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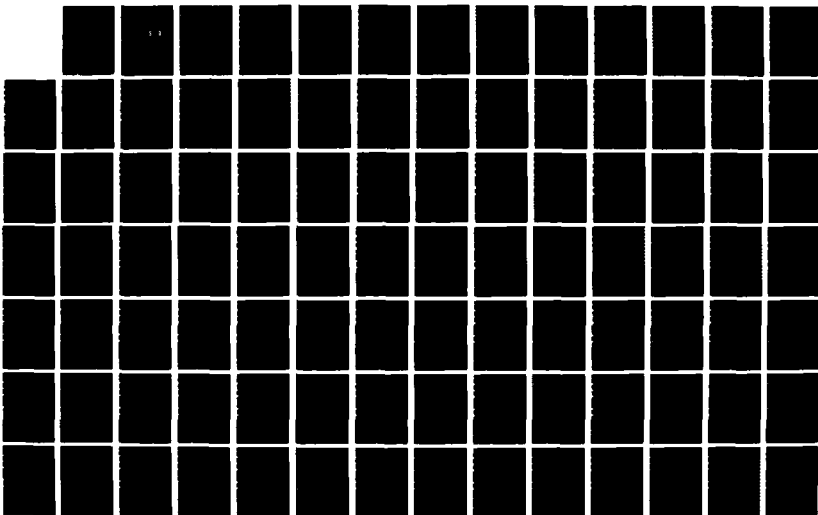
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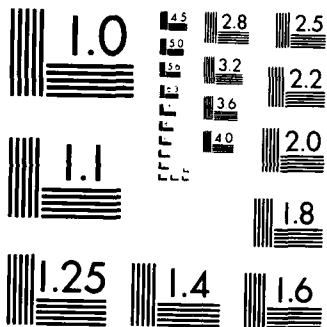
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Applied Computation Theory

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Margaret Lefevre Weaver

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

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For k -tuples of integers $X = (x_1, x_2, \dots, x_k)$ and $Y = (y_1, y_2, \dots, y_k)$, let $|X - Y| = \sum_{i=1}^k |x_i - y_i|$. The k -dimensional bandwidth problem for a graph G is to label the vertices v_i of G with distinct k -tuples of integers $f(v_i)$ so that the quantity $\max \{ |f(v_i) - f(v_j)| : (v_i, v_j) \in E(G) \}$ is minimized. We find bounds on the k -dimensional bandwidth of a graph in terms of other graph parameters and we find the bandwidth and k -dimensional bandwidth of several classes of graphs.

For a given nontrivial graph H , an H -forbidden coloring of a graph G is an assignment of colors to the vertices of G so that G contains no monochromatic subgraph isomorphic to H . The H -forbidden chromatic number of G is the minimum number of colors in an H -forbidden coloring of G . An H -required coloring of G is an assignment of colors to the vertices of G such that every induced monochromatic subgraph of G is a subgraph of H . The H -required chromatic number of G is the minimum number of colors in an H -required coloring of G . We find triangle-free graphs with arbitrarily large star-required chromatic numbers and we seek an analogue to Brooks' Theorem for the P_2 -required chromatic number, where P_2 is the path containing two vertices. We also find the generalized chromatic numbers of several classes of graphs, including the Cartesian product of cycles and the complete multipartite graphs, when the forbidden or required configurations are stars or paths.

GRAPH LABELINGS

BY

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THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
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GRAPH LABELINGS

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University of Illinois at Urbana-Champaign, 1988
Douglas B. West, Advisor

Given an ordering of the vertices of a graph around a circle, a *page* is a collection of edges forming non-crossing chords. A *book embedding* is a circular permutation of the vertices together with a partition of the edges into pages. The *pagenumber* $t(G)$ is the minimum number of pages in a book embedding of G . We present a general construction showing $t(K_{m,n}) \leq \lfloor (m + 2n)/4 \rfloor$, which we conjecture to be optimal. We prove a result suggesting this is optimal for $m \geq 2n - 3$. For the most difficult case, $m = n$, we consider vertex permutations that are *regular*, i.e. place the vertices from each partite set into runs of equal size. Book embeddings with such orderings require $\lfloor (7n - 2)/9 \rfloor$ pages, which is achievable. The general construction ~~uses~~ **fewer** pages, but with an irregular ordering.

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DEDICATION

To my husband, Carl
and
my parents, Mr. and Mrs. F. H. J. Lefevre.

ACKNOWLEDGEMENT

I would like to thank my advisor, Douglas B. West, for his help and encouragement during the research for and the preparation of this thesis.

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CHAPTER 1

INTRODUCTION

Many problems in graph theory can be stated in the following general framework: Label the vertices or edges of a graph G subject to certain conditions in order to optimize some measure of efficiency. In this thesis, we will consider three problems that fit into this scheme: (1) the pagenumber of a graph, (2) the bandwidth of a graph, and (3) the chromatic number of a graph.

The pagenumber problem can be described as follows. Place the vertices along a line, called the spine, in some order. All edges must be placed in a half-plane bounded by the spine. Label the edges so that no two edges that cross receive the same label. If two edges receive the same label, they are said to lie on the same page. If k labels suffice to label all of the edges then there is a k -page book embedding of the graph. The goal is to find the minimum number of pages in any book embedding of a graph. This quantity is called the *pagenumber* of the graph.

In Chapter 2, we will study the pagenumber of the complete bipartite graph, which can be defined as follows. The *complete bipartite graph*, denoted by $K_{m,n}$, contains $m + n$ vertices split into two subsets, one of size m and one of size n . Every vertex in one of the subsets is adjacent to every vertex in the other subset, but to no vertex in its own subset. We will obtain an upper bound for the pagenumber of $K_{m,n}$, and lower bounds for some special types of vertex orderings.

Suppose G is a graph with n vertices. The bandwidth problem takes the following form. Label the vertices v_i of G with distinct integers $f(v_i)$ so as to minimize the maximum value of $|f(v_i) - f(v_j)|$ over all pairs of adjacent vertices. This minimum value is called the *bandwidth* of G .

In Chapter 3, we will generalize the idea of the bandwidth of a graph to labeling the vertices by k -dimensional vectors. We will find some bounds on the generalized bandwidth in terms of other parameters, and consider the generalized bandwidth of several classes of graphs, which will help to show that our bounds are good.

The final parameter we will consider is the chromatic number of a graph, which can be defined in the following way. Label the vertices of a graph G with colors so that no two adjacent vertices receive the same color. The *chromatic number* of G is the minimum number of colors in a legal coloring of G .

In chapter 4, we will generalize the idea of the chromatic number in two ways. First, we will allow certain subgraphs of a graph to have their vertices colored monochromatically. Second, we will forbid a certain subgraph of a graph to have its vertices colored monochromatically. We will consider two famous theorems concerning bounds on the chromatic number in terms of other graph parameters and try to extend them to generalized chromatic numbers. Then we will find the generalized chromatic numbers for several classes of graphs.

Throughout this thesis we will consider only graphs that are finite, simple, and undirected. We will now detail some definitions and notation.

Suppose G and H are graphs with vertex sets $V(G)$ and $V(H)$, respectively. Then the *Cartesian product* of G and H , denoted by $G \times H$, has vertex set $\{(u, v) : u \in V(G), v \in V(H)\}$. Vertices (u, v) and (x, y) are adjacent if either (1) $u = x$ and v is adjacent to y in H or (2) $v = y$ and u is adjacent to x in G .

There are several classes of graphs that will appear throughout this thesis. We define these now.

- (1) The *complete graph on n vertices* or the *n -clique*, denoted K_n , is the graph on n vertices in which each pair of distinct vertices is joined by an edge. If G is any graph, the size of the largest clique contained in G is denoted by $\omega(G)$.
- (2) A *p -partite graph* is one whose vertex set can be partitioned into p subsets so that no edge has both ends in any one subset. The *complete p -partite graph*, denoted $K_{n_1, n_2, \dots, n_p}$, has n_i vertices in the i th subset and each vertex is adjacent to every vertex that is not in the same subset. The complete bipartite graph described earlier is a special case of the p -partite graph where $p = 2$.

(3) The n -cube, denoted Q_n , has vertex set corresponding to all n -tuples whose entries are zeroes and ones. Two vertices are adjacent if their n -tuples differ in exactly one coordinate. Q_n is the Cartesian product of n copies of K_2 .

(4) The n -star, denoted S_n , has vertex set $\{x, y_1, y_2, \dots, y_{n-1}\}$ with x adjacent to y_i for all $1 \leq i \leq n-1$. The y_i 's are called *leaves*.

(5) The n -path, denoted P_n , has vertex set $\{x_1, x_2, \dots, x_n\}$ with x_i adjacent to x_{i+1} for $1 \leq i \leq n-1$.

(6) The n -cycle, denoted C_n , has vertex set $\{x_0, x_1, \dots, x_{n-1}\}$ with x_i adjacent to x_{i+1} for $0 \leq i \leq n-1$, where the addition is performed modulo n .

