdot(\bar{Y}\nabla Y - Y\nabla\bar{Y}) + (\bar{Y} - Y)^2(\nabla^2\bar{\psi} - \nabla^2\psi) + 2D_Y\nabla\bar{Y}\nabla Y$$

$$+ \bar{Y}f_o(Y) + Yf_o(\bar{Y}) \quad . \tag{3.38}$$

The Poisson bracket relations for the system are of the form given in Equations (3.18) and (3.19), with T replaced by Y. We may transform Y and $\bar{Y}$ using the relations

$$Y = \sum_k Y_k e^{-i\omega_{Yk}t + ik\cdot r} \tag{3.39}$$

$$\bar{Y} = \sum_k \bar{Y}_k e^{i\omega_{Yk}t - ik\cdot r} \quad , \tag{3.40}$$

with

$$\omega_{\gamma k} = v \cdot k + iD_{\gamma} k^2 \quad . \tag{3.41}$$

The description of the total density ρ is somewhat different than that for the species densities ψ, since there is no diffusion term. Nevertheless, we will still need to construct a Lagrangian density in terms of adjoint fields. First we show the results of an attempt to formulate a description in terms of a single field. The continuity equation

$$\dot{\rho} + \nabla \cdot (\rho v) = 0 \tag{3.42}$$

may be written as

$$\dot{\rho} + \nabla \rho \cdot (\nabla \psi - \nabla \bar{\psi}) + \rho (\nabla^2 \psi + \nabla^2 \bar{\psi}) \quad . \tag{3.43}$$

Equation (3.43) can be obtained from the Lagrangian density $\mathcal{L}_{\rho}$, given by

$$\mathcal{L}_{\rho} = \frac{c^3 \nu}{2\kappa} [\rho \dot{\rho} + \rho \nabla \rho \cdot (\nabla \psi - \nabla \bar{\psi})] \quad . \tag{3.44}$$

We see that for a change the density field is purely real. L_{ρ} makes no net contribution to the velocity equations and the equation of motion resulting from it is unaffected by a variation of the velocity fields. Unfortunately, however, the momentum conjugate to ρ, π_{ρ}, is given by

$$\pi_{\rho} = \frac{c^3 \nu}{2\kappa} \rho \quad ! \tag{3.45}$$

Since we cannot formulate Poisson bracket relations with field variables that are self conjugate, we must invoke adjoint fields, even though there are no second-order derivative terms in Equation (3.42). We see then that Equation (3.42) is a special case of Equation (3.31) with $D_{\rho} = 0$.

4. APPLICATION OF THE RENORMALIZATION GROUP

by John Erdei

In this section we will present initial results on the applicability of the renormalization group to our model. We note that both the static and dynamic forms of RGT are of interest. The dynamic version can be applied directly to the results of Appendix E, while the use of the static renormalization group requires the recasting of the model in a form similar to that of Landau-Ginzburg Theory. Such a technique would be used to study the steady-state features of a given flow configuration. Keeping in line with the thrust of the previous work, we will use the equations of motion from Appendix E to implement the dynamic renormalization group procedure.

Our intention is to introduce a perturbative expansion for the equations of motion given in Appendix E. The goal is to categorize the fluctuation integrals which contribute to the renormalization of the viscosity. Once the form of the fluctuation integral is identified, we can examine the applicability of the renormalization group procedure and determine a direction for future work. Due to the complexity of the model and the number of integrals contributing to the renormalization of the viscosity, in this report we will display the derivation of one of the contributing fluctuation integrals. Two possible expositions of the renormalization group can be considered; one involves the averaging of fluctuations over all wavelengths. The static form of this procedure is described in Amit,[17] while the dynamic version can be found in Lovesey.[18] The second allows for the averaging of fluctuation over some defined length scales. This system is in line with the iterative form of RGT developed by Wilson,[19] and applied to the Navier-Stokes equation by a variety of authors.[20-22]

4.1 Equations of Motion

We will use the equations of motion (E.20) to (E.25) to generate a correction to the physical parameters in the systems, and then examine the results for the application of the renormalization group. The complexity of these equations of motion is indeed great, and for this report we will display a portion of the results of a perturbative approach to fluctuations. As a system undergoes some transition to turbulence, it becomes dominated by the formation of vortices. The vortices are associated with the local rotation of the fluid. As such, we will take the interesting fields to be the set a in the following exposition. Recall that these fields are related to the solenoidal part of the velocity field A. We begin with the set of Equations (E.20) to (E.25). We will introduce the notation that any wave vector is now placed as an argument for each field, leaving subscripts to denote Cartesian components. With this notation, we have:

$$\dot{a}_i(k) = -i\omega_{ak}a_i(k) + f(k)\sum_q \epsilon_{ilm}k_l[A_m(q,k)\eta^\dagger(q+k)\eta(q) + B_m(q,k)\eta(q+k)\zeta^\dagger(q)$$

$$+ C_m(q,k)\eta(q)\zeta(q+k) + D_m(q,k)\zeta^\dagger(q)\zeta(q+k)] \quad , \tag{4.1}$$

$$\dot{a}_i^\dagger(k) = i\omega_{ak}a_i^\dagger(k) + f(k)\sum_q \epsilon_{ilm}k_l[A_m(q,k)\eta^\dagger(q+k)\eta(q) + B_m(q,k)\eta(q+k)\zeta^\dagger(q)$$

$$+ C_m(q,k)\eta(q)\zeta(q+k) + D_m(q,k)\zeta^\dagger(q)\zeta(q+k)] \quad , \tag{4.2}$$

$$\dot{\eta}(q) = -i\omega_{\eta q}\eta(q) + \sum_{k'} f(k')\epsilon_{npq}k'_p(a_q(k') - a_q^\dagger(k'))[A_n(q,k')\eta(q-k')$$

$$+ B_n(q,k')\zeta^\dagger(q+k)] \quad , \tag{4.3}$$

$$\dot{\eta}^{\dagger}(q) = i\omega_{\eta q}\eta^{\dagger}(q) - \sum_{k'} f(k')\epsilon_{npq}k'_p\,[a_q(k') - a_q^{\dagger}(k')][A_n(q,k')\eta^{\dagger}(q+k')$$

$$+ C_n(q,k')\zeta(q+k')] \quad , \tag{4.4}$$

$$\dot{\zeta}(q) = -i\omega_{\zeta q}\zeta(q) + \sum_{k'} f(k')\epsilon_{rst}k'_s[a_t(k') - a_t^{\dagger}(k')][B_r(q,k')\eta^{\dagger}(q+k')$$

$$+ D_r(q,k')\zeta(q)] \quad , \tag{4.5}$$

and

$$\dot{\zeta}^{\dagger}(q) = i\omega_{\zeta q}\zeta^{\dagger}(q) + \sum_{k'} f(k')\epsilon_{rst}k'_s[a_t(k') - a_t^{\dagger}(k')][C_r(q,k')\eta^{\dagger}(q+k')$$

$$+ D_r(q,k'')\zeta^{\dagger}(k'-q)] \quad . \tag{4.6}$$

We will remove the time derivatives in these equations by introducing an appropriate frequency, so that, for example, we can set $\dot{a} = -i\omega a$. As a technical convenience, we also introduce a random forcing term into each equation. Using these steps, we write

$$a_j(k) = G^a_{ij}(k)F^a_i(k) + G^a_{ij}(k)f(k)\sum_q \epsilon_{ilm}k_l[A_m(q,k)\eta^{\dagger}(q+k)\eta(q)$$

$$+ B_m(q,k)\eta(q+k)\zeta^{\dagger}(q) + C_m(q,k)\eta(q)\zeta(q+k)$$

$$+ D_m(q,k)\zeta^{\dagger}(q)\zeta(q+k)] \quad , \tag{4.7}$$

$$a_j^{\dagger}(k) = G^{a\dagger}_{ij}(k)F^{a\dagger}_i(k) + G^{a\dagger}_{ij}(k)f(k)\sum_q \epsilon_{ilm}k_l[A_m(q,k)\eta^{\dagger}(q+k)\eta(q)$$

$$+ B_m(q,k)\eta(q+k)\zeta^{\dagger}(q) + C_m(q,k)\eta(q)\zeta(q+k)$$

$$+ D_m(q,k)\zeta^{\dagger}(q)\zeta(q+k)] \quad , \tag{4.8}$$

$$\eta(q) = G^{\eta}(q)F^{\eta}(q) + G^{\eta}(q)\sum_{k'} f(k')\epsilon_{npq}k'_p[a_q(k') - a_q^{\dagger}(k')][A_n(q,k')\eta(q-k') + B_n(q,k')\zeta^{\dagger}(q+k')] \quad , \tag{4.9}$$

$$\eta^{\dagger}(q) = G^{\eta^{\dagger}}(q)F^{\eta^{\dagger}}(q) + G^{\eta^{\dagger}}(q)\sum_{k'} f(k')\epsilon_{npq}k'_p[a_q(k') - a_q^{\dagger}(k')] + [A_n(q,k')\eta^{*}(q+k') + C_n(q,k')\zeta(q+k')] \quad , \tag{4.10}$$

