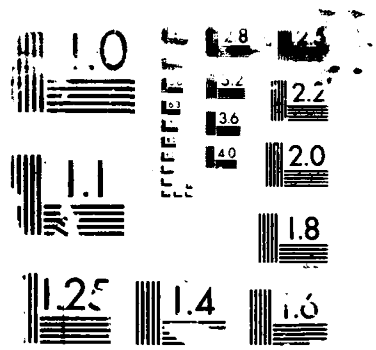
AD-A189 124 FREE BOUNDARY PROBLEMS ARISING IN THE CONRROL OF A 1/1
FLEXIBLE ROBOT ARM (U) MARYLAND UNIV BALTIMORE COUNTY
CATONSVILLE DEPT OF MATHEMATIC T I SEIDMAN 24 SEP 87
UNCLASSIFIED AFOSR-TR-87-1558 AFOSR-87-0190 F/G 6/3 NL


1.0
2.8
3.6
4.0
2.2
2.0
1.1
1.8
1.4

DTIC FILE COPY

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

1a. AD-A189 124	1b. RESTRICTIVE MARKINGS
2a.	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.
2b. DECLASSI...	
4. PERFORMING ORGANIZATION REPORT NUMBER(S)	5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR- 87-1558
6a. NAME OF PERFORMING ORGANIZATION / 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION
6c. ADDRESS (City, State, and ZIP Code)	7b. ADDRESS (City, State, and ZIP Code)
8a. NAME OF FUNDING/SPONSORING ORGANIZATION / 8b. OFFICE SYMBOL (If applicable) NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFOSR-87-0190
8c. ADDRESS (City, State, and ZIP Code)	10. SOURCE OF FUNDING NUMBERS: PROGRAM ELEMENT NO. / PROJECT NO. / TASK NO / WORK UNIT ACCESSION NO.

11. TITLE (Include Security Classification)

12. PERSONAL AUTHOR(S)

13a. TYPE OF REPORT	13b. TIME COVERED FROM ____ TO ____	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

DTIC ELECTE NOV 0 9 1987 D E

DTIC COPY INSPECTED

Accession For: NTIS GRA&I, DTIC TAB, Unannounced, Justification; By; Distribution/; Availability Codes; Avail and/or Dist Special; A-1

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT ☐ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION
22a. NAME OF RESPONSIBLE INDIVIDUAL	22b. TELEPHONE (Include Area Code) / 22c. OFFICE SYMBOL

DD Form 1473, JUN 86 — Previous editions are obsolete. — SECURITY CLASSIFICATION OF THIS PAGE

AFOSR-TR- 87-1558

September 24, 1987

FREE BOUNDARY PROBLEMS ARISING IN THE CONTROL OF A FLEXIBLE ROBOT ARM[1,2]

Thomas I. Seidman
Department of Mathematics and Statistics
University of Maryland Baltimore County
Catonsville, MD 21228 USA
(BITNET: seidman@umbc, arpanet: seidman@umbc3)

ABSTRACT: Modeling of a flexible robot arm mounted at a prismatic joint leads to a moving boundary problem. In a control-theoretic context with joint motion as control, these become *free boundary problems*. Some problems are discussed for transverse vibration (the beam equation) but the principal result is an *exact controllability theorem* for longitudinal vibration (the wave equation).

Key words: *free boundary, robot arm, prismatic joint, beam equation, feedback, wave equation, exact controllability*

1. INTRODUCTION

We consider the modeling of a robot arm consisting of a flexible beam with a *prismatic joint* — that is, a mounting of the arm in which one end is clamped but permits controlled in-out motion so the extension of the arm (i.e., the relevant portion of the beam) has variable length $\ell = \ell(t)$. In this case 'flexible' means only that the dynamics of the beam must be modeled by a partial differential equation: treatment as a rigid body is an inadequate approximation in that vibration (i.e., time-dependent deviation from the nominal rigid-body position) is a significant aspect of the problem. The space-time domain on which we consider the dynamics is thus

$$(1.1) \qquad \mathcal{Q} = \mathcal{Q}_T : \quad \{(t,s) : 0 < t < T;\ 0 < s < \ell(t)\}$$

[1]This research has been partially supported by the U.S. Air Force Office of Scientific Research under grant #AFOSR-87-0190

[2]This material was presented at the international colloquium: *Free Boundary Problems: Theory and Applications* held at Irsee (Bavaria, Germany) June 11-20, 1987. The present text will appear in the proceedings of that conference, edited by K.-H. Hoffmann and J. Sprekels.

where s denotes length (in *material coordinates*, measured from the tip of the arm). This is certainly a *moving boundary problem*. From a control-theoretic point of view, we note that the significant question is precisely the determination of the control $\ell(\cdot)$ so we will have a *free boundary problem*.