We will be particularly interested in Cartesian products of cycles or paths. A Cartesian product of paths is called a *grid graph* in computational literature and $C_m \times C_n$ is sometimes called a *toroidal grid* or *discrete torus*.

CHAPTER 2

PAGENUMBER OF COMPLETE BIPARTITE GRAPHS

Section 2.1: Introduction

Recent work on the efficient layout of networks in VLSI [12, 13] has sparked interest in *book embeddings* of graphs, which we now define.

DEFINITION 2.1: Let G be a graph. Place the vertices of G along a line called the spine of the book, in some order. The edges are then embedded without crossing in half-planes bounded by the line, which are called *pages* of the book embedding. The *pagenumber* or *book thickness* of G , denoted by $t(G)$, is the minimum number of pages in any book embedding of G .

For analysis, an equivalent formulation is often more helpful. The ordering on the spine can be viewed as an ordering of the vertices on a circle, and then the pages are collections of non-crossing chords. The early paper of Bernhart and Kainen [4] characterized the graphs with $t(G) = 1$ as the outerplanar graphs, and those with $t(G) = 2$ as the subgraphs of planar Hamiltonian graphs. Although graphs with pagenumber 3 may have arbitrarily high genus [4], the maximum pagenumber of planar graphs was a lively subject of investigation (e.g. [9, 21]) until Yannakakis [34] determined it to be 4.

Book embeddings have also been studied for special classes of graphs. Games [17] considered several networks important in VLSI. Bernhart and Kainen showed $t(K_n) = \lceil n/2 \rceil$ (the pages are $\lceil n/2 \rceil$ rotations of a simple path), but $t(K_{n,n})$ has proved far more elusive. They gave a construction and elementary argument to show $n/2 \leq t(K_{n,n}) \leq n-1$, for $n \geq 4$.

In this chapter we begin a systematic study of book embeddings of $K_{m,n}$. Let the two parts of $K_{m,n}$ be called X and Y . Suppose that the vertices of $K_{m,n}$ are arranged around a circle in some order. We can view the vertices as a set of vertices from X , followed by a set of vertices from Y , followed by a set of vertices from X , etc., until we get back to the place where we started. Call

each of these sets a *run* of vertices. Any circular vertex ordering for $K_{m,n}$ has some number of runs of vertices from each partite set; an *r*-bucket ordering is a circular ordering of m X's and n Y's with r runs of vertices from X and r runs of vertices from Y . Let $t(m, n) = t(K_{m,n})$, and let $t_r(m, n)$ be the minimum number of pages in a book embedding of $K_{m,n}$ using an *r*-bucket ordering. An *r*-bucket ordering is *regular* if for each partite set the runs all have the same size.

Given a vertex ordering, a set of edges forming a pairwise crossing set of chords is called a *twist* and the edges must go on separate pages. The lower bound of [4] can be obtained by observing that any vertex ordering of $K_{n,n}$ has a twist of size $n/2$ (any division of the ordering into halves has at least $n/2$ vertices of one type in one half and at least $n/2$ vertices of the other type in the other half). Intuition suggests that lack of large twists permits efficient embeddings. Every vertex ordering has a twist of size $n/2$, and only the regular 2-bucket ordering of $K_{n,n}$ has no larger twist. We will show that optimal embeddings for this ordering have $\lceil (7n - 2)/9 \rceil$ pages. More generally, regular embeddings of $K_{m,n}$ require at least $\min \{ n, \lceil (5n + 2m - 2)/9 \rceil \}$ pages.

Surprisingly, intuition fails to produce the best result here; the regular 2-bucket ordering does not achieve $t(m, n)$. We can embed $K_{n,n}$ in $\lceil 3n/4 \rceil$ pages, and more generally we will provide a 2-bucket embedding of $K_{m,n}$ in $\lceil (m + 2n)/4 \rceil$ pages. We will then obtain a result toward optimality of the construction: $t_2(2n-3, n) = n$. Bernhart and Kainen used the pigeonhole principle to show that $t(m, n) = n$ for $m > n(n-1)$; our result suggests that this can be improved to $t(m, n) = n$ when $m > 2n - 4$. The main difficulty in doing so, and indeed in making significant further progress on the computation of $t(m, n)$, is showing that optimality is achieved by a 2-bucket ordering.

Section 2.2: Elementary Results

To prove their upper bound of $t(n, n) \leq n - 1$ for $n \geq 4$, Bernhart and Kainen provided an inductive construction, using a 2-bucket ordering, in which they added 4 to n at each step, and thus gave explicit constructions for $n = 4, 5, 6, 7$ as a basis. Applying the following more general lemma with $r = 2$, one need only supply constructions for $n = 4, 5$ to get the same result.

LEMMA 2.2: $t_r(m, n) \leq t_r(m - r, n - r) + r$.

PROOF: Consider an optimal r -bucket embedding of $K_{m \rightarrow n \rightarrow r}$. Extend each bucket of X's and each bucket of Y's by adding one vertex at the clockwise end of the ordering. We need only add r pages that embed all edges involving at least one of these new vertices. Each of these pages contains a complete matching on the $2r$ new vertices, consisting of "parallel" edges in a configuration that rotates from page to page. In addition, we now include edges joining each new vertex to each old vertex in the bucket containing its mate. Since there is a new vertex in each bucket, the new pages will contain edges from each new vertex to each old vertex of the other type. Figure 2.1 illustrates the new pages when $r = 2$. $\square$

Next we restate the pigeonhole argument for general m, n .

LEMMA 2.3: $t(kn, n) \geq nk/(k + 1)$.

PROOF: We show every ordering has a twist of the desired size. Partition the vertices into $k + 1$ groups of n consecutive vertices. With kn Y's and n X's, one of these groups must have at least $nk/(k + 1)$ Y's. Since the group size is n , there must be at least $nk/(k + 1)$ X's outside this group. This set of X's yields a twist with the specified set of Y's. $\square$

Finally, another simple counting lemma disposes of regular orderings with more than two buckets. For vertices around a circle, let the *distance* between vertices be the number of spaces between vertices that must be crossed to get from one to the other (at most n in a set of $2n$ vertices), and let the *length* of an edge be the distance between its endpoints.

LEMMA 2.4: At most $\lfloor p/k \rfloor$ edges of length exactly k can appear on the same page in a book embedding of a graph on p vertices. In particular, any regular embedding of $K_{m,n}$ with $r > 2$ buckets requires at least n pages, where $n \leq m$.

PROOF: Edges of the same length cannot fit "inside" each other on a page, and hence must cut off disjoint arcs of the circumference, possibly meeting at a vertex. The total circumference is p , so there can be at most $\lfloor p/k \rfloor$ of these edges.

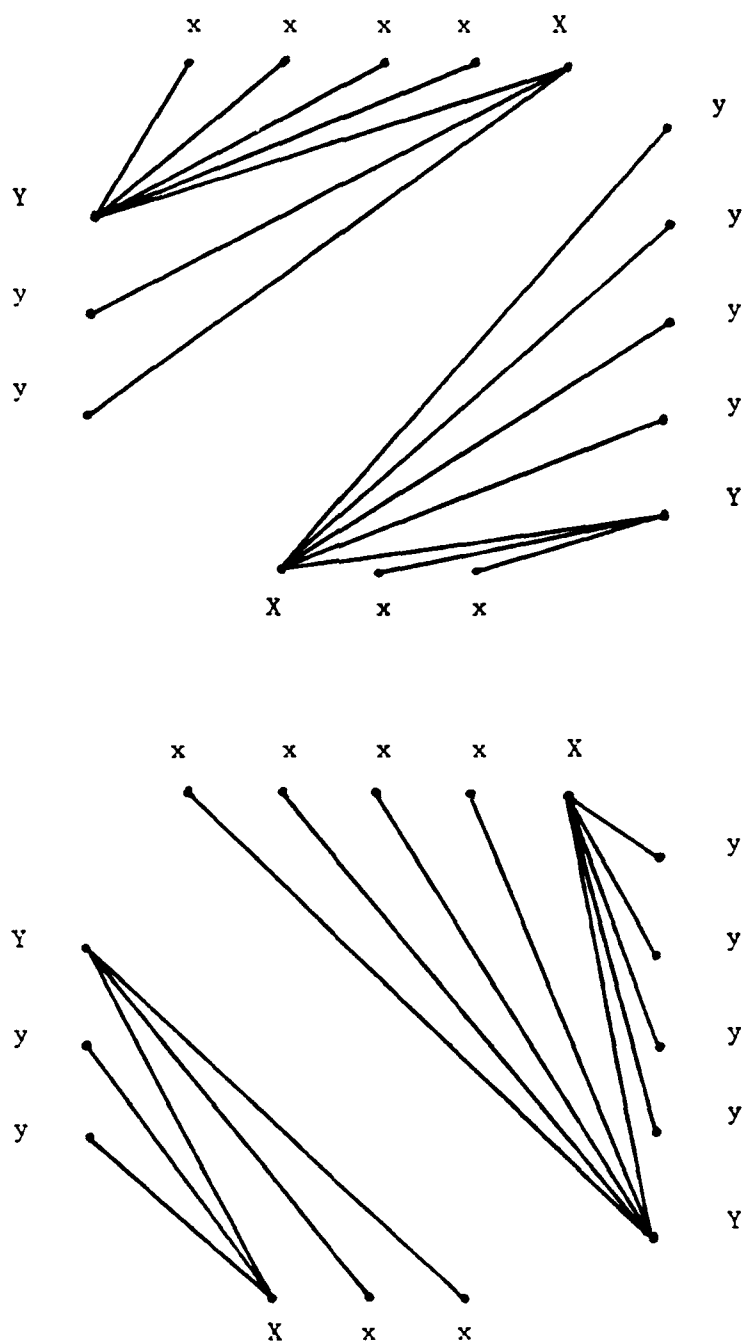


Figure 2.1: Additional pages for inductive construction with $r = 2$.