$$\zeta(q) = G^{\zeta}(q)F^{\zeta}(q) + G^{\zeta}(q)\sum_{k'} f(k')\epsilon_{rst}k'_s[a_t(k') - a_t^{\dagger}(k')] + [B_t(q,k')\eta^{\dagger}(q+k') + D_t(q,k')\zeta(q)] \quad , \tag{4.11}$$

and

$$\zeta^{\dagger}(q) = G^{\zeta^{\dagger}}(q)F^{\zeta^{\dagger}}(q) + G^{\zeta^{\dagger}}(q)\sum_{k'} f(k')\epsilon_{rst}k'_s(a_t(k') - a_t^{\dagger}(k')) + [C_r(q,k')\eta(q+k') + D_r(q,k')\zeta^{\dagger}(k'-q)] \quad ; \tag{4.12}$$

where

$$G^{a}_{ij}(k) = \frac{\delta_{ij}}{(-i\omega + i\omega_{ak})} \quad , \tag{4.13}$$

$$G^{a^{\dagger}}_{ij}(k) = \frac{\delta_{ij}}{(i\omega - i\omega_{ak})} \quad , \tag{4.14}$$

$$G^{\eta}(q) = \frac{1}{(-i\omega + i\omega_{\eta q})} \quad , \tag{4.15}$$

$$G^{\eta^{\dagger}}(q) = \frac{1}{(i\omega - i\omega_{\eta q})} \quad , \tag{4.16}$$

$$G^{\zeta}(q) = \frac{1}{(-i\omega + i\omega_{\zeta q})} \quad , \tag{4.17}$$

and

$$G^{\zeta\dagger}(q) = \frac{1}{(i\omega - i\omega_{\zeta q})} \tag{4.18}$$

are the bare Green's functions for the system. We will be interested in working with Equation (4.1). Here

$$\omega_{ak} = i\nu k^2 \quad . \tag{4.19}$$

4.2 Fluctuation Integrals

We will impose the condition

$$\langle F_i(k)F_j(k')\rangle = \delta_{ij}Q(k)\delta(k-k') \quad , \tag{4.20}$$

where < > denotes an average, and, if desired, some function of k can be defined through Q(k).[21] We will simply take Q(k) to be a constant. We will introduce an iterative perturbation solution to Equation (4.7) by averaging over the variables η and ζ. To do so, we introduce Equations (4.8) to (4.12) into Equation (4.1).

It is clear that even with the simplification of only treating the equations for the variables a, the complexity of the iteration is formidable. For the purposes of this report, we will examine one specific term in the series, and show its relationship to the formal renormalization of ν. Under the described iteration procedure, one can show (See Appendix G) that Equation (4.7) can be recast into the form

$$\begin{aligned}\langle a_j(k)\rangle = {} & \langle G^a_{ij}(k)F^a_i(k)\rangle \\ & + \langle G^a_{wj}(k)\epsilon_{wlm}k_l\epsilon_{rst}k_s\Big[\sum_q G^{\eta}(q+k)G^{\eta\dagger}(q)G^{\eta}(q+k) \\ & [f(k)A_m(a,k)][f(k)A_r(q,k)]G^a_{it}(k)\Big]F^a_i(k)\rangle + \ldots\ldots\end{aligned} \tag{4.21}$$

or

$$a_j(k) = (G^a_{ij}(k) + G^a_{it}(k)\epsilon_{wlm}k_l\epsilon_{rst}k_s$$

$$\sum_q G^\eta(q+k)G^{\eta\dagger}(q)G^\eta(q+k)f(k)A_m(q,k)f(k)A_r(q,k)$$

$$G^a_{it}(k))F^a_i(k) + \ldots\ldots \tag{4.22}$$

We can therefore define a (partially) renormalized Green's function in the form

$$\tilde{G}^a_{ij}(k) = G^a_{ij}(k) + G^a_{it}(k)\epsilon_{wlm}\epsilon_{rst}k_lk_sf(k)G_{it}(k)$$

$$\sum_q A_m(q,k)A_r(q,k)G^\eta(q+k)G^{\eta\dagger}(q)G^\eta(q+k). \tag{4.23}$$

One of the effects of the non-linear coupling between fields is to alter the matrix structure of the bare Green's function, so that it becomes non-diagonal. The off-diagonal elements indicate that the viscosity matrix, which was originally diagonal in the linear theory, becomes a non-diagonal viscosity matrix. This can be seen since one can invert Equation (4.23) and identify the viscosity matrix. In the linear theory,

$$[G^a_{ij}(k)]^{-1} = \delta_{ij}(i\omega + i\omega_{ak}) \tag{4.24}$$

$$= -i\omega\delta_{ij} - [\nu\delta_{ij}]k^2 \tag{4.25}$$

which defines the viscosity matrix $\nu\delta_{ij}$. When the (partially) renormalized Green's function given in Equation (4.23) is inverted and the renormalized viscosity identified, the viscosity matrix will no longer be diagonal, i.e.,

$$[\tilde{G}^a_{ij}(k)]^{-1} = -i\omega\delta_{ij} - \tilde{\nu}_{ij}k^2. \tag{4.26}$$

One might associate the appearance of off-diagonal elements in the renormalized viscosity with the generation of a turbulent eddy viscosity matrix. The exact nature of these off-diagonal elements requires further study.

We see from Equation (4.23) that a complete description of the viscosity matrix requires the evaluation of the sum

$$I_{mr}(k) = \sum_{q} A_m(q,k)A_r(q,k)G^{\eta}(q+k)G^{\eta\dagger}(q)G^{\eta}(q+k). \tag{4.27}$$

This sum typifies one of the contributions of the η and ζ fields to the study of the properties of the a fields. It would be convenient for future evaluations to replace this sum with an integral,

$$\sum_{k} \rightarrow \int d^3k g(k), \tag{4.28}$$

where g(k) is an appropriately defined density of states. Then the sum given by Equation (4.27) can be written in the form

$$I_{mr}(q) = \int d^3q g(q)$$

$$\frac{[(2q_m + k_m)\omega_a(k) + (q_m + k_m)\omega_{\eta}(q+k) + q_m\omega_{\eta}(q)]}{[\omega - \omega_{\eta}(q+k)]^2}$$

$$\frac{[(2q_r + k_r)\omega_a(k) + (q_r + k_r)\omega_{\eta}(q+k) + q_r\omega_{\eta}(q)]}{[\omega - \omega_{\eta}(q)]} \tag{4.29}$$

or,

$$I_{mr}(q) = 2\int_0^1 dx \int d^3q g(q)[(2q_m + k_m)\omega_a(k) + (q_m + k_m)\omega_{\eta}(q+k)+q_m\omega_{\eta}(q)]$$

$$\frac{[(2q_r + k_r)\omega_a(k) + (q_r + k_r)\omega_{\eta}(q+k) + q_r\omega_{\eta}(q)]}{[\omega - (1 - x)\omega_{\eta}(q) - x\omega_{\eta}(q+k)]^3} \tag{4.30}$$

where we have used the Feynman method of folding denominators[1]

$$\frac{1}{a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_n^{\alpha_n}} = \frac{\Gamma(\alpha_1 + \alpha_2 + \ldots \alpha_n)}{\Gamma(\alpha_1)\,\Gamma(\alpha_2) \ldots \Gamma(\alpha_n)} \int dx_1 dx_2 \ldots dx_{n-1}$$

$$\frac{x_1^{\alpha_1 - 1}\, x_2^{\alpha_2 - 1} \ldots x_{n-1}^{\alpha_{n-1} - 1} (1 - x - x - \ldots - x_{n-1})^{\alpha_n - 1}}{[x_1 a_1 + x_2 a_2 + \ldots + x_{n-1} a_{n-1} + (1 - x_1 - x_2 - \ldots - x_{n-1}) a_n]^{\alpha_1 + \ldots + \alpha_n}} \tag{4.31}$$

and

$$0 \leq x_i \leq 1 \; ; \quad x_1 + x_2 + \ldots + x_{n-1} \leq 1. \tag{4.32}$$

At this point, the requirement of a renormalization group calculation can be based on two points. If the integral $I_{mr}(q)$ suffers from infrared ($q \to 0$) divergences, then the renormalization group method can be used to discuss the removal of divergences. One may also incorporate a procedure which allows for the treatment of the fluctuations in the η and ζ fields only over a prescribed set of length scales. Assuming a constant density of wavevectors, the integral $I_{mr}(q)$ indicates a q-dependence of the form

$$q^{d-6} \; , \tag{4.33}$$

which suggests that the integrals will suffer a power-law divergence for spatial dimensions below six. A renormalization group treatment which employs an ϵ-expansion technique would be valid only around six dimensions, and would not be reliable for a three-dimensional system. Similarly, one would have to determine if the iteration is defined such that higher-order-fluctuation integrals do not introduce higher-order divergences. If so, the renormalization group procedure will fail.