In the next section we analyze purely longitudinal vibration of such a beam. This is not, of course, the problem of greatest practical concern but we will be able to treat it with some degree of completeness — showing *exact controllability*, subject only to some rather mild conditions. The more significant problem of treating transverse vibration in this context requires more extensive analysis than can be presented here. A brief discussion of such problems[3] is presented in Section 3. For our present purposes we take the beam to be (nominally) horizontal along the x-axis of the 'laboratory coordinate system' with the joint rigidly fixed[4] at the origin. Included in our model is a load of mass m supported at the free end of the arm. We assume that the system is (except for the boundary condition at the joint) conservative — dissipative mechanisms such as friction and, e.g., air resistance can be neglected.

2. EXACT CONTROLLABILITY

Here we restrict our consideration to the purely *longitudinal* vibrations of the arm. Suppose we let $x(t,s)$ be the actual position (in the laboratory coordinate system) of the material point s at time t. The formulation of the dynamics takes a much simpler form if we take the unknown to be $v := [s+x]$ so, e.g., $v = \ell$ at the joint. For a hyperelastic stress-strain law[5] one can easily show, using Hamilton's Principle of Stationary Action and a bit of manipulation, that v must satisfy the one-dimensional wave equation:

$$\rho\, v_{tt} = \psi(\cdot, v_s)_s \qquad \text{on } Q_T \tag{2.1}$$

with the boundary conditions:

$$v = \ell \qquad \text{at the joint: } s = \ell = \ell(t), \tag{2.2}$$

$$m\, v_{tt} = \psi(v_s) \qquad \text{at the tip: } s = 0. \tag{2.3}$$

[3] Work on such problems has been initiated by P.K.C. Wang and, indeed, [2] should be acknowledged as the principal stimulus to the present work. Many interesting and important problems remain open.

[4] In practice one would expect that the joint may, itself, be mounted on a moving support (e.g., on another arm) and/or that there is coupling with a controlled rotation around a vertical axis. One might also wish to include the possibility that the mounting is *nominally* rigid yet subject to vibration giving 'noise' in the boundary condition there; compare [1].

[5] We make the usual assumption that the *elastic potential* $\Psi = \Psi(s,r)$ is convex in r with minimum at $\Psi(s,0) = 0$. We set $\psi(s,r) = \Psi_r(s,r)$ and let $\rho(s)$ be the mass density.

To this we adjoin specification of initial conditions:

(2.4) $$\ell(0) = \ell^0 \text{ and } \quad v = v^0,\ v_t = w^0 \text{ on } [0, \ell^0] \text{ at } t = 0,$$

subject to (2.2): $v^0(\ell^0) = \ell^0$ as a *compatibility condition.*

Suppose, in addition to (2.4), one were also to specify a target state for the terminal time $t = T$:

(2.5) $$\ell(T) = \ell^T \text{ and } \quad v = v^T,\ v_t = w^T \text{ on } [0, \ell^T] \text{ at } t = T,$$

again subject to (2.2). Our object is to prove *exact controllability*:

> There should be a time $T > 0$ and an admissible control function $\ell(\cdot)$ on $[0, T]$ such that (2.1), (2.4), (2.2), (2.3) give (2.5)

under suitable (mild) hypotheses. By "admissible" we mean that $\ell(0) = \ell^0$, $\ell(T) = \ell^T$, and[6]

(2.6) $$|\ell(t_1) - \ell(t_2)| \leq \kappa |t_1 - t_2| \qquad \text{with } \kappa < 1$$

which will ensure that the free boundary curve: $\Gamma := \{s = \ell(t)\}$ is 'time-like' so a characteristic $[s \pm t = \mathit{const.}]$ can intersect Γ only once (transversely).