Now assume $m \geq n$. A regular embedding with an odd number of buckets has n edges of length $(m + n)/2$: they are diameters of the circular configuration and form a twist. (Alternatively, two edges of length n on the same page must share both endpoints, but then they are the same edge.)

In a regular embedding with an even number of buckets, the diameters are not edges, but each Y vertex has an edge to the vertex at distance $k = (m + n)/2 - n/r$ in each direction. There are $2n$ such edges. Note that $(m + n)/2 > k \geq n(1 - 1/r)$, which exceeds $2n/3$ when $r > 3$. Hence no three of these edges can lie on the same page and there must be at least $2n/2 = n$ pages. (When $r = 2$, this repeats the lower bound of $n/2$.) $\square$

Section 2.3: The Encoding and the Construction

There is an encoding that makes book embeddings of complete bipartite graphs somewhat easier to visualize and describe. Label the X vertices with indices $1, \dots, m$ in clockwise order; these indices will correspond to the rows of an $m \times n$ grid. Label the Y 's similarly, corresponding to the columns of the grid. For a complete bipartite graph, location i, j of the grid corresponds uniquely to the edge from X_i to Y_j . Placing this edge on page c is equivalent to placing label c or color c at that location of the grid. The usage of a single color in the grid must satisfy restrictions equivalent to forbidding crossings on a page.

For convenience, we assume that X_1 and Y_1 begin runs (buckets), and that X_1 begins the run clockwise following Y_1 . This partitions the grid into subgrids such that the (i, j) th subgrid corresponds to the edges from the i th run of X 's to the j th run of Y 's. The numbering is illustrated in Figure 2.2, along with two feasible pages for the 2-bucket case. In the 2-bucket case, we call the subgrids *quadrants*, and we label the runs as $Y_1, Y_2, \dots, Y_q, X_1, X_2, \dots, X_p, Y_{q+1}, Y_{q+2}, \dots, Y_m$, and $X_{p+1}, X_{p+2}, \dots, X_n$.

Now we translate the non-crossing condition into the grid encoding. First consider the 2-bucket case; here it is easy to visualize what constitutes a legal page. Each quadrant corresponds to a pair of neighboring runs in the ordering. Edge pairs with endpoints in all four runs belong

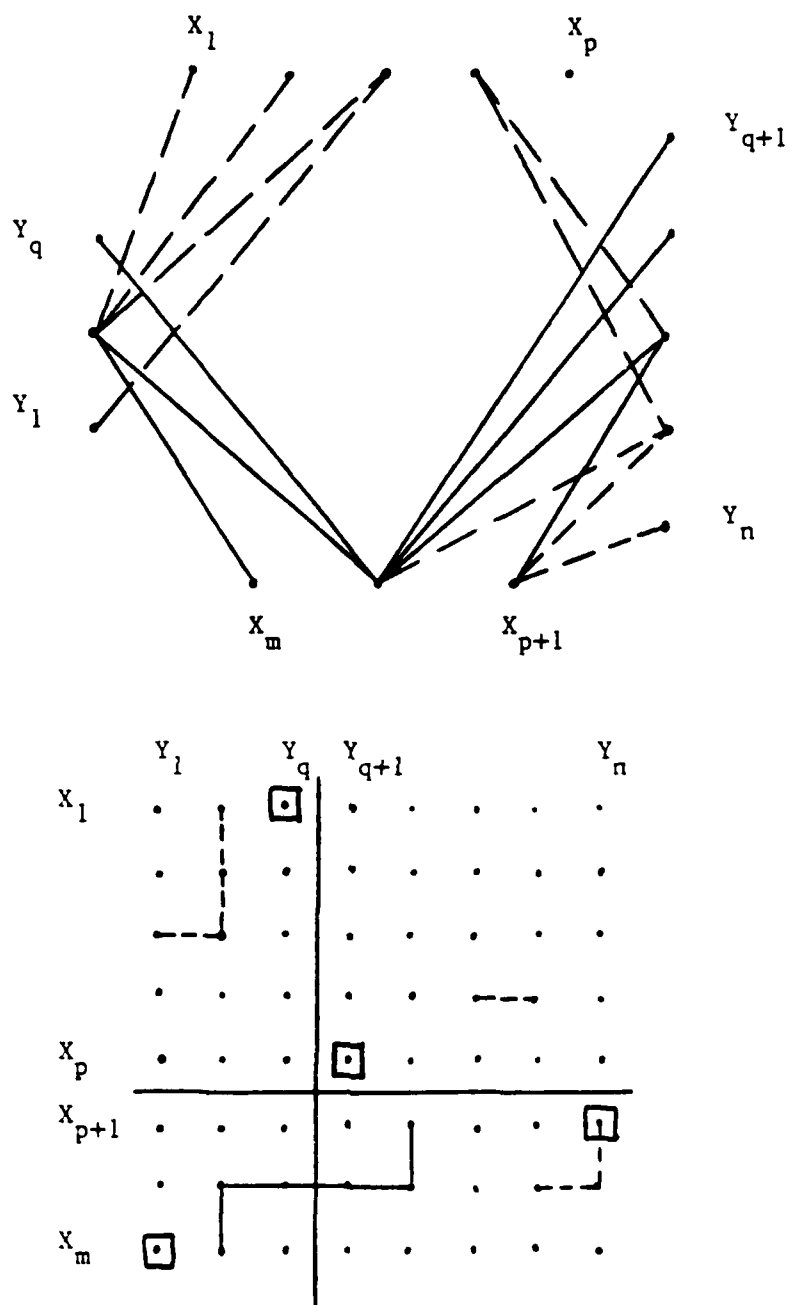


Figure 2.2: Encoding of the edges in two-bucket embeddings.

to diagonally opposite subgrids and never conflict, so we need only consider positions within the same quadrant or neighboring quadrants. Each quadrant has a "free" position corresponding to an edge of length 1; it crosses no other and can be added to any page. These positions are marked in Figure 2.2 with a $\square$. Edges encoded in two neighboring quadrants all have endpoints in a single run; suppose this run is in X . If (x, y) and (x', y') are two such edges on a page and we move clockwise within the run to reach x' from x , then we must move counterclockwise (or stay fixed) outside the run to reach y' from y ; when i increases j cannot. In terms of the encoding, this means that in two neighboring quadrants a single color must occupy a subset of a lattice path between the two free corners. The grid is actually a discrete torus; for the pairs of quadrants involving the upper right quadrant, the lattice path wraps around in columns or rows.

For more buckets, the condition generalizes naturally. Again suppose two edges have endpoints in four distinct runs. For the corresponding subgrids, either all pairs of positions lead to crossing edges or all pairs lead to non-crossing edges. For example, in the 3-bucket case any pair of edges between disjoint pairs of diametrically opposite runs cross, but any pair of edges between disjoint pairs of consecutive runs are always compatible. This describes the situation when the subgrids containing two positions are not in the same row of subgrids or the same column of subgrids. When they are in the same row [column], the edges have endpoints in the same run S , and the order of the endpoints yields a lattice path condition in the same manner as before. In each row [column] of subgrids, there are two subgrids with a free position, corresponding to the edges between S and the two neighboring runs from the other partite set. The free positions correspond to consecutive vertices in the other partite set and therefore appear in cyclically consecutive columns [rows]. The edges of a single page that are incident to S correspond to a subset of a lattice path between these two free positions. The path wraps around in columns [rows] to avoid stepping directly between the consecutive columns [rows] containing the free positions.

These lattice paths can be described technically in terms of the positions (i, j) , but we prefer to avoid that so as to make more geometric arguments. Having specified the non-crossing conditions

for the grid coloring, we henceforth often avoid verbiage by using the word "edge" to describe either an edge of $K_{m,n}$ or the corresponding grid position, and we can refer to a legal k -coloring of the grid as a k -page book embedding. In the remainder of the chapter, we discuss only 2-bucket vertex orderings. It is convenient, therefore, to refer to a k -page book embedding using a 2-bucket ordering as a k -page 2-embedding, and a k -page book embedding using a regular 2-bucket ordering as a regular k -page 2-embedding. A k -page 2-embedding of a subgraph of $K_{m,n}$ is a *partial* k -page 2-embedding.

Using the encoding, we can give a simple description of an efficient 2-embedding for general $K_{m,n}$.