The procedure may be used, however, if one assumes an upper and lower cut-off in wave vector. For example, shell integrations may be used to integrate over length scales from a long wavelength on the order of the size of the appropriate region, down to the Kolomogorov scale. We could then carry out integrals of the type

$$\int d^3q \rightarrow \int_{\Delta_i}^{\Delta_{i+1}} d^3q \quad , \tag{4.34}$$

where the Δ's define the wave-vector shell. The shell integration could then be iterated over the shells (i.e., over values of i) until the entire wave-vector region is covered, or the effects of fluctuations on a specific scale can be treated by examining only one of the shell integrations. This method would not result in divergent integrals due to wave-vector dependences. Since the interactions between the a fields and fluctuations of specific lengths are of interest, this is the technique we would pursue.

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SUMMARY AND SUGGESTIONS FOR FUTURE WORK

We now review the development of the work described in Parts I through III and explore the implications for further study.

Summary

The Lagrangian for corresponding to the Navier-Stokes equation was presented in Part I, Section 2, and the basis for deriving it is given in Appendix A. Equations of motion for the linearized system were derived using Euler-Lagrange relations. The Hamiltonian and Poisson bracket relations were derived from the Lagrangian, and equations of motion were again obtained. The equations of motion obtained from the Hamiltonian and the Poisson bracket relations were identical to those obtained from the Lagrangian, showing consistency in the theory. The relationship between the above equations and the Navier-Stokes equation was also demonstrated. The entire development given in Section 2 of Part I depends upon the assumption that the velocity potentials may be expressed in terms of the sums of hypercomplex fields and their adjoint fields. The justification for this assumption is discussed in Appendix C. It is not clear whether the postulates given in Equations (2.5) and (2.6) of Part I contain a physical basis or are merely a mathematical device. However, without these postulates, the entire theory falls through. In Section 3 of Part I the symmetries of the Lagrangian were examined to derive conservation laws and to determine those variables which are conserved with respect to the basis Hamiltonian, H_o. The derivation of the important stress-energy tensor is given in Appendix D. The form given in Appendix D pertains only to the scalar fields. A four-vector formalism was derived to include the solenoidal fields in the September 1985 monthly report, but was not included in this report as it was

judged to be unwieldy. The symmetry properties of the Lagrangian and conserved variables can be used to analyze the transition from laminar to turbulent flow.

The Hamiltonian was diagonalized in Section 4 of Part I. It was found that two different fields are required to diagonalize the scalar fields. This suggests an analog with the description of a charged meson in quantum field theory or perhaps the electron-hole description of a metal. However, the physical significance, if any, behind the need for two different diagonalized fields remains elusive. One possibility is that the fields describe sound waves in which local compressions are in phase with the maxima in the local velocity, and that the ζ fields describe sound waves in which local condensations are out of phase with the local velocity.

Green's functions were derived in Section 5 of Part I. The form of the propagators for the potential fields resemble phonon propagators, while the propagator for the vortex fields resembles that of a nonrelativistic electron. This suggests that there may be an analogy between fluid dynamics and the electron-phonon problem. The development of Part I contains all the elements needed for a field theory. Part I thus describes the core work of this report. Its consequences were explored in Parts II and III.

Part II illustrated how the canonical transform procedure common to quantum field theory can be used to investigate mode coupling and reveal analogies with various critical systems which are relatively well understood. The discussion of Part II suggests that the many-body formalism can be useful not only for describing the interactions between distrubances in fluid, but can also be used as the basis for other descriptions of turbulence, such as the "deterministic" approaches currently in vogue.

Part III is essentially a preview of more advanced calculations which can be based on the developments of Parts I and II. Rules for diagram calculations were given in Section 1 of Part III, and simple illustrative calculations were given in Section 2. The rules were made by analog with the rules for condensed-matter theory, and may need to be modified following a thorough review.

Section 3 of Part III showed how the concepts of Part I may be applied to describe additional phonemena, especially the behavior of passive scalars. The heat-budget equation was used as an example. This treatment shows that the concepts used in this report can be applied to the solution of a wide range of partial differential equations, including but not limited to other dissipative motion. Section 4 gave an example of a renormalization group calculation which was done by John Erdei. This extensive calculation illustrates how the renormalization group can be used to include multiple scales in a calculation.

Conclusions and Suggested Research

The formalism developed in this report is internally consistent, and shows strong parallels with field theories for other systems, as was intended. It appears to satisfy the objective of deriving a Hamiltonian for the Navier-Stokes equation that can be used to carry out many-body calculations of turbulent flows. This work indicates that all of the standard techniques of field theory, including canonical tranformations to rediagonalize the field variables, the use of propagators, and diagram calculations can be brought to bear on turbulent systems using this or an allied formalism, notwithstanding that the development is entirely classical, and includes dissipation. Indeed, the approach taken here is applicable to a wide range of other partial differential equations.

A number of loose ends remain in the presentation of the basic theory. In particular, the Green's functions and rules for diagrams should be thoroughly reviewed and derived on a rigorous basis. In addition, the symmetries of the Lagrangian should be analyzed further, particularly in regard to local gauge invariance.

To merit serious attention, any theory must give reasonably accurate calculations of experimental results. Therefore, the present formalism should be tested as soon as possible before extensive work is based upon it. Since almost all calculations will involve Green's functions, diagrams, or probability density functions and partition functions, these quantities should be rigorously derived and checked for internal consistency. The nonlinear expansion terms in the pressure gradient term were not studied in terms of the diagonalized field variables. The contribution of these terms to mode coupling should be examined.

Once the foundations of the theory have been checked, calculations should be made of those aspects of turbulence which are well understood. Turbulence spectra should be predicted, the propagation of sound through a turbulent fluid should be analyzed, and expressions for the lower-order velocity moments should be derived. Finally, attempts should be made to predict the transport of mass, momentum, and heat for a turbulent fluid in a simple geometry.

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APPENDICES

A THROUGH G

APPENDIX A

EULER-LAGRANGE EQUATIONS FOR FIELDS

Consider the functional

$$L = \int_a^b dt d^3x \, \mathcal{L} \quad , \tag{A.1}$$

where

$$\mathcal{L} = \mathcal{L} (\psi, \dot{\psi}, \nabla\psi, \nabla\dot{\psi}, \nabla^2\psi, A, \dot{A}, \nabla\times A, \nabla\times\dot{A}), \tag{A.2}$$

then

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \int_a^b dt d^3x \left\{ \frac{\partial \mathcal{L}}{\partial \psi} \frac{\partial \psi}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \frac{\partial \dot{\psi}}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \nabla\psi} \frac{\partial \nabla\psi}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \nabla\dot{\psi}} \frac{\partial \nabla\dot{\psi}}{\partial \alpha} \right. \tag{A.3}$$

$$\left. + \frac{\partial \mathcal{L}}{\partial \nabla^2\psi} \frac{\partial \nabla^2\psi}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial A} \frac{\partial A}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \dot{A}} \frac{\partial \dot{A}}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial (\nabla\times A)} \frac{\partial (\nabla\times A)}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial (\nabla\times\dot{A})} \frac{\partial (\nabla\times\dot{A})}{\partial \alpha} \right\} .$$

Now

$$\int_a^b dt d^3x \, \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \frac{\partial \dot{\psi}}{\partial \alpha} = \int d^3x \, \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \frac{\partial \psi}{\partial \alpha} \Bigg|_{t_a}^{t_b} - \int_a^b d^3x dt \, \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}} \right) \frac{\partial \psi}{\partial \alpha} \quad . \tag{A.4}$$

Similarly,

$$\int_a^b dt d^3x \, \frac{\partial \mathcal{L}}{\partial \nabla\psi} \frac{\partial \nabla\psi}{\partial \alpha} = \int dt \, \frac{\partial \mathcal{L}}{\partial \nabla\psi} \frac{\partial \psi}{\partial \alpha} \Bigg|_{x_a}^{x_b} - \int_a^b dt d^3x \, \nabla \left(\frac{\partial \mathcal{L}}{\partial \nabla\psi} \right) \frac{\partial \psi}{\partial \alpha} \quad , \tag{A.5}$$

and

$$\int_a^b dt d^3x \, \frac{\partial \mathcal{L}}{\partial (\nabla^2\psi)} \frac{\partial (\nabla^2\psi)}{\partial \alpha} = \int dt \, \frac{\partial \mathcal{L}}{\partial (\nabla^2\psi)} \frac{\partial (\nabla\psi)}{\partial \alpha} \Bigg|_{x_a}^{x_b} - \int_a^b dt d^3x \, \nabla \left(\frac{\partial \mathcal{L}}{\partial \nabla^2\psi} \right) \frac{\partial (\nabla\psi)}{\partial \alpha} \tag{A.6}$$

$$= \int_a^b dt d^3x \, \nabla^2 \left(\frac{\partial \mathcal{L}}{\partial (\nabla^2\psi)} \right) \frac{\partial \psi}{\partial \alpha} - \int dt \, \nabla \left(\frac{\partial \mathcal{L}}{\partial (\nabla^2\psi)} \right) \frac{\partial \psi}{\partial \alpha} \Bigg|_{x_a}^{x_b}$$