For simplicity of exposition we present in detail only the linear case[7] for which (2.1) reduces to

(2.7) $$v_{tt} = v_{ss} \qquad \text{on } \mathcal{Q}_T.$$

In this case the *method of characteristics* gives the simple decomposition

(2.8) $$v(t, s) = f(s + t) + g(s - t)$$

into left- and right-moving waves. Note that differentiating (2.8) gives the identities:

(2.9) $$f'(s + t) = [v_s(t, s) + v_t(t, s)]/2,$$
$$g'(s - t) = [v_s(t, s) - v_t(t, s)]/2.$$

The representation (2.8) gives the general (weak) solution of (2.7) and, in terms of f, g, the boundary conditions (2.2), (2.3) now become:

(2.10) $$f(\ell(t) + t) + g(\ell(t) - t) = \ell(t),$$

(2.11) $$m[f''(s) + g''(-s)] = f'(s) + g'(-s).$$

---

[6] Physically, this corresponds to a requirement that the controlled motion at the joint must always be slower than the wave propagation speed along the beam. For smooth $\ell(\cdot)$ it is equivalent to $|\dot{\ell}(t)| \leq \kappa < 1$.

[7] I.e., $\Psi(s, r) = \frac{1}{2}\beta(s) r^2$ so $\psi = \beta r$. We further simplify by assuming the beam is homogeneous so, in suitable units, $\rho = \beta = 1$.

Note that we can differentiate (2.10) with respect to t to obtain an ODE for ℓ:

$$\dot{\ell} = \frac{f'(\ell+t) - g'(\ell-t)}{1 - [f'(\ell+t) - g'(\ell-t)]}. \tag{2.12}$$

From the inequality: $|u-v| + |u+v| \le 2\max\{|u|,|v|\}$ it would follow that (2.12) is well-behaved — indeed, giving (2.6) — if we were always to have

$$|f'|, |g'| \le \kappa/2 \tag{2.13}$$

where defined.

The initial data v^0, w^0 in (2.4) determine f, g on $[0, \ell^0]$ using (2.9) with $t = 0$. In particular, we have g' determined on $\mathcal{D}_0^g := \{(t,s) : 0 \le s - t \le \ell^0\}$. We then can use (2.11) as an ODE for g' to obtain:

$$\begin{aligned} g'(s) &= \tfrac{1}{2}[v_s^0 + w^0](-s) + e^{s/m} w^0(0) \\ &\qquad + \tfrac{1}{m}\int_0^{-s} e^{(s+r)/m}[v_s^0 + w^0](r)\,dr \end{aligned} \tag{2.14}$$

for $-\ell^0 \le s \le 0$, which defines g' from (2.4) on $\mathcal{D}_1^g := \{(t,s) : -\ell^0 \le s - t \le 0\}$. Finally, we note that $v(\ell^0, 0)$ is determined by the data of (2.4).

Similarly, we note that the target data in (2.5) directly determine f' on $\mathcal{D}_0^f := \{(t,s) : T \le s + t \le \ell^T + T\}$, using (2.9) with $t = T$, and give,[8] now using (2.11) as an ODE for f':

$$\begin{aligned} f'(T-\tau) &= \tfrac{1}{2}[v_s^T + w^T](\tau) + e^{-\tau/m} w^T(0) \\ &\qquad + \tfrac{1}{m}\int_0^{\tau} e^{-r/m}[v_s^T + w^T](r)\,dr \end{aligned} \tag{2.15}$$

for $0 \le \tau \le \ell^T$, which defines f' from (2.5) on $\mathcal{D}_1^f := \{(t,s) : T - \ell^T \le s + t \le T\}$. From (2.5) we also, then, determine $v(T - \ell^T, 0)$.