THEOREM 2.5: $t(K_{m,n}) \leq t_2(m,n) \leq \min \{ \lfloor (m+2n)/4 \rfloor, n \}$, where $m \geq n$. In particular, $t(K_{n,n}) \leq \lfloor 3n/4 \rfloor$.

PROOF: Since we can always use a separate page for each vertex of Y , we need only exhibit a $\lfloor (m+2n)/4 \rfloor$ -page 2-embedding when $n \leq m \leq 2n$. Note that if m is odd, $\lfloor (m+2n)/4 \rfloor = \lfloor (m+1+2n)/4 \rfloor$. Hence we can obtain the desired embedding when m is odd by deleting a vertex from the embedding for $K_{m+1,n}$, and we may assume henceforth that m is even. Similarly, when m is even we have $\lfloor (m+2n)/4 \rfloor = \lfloor (m+2(n+1))/4 \rfloor$ if n has opposite parity from $m/2$. Hence we may also assume n has the same parity as $m/2$, obtaining the desired embedding by deleting a vertex from the embedding for $K_{m,n+1}$ if not.

We give an explicit 2-embedding. Let $q = (2n - m)/4$. Split X evenly into runs of length $m/2$. Split Y into runs of length q and $n - q$. Since $q \leq n/4$, this produces two large lower quadrants and two small upper quadrants, as indicated in Figure 2.3 for $m, n = 10, 9$, where for aesthetic reasons we have drawn the encoding with the roles of X and Y interchanged. We therefore index rows by j and columns by i in describing this construction.

The edges of a page in a quadrant appear as a path in Figure 2.3. We use three types of pages (colors). Type 1 colors (solid paths) appear in the two lower quadrants. Type 2 colors (dashed

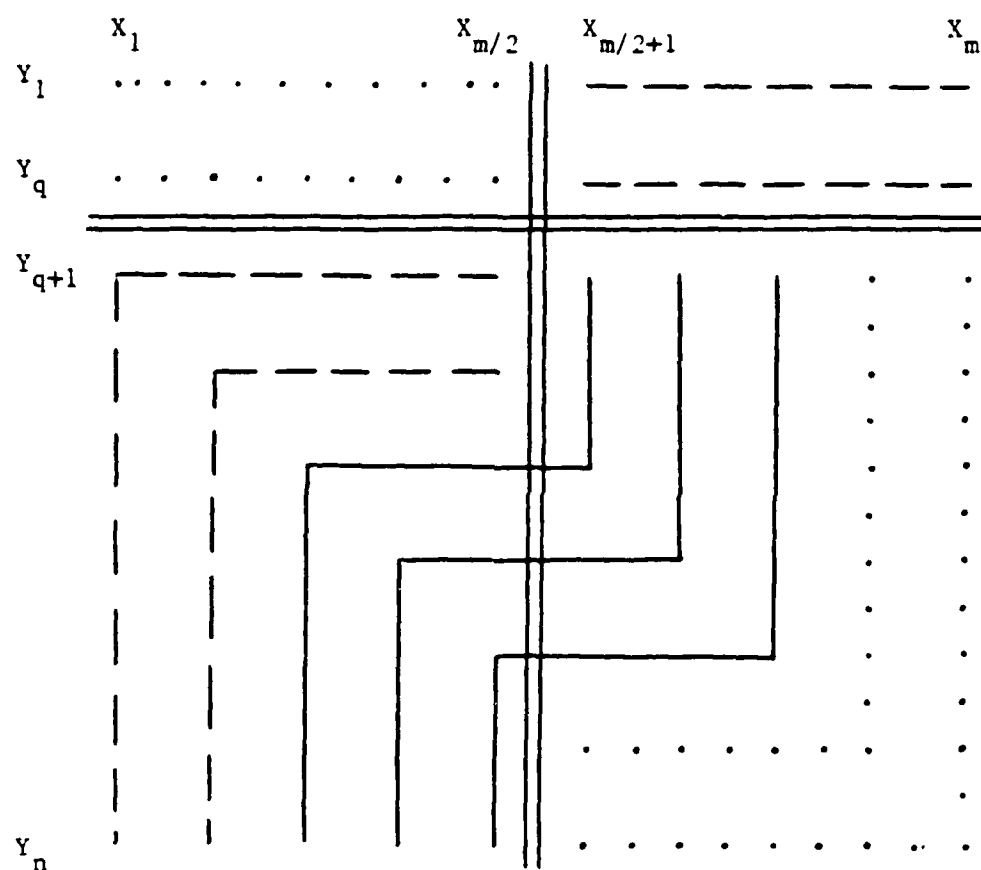


Figure 2.3: An example of the general construction.

paths) appear in the upper right and lower left quadrants. Type 3 colors (dotted paths) appear in the upper left and lower right quadrants. We use q colors of each of Type 2 and Type 3 and $m/2 - q$ colors of Type 1. The total number of colors is $m/2 + q = (m + 2n)/4$.

For $j=1, 2, \dots, q$, the positions occupied by the j th Type 3 color in the upper left are $\{(j, 1), (j, 2), \dots, (j, m/2)\}$, and those of the j th Type 2 color in the upper right are $\{(j, m/2+1), \dots, (j, m)\}$. In the lower left quadrant, the j th Type 2 color occupies $\{(n, j), \dots, (q + j, j), \dots, (q + j, m/2)\}$. These two "segments" in the grid correspond to edges in $K_{m,n}$ incident to X_j or Y_{q+j} . Next, for $1 \leq j \leq m/2 - q$ the edges of the j th Type 1 color are all incident to Y_{2q+j} , X_{q+j} , or $X_{m/2+j}$, and they occupy grid positions

$$\{(n, q + j), \dots, (2q + j, q + j), \dots, (2q + j, m/2 + j), \dots, (q + 1, m/2 + j)\}.$$

Finally, the j th Type 3 color occupies grid positions

$$\{(m/2 + q + j, m/2 + 1), \dots, (m/2 + q + j, m - q + j), \dots, (q + 1, m - q + j)\}.$$

all incident to $Y_{m/2+q+j}$ or X_{m-q+j} .

By construction, the pages comprise disjoint sets of non-crossing edges. The horizontal portions of the colors in the lower quadrants march successively from row $q + 1$ to row $m/2 + 2q$. Since $m/2 + 2q = n$, all edges are colored, which completes the proof. $\square$

Section 2.4: Reductions for the Two-bucket Problem

In defining the encoding, we noted that each quadrant has a "free" position corresponding to an edge of length 1, which we henceforth call HOME. Given any book embedding of the rest of $K_{m,n}$, HOME can be added to any page. We want to identify other positions we may ignore in considering colorings.

For the remainder of the chapter, we return to the convention of rows indexed $1 \leq i \leq m$, columns indexed $1 \leq j \leq n$, and vertex ordering $Y_1, Y_2, \dots, Y_q, X_1, X_2, \dots, X_p, Y_{q+1}, Y_{q+2}, \dots, Y_n, X_{p+1}, X_{p+2}, \dots, X_m$, as in Figure 2.2. Think of the X 's as being on the top and bottom of the ordering and the Y 's on the left and right of the ordering, so the top quadrants have p rows and the left quadrants have q columns. When $m = n$, we assume $p \leq q \leq n/2$.

Let the edge farthest from HOME in a quadrant be called AWAY. The *distance* between two edges $(i, j), (i', j')$ in a quadrant is $|i - i'| + |j - j'|$. The distance of an edge from HOME is one less than its length. The edges of a given length in a quadrant form a *diagonal*; they have the same distance from HOME and have a constant value of $|i - j|$. Edges getting the same color belong to a lattice path from HOME to AWAY that increases length by one at each step. Note that the edges of a diagonal form a twist and must get different colors; in fact, any set of positions from two adjacent quadrants that have no pair on any lattice path between the free positions forms a twist.

A diagonal whose size is the number of rows or columns of the quadrant is called a *full diagonal*. The full diagonal farthest from HOME is the *major diagonal*; it always contains a corner of the quadrant. Let M be the length of edges in the major diagonal; let L be the length of AWAY. The edges of length $M + j$ in the quadrant form the *jth superdiagonal*, denoted D_j . The triangular set of grid positions consisting of the major diagonal and all superdiagonals is called the *essential triangle*. This name suggests that if the long edges are properly colored, then the rest come for free, which is justified by Lemma 2.7. Lemmas 2.7 and 2.8 say that we need only find a partial k -page 2-embedding for a particular subgraph of $K_{m,n}$ to have a k -page 2-embedding of $K_{m,n}$. Lemma 2.9 allows us to make further assumptions about what that embedding looks like. The extensions and recolorings needed to prove these lemmas rest on the following simple observation.

LEMMA 2.6: Consider an edge $e = xy$ in quadrant Q in a 2-bucket ordering of $K_{m,n}$. Let S denote the set of edges in Q that are shorter than e and do not cross e . Let T denote all other edges that do not cross e . Then no edge of S crosses any edge of T .