Also,

$$\int_a^b dtd^3x \frac{\partial \mathcal{L}}{\partial(\nabla\dot{\psi})} \frac{\partial(\nabla\dot{\psi})}{\partial\alpha} = \int dt \frac{\partial \mathcal{L}}{\partial(\nabla\dot{\psi})} \frac{\partial\dot{\psi}}{\partial\alpha}\Bigg|_{x_a}^{x_b} - \int_a^b dtd^3x \nabla\frac{\partial \mathcal{L}}{\partial(\nabla\dot{\psi})} \frac{\partial\dot{\psi}}{\partial\alpha} \tag{A.7}$$

$$= \int_a^b dtd^3x \frac{\partial}{\partial t} \nabla\left(\frac{\partial \mathcal{L}}{\partial(\nabla\dot{\psi})}\right)\frac{\partial\psi}{\partial\alpha} - \int d^3x \nabla\left(\frac{\partial \mathcal{L}}{\partial(\nabla\dot{\psi})}\right) \frac{\partial\psi}{\partial\alpha}\Bigg|_a^b$$

Using Equations (A.3) to (A.7), the condition

$$\delta L/\delta\alpha = 0 \tag{A.8}$$

leads to the equation

$$\frac{\partial \mathcal{L}}{\partial\psi} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial\dot{\psi}} - \nabla\frac{\partial \mathcal{L}}{\partial\nabla\psi} + \nabla^2 \frac{\partial \mathcal{L}}{\partial(\nabla^2\psi)} + \frac{\partial \mathcal{L}}{\partial t}\nabla\frac{\partial \mathcal{L}}{\partial\nabla\dot{\psi}} = 0 \ . \tag{A.9}$$

Relations similar to (A.4) to (A.7) hold for the vector fields, A. For example,

$$\int_a^b dtd^3x \frac{\partial \mathcal{L}}{\partial\dot{A}} \frac{\partial\dot{A}}{\partial\alpha} = \int d^3x \frac{\partial \mathcal{L}}{\partial\dot{A}} \frac{\partial A}{\partial\alpha}\Bigg|_{t_a}^{t_b} - \int dtd^3x \frac{\partial \mathcal{L}}{\partial t}\left(\frac{\partial}{\partial\dot{A}}\right)\frac{\partial A}{\partial\alpha} \tag{A.10}$$

and

$$\int_a^b dtd^3x \frac{\partial \mathcal{L}}{\partial(\nabla\times A)} \frac{\partial(\nabla\times A)}{\partial\alpha} = \int dt \frac{\partial \mathcal{L}}{\partial(\nabla\times A)}\frac{\partial A}{\partial\alpha}\Bigg|_{x_a}^{x_b} - \int dtd^3x \nabla\times\frac{\partial \mathcal{L}}{\partial(\nabla\times A)} \frac{\partial A}{\partial\alpha} \ . \tag{A.11}$$

Continuing in a parallel manner to that used to derive Equation (A.6) to (A.9), the condition (A.8) gives

$$\frac{\partial \mathcal{L}}{\partial A} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\dot{A})} - \nabla\times\frac{\partial \mathcal{L}}{\partial(\nabla\times A)} + \frac{\partial}{\partial t}\nabla\times\frac{\partial \mathcal{L}}{\partial(\nabla\times\dot{A})} = 0 \ . \tag{A.12}$$

APPENDIX B

OPERATOR FORMALISM

As noted earlier, the solution to any partial differential equation (PDE) is a field.[B1] We add that it is a field under scalar addition and multiplication over the domain of the independent variables of the PDE. In our case, this domain is $\{r,t\}$. In passing, we also note that the field variables for each PDE form an Abelian group under scalar addition[B2] and that the Poisson bracket relations for these fields and their conjugate fields define a Lie algebra.[B3] Of more immediate interest is the fact that any function of a field variable defines a vector. In particular, the elements of the fields themselves comprise a vector through the relation

$$f(\phi) = \phi \quad , \tag{B.1}$$

where we now use the letters ϕ, ψ, and χ to denote general field variables. Now for every vector field there exists a dual vector field. We may think of vectors defined by Equation (B.1) as column vectors and the dual vectors as row vectors.[B2] Then the dual vectors are formed by taking the transpose of the orginal vectors.

We may take the outer or tensor product of a vector and its dual vector to form a tensor that can be represented by a matrix. Dirac invented a handy notation to take advantage of the above properties of field variables.[B4] His application was confined to the solution of the wave equation for the probability amplitude in quantum mechanics, but we may use it quite generally. We denote a column vector by the "ket" $|\phi\rangle$, and its dual row vector by the "bra" $\langle\phi|$. Then the tensor product of a column vector times a row vector is denoted by $|\phi\rangle\langle\psi|$. Part of the utility of this matrix formalism is due to the isomorphism that exists between the properties of linear operators operating on

fields and the properties of matrix multiplication. This happy correspondence is further strengthened by the fact that the isomorphism extends to the transformation properties of these quantities. However, if the matrix formation is to be truly useful, we must define an inner product, which we denote by $\langle\phi|\psi\rangle$, that has the following properties:[B2]

to linearity

$$\langle c\phi + d\psi|\chi\rangle = c\langle\phi|\chi\rangle + d\langle\psi|\chi\rangle \quad , \tag{B.2}$$

skew-symmetry

$$\langle\phi|\psi\rangle^* = \langle\psi|\phi\rangle \quad , \tag{B.3}$$

and positiveness

$$\langle\phi|\phi\rangle > 0, \text{ real}, \phi \neq 0 \quad . \tag{B.4}$$

In Equations (B.2) to (B.4) we have denoted complex scalars by the letters c and d. If the fields ϕ and ψ are real, then the skew-symmetry property collapses to simple symmetry. The properties defined by Equations (B.2) and (B.3) give the inner product the property of bilinearity.

If we can find an operation that meets the requirements of Equations (B.2) to (B.4) we will have vector fields that form a unitary space,[B2] and will have a complete formalism that can be used to calculate quantities of interest. Many linear operations, such as conditional probability density functions, satisfy Equations (B.2) and (B.4). However, it is more difficult to satisfy Equation (B.3). The requirement of Equation (B.3) is met by the definition

$$\langle\phi(a)|\psi(a)\rangle = \int da\ \phi^*(a)\psi(a)f(a) \quad , \tag{B.5}$$

where a is the common domain of the fields ϕ and ψ and f(a) is the steady-state probability density function for the occurrence of given values of a. Equation (B.5) defines a correlation function, which is a very useful quantity in

probabilistic descriptions of many-body systems. The utility of the definition (B.5) was first pointed out by Zwanzig [B5-B12] and Mori[B13-B17] and has led to the development of projection operator and memory function calculations.[B18-B20]

Actually, the definition (B.5), aside from the weighting factor f(a) that takes account of the possibility expressing the fields ϕ and ψ over an arbitrary transformed domain for which all values a may not be equally probable, is the same as a standard mathematical definition of the inner product.[B2]

We see that we may construct a matrix formalism that is completely parallel to the well known one of quantum mechanics. This formalism is quite general and not at all peculiar to a quantum system. Indeed, almost all of the development stemmed from the basic fact that the solution to any PDE is a field. The only aspect of our development that presented any serious restriction was the skew-symmetry requirement for the inner product.

Since they are vectors, $|\phi\rangle$ and $\langle\phi|$ satisfy the relations

$$|\phi\rangle + |\psi\rangle = |\psi\rangle + |\phi\rangle \quad , \tag{B.6}$$

$$\langle\phi| + \langle\psi| = \langle\psi| + \langle\phi| \quad , \tag{B.7}$$

$$|\phi\rangle + (|\psi\rangle + |\chi\rangle) = (|\phi\rangle + |\psi\rangle) + |\chi\rangle \quad , \tag{B.8}$$

$$\langle\phi| + (\langle\psi| + \langle\chi|) = (\langle\phi| + \langle\psi|) + \langle\chi| \quad , \tag{B.9}$$

$$c(|\phi\rangle + |\psi\rangle) = c|\phi\rangle + c|\psi\rangle \quad , \tag{B.10}$$

$$c(\langle\phi| + \langle\psi|) = c\langle\phi| + c\langle\psi| \quad , \tag{B.11}$$

and

$$1|\phi\rangle = |\phi\rangle \quad , \quad 1\langle\phi| = \langle\phi| \quad . \tag{B.12}$$

Furthermore, the dual of $c|\phi\rangle$ is $c^*\langle\phi|$. We write this as

$$(c|\phi\rangle)^\dagger = c^*\langle\phi| \quad . \tag{B.13}$$

Let us denote the linear tensor operators $|\phi\rangle\langle\phi|$, $|\psi\rangle\langle\psi|$, and $|\chi\rangle\langle\chi|$ respectively by Φ, Ψ, and χ. Since the transformation properties of these operators are isomorphic to those of matrices, we also have the relations

$$c(d\Phi) = (cd)\Phi \quad , \tag{B.14}$$

$$(c + d)\Phi = c\Phi + d\Phi \quad , \tag{B.15}$$

$$c(\Phi + \Psi) = c\Phi + c\Psi \quad , \tag{B.16}$$

$$\Phi(\Psi\chi) = (\Phi\Psi)\chi \quad , \tag{B.17}$$

$$(\Phi + \Psi)\chi = \Phi\chi + \Psi\chi \quad , \tag{B.18}$$

$$\Phi(\Psi + \chi) = \Phi\Psi + \Phi\chi \quad , \tag{B.19}$$

$$(c\Phi)\Psi = c(\Phi\Psi) \quad , \tag{B.20}$$

and

$$\Phi(c\Psi) = c(\Phi\Psi) \quad . \tag{B.21}$$

Furthermore, there will exist an identity operator, I, such that

$$I\Phi = \Phi = \Phi I \quad . \tag{B.22}$$

The combination of Equations (B.6) through (B.13) and (B.14) through (B.22) leads to the usual linear relationships.[B4,B21,B22] The operation of a bilinear operator on a vector will yield a new vector of the same type,

$$\Phi|\psi\rangle = |\chi\rangle, \ \langle\psi|\Phi = \langle\chi| \quad . \tag{B.23}$$