We now choose Cauchy data on the segment: $\mathcal{G} := \{(t,0) : \ell^0 \le t \le T - \ell^T\}$, taking $v(t,0)$ linear in t, making v_t constant:

$$v_t(t,0) = c := \frac{v(T - \ell^T, 0) - v(\ell^0, 0)}{T - [\ell^0 + \ell^T]} \tag{2.16}$$

on $\mathcal{G}$. This gives $v_s(t,0) = m v_{tt} = 0$ by (2.3) with $\psi(r) = r$. Using (2.9), we now have

$$\begin{aligned} f' &= c/2 \quad \text{on } \mathcal{D}_2^f := \{(t,s) : \ell^0 \le s + t \le T - \ell^T\}, \\ g' &= -c/2 \quad \text{on } \mathcal{D}_2^f := \{(t,s) : \ell^T - T \le s - t \le -\ell^0\}. \end{aligned} \tag{2.17}$$

[8] To avoid any possibility of conflict we assume that T is chosen so $T > \ell^0 + \ell^T$, giving $-\ell^0 < \ell^T - T$ and $\ell^0 < T + \ell^T$; later we will strengthen this requirement to (2.18). Note that the computation, here, from (2.5) is independent (to within translation in t) of the particular choice of T.

Note that if we require that

$$T \geq [\ell^0 + \ell^T] + |v(T - \ell^T, 0) - v(\ell^0, 0)|/\kappa, \tag{2.18}$$

then (2.16), (2.17) give (2.13) on $\mathcal{D}_2^f, \mathcal{D}_2^g$. If we now impose the requirement that

$$\begin{aligned} |v_s^0|, |w^0| &\leq \kappa/4 \ \text{ on } [0, \ell^0], \\ |v_s^T|, |w^T| &\leq \kappa/4 \ \text{ on } [0, \ell^T], \end{aligned} \tag{2.19}$$

then (2.13) holds on $\mathcal{D}_0^f, \mathcal{D}_0^g$ directly from (2.9) and it easily follows from (2.14), (2.15), that we will also have (2.13) on $\mathcal{D}_1^f, \mathcal{D}_1^g$.

Theorem 1: *Suppose one is given the data of* (2.4), (2.5) *subject to compatibility with* (2.2) *and the condition* (2.19) *with* $\kappa < 1$. *(Assume* w^0, w^T *are continuous and* v^0, $v^T \in C^1$.*) Then there exists* $T > 0$ *and an admissible control function* $\ell(\cdot) \in C^1[0, T]$ *such that the solution* v *of* (2.7), (2.4), (2.2), (2.3) *also satisfies* (2.5).

PROOF: As above, solve (2.7), (2.4), (2.3) on the triangle $\Delta_0 = \{(t, s) : 0 \leq t, s;\ s + t \leq \ell^0\}$ and (2.7), (2.5), (2.3) on $\Delta_T = \{(t, s) : 0 \leq T - t, s;\ s - t \leq \ell^T - T\}$ — obtaining, in particular, $v(\ell^0, 0)$ and $v(T - \ell^T, 0)$. Then choose T subject to (2.18) and specify Cauchy data on $\mathcal{G}$ as above — i.e., (2.16) and $v_s = 0$. Thus, f is determined on $\mathcal{D}^f = \mathcal{D}_0^f \cup \mathcal{D}_1^f \cup \mathcal{D}_2^f$ and g on $\mathcal{D}^g = \mathcal{D}_0^g \cup \mathcal{D}_1^g \cup \mathcal{D}_2^g$ with (2.13) satisfied on $\mathcal{S} = \mathcal{D}^f \cap \mathcal{D}^g$ as in the discussion above. Note that our assumptions give f', g' continuous on $\mathcal{S}$.

Starting with the initial condition: $\ell(0) = \ell^0$, solve[9] (2.12) to obtain Γ. This solution curve satisfies (2.10) and (2.6). Note that the solution curve Γ cannot leave $\mathcal{S}$ before $t = T$ by, e.g., crossing the characteristic: $[s + t = \ell^T + T]$ — say, at a point (ℓ_*, t_*) — since this would give, by (2.10):

$$\begin{aligned} \ell^T + T - t_* &= \ell_* = f(\ell_* + t_*) + g(\ell_* - t_*) \\ &= f(\ell^T + T) + g(\ell^T + T - 2t_*) \\ &= [\ell^T - g(\ell^T - T)] + g(\ell^T + T - 2t_*), \end{aligned}$$

contradicting the fact that $|g'| \leq \kappa/2 < 1/2$; similarly, Γ cannot leave $\mathcal{S}$ by crossing $[s - t = \ell^T - T]$ since $|f'| < 1/2$. It follows that $\ell(\cdot)$ is well-defined on $[0, T]$ with $\ell(T) = \ell^T$.