PROOF: This statement is a simpler instance of the reasoning used earlier to describe legal usage of a color in terms of lattice paths. From the vertex ordering define two segments A, B of vertices, each with endpoints x, y . No edge of S crosses any edge of T because all endpoints of edges of S lie in one of A, B , and all endpoints of edges of T lie in the other. $\square$

Note that $S \cup \{e\}$ and $T \cup \{e\}$ are rectangles in the discrete torus. S is the rectangle whose opposite corners are e and HOME of Q . T is the rectangle whose opposite corners are e of Q and AWAY of the quadrant diagonally opposite Q . $K_{m,n} - S - T - \{e\}$ are the edges crossed by e . Typically, usage of Lemma 2.6 is to extend partial embeddings. If a legal page contains e and no edge of S , then we can add an edge of S .

LEMMA 2.7: Any partial k -page 2-embedding including the essential triangles extends to a k -page 2-embedding of $K_{m,n}$.

PROOF: The edges on the major diagonal of a quadrant have distinct colors; let C be the ordered list of these colors. Complete the coloring of the quadrant by using C in order on each successive diagonal closer to HOME. When the diagonals begin to get shorter, at each step delete a color from one end of C . By Lemma 2.6, the choices on each successive diagonal yield no crossings. $\square$

We call the edges of the essential triangles the *essential edges*. For some vertex orderings, we can ignore additional positions. Consider two adjacent quadrants A, B . If the dimensions are suitable, a color close to HOME in A is forbidden from the essential triangle in B (and hence all B , by Lemma 2.7), which allows us to use it more freely in A . The application of this lemma to a 2-bucket ordering of $K_{n,n}$, where we may assume $p \leq q \leq n/2$, appears in Figure 2.4. The dots in Figure 2.4 designate the only positions that need to be colored.

LEMMA 2.8: Given a 2-bucket ordering, let A, B be two adjacent quadrants, together comprising c columns [rows]. Suppose A, B have a, b rows [columns], and suppose $a < c$. Let $s = a - \max\{0, c - b\}$. Then any partial k -page 2-embedding whose edges in B include $D_0 \cup D_1 \cup \dots \cup D_s$ extends to a partial k -page 2-embedding including all of B .

PROOF: We may assume that A and B are vertically adjacent and comprise c columns. Any position in D_s is at least a columns from HOME in B , but all points of the essential triangle in A are less than a columns from AWAY in A . Hence no color in $C = D_s$ appears in the essential

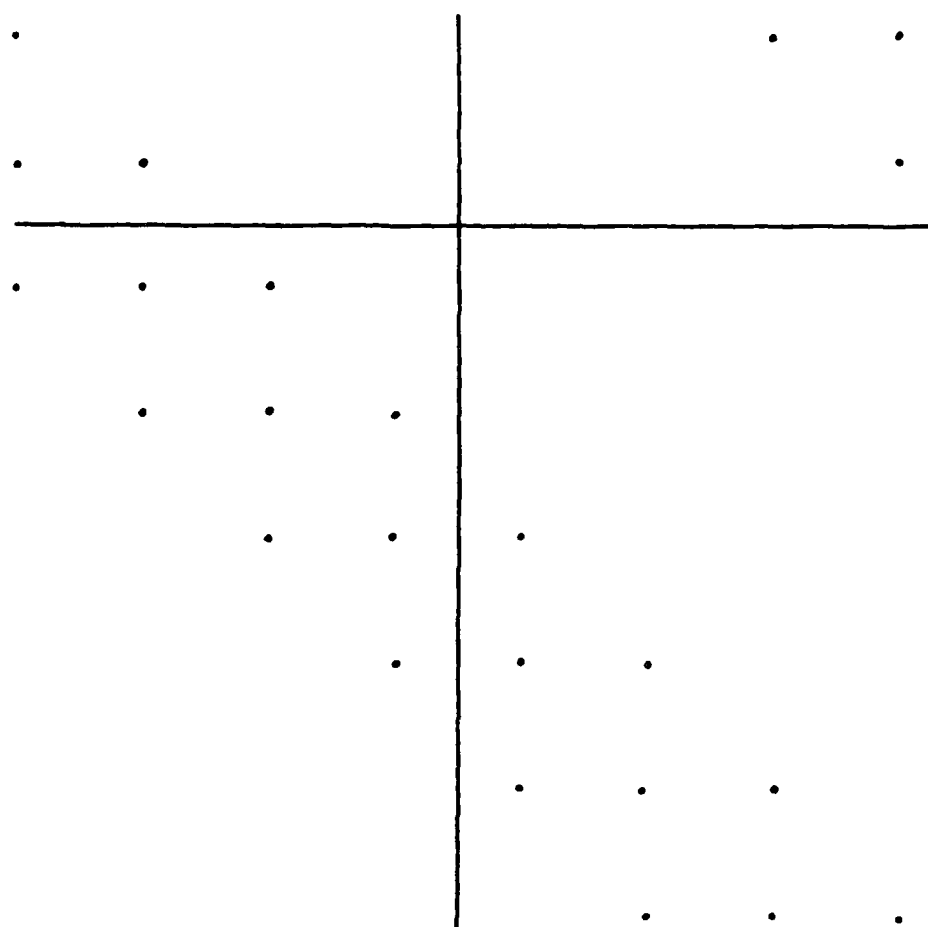


Figure 2.4: Essential edges for a two-bucket ordering of $K_{8,8}$.

triangle of A . This means that the color α in row i of C can be used on row i of D_j in B without penalty for all $j > s$. These positions exert no constraint on usage of α in the quadrant horizontally adjacent to B that is not already exerted by the presence of α in C . This extension of the colors on C completes the coloring of the essential triangle in B , and we apply Lemma 2.7. $\square$

For a 2-embedding of $K_{m,n}$, Lemma 2.8 allows us to color only a continuous band of positions on the discrete torus, with $p + 1$ positions in every row and column. For general $K_{m,n}$, Lemma 2.8 can be very effective; if $a + b \leq c$, then we need only color the major diagonals in A and B . For regular 2-embeddings of $K_{m,n}$ with $m < 2n$ ($m \geq 2n$ yields a twist of size n), we get two bands with $n + 1 - m/2$ positions in each row.

We could consider only partial k -page 2-embeddings of the edges remaining after applying Lemma 2.8, but it will be convenient to retain the full essential triangles, which explains why we called these the essential edges of $K_{m,n}$. Next, we further restrict partial k -page 2-embeddings of the essential edges to a canonical form. In any book embedding, define the longest edge of a quadrant on a given page to be the *leading edge* of that page in that quadrant. A *staircase embedding* of $K_{m,n}$ is a 2-embedding in which each quadrant has exactly one leading edge of each length $M, \dots, L$, which are the lengths of edges in the essential triangle. Since there is only one leading edge of each length in a staircase embedding, a page whose leading edge has length $M + j$ must also have edges of length $M, \dots, M + j - 1$.

LEMMA 2.9: *If the essential edges of $K_{m,n}$ have a partial k -page 2-embedding, then $K_{m,n}$ has a k -page staircase embedding.*

PROOF: Select a quadrant Q . We recolor to obtain the desired embedding iteratively, in decreasing order of length. The edges of D_{r-M} have length r . For the single edge of length L there is nothing to prove; it is the leading edge of that page. Now assume $L > r \geq M$ and the edges of length exceeding r in Q are embedded in pages $c_1, c_2, \dots, c_{L-r}$ with one leading edge of each length. Since D_{r-M} has $L - r + 1$ edges, there is some color c_{L-r+1} on D_{r-M} that does not appear among

$c_1, c_2, \dots, c_{L \rightarrow}$ and thus is the leading edge of its page in Q . We need only show that the other edges of $D_{r \rightarrow M}$ can be given colors $c_1, c_2, \dots, c_{L \rightarrow}$. In fact, there is a unique way to do this: the order of $c_1, c_2, \dots, c_r$ on $D_{L \rightarrow M \rightarrow}$ must be the same as their order on $D_{L \rightarrow M \rightarrow +1}$, as indicated in Figure 2.5 for the two possible orientations of the essential triangle. By Lemma 2.6, this introduces no crossings, except possibly with the edges yet to be recolored. Since only edges of colors not yet used will be unchanged on later diagonals, any such crossing will be corrected. $\square$

Section 2.5: Regular Two-bucket Embeddings

Additional arguments depend on knowing the sizes of the runs, so in this section we consider only regular 2-embeddings of $K_{m,n}$. Lemma 2.9 allows us to assume the optimal embedding is a staircase embedding. Since a staircase embedding is completely determined by the choice of leading edges for each page, it is natural to consider what sets of lengths are allowable for the leading edges. In each quadrant of a regular embedding, the length of edges on the major diagonals is $M = m/2$, and the length of AWAY is $(m + n)/2 - 1$. The length of a page is the sum of the lengths of its leading edges. The bound on the length of a page depends on the number of quadrants it appears in.