This is often taken as the definition of a bilinear operator.[B4,B21,B22] We also have

$$\Phi(|\psi\rangle + |\chi\rangle) = \Phi|\psi\rangle + \Phi|\chi\rangle \quad , \tag{B.24}$$

$$(\langle\psi| + \langle\chi|)\Phi = \langle\psi|\Phi + \langle\chi|\Phi \quad , \tag{B.25}$$

$$\Phi(c|\psi\rangle) = c\Phi|\psi\rangle \quad , \tag{B.26}$$

$$(\langle\psi|c)\Phi = \langle\psi|\Phi c \quad , \tag{B.27}$$

$$(\Phi + \Psi)|\chi\rangle = \Phi|\chi\rangle + \Psi|\chi\rangle \quad , \tag{B.28}$$

$$\langle\chi|(\Phi + \Psi) = \langle\chi|\Phi + \langle\chi|\Psi \quad , \tag{B.29}$$

$$(\phi\Psi)|\chi\rangle = \phi(\Psi|\chi\rangle) \quad , \tag{B.30}$$

and

$$\langle\chi|(\phi\psi) = (\langle\chi|\phi)\Psi \quad . \tag{B.31}$$

However, we do not necessarily have commutation of operators;

$$\phi\Psi \neq \Psi\phi \quad . \tag{B.32}$$

Note that all of the properties (B.6) through (B.32) are quite general, and beginning with equations like Equation (B.1) can be derived for any field. The critical properties (B.2) to (B.4) depend on the definition chosen for the inner product.

We now define the length $|\phi|$ of a vector $|\phi\rangle$ or $\langle\phi|$ by[B2]

$$|\phi| = \langle\phi|\phi\rangle^{\frac{1}{2}} \quad . \tag{B.33}$$

If all vectors have a finite length, then Equations (B.2) through (B.33) define a Hilbert space. If in addition, our definition of the inner product is such that the rules (B.2) to (B.4) apply, we must have:[B2]

$$|c\phi| = |c||\phi| \quad , \tag{B.34}$$

$$|\phi| > 0 \quad , \quad \phi \neq 0 \quad , \tag{B.35}$$

$$|\langle\phi|\psi\rangle| \leq |\phi||\psi| \quad , \tag{B.36}$$

$$|\phi + \psi| \leq |\phi| + |\psi| \quad , \tag{B.37}$$

$$|\phi - \phi| = 0 \quad , \tag{B.38}$$

$$|\phi - \psi| > 0 \quad , \quad \phi \neq \psi \quad , \tag{B.39}$$

and

$$|\phi - \psi| + |\psi - \chi| \geq |\phi - \chi| \quad . \tag{B.40}$$

Equations (B.34) through (B.40) define a Euclidian vector space.

From Equations (B.13) to (B.21) we see that we must also have

$$\langle\phi|(c|\psi\rangle) = c\langle\phi|\psi\rangle \quad , \tag{B.41}$$

$$\langle\phi|(|\psi\rangle + |\chi\rangle) = \langle\phi|\psi\rangle + \langle\phi|\chi\rangle \quad , \tag{B.42}$$

$$(\langle\phi| + \langle\psi|)|\chi\rangle = \langle\phi|\chi\rangle + \langle\psi|\chi\rangle \quad . \tag{B.43}$$

These Equations (B.41) to (B.43), as well as many of the equations above are given as postulates by Fujita,[B21] but we see that they follow directly from the properties of fields and our definition (B.5).

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APPENDIX C

HYPERCOMPLEX DESCRIPTION AND COUPLED FIELDS

We now elucidate the incongruous situation which occurs whenever we have first-order time or space derivatives in our equations. This occurs, for example, when we juxtapose the equations of motion for the ψ and $\bar{\psi}$ fields or the A and $\bar{A}$ fields, and is most evident in the disquieting form of Equations (2.38) and (2.39) in Part I:

$$\dot{\Psi} = \gamma_o \dot{\psi} - \dot{\bar{\psi}} \gamma_o \quad , \tag{C.1}$$

and

$$\nabla\Psi = \gamma_k \nabla\psi - \nabla\bar{\psi}\gamma_k \quad . \tag{C.2}$$

We may find some relief from the unfamiliar appearance of such equations by noting that the adjoint fields, $\bar{\psi}$, defined by

$$\bar{\psi} = \psi^* \gamma_A \quad , \tag{C.3}$$

where γ_A is a suitable hypercomplex number, behave differently than the more familiar fields, such as ψ^*. The use of hypercomplex numbers is common in particle physics, where they are used in a four-dimensional context to describe degrees of freedom of the system, such as spin, that are not explicitly manifest in the equations of motions. The physics underlying the use of hypercomplex numbers in our equations is not yet clear, but a hypercomplex description gives us extra freedom to achieve a fully self-consistent system of equations. We first give a brief review of hypercomplex numbers, and then illustrate how they may be used to deal with anomalies in the equations of motion.

Hypercomplex numbers are an extension of complex numbers and are often represented by matrices, as their multiplication table can be illustrated by

multiplication of the appropriate matrices.[C1] In our context, they are understood to operate in a four-space apart from the space-time space described above. We are especially interested in four such numbers,

$$\gamma_0, \gamma_1, \gamma_2, \gamma_3 = \gamma_0, \gamma = \gamma_\mu \quad , \tag{C.4}$$

which we will now associate with the time and space derivative

$$\partial_\mu = \partial_t, \partial_x \quad , \tag{C.5}$$

as follows:

$$\partial_t, \partial_x \rightarrow \gamma_o \partial_t, \gamma_k \partial_x \quad . \tag{C.6}$$

We have a wide degree of latitude in choosing matrices to describe the numbers γ_μ; the choice will rest on those aspects of the system we wish to highlight.[C2] The Pauli-Dirac representation:

$$\gamma_0 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \quad , \tag{C.7}$$

$$\gamma_k = \begin{bmatrix} & \sigma \\ -\sigma & \end{bmatrix} \quad , \tag{C.8}$$

where the σ_i are the Pauli spin matrices,

$$\sigma_1 = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} , \sigma_2 = \begin{bmatrix} & -i \\ i & \end{bmatrix} , \sigma_3 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \quad , \tag{C.9}$$

is the most commonly used system in particle physics, and emphasizes parity conservation. The Weyl representation:

$$\gamma_0 = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} , \quad \gamma = \begin{bmatrix} & \sigma \\ -\sigma & \end{bmatrix} \quad , \tag{C.10}$$

is also common to particle physics, especially when considering the conservation of chirality.

We will define a representation that will emphasize symmetry properties under time reversal:

$$\gamma_0 = \begin{bmatrix} \bar{} & 1 \\ 1 & \bar{} \end{bmatrix}, \quad \gamma_k = \begin{bmatrix} \sigma & \\ & \bar{\sigma} \end{bmatrix} \quad . \tag{C.11}$$

We find

$$\gamma_0^2 = \begin{bmatrix} \bar{1} & \\ & \bar{1} \end{bmatrix} = -\,I \quad , \tag{C.12}$$

$$\gamma_k^2 = \begin{bmatrix} \sigma^2 & \\ & \sigma^2 \end{bmatrix} = I \quad , \tag{C.13}$$

and

$$\gamma_0\gamma_k = -\,\gamma_k\gamma_0 \quad . \tag{C.14}$$

We now postulate a pair of fields ϕ and χ, coupled by the first-order hypercomplex equations.[C2,C3]

$$(\gamma_0\partial_t + c\gamma_k\partial_x)\phi = \tfrac{1}{4}D(\gamma_0\partial_t - c\gamma_k\partial_x)\gamma_k\partial_x(\phi - \chi) \quad , \tag{C.15}$$

and

$$(\gamma_0\partial_t - c\gamma_k\partial_x)\chi = \tfrac{1}{4}D(\gamma_0\partial_t + c\gamma_k\partial_x)\gamma_k\partial_x(\phi - \chi) \quad . \tag{C.16}$$

Operating on the first of these equations by $\gamma_0\partial_t + c\gamma_k\partial_x$ and on the second by $-\gamma_0\partial_t + c\gamma_k\partial_x$

$$(\partial_t^2 - c^2\partial_x^2)\phi = \tfrac{1}{4}D(\partial_t^2 - 2\gamma_k\gamma_0\partial_x\partial_t + c^2\partial_x^2)\gamma_k\partial_x(\phi - \chi) \tag{C.17}$$

and

$$(\partial_t^2 - c^2\partial_x^2)\chi = \tfrac{1}{4}(\partial_t^2 + 2\gamma_k\gamma_0\partial_x\partial_t + c^2\partial_x^2)\gamma_k\partial_x(\phi - \chi) \quad . \tag{C.18}$$