This specification of $T, \ell(\cdot)$ defines $\mathcal{Q}_T$. In view of (2.6), the geometry of Γ ensures that $\mathcal{Q}_T \subset \Delta_0 \cup \Delta_T \cup \mathcal{S}$ so f, g are both defined on all of $\mathcal{Q}_T$. Now (2.8) gives a (weak) solution v of (2.7) and the construction ensures that we also have (2.4), (2.5), (2.2), (2.3) as desired. □

---

[9] A solution exists since the right hand side of (2.12) is continuous in ℓ (as long as one remains in $\mathcal{S}$). Our assumptions do not give a Lipschitz condition; thus uniqueness — not really needed for the Theorem — would have to be obtained by a different argument: noting that (2.13) makes the right hand side of (2.10) contractive in ℓ.

Remarks: The regularity required for the data — continuity of v_s^0, w^0, v_s^T, w^T — can be weakened by a limit argument. In this case one retains (2.6) although $\dot{\ell}$ may no longer be continuous. There is also no really new difficulty in considering spatially variable coefficients: $\rho v_{tt} = (\beta v_s)_s$. The argument must be modified in detail[10] but is essentially similar in spirit.

It is worth noting that the construction in the proof above can be viewed from a slightly different perspective. After solving on Δ_0 and Δ_T one fills in Cauchy data on a 'gap' $\mathcal{G}$ so as to consider an *initial value problem* with the roles of t and s reversed — taking, say, v^0, v^T (perhaps suitably extended to larger s) as *boundary data*. The choice of data on $\mathcal{G}$ and the bounds (2.19) are such as to ensure that $|v_s| \le \kappa < 1$ which then makes the right hand side of (2.10) contractive in ℓ so (2.10) determines $\ell(t)$ uniquely for each t. (One must also, of course, verify that this gives $\ell(\cdot)$ as an 'admissible control function'.)

Stated this way, it is clear that the argument does not really depend on the linearity of (2.7) or on the possibility of a decomposition (2.8). Without providing details, we note only that one can treat (2.1) from this point of view under quite mild conditions on $\psi(\cdot)$. If we set $\phi := \psi^{-1}$, $w = v_t$ and $u = \psi(v_s)$ so $v_s = \phi(u)$ we can rewrite (2.1) in terms as a system:

$$u_s = \rho w_t \qquad w_s = \phi(u)_t \tag{2.20}$$

suggesting the role-reversed problem. The crux of the argument, now, is to show that 'reasonable bounds' on the initial and target data can be imposed to ensure that one will have (for a suitable construction of $\mathcal{G}$ and of the data there, satisfying $m w_t = u$) the necessary bound: $|v_s| = |\phi(u)| \le \kappa$ to be able to apply the Contractive Mapping Theorem to (2.10). □

3. TRANSVERSE VIBRATION

In this section we present an extremely brief introduction to the class of problems which might be considered. As above, we consider the robot arm as a nominally horizontal beam with a load m at the tip and with no dissipation or body forces (e.g., ignoring friction, gravity, ...). Here, however, we consider *transverse vibrations*. For simplicity we consider only vertical vibration, so the principal unknown is the vertical deflection $u = u(t,s)$. Thus, the 'laboratory coordinates' of a material point s will be $[x,y,z] = [\ell(t) - s, 0, u(t,s)]$. Further, we ignore shear and work with a linearized

[10] E.g., the characteristics are no longer straight lines and the propagation along them may no longer be constant. We now must take κ to be a constant less than the *minimum* sound speed and impose a more stringent condition (2.19).

theory.[11]