LEMMA 2.10: In a regular staircase embedding of $K_{m,n}$, a page with edges in 4, 3, 2, 1 quadrants has length at most $m + n, m + n - 1, m + n - 2$, or $(m + n - 2)/2$, respectively, yielding at most 4, $n - m/2 + 2, n$, or $n/2$ essential edges. A page with edges in two adjacent quadrants has length at most $m + n/2 - 1$, with at most $n/2 + 1$ essential edges. If $m \neq n$, no page appears in four quadrants. Finally, the j longest pages using two opposite quadrants have total length at most $(m + n)j - (j^2 + 2j + \epsilon(j))/2$ and yield at most $(n + 2)j - (j^2 + 2j + \epsilon(j))/2$ essential edges, where $\epsilon(j) = 0$ if j is even and $\epsilon(j) = 1$ if j is odd.

PROOF: Consider the circular vertex ordering. The leading edges of a page must enclose disjoint portions of the ordering. If each of the leading edges starts at the same vertex where the previous one ends, the total length of $m + n$ can be attained. However, this is only possible with edges

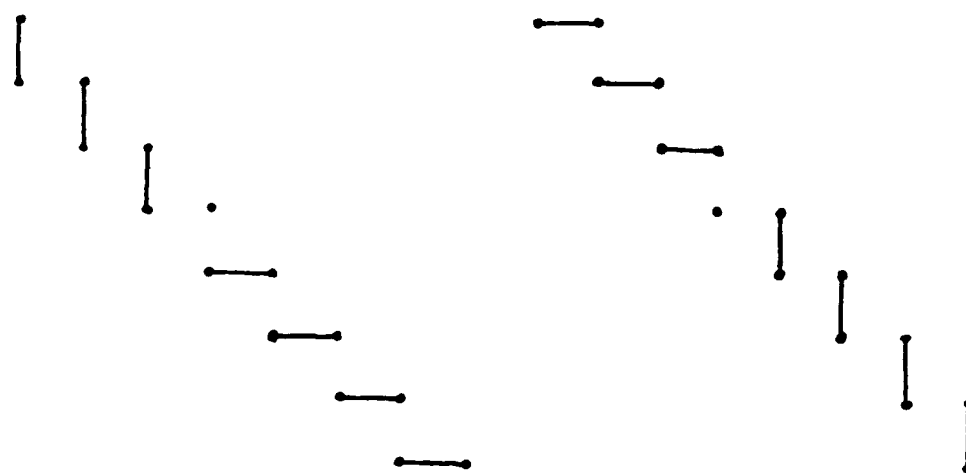


Figure 2.5: Extension of staircase embeddings.

from all four quadrants. With three, no edge encloses the free edge corresponding to HOME of the unused quadrant. With two quadrants, two such spots are missing, and if the two quadrants are adjacent, then in addition one entire run of the ordering must be subtracted from the length. For one quadrant, the bound is the length of AWAY. To count the essential edges obtained, we must subtract $s(m/2-1)$ from the length of a page in s quadrants, because the essential edges begin with the major diagonal, whose edges have length $m/2$. A s -quadrant page thus has length at least $sm/2$, which forbids 4-quadrant pages when $m > n$ and 3-quadrant pages when $m \geq 2n$.

We must be careful about the length of pages in two opposite quadrants. Length $m + n - 2$ requires two edges of length $(m + n - 2)/2$, but there are only four such edges, so there are at most two such pages. In general, each quadrant has exactly one leading edge of each length $m/2, \dots, (m + n - 2)/2$. A page in two opposite quadrants has length $(m + n)/2 - s + (m + n)/2 - t$, where its leading edges are the s th and t th longest leading edges in those quadrants. The total "loss" will be least if the leading edges of these j pages are the $\lfloor j/2 \rfloor$ longest in one pair of quadrants and the $\lfloor j/2 \rfloor$ longest in the other pair. The resulting count of

$$(m + n)j - 2 \binom{\lfloor j/2 \rfloor + 1}{2} - 2 \binom{\lfloor j/2 \rfloor + 1}{2}$$

simplifies to the formula claimed above. $\square$

In the four essential triangles there are $n(n+2)/2$ positions, of which $2n$ positions are on major diagonals. The limits on what a page can contribute to these two counts give us a lower bound on the number of pages needed.

THEOREM 2.11: A regular embedding of $K_{m,n}$ requires at least $\min \{ n, \lfloor (5n + 2m - 2)/9 \rfloor \}$ pages.

PROOF: Consider a k -page 2-embedding; we may assume it is a staircase embedding. Let

$a = (a_1, a_2, a_3, a_4)$, where a_i is the number of pages appearing in i quadrants. We want to minim-

ize $k = \sum_{i=1}^4 a_i$ subject to two conditions. The requirement from diagonal edges is $\sum_{i=1}^4 ia_i = 2n$, and

we need a quadratic inequality $f(a_1, a_2, a_3, a_4) \geq n(n+2)/2$ enforcing the requirement that all essential positions be covered. The function f simply adds up the limits obtained in Lemma 2.10 for the usage of pages among the essential edges. This is a purely numerical argument and ignores whether the configuration is realizable, but the resulting value of k is a lower bound on the number of pages in a regular embedding of $K_{m,n}$.

There are two simplifications we can make in f . First, we may assume $a_4 = 0$. Lemma 2.10 already guarantees this if $m \neq n$. Consider $K_{n,n}$. Except for pages in two opposite quadrants, we get at most $n/2 + 2$ essential edges per page, which requires $n(n+2)/(n+4) \geq n-2$ pages if $a_2 = 0$. Hence we may assume $a_2 > 0$. Choose the smallest value of a_4 in an a which minimizes k . If $a_4 > 0$, we can add $-1, +2, -1$ to a_2, a_3, a_4 to preserve the diagonal requirement and increase f by at least $-n + n + 4 - 4 = 0$, because any page counted by a_2 contributes at most n essential positions. The resulting a is also a solution.

Now consider a_1 . If $a_3 = 0$, then the diagonal requirement forces $k \geq n$. Hence we may assume $a_3 > 0$, and if $a_1 > 0$ we add $-1, +2, -1$ to a_1, a_2, a_3 to obtain an a that satisfies the diagonal requirement, has the same value of k , and increases f by at least $-n/2 + 2(n/2 + 1) - (n - m/2 + 2) \geq 0$. The last statement rests on a closer look at the contribution from a_2 . When the increase from $a_2 = j - 1$ to $a_2 = j$ in the contribution $(n+2)j - (j^2 + 2j + e(j))/2$ from using pages in opposite quadrants falls below $n/2 + 1$, we switch to pages in adjacent quadrants, and thereafter count a possible contribution of $n/2 + 1$ to f for each such page. Therefore, the contribution to f from each increase in a_2 is at least $n/2 + 1$.

We are left with a minimization problem in two variables. Letting $y = a_3$ and $z = a_2$, we want to minimize $y + z$ subject to $3y + 2z = 2n$ and $f(y, z) \geq n(n+2)/2$. The equality constraint allows us to reduce to one variable by setting $y = 2(n - z)/3$. The bound on pages is now $(2n + z)/3$, so we want to find the smallest value of z satisfying $f(z) \geq n(n+2)/2$. We have

$$f(z) = (n - m/2 + 2)(2(n - z))/3 + (n + 2)z - (z^2 + 2z + \epsilon(z))/2$$

for $z \leq 2\lfloor n/4 \rfloor$. If $f(z) \geq n(n + 2)/2$ somewhere in this range, we need not consider the modification needed for $z > n/2$. Treating $\epsilon(z)$ as a constant, we solve the quadratic equation $f(z) = n(n + 2)/2$. It is helpful to set $m = \alpha n$ and clear fractions to obtain

$$3z^2 - 2[n(1 + \alpha) - 1]z + (2\alpha - 1)n^2 - 2n + 3\epsilon = 0.$$

The solution to this simplifies to

$$z = \frac{n + \alpha n - 1 + \sqrt{[n(\alpha - 2) - 1]^2 - 9\epsilon}}{3}.$$

Ignoring ϵ yields $z = (2m - n - 2)/3$ and $y = (8n - 4m + 4)/9$. Both y and z must be integers: if $\lfloor (2m - n - 2)/3 \rfloor$ is even, then ϵ may allow z to be one smaller, but then y may be larger. In all cases we find that the number of colors must be at least $\lfloor (5n + 2m - 2)/9 \rfloor$. $\square$

In fact, satisfying these counting requirements is also sufficient for the existence of regular k -page staircase embeddings. To see this, we define matrices that will summarize the potential usage of pages in quadrants in a staircase embedding. Given a k -page staircase embedding, we want the j th entry of the i th row of the corresponding matrix to be the number of essential edges page i contributes to quadrant j . Lemma 2.10 has given us some necessary requirements for this matrix. With those in mind, we define a class of matrices.

DEFINITION 2.12: A k -book for (m, n) (with $m \geq n$) is a k by 4 matrix in which each column consists of $\{1, 2, \dots, n/2\}$ and $k - n/2$ 0's, and each row $x = (x_1, x_2, x_3, x_4)$ satisfies the following:

- 1) If x has four non-zero terms, they are all 1. (This is allowed only if $m = n$.)
- 2) If x has three non-zero terms, then $\sum_{i=1}^4 x_i \leq n - m/2 + 2$.
- 3) If x has two non-zero terms and $\{x_1, x_4\}$ or $\{x_2, x_3\}$ non-zero, then $\sum_{i=1}^4 x_i \leq n/2 + 1$.