Subtracting Equation (C.18) from Equation (C.17), leads to

$$(\partial_t^2 - c^2\partial_x^2)\,(\phi - \chi) = - D\gamma_k\gamma_0\gamma_k\partial_x\partial_t\partial_x(\phi - \chi)$$

$$= D\gamma_0\partial_x\partial_t\partial_x(\phi - \chi) \quad . \tag{C.19}$$

If we now make the connection

$$\psi = \begin{pmatrix} \phi \\ -\chi \end{pmatrix} \tag{C.20}$$

we retrieve our equation of motion for ψ corresponding to L_ψ:

$$(\partial_t^2 - c^2\partial_x^2)\psi = D\gamma_0\partial_x\partial_t\partial_x\psi \quad . \tag{C.21}$$

From Equation (C.21) we find the corresponding Equation for ψ^A; we take the hermetian conjugate of Equation (C.21) :

$$(\partial_t^2 - c^2\partial_x^2)\psi^* = D\partial_x\partial_t\partial_x\psi^*\gamma_o \quad . \tag{C.22}$$

and multiply on the right by γ_A. If we choose γ_A such that

$$\gamma_0\gamma_A = - \gamma_A\gamma_0 \quad , \tag{C.23}$$

we obtain

$$(\partial_t^2 - c^2\partial_x^2)\bar{\psi} = - D\partial_x\partial_t\partial_x\bar{\psi}\gamma_0 \quad . \tag{C.24}$$

Note that the contrasting forms of Equations (C.21) and (C.24) depend on the use of hypercomplex numbers and the relation (C.23). A suitable choice for γ_A is

$$\gamma_A = \begin{bmatrix} \bar{1} & \\ & 1 \end{bmatrix} \quad .$$

This choice yields the relations -

$$\gamma_0\gamma_A = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} = -\gamma_A\gamma_0 \quad , \tag{C.25}$$

and

$$\gamma_A\gamma_k = -\sigma_k I = \gamma_k\gamma_A \tag{C.26}$$

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APPENDIX D

THE SCALAR STRESS TENSOR

Since the Lagrangian density contains second-order derivatives of the fields, it is necessary to extend the standard derivation of the stress-energy tensors. Denoting general derivatives of the fields ψ and $\bar{\psi}$ by ψ_α and $\bar{\psi}_\alpha$, the variation in the Lagrangian density, $\delta\mathcal{L}$, is given by

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \sum_\nu \frac{\partial\mathcal{L}}{\partial\psi_\nu}\delta\psi_\nu + \sum\sum_{\nu\rho} \frac{\partial\mathcal{L}}{\partial\psi_{\nu\rho}}\delta\psi_{\nu\rho} + \text{adjoint terms.} \tag{D.1}$$

In the following discussion, specific mention of the adjoint terms will be dropped, i.e., the existence of parallel adjoint terms will be implied. From Appendix A, the minimum in the variation in satisfies an equation similar to that of (A.12):

$$0 = \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi - \frac{\partial}{\partial x_\sigma}\left(\frac{\partial\mathcal{L}}{\partial\psi_\sigma}\right)\delta\psi - \frac{\partial}{\partial x_\sigma}\frac{\partial\mathcal{L}}{\partial\psi_{\sigma\rho}}\delta\psi_\rho, \tag{D.2}$$

where x_ν is one of the components of the four vector

$$x = (r,t) \quad . \tag{D.3}$$

Substitution of Equation (D.2) in Equation (D.1) gives

$$\delta\mathcal{L} = \sum_\sigma \frac{\partial}{\partial x_\sigma}\left[\frac{\partial\mathcal{L}}{\partial\psi_\sigma}\delta\psi + \sum_\rho \frac{\partial\mathcal{L}}{\partial\psi_{\sigma\rho}}\delta\psi_\rho\right] \quad . \tag{D.4}$$

If the transformations of the fields are such that

$$\psi(x) \rightarrow \psi'(x) = \psi(x) + \delta\psi(x) \quad , \tag{D.5}$$

and

$$\psi(x) \rightarrow \psi'_\rho(x) = \psi_\rho(x) + \delta\psi_\rho(x) \quad , \tag{D.6}$$

then the total variation in the fields, including transformation to new coordinates will be given by

$$\delta_T\psi(x) = \psi'(x) - \psi(x), \tag{D.7}$$

and

$$\delta_T \psi_\rho(x) \equiv \psi'_\rho(x') - \psi_\rho(x). \tag{D.8}$$

Consequently,

$$\delta_T \psi(x) = \delta\psi(x') + \sum_\nu \frac{\partial \psi}{\partial x_\nu} \delta x_\nu \ . \tag{D.9}$$

and

$$\delta_T \psi_\rho(x) = \delta\psi_\rho(x') + \sum_\nu \frac{\partial \psi_\rho}{\partial x_\nu} \delta x_\nu \ . \tag{D.10}$$

Substituting Equations (D.9) and (D.10) into Equation (D.4) yields

$$\delta \mathcal{L} = \sum_\sigma \frac{\partial}{\partial x_\sigma} \left[\left(\frac{\partial \mathcal{L}}{\partial \psi_\sigma} \delta_T \psi + \sum_\rho \frac{\partial \mathcal{L}}{\partial \psi_{\sigma\rho}} \delta_T \psi_\rho \right) - \sum_\nu \left(\frac{\partial \mathcal{L}}{\partial \psi_\sigma} \psi_\nu + \sum_\rho \frac{\partial \mathcal{L}}{\partial \psi_{\sigma\rho}} \psi_{\nu\rho} \right) \delta x_\nu \right] \tag{D.11}$$

The coefficient of δx_ν in Equation (D.11) is the tensor $T_{\nu\sigma}$ given in Equation (3.10) of Part I. For pure translations δ_T of ψ and ψ_ρ equals zero and Equations (3.1) to (3.4) give Equations (3.16) and (3.17).

APPENDIX E

In the special case where the approximation of constant vorticity,

$$\nabla x \mathcal{A} \simeq 0 \tag{E.1}$$

may be made, the interaction Hamiltonian, $\mathcal{H}_I$, can be written as

$$\mathcal{H}_I = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 \ , \tag{E.2}$$

where

$$\mathcal{H}_1 = \frac{\rho}{8c^2} (\nabla x \bar{A} - \nabla x A) \cdot (\nabla \dot{\bar{\psi}} \gamma_o \psi + \bar{\psi} \gamma_o \nabla \dot{\psi}) \ , \tag{E.3}$$

$$\mathcal{H}_2 = \frac{\rho}{8c^2} (\nabla x \dot{\bar{A}} \gamma_o + \gamma_o \nabla x \dot{A}) \cdot (\nabla \bar{\psi} \psi - \bar{\psi} \nabla \psi) \ , \tag{E.4}$$

and $\mathcal{H}_3$ is the same as $\mathcal{H}_2$ in Section 2.

E.1 Revised Canonical Transforms

We first note that

$$\bar{\psi} \nabla \psi - \nabla \bar{\psi} \psi = i \sum\sum_{pq} \frac{c^2}{\rho \omega_q} [(p + q)(\eta_q^\dagger \eta_p - \zeta_p^\dagger \zeta_q)$$

$$+ (p - q)(\eta^\dagger \zeta^\dagger + \eta_p \zeta_q)] \ , \tag{E.5}$$

and

$$\nabla \dot{\bar{\psi}} \gamma_o \psi + \bar{\psi} \gamma_o \nabla \dot{\psi} = \gamma_o \sum\sum_{pq} \frac{c^2}{\rho \omega_q} [(p\omega_{\eta p} + q\omega_{\eta q}) \eta_p^\dagger \eta_q + (p\omega_{\zeta q} + q\omega_{\zeta q}) \zeta_p^\dagger \zeta_q$$

$$+ (p\omega_{\eta p} + q\omega_{\zeta q})(\eta_p^\dagger \zeta_q^\dagger + \eta_p \zeta_q)] \ . \tag{E.6}$$

Therefore,

$$H_1 = \int d^3r \, \mathcal{H}_1$$

$$= \frac{-i\gamma_o \rho}{8c^2} \sum_{qk} \frac{c^3}{\omega_q} \sqrt{\frac{2}{\rho\omega_{ak}}} kx(a_k - a_{\underline{k}}^\dagger)$$

$$\cdot \; [(q + k)\omega_{\eta q+k} + q\omega_{\eta q}]\eta_{q+k}^\dagger \eta_q$$

$$- \; [(q + k)\omega_{\zeta q+k} + q\omega_{\zeta q}]\zeta_{\underline{q+k}}^\dagger \zeta_{\underline{q}}$$

$$+ \; [q\omega_{\eta q} - (q + k)\omega_{\zeta q+k}]\eta_q \zeta_{\underline{\underline{q+k}}}$$

$$+ \; [(q + k)\omega_{\eta q+k} - q\omega_{\zeta q}]\eta_{q+k}^\dagger \zeta_{\underline{\underline{q}}}^\dagger \qquad \text{(E.7)}$$