We begin with the 'clamped' boundary conditions at the prismatic joint:

$$(3.1) \qquad u = u_s = 0 \qquad \text{at the joint: } s = \ell(t).$$

Next, in this setting a geometric analysis shows that the stress deformation for a segment ds of the beam is proportional to [radius of curvature]$^{-2}$ so the strain energy (elastic potential) is $\approx \frac{1}{2}\beta\, u_{ss}^2 ds$ where β (sometimes written as EI) corresponds to stiffness. The corresponding kinetic energy is:

$$\frac{1}{2}\int_0^{\ell(t)} [\rho\,(\dot\ell^2 + u_t^2) + \alpha\, u_{st}^2]\, ds + \tfrac{1}{2}[m\,(\dot\ell^2 + u_t^2) + a\, u_{st}^2]|_{s=0}$$

where α is the cross-sectional moment. By Hamilton's Principle, we obtain the Euler-Bernoulli beam equation[12]

$$(3.2) \qquad \rho\, u_{tt} + (\alpha\, u_{stt})_s + (\beta\, u_{ss})_{ss} = 0 \qquad \text{on } \mathcal{Q}_T$$

(where $\mathcal{Q}_T$ is as above in (1.1)) and the other boundary conditions:

$$(3.3) \qquad m\, u_{tt} + \alpha\, u_{stt} = (\beta\, u_{ss})_s, \quad a\, u_{stt} + \beta\, u_{ss} = 0 \quad \text{at the tip: } s = 0.$$

The system, if the control function $\ell(\cdot)$ were specified, would consist of (3.2) with (3.1) and (3.3) together with initial conditions (2.4). We note here, however, that the type of problem having particular control-theoretic significance is that which arises when $\ell(\cdot)$ is determined in *feedback form*:

$$(3.4) \qquad \ell(t) = \Lambda[u(t,\cdot)],$$

i.e., one couples (3.4) with the other conditions as an implicit characterization of the free boundary Γ.

In (3.4), we consider $\Lambda[\cdot]$ as some suitable functional — presumably a function $\Phi(\omega)$ for some (finite) set of implementable observations:[13] $\omega = \mathrm{C}[u]$. For example, one might take Λ to be of the form $\Lambda[u] = \Phi(\mathrm{C}[u])$ with

$$(3.5) \qquad \mathrm{C} : u \mapsto \{u|_{s=0}, u_{tt}|_{s=s_*}\} =: \omega.$$

[11]Use of a linearized stress-strain relation is a consequence of the fundamental beam assumption: 'infinitesimal' cross-section, giving infinitesimal deformations. The significant assumption is the the simplifying but rather restrictive one that the arm always remains straight enough to ignore the so-called 'geometric nonlinearities' — essentially, that one can adequately approximate [radius of curvature]$^{-2}$ by u_{ss}^2, linear velocity by $|\dot\ell \mathbf{i} + u_t \mathbf{k}|$, angular velocity $[\arctan u_s]_t$ by $-u_{st}$, etc.

[12]Note that the equation is just the standard beam equation when we use s as the independent variable. This formulation is somewhat different from that of [2].

[13]Alternatively, we might consider $\Lambda = \Phi(z)$ where z is the solution of an auxiliary system of the form: $\dot z = F(z,\omega)$.

Such an *observation operator* C might be implementable by, say, visual observation of the tip position and placing an accelerometer on the beam at s_*, allowing for the (known) effect of the joint motion. This would give, as an example,

$$\ell(t) = \Phi(u(t,0), u_{tt}(t,s_*)) \tag{3.6}$$

for some given function Φ.

Once a specific form of the feedback functional $\Lambda[\cdot]$ has been prescribed, the problem is analysis[14] of the full system: (3.2), (2.4), (3.1), (3.3), and, e.g., (3.6) to determine existence and uniqueness as well as such properties of control-theoretic interest as stability.

References

[1] W.J. Bottega, *Dynamics and stability of support-excited beam-columns with end mass*, Dyn. and Stab. of Syst. **1** (1986), pp. 201–215.

[2] P.K.C. Wang, *Some nonlinear problems in the control of distributed systems*, in *Proc. IFIP WG 7.2 Working Conf. on Control of Systems Gov. by PDE* (I. Lasiecka and R. Triggiani, eds.), Springer-Verlag, *to appear*.

[14]From the control-theoretic viewpoint the interesting analysis *begins* after this analysis has been done: to determine the feedback functionals with desirable properties and, perhaps, to make an optimal choice.

END
DATE
FILMED
3-88
DTIC