4) If x has two non-zero terms and $\{x_1, x_2\}$ or $\{x_3, x_4\}$ non-zero, then

$$\sum_{i=1}^4 x_i \leq n - m/2 + 1.$$

Note that there is no explicit restriction when one term or two terms with indices of the same parity are non-zero. However, the fact that each column contains each of $\{1, 2, \dots, n/2\}$ exactly once constrains the total sum in rows of this type.

LEMMA 2.13: There exists a k -page regular staircase embedding of $K_{m,n}$ if and only if there exists a k -book for (m, n) .

PROOF: Necessity was mostly shown in Lemma 2.10. Given a k -page regular staircase embedding, form a k by 4 matrix by recording in row i the number of essential edges in page i in each of the four quadrants, with quadrants indexed clockwise from the upper left of the grid encoding. Since a staircase embedding has for each quadrant one page contributing i essential edges for each $1 \leq i \leq n/2$, the column constraint for k -books holds. As discussed in the proof of Lemma 2.10, the quadrants in which page i appears restricts its length and thus the number of essential edges it has, in such a way that the rows of the matrix satisfy the conditions for a k -book. The distinction between the two types of adjacent quadrants arises from whether the relevant vertices omit a run of length $n/2$ or $m/2$ from the ordering. Note that the column conditions on the k -book enforce the global limitation on pages using opposite quadrants.

Sufficiency is not much harder. By the argument in Lemma 2.9, a staircase embedding is determined by the placement of the leading edges in each page. No matter where on the diagonals the leading edges are placed, the quadrant can be completed in exactly one way. Lemma 2.6 guarantees that the resulting assignment of edges to pages yields no crossings if and only if for each page the leading edges have no crossings. Therefore, it suffices to show that for any row x of a k -book, there is a way to assign non-crossing edges from the specified diagonals in each quadrant: these diagonals are D_{x_j-1} from quadrant j if $x_j > 0$, otherwise no edge from quadrant j .

If $m = n$ and x has four non-zero terms, we can choose any four non-crossing edges of length $m/2$, such as those corresponding to the lower right corner of each quadrant in the grid encoding - positions $(n/2, n/2)$, $(n/2, n)$, (n, n) , $(n, n/2)$. If x has two non-zero terms with indices of the same parity, there are no crossings between edges of these quadrants, and we can choose the positions arbitrarily from the specified diagonals. Similarly, the choice is arbitrary when x has only one non-zero term.

For pages on three quadrants or two adjacent quadrants, we want edges of lengths $x_j + m/2 - 1$, for the non-zero x_j . By reflection and rotation of the circular vertex ordering, we may assume x_1, x_2, x_3 are non-zero, or x_1, x_2 are non-zero, or x_1, x_4 are non-zero (recall we indexed the quadrants clockwise from the upper left, starting with the quadrant of edges between $Y_1, \dots, Y_{m/2}$ and $X_1, \dots, X_{n/2}$).

If x_1, x_2 are non-zero, choose $(m/2 - n/2 + x_1, 1)$ and $(n/2 + 1 - x_2, n)$. The condition $x_1 + x_2 \leq n - m/2 + 1$ is the same as $m/2 - n/2 + x_1 \leq n/2 + 1 - x_2$, so there is no crossing. If x_1, x_4 are non-zero, choose $(m/2, n/2 + 1 - x_1)$ and $(m/2 + 1, x_4)$. The condition $x_1 + x_4 \leq n/2 + 1$ is the same as $n/2 + 1 - x_1 \geq x_4$, so again there is no crossing.

If x_1, x_2, x_3 are non-zero, choose the grid positions $(m/2 - n/2 + x_1, 1)$ and $(m, n + 1 - x_3)$ from quadrants 1 and 3; these are the edges of the desired lengths that exert the least influence on quadrant 2. The distance between the endpoints $X_{m/2 - n/2 + x_1}$ and $Y_{n + 1 - x_3}$ of these edges is $n/2 - x_1 + n + 1 - x_3 - n/2$, which is at least $m/2 - 1 + x_2$ because $\sum_{i=1}^4 x_i \leq n - m/2 + 2$. Hence there is room for an edge of length $m/2 - 1 + x_2$ in quadrant 2. We can choose any position from D_{x_2-1} lying on a lattice path from HOME to $(m/2 - n/2 + x_1, n + 1 - x_3)$ in quadrant 2. $\square$

For regular embeddings, we have reduced the question of finding k -page embeddings to that of constructing k -books. In principle, the same analysis can be followed for any 2-bucket ordering. Again there is an optimal staircase embedding, and it suffices to specify compatible lengths for

the leading edges of the pages. The difficulty is that the corresponding constraints in Lemma 2.10 and the definition of k -books become more difficult to keep track of. In the regular case, however, we can show that the bound of Theorem 2.11 for $K_{n,n}$ can be achieved. We presume it is also achievable for $K_{m,n}$, but have not worked out the details of the k -book construction.

THEOREM 2.14: The optimal regular embedding of $K_{n,n}$ has $\lfloor (7n - 2)/9 \rfloor$ pages.

PROOF: We may assume n is even; Theorem 2.11 gives the lower bound. By Lemma 2.2, we need only present constructions of k -books for four of the nine even congruence classes of $n \bmod 18$ - the places where $\lfloor (7n - 2)/9 \rfloor = \lfloor (7(n - 2) - 2)/9 \rfloor + 1$. These are $n \equiv 0, 4, 8, 14 \bmod 18$. Given the reductions in the proof of Theorem 2.11, it is not surprising that we use only pages appearing in three quadrants or in two opposite quadrants, and that we use approximately $n/3$ pages in two opposite quadrants and $4n/9$ pages in three quadrants. More precisely, we use values of y and z as indicated in Table 2.1, for the four important congruence classes.

Table 2.1. Number of pages in staircase embedding for $K_{n,n}$.

n	$\lfloor (7n - 2)/9 \rfloor$	y	z
$18p$	$14p$	$8p$	$6p$
$18p + 4$	$14p + 3$	$8p + 2$	$6p + 1$
$18p + 8$	$14p + 6$	$8p + 4$	$6p + 2$
$18p + 14$	$14p + 11$	$8p + 6$	$6p + 5$

With y and z chosen appropriately, Table 2.2 contains a single construction of a $\lfloor (7n - 2)/9 \rfloor$ -book that works in each case. We use 10 types of pages. The last two types are those in two opposite quadrants, about $z/2$ of each. There are about $y/8$ (rounded in various ways) of each type of page in three quadrants. These eight types come in four pairs. Type j and Type j' . Recall that the values x_j in the row for a page are the number of essential edges it receives from the four quadrants. The pages of Type j have a small odd number of essential edges from the j th quadrant (numbered cyclically), and the pages of Type j' have a small even number of essential edges from the j th coordinate. Note that y is even in each case.

Table 2.2. A $\lfloor (7n - 2)/9 \rfloor$ -book for $K_{n,n}$

Page Type	Index $1 \leq i$	Leading edge in quadrant j			
		$j = 1$	$j = 2$	$j = 3$	$j = 4$
1	$i \leq \lfloor y/8 \rfloor$	$2i - 1$	$y/2 + \lfloor y/8 \rfloor + 1 - i$	0	$y/2 + 1 - i$
1'	$i \leq \lfloor y/4 \rfloor - \lfloor y/8 \rfloor$	$2i$	$y/2 + \lfloor y/4 \rfloor + 1 - i$	0	$y/2 - \lfloor y/8 \rfloor + 1 - i$
2	$i \leq \lfloor y/4 \rfloor - \lfloor y/8 \rfloor$	$y/2 + 1 - i$	$2i - 1$	$\lfloor 3y/4 \rfloor - \lfloor y/8 \rfloor + 1 - i$	0
2'	$i \leq \lfloor y/8 \rfloor$	$\lfloor y/4 \rfloor + \lfloor y/8 \rfloor + 1 - i$	$2i$	$y/2 + \lfloor y/4 \rfloor + 1 - i$	0
3	$i \leq \lfloor y/8 \rfloor$	0	$y/2 + 1 - i$	$2i - 1$	$y/2 + \lfloor y/8 \rfloor + 1 - i$
3'	$i \leq \lfloor y/4 \rfloor - \lfloor y/8 \rfloor$	0	$y/2 - \lfloor y/8 \rfloor + 1 - i$	$2i$	$y/2 + \lfloor y/4 \rfloor + 1 - i$
4	$i \leq \lfloor y/4 \rfloor - \lfloor y/8 \rfloor$	$\lfloor 3y/4 \rfloor - \lfloor y/8 \rfloor + 1 - i$	0	$y/2 + 1 - i$	$2i - 1$
4'	$i \leq \lfloor y/8 \rfloor$	$y/2 + \lfloor y/4 \rfloor + 1 - i$	0	$\lfloor y/4 \rfloor + \lfloor y/8 \rfloor + 1 - i$	$2i$
13	$i \leq \lfloor z/2 \rfloor$	$y/2 + \lfloor y/4 \rfloor + i$	0	$y/2 + \lfloor y/4 \rfloor + i$	0
24	$i \leq \lfloor z/2 \rfloor$	0	$y/2 + \lfloor y/4 \rfloor + i$	0	$y/2 + \lfloor y/4 \rfloor + i$

To verify this construction, we must show that the pages obey the length limits and that each column contains $1, 2, \dots, n/2$ once each. The main idea for the latter is that the 3-quadrant page types, when taken in pairs j, j' , cover a consecutive segment of $1, 2, \dots, n/2$. For this to hold in quadrant j , with Type j having small odd x_j and Type j' having small even x_j , the number of Type j pages must exceed the number of Type j' by 0 or 1. We can discuss the four congruence cases at once via a notational convenience. In specifying a formula, a string of four digits indicates the values to be used in the four cases $n \equiv 0, 4, 8, 14 \pmod{18}$. For example, all the information in Table 2.1 above is encoded by writing $n/2 = 9p + 0247$, $y = 8p + 0246$, and $z = 6p + 0125$. The computations we need appear in Table 2.3.