Proceeding in the same vein, we find

$$H_2 = \int d^3r \; \mathcal{H}_2$$

$$= \frac{-i\rho}{8c^2\gamma_o} \sum_{qk} \frac{c^3}{\rho\omega_q} \sqrt{\frac{2}{\rho\omega_{ak}}} \omega_{ak} kx(a_k - a_{\underline{k}}^\dagger) \cdot (2q + k)$$

$$(\eta_{q+k}^\dagger \eta_q + \eta_{q+k}^\dagger \zeta_{\underline{q}}^\dagger + \eta_q \zeta_{\underline{\underline{q+k}}} + \zeta_{\underline{q}}^\dagger \zeta_{\underline{\underline{q+k}}}) \quad . \qquad \text{(E.8)}$$

It is convenient to add Equations (E.7) and (E.8) to obtain a quantity denoted by H_4:

$$H_4 = H_1 + H_2$$

$$= \frac{-ic\gamma_o}{4\sqrt{2\rho}} \sum_{qk} \frac{1}{\omega_q \sqrt{\omega_{ak}}} kx(a_k - a_{\underline{k}}^\dagger)$$

$$\cdot(A\eta^{\dagger}_{q+k}\eta_q + B\eta^{\dagger}_{q+k}\zeta^{\dagger}_{\underline{q}} + C\eta_q\zeta_{\underline{\underline{q+k}}} + D\zeta^{\dagger}_{\underline{q}}\zeta_{\underline{\underline{q+k}}}) \quad , \tag{E.9}$$

where the vector functions of q and k, A, B, C, and D, are given by

$$A(q, k) = (2q + k)\omega_{ak} + (q + k)\omega_{\eta q+k} + q\omega_{\eta q} \quad , \tag{E.10}$$

$$B(q, k) = (2q + k)\omega_{ak} + (q + k)\omega_{\eta q+k} - q\omega_{\zeta q} \quad , \tag{E.11}$$

$$C(q, k) = (2q + k)\omega_{ak} + q\omega_{\eta q} - (q + k)\omega_{\zeta q+k} \quad , \tag{E.12}$$

and

$$D(q, k) = (2q + k)\omega_{ak} - q\omega_{\zeta q} - (q + k)\omega_{\zeta q+k} \quad . \tag{E.13}$$

The required "canonical" transforms needed to diagonalize H_4 are

$$c_{\underline{q}} = [(Q_q + \omega_{ak}q + \omega_{\eta q}q)\eta_{\underline{q}} + (Q_q - \omega_{ak}q - \omega_{\zeta q}q)\zeta^{\dagger}_{q}]/\sqrt{2Q}_q \quad , \tag{E.14}$$

$$c^{\dagger}_{q} = [(Q_q + \omega_{ak}q + \omega_{\eta q}q)\eta^{\dagger}_{q} + (Q_q - \omega_{ak}q - \omega_{\zeta q}q)\zeta_{\underline{q}}]/\sqrt{2Q}_q \quad , \tag{E.15}$$

$$d_q = [(Q_q - \omega_{ak}q - \omega_{\eta q}q)\eta_q + (Q_q + \omega_{ak}q + \omega_{\zeta q}q)\zeta^{\dagger}_{\underline{q}}]/\sqrt{2Q}_q \quad , \tag{E.16}$$

and

$$d^{\dagger}_{q} = [(Q_q - \omega_{ak}q - \omega_{\eta q}q)\eta^{\dagger}_{q} + (Q_q + \omega_{ak}q + \omega_{\zeta q}q)\zeta_{\underline{q}}]/\sqrt{2Q}_q \quad . \tag{E.17}$$

The resulting contribution to H_β will lead to a Hamiltonian of the form

$$H_\beta = \sum_k[\omega_{ak}a^{\dagger}_k a_k - \frac{ic\gamma_o}{4\sqrt{2\rho\omega_{ak}}}kx(a_k - a^{\dagger}_{\underline{k}})\sum_k(c^{\dagger}_{q+k}c_q - d^{\dagger}_{q+k}d_q)/\omega_q \tag{E.18}$$

$$= \sum_k[\omega_{ak}a^{\dagger}_k a_k - if\gamma_o(a_k - a^{\dagger}_{\underline{k}})\sum_q(n_{cqk} - n_{dqk})/\omega_q] \quad . \tag{E.19}$$

E.2 Order Parameter

An alternative means of gaining insight into the behavior of the flow system is to examine the behavior of variables analogous to the model of the lasing transition studied by Haken[E1-E6] and others.[E7-E14] In this formulation, a variable related to a_k and a_k^* at low values of k will behave much like an order parameter in the Landau-Ginzberg model of critical behavior.[E15-E18] We begin with a truncated Hamiltonian, omitting H_{NL} and H_3 for the time being. We find the following equations of motion for the normal modes of H_o:

$$\dot{a}_{\underline{k}} = -i\omega_{ak} a_{\underline{k}} + f_k(kx\hat{e}) \sum_q [A\eta^\dagger_{q+k}\eta_q + B\eta_{q+k}\zeta^\dagger_{\underline{q}} + C\eta_q \zeta_{\underline{\underline{q+k}}} + D\zeta^\dagger_{\underline{q}}\zeta_{\underline{\underline{q+k}}}] \quad (E.20)$$

$$\dot{a}^\dagger_k = i\omega_{ak} a^\dagger_k + f_k(kx\hat{e}) \sum_q [A\eta^\dagger_{q+k}\eta_q + B\eta_{q+k}\zeta^\dagger_{\underline{q}} + C\eta_q \zeta_{\underline{\underline{q+k}}} + D\zeta^\dagger_{\underline{q}}\zeta_{\underline{\underline{q+k}}}] \,, \quad (E.21)$$

$$\dot{\eta}_q = i\omega_{\eta q}\eta_q + \sum_k f_k kx(a_k - a^\dagger_{\underline{k}})[A\eta_{q-k} + B\zeta^\dagger_{\underline{\underline{q+k}}}] \,, \quad (E.22)$$

$$\dot{\eta}^\dagger_q = i\omega_{\eta q}\eta^\dagger_q - \sum_k f_k kx(a_k - a^\dagger_{\underline{k}})[A\eta^\dagger_{q+k} + C\zeta_{\underline{\underline{q+k}}}] \,, \quad (E.23)$$

$$\dot{\zeta}_{\underline{q}} = -i\omega_{\zeta q}\zeta_q + \sum_k f_k kx(a_k - a^\dagger_{\underline{k}})[B\eta^\dagger_{q+k} + D\zeta_{\underline{q}}] \,, \quad (E.24)$$

and

$$\dot{\zeta}^\dagger_{\underline{q}} = i\omega_{\zeta q}\zeta^\dagger_{\underline{q}} + \sum_k f_k kx(a_k - a^\dagger_{\underline{k}})[C\eta_{q+k} + D\zeta^\dagger_{k-q}] \,. \quad (E.25)$$

Equations (E.22) to (E.25) are now used to find the equations of motion for $N_{\eta q}$ and $N_{\zeta q}$, as well as for two new variables, Δ_q and σ_q, defined by

$$\sigma_q = \eta_q \zeta_{\underline{q}} \quad (E.26)$$

and

$$\sigma_q^\dagger = \eta_q^\dagger \zeta_{\overline{q}}^\dagger \quad . \tag{E.27}$$

Note that

$$[n_q, \sigma_{q'}] = \sigma_q \delta_{q,q'} \quad , \tag{E.28}$$

$$[n_q, \sigma_{q'}^\dagger] = \sigma_q^\dagger \delta_{q,q'} \quad , \tag{E.29}$$

and

$$[\sigma_q, \sigma_{q'}^\dagger] = N_q \delta_{q,q'} \quad . \tag{E.30}$$

The equation of motion for N_q is obtained by multiplying Equation (E.22) by $\eta_q^\dagger$ and Equation (E.23) by η_q, and adding. We adopt the convention that the appearance of terms with k in their subscripts implies a sum of terms with both $\pm|k|$. We find

$$\dot{n}_{\eta q} = \sum_k f_k kx(a_k - a_{\overline{k}}^\dagger) \cdot [C\eta_q \zeta_{\underline{\quad}_{q+k}} - B\eta_q^\dagger \zeta_{\underline{\quad}_{q+k}}^\dagger]. \tag{E.31}$$

Similarly, letting

$$F_k = f_k kx(a_k - a_k^\dagger), \tag{E.32}$$

$$\dot{n}_{\zeta_q} = \sum_k F_k \cdot [C\zeta_{\underline{\quad}_q} \eta_{q+k} - B\zeta_{\underline{\quad}_q}^\dagger \eta_{q+k}^\dagger] \quad , \tag{E.33}$$

$$\dot{\sigma}_q = -icq\sigma_q + \sum_k F_k \cdot [A\zeta_{\underline{\quad}_q} \eta_{q-k} + B(\zeta_{\underline{\quad}_{q+k}}^\dagger \zeta_{\underline{\quad}_q} + \eta_{q+k}^\dagger \eta_q) + D\eta_q \zeta_{\underline{\quad}}] \quad , \tag{E.34}$$

and

$$\dot{\sigma}_q^\dagger = icq\sigma_q - \sum_k F_k \cdot [A\eta_{q+k}^\dagger \zeta_{\underline{\quad}_q}^\dagger + C(\zeta_{\underline{\quad}_q}^\dagger \zeta_{\underline{\quad}_{q+k}} + \eta_q^\dagger \eta_{q+k}) + D\eta_q^\dagger \zeta_{k-q}^\dagger] \quad . \tag{E.35}$$

We now take the limit k→o and find with

$$n_q = n_{\eta q} + n_{\zeta q} \tag{E.36}$$

the following approximate equations:

$$\dot{n}_q \approx \lim_{k\to 0} F_k(C\sigma_q - B\sigma_q^\dagger), \tag{E.37}$$

$$\dot{\sigma}_q \approx -icq\sigma_q + \lim_{k\to 0} F_k\cdot[(A+D)\sigma_q + Bn_q] \quad , \tag{E.38}$$

and

$$\dot{\sigma}_q^\dagger \approx icq\sigma_q^\dagger - \lim_{k\to 0} F_k\cdot[(A+D)\sigma_q^\dagger + Cn_q] \quad . \tag{E.39}$$

Equations (E.37) to (E.39) can be used as the starting point for two different treatments of critical behavior. The first is a computer study of a transition to chaos similar to that of the Lorentz model of Benard convection.[E19,E20] The second parallels a model of the lasing transition studied extensively by Haken.[E1-E6]

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APPENDIX F

TRANSFORMATION OF THE INTERACTION HAMILTONIAN

In this appendix, we show how Hamiltonians such as H_β given in Equation (3.1) may be transformed to the form given in Equation (3.6). The treatment follows closely that given by Kittel for the electron-phonon interaction.[F1] Consider the truncated Hamiltonian H_T, given by a diagonal basis Hamiltonian H_D and a perturbation part H':

$$H_T = H_D - \frac{if'}{\omega_q} H' \quad , \tag{F.1}$$

where for example

$$H_D = \omega_{ak} a_k^\dagger a_k \tag{F.2}$$

and

$$H' = kx(a_k^\dagger - a_{\bar{k}}) c_{q+k}^\dagger c_q \quad . \tag{F.3}$$

The key step in transforming Equation (F.3) is to postulate the existence of a function S such that

$$\dot{S} = [S, H_D] = - \frac{if'}{\omega_q} H' \quad . \tag{F.4}$$

Equation (F.4) can be shown to be satisfied using the interaction picture.[F1] Although we are discussing a fully classical system, the concept of Schroedinger, Heisenberg, and interaction picture can still be applied[F2], as implied by the discussion in Appendix B.

Integrating Equation (F4) gives

$$S(o) = - \frac{if'}{\omega_q} \int_{-D}^{o} dt \; H'(t) \quad . \tag{F.5}$$

Substituting Equation (F.5) into Equation (F.4) and then substituting the resulting form of Equation (F.4) into Equation (F.1) gives

$$H_T = H_D - \frac{if'}{2\omega_q}\,[H'(o),S(o)] \quad , \tag{F.6}$$

or

$$H_T = H_D + \frac{f'^2}{2\omega_q^2}\int_{-D}^{o} dt[H'(t),H'(o)] \quad . \tag{F.7}$$

From Equation 3.1

$$f' = \gamma_o f_k \, . \tag{F.8}$$

Therefore, the perturbative term in Equation (3.1) in the fields c_q can be put in the form ($\gamma_o^2 = -1$)

$$H_c' = \frac{f_k^2}{2\omega_q^2}\int_{-D}^{o} dt\Big[kx[a_k(t) - a_{\bar{k}}^{\dagger}(t)]c_{q+k}^{\dagger}(t)c_q(t),k'x(a_{k'} - a_{\bar{k}'}^{\dagger})c_{q'+k'}^{\dagger}c_{q'}\Big]. \tag{F.9}$$

This may be written as

$$H_c' = \frac{kk'}{2}\left(\frac{f_k}{\omega_q}\right)^2\int_{-D}^{o} dt\; e^{i(\omega_{cq+k}-\omega_{cq})t}\, c_{q+k}^{\dagger}c_q c_{q'+k'}^{\dagger}c_{q'}$$

$$\left(\left[a_k,a_{\bar{k}'}^{\dagger}\right] e^{-i\omega_{ak}t} + \left[a_{\bar{k}}^{\dagger},a_{k'}\right] e^{i\omega_{ak'}t}\right) \quad . \tag{F.10}$$

The Fourier transformation yields, upon contracting the Poisson brackets,

$$H_c' = \frac{1}{2}\left(\frac{kf_k}{\omega_q}\right)^2\left(\frac{c_{q+k}^{\dagger}c_q c_{q'-k}^{\dagger}c_{q'}}{\omega_{cq+k} - \omega_{cq} + \omega_{ak}} - \frac{c_{q+k}^{\dagger}c_q c_{q'-k}^{\dagger}c_{q'}}{\omega_{cq+k} - \omega_{cq} - \omega_{ak}}\right) . \tag{F.11}$$

Performing the addition and making a similar transformation for the d fields gives Equation (3.6).

One difficulty occurs in that there are several nonlinear contributions to H of the form described above. Transformations using functions S of the type described in Equations (F.4) and (F.5) must be applied to the various nonlinear contributions simultaneously. This can be achieved through the use of projection operators, again using the concepts described in Appendix B.

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APPENDIX G

In this appendix, we will display the steps required to generate the specific term shown in Equation (4.21) of Part III. We begin with the set of Equations (4.7) to (4.12). For the particular term of interest, we replace the $\eta^\dagger\eta$ term in Equation (4.7) with expressions (4.9) and (4.10). We find that

$$\sum \epsilon_{ilm} k_l A_m(q,k) \eta^\dagger(q+k)\eta(q) =$$

$$\sum_q \epsilon_{ilm} k_l A_m(q,k) G^\eta(q+k) G^{\eta\dagger}(q) F^\eta(q+k) F^\eta(q)$$

$$- \sum_q \sum_{k''} \epsilon_{ilm} k_l A_m(q,k) G^\eta(q+k) G^{\eta\dagger}(q) F^\eta(q+k) f(k'') \epsilon_{rst} k_s$$

$$[a_t(k'') - a_t^\dagger(k'')][A_r(q,k'')\eta(q+k'') + B_r(q,k'')\zeta^\dagger(q+k'')]$$

$$+ \sum_q \sum_{k'} \epsilon_{ilm} k_l A_m(q,k) G^{\eta\dagger}(q) G^\eta(q+k) F^\eta(q) f(k') \epsilon_{npq} k_p$$

$$[a_q(k') - a_q^\dagger(k')][\, A_n(q+k,k')\eta(q+k-k') + B_n(q+k,k')\zeta^\dagger(q+k+k')\,]$$

$$+ O(\, f^2(k)\,). \qquad (G.1)$$

In this expression we will examine the second term on the right side, looking at the term proportional to A_r. Using Equation (4.9) once again, but this time truncating at the linear term, we find

$$\sum_q \sum_{k''} f(k'') G^\eta(q+k) G^{\eta\dagger}(q) G^\eta(q+k'') F^\eta(q+k) F^\eta(q+k'')$$

$$\epsilon_{ilm} k_l A_m(q,) \epsilon_{rst} k''_s \, a_t(k'') A_r(q,k'')$$

$$+ \text{the same term with } a_t(k'') \to a_t^\dagger(k''). \qquad (G.2)$$

We will now take advantage of this equation, and employ the relation

$$\langle F^{\eta}(q+k)F^{\eta}(q+k")\rangle = \delta(q+k-q-k")$$

$$= \delta(k-k") \qquad (G.3)$$

leaving us with, after the sum over k" is carried out,

$$\langle f(k)k_l\epsilon_{rst}k_s\left[\sum_q G^{\eta}(q+k)G^{\eta\dagger}(q)G^{\eta}(q+k)A_m(q,k)A_r(q,k)\right]a_t(k)\rangle \quad . \qquad (G.4)$$

replacing a_t with a lowest order contribution, we find that Equation (4.7) becomes

$$\langle a_j(k)\rangle \langle \Big[G^a_{ij}(k) +$$

$$G^a_{wj}(k)\epsilon_{wlm}k_l\epsilon_{rst}k_s\sum_q G^{\eta}(q+k)G^{\eta\dagger}(q)G^{\eta}(q+k)$$

$$f(k)A_m(q,k)f(k)A_r(q,k)G^a_{it}(k)\Big]F^a_i(k)\rangle \quad . \qquad (G.5)$$

Thus, we write to lowest order,

$$\langle a_j(k)\rangle = \langle \tilde{G}^a_{ij}(k)\ F^a_i(k) + \ldots \qquad (G.6)$$

where $\tilde{G}^a_{ij}(k)$ is now given by Equation (23). Also, it is seen in Equation (4.5) that the sum over internal wave-vectors contributes a term of the form

$$I_{mr}(k) = \sum_q A_m(q,k)A_r(q,k)G^{\eta}(q,k)G^{\eta\dagger}(q)G^{\eta}(q+k) \quad . \qquad (G.7)$$

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