Comparison of the entries for $\lfloor y/8 \rfloor$ and $\lfloor y/8 \rfloor$ with the entries immediately below them shows that in each case the number of pages of Type j is the number of pages of Type j' plus 0 or 1, so combining them yields a consecutive sequence of small leading edges. In the other quadrants, they also occupy consecutive segments, as indicated in Table 2.4.

Table 2.3. Functions of y and z in various congruence classes.

$y=8p+0246$	$z=6p+0125$	$y+\lfloor y/8 \rfloor +1=9p+1468$
$y/2=4p+0123$	$n/2=9p+0247$	$y+\lfloor y/4 \rfloor -\lfloor y/8 \rfloor +1=9p+1358$
$\lfloor y/4 \rfloor =2p+0112$	$\lfloor y/4 \rfloor =2p+0011$	$y+\lfloor y/4 \rfloor -\lfloor y/8 \rfloor +1=9p+1368$
$\lfloor y/8 \rfloor =p+0111$	$\lfloor y/8 \rfloor =p$	$y+\lfloor y/8 \rfloor +2=9p+2468$
$\lfloor y/4 \rfloor -\lfloor y/8 \rfloor =p+0001$	$\lfloor y/4 \rfloor -\lfloor y/8 \rfloor =p+0011$	
$\lfloor z/2 \rfloor =3p+0012$	$\lfloor z/2 \rfloor =3p+0113$	
$\lfloor y/4 \rfloor +\lfloor z/2 \rfloor =5p+0124$	$\lfloor y/4 \rfloor +\lfloor z/2 \rfloor =5p+0124$	
$y/2+\lfloor y/4 \rfloor +\lfloor z/2 \rfloor =9p+0247$	$y/2+\lfloor y/4 \rfloor +\lfloor z/2 \rfloor =9p+0247$	

Table 2.4. Sequences of leading edges in combined Types.

Types	#	Range of leading edges in quadrant j			
1U1'	$\lfloor y/4 \rfloor$	$1 \leftrightarrow \lfloor y/4 \rfloor$	$y/2+1 \leftrightarrow y/2+\lfloor y/4 \rfloor$	-	$\lfloor y/4 \rfloor +1 \leftrightarrow y/2$
2U2'	$\lfloor y/4 \rfloor$	$\lfloor y/4 \rfloor +1 \leftrightarrow y/2$	$1 \leftrightarrow \lfloor y/4 \rfloor$	$y/2+1 \leftrightarrow y/2+\lfloor y/4 \rfloor$	-
3U3'	$\lfloor y/4 \rfloor$	-	$\lfloor y/4 \rfloor +1 \leftrightarrow y/2$	$1 \leftrightarrow \lfloor y/4 \rfloor$	$y/2+1 \leftrightarrow y/2+\lfloor y/4 \rfloor$
4U4'	$\lfloor y/4 \rfloor$	$y/2+1 \leftrightarrow y/2+\lfloor y/4 \rfloor$	-	$\lfloor y/4 \rfloor +1 \leftrightarrow y/2$	$1 \leftrightarrow \lfloor y/4 \rfloor$
13	$\lfloor z/2 \rfloor$	$y/2+\lfloor y/4 \rfloor +1 \leftrightarrow n/2$	-	$y/2+\lfloor y/4 \rfloor +1 \leftrightarrow n/2$	-
24	$\lfloor z/2 \rfloor$	-	$y/2+\lfloor y/4 \rfloor +1 \leftrightarrow n/2$	-	$y/2+\lfloor y/4 \rfloor +1 \leftrightarrow n/2$

Examination of the columns of Table 2.4 shows that the column conditions hold. The fact that Types 13 and 24 with long edges take them precisely up to $n/2$ follows from the last line of

Table 2.3. Finally, consider the length condition. We must compute $\sum_{i=1}^4 x_i$ for each page in three

quadrants. This is $y+\lfloor y/8 \rfloor +1$ for pages of Type 1 or 3, $y+\lfloor y/4 \rfloor -\lfloor y/8 \rfloor +1$ for pages of Type 1' or 3', $y+\lfloor y/4 \rfloor -\lfloor y/8 \rfloor +1$ for pages of Type 2 or 4, and $y+\lfloor y/8 \rfloor +2$ for pages of Type 2' or 4'. In the last column of Table 2.3, the computation is completed to show that each of these is at most $n/2+2$. $\square$

Section 2.6: Non-regular Two-bucket Embeddings

Our next result begins to study the question of the lower bound on the pagenumber of $K_{n,n}$ if the buckets do not all have size $n/2$.

THEOREM 2.15: Suppose n is even. Let $n/3 \leq p < n/2$. If we split $K_{n,n}$ into buckets of sizes $p, n/2, n-p$, and $n/2$, then we must use at least $\lceil 3n/4 \rceil$ pages to embed $K_{n,n}$.

PROOF: Assume there is a coloring for this ordering with c colors. By the same arguments as before, we may assume that no color appears in only one quadrant and no color appears in all four quadrants. We will color only the essential positions in our grid. Let d_{ij} be the number of colors that appear only in quadrants i and j . Let d be the sum of the d_{ij} 's. Let t_1 be the number of colors appearing in quadrants 3 and 4, and either 1 or 2. Let t_2 be the number of colors appearing in quadrants 1 and 2, and either 3 or 4. Let $t = t_1 + t_2$. Suppose we have a color appearing in only quadrants 1 and 3. Such a color can account for at most p positions in quadrant 1 and $n/2$ positions in quadrant 3. Each successive color can account for two fewer positions in quadrants 1 and 3. Therefore, the j th largest color appearing in only quadrants 1 and 3 can account for at most $n/2 + p + 2 - 2j$ positions in our grid. The same result holds for colors appearing only in quadrants 2 and 4. Each other color that appears in only two quadrants can account for at most $p + 1$ positions. Note that $p + 1 \leq n/2 + p + 2 - 2j$ if $j \leq (n + 2)/4$. Suppose a color appears in quadrants 1, 3, and 4. Then that color cannot appear in columns $p + 1, p + 2, \dots, n/2$ of quadrant 4. Therefore, it can account for at most $2p - n/2 + 2$ positions in the grid. Suppose a color appears in quadrants 1, 2, and 4. Then that color cannot appear in columns $p + 1, p + 2, \dots, n/2$ of quadrant 4 and it can account for at most $p + 2$ positions.

For any c -page embedding with this ordering, we have $t + d = c$. Using the fact that there are $n + 2p$ positions on the major diagonals in the essential triangles, we have $3t + 2d = n + 2p$. Solving this system we find that $t = 2p - 2c + n$ and $d = 3c - 2p - n$. In quadrants 3 and 4, the major diagonals must share $n - c$ colors, giving $t_1 = n - c - d_{34}$ and $t_2 = 2p - c + d_{34}$. The fact that there are n elements on the major diagonals in quadrants 3 and 4, implies that

$n = t + t_1 + d - d_{12} + d_{34}$. Since $t + d = c$ and $t_1 + d_{34} = n - c$, this simplifies to $d_{12} = 0$. We also have that $d_{14} + d_{23} = 3c - 2p - n - d_{13} - d_{24} - d_{34}$.

Let g be the maximum number of essential positions that can be colored with c colors, given $d_{13} = x$, $d_{24} = y$, and $d_{34} = z$. Colors that appear in three quadrants, including quadrants 3 and 4, can account for at most $(n - c - z)(2p - n/2 + 2)$ essential positions. Colors that appear in three quadrants, including quadrants 1 and 2, can account for at most $(2p - c + z)(p + 2)$ essential positions. Colors that appear in only quadrants 3 and 4 can account for at most $z(p + 1)$ essential positions. Other colors that appear in only two adjacent quadrants can account for at most $(3c - 2p - n - x - y - z)(p + 1)$ essential positions. Colors that appear in only quadrants 1 and 3 can account for at most $\sum_{j=1}^x (n/2 + p + 2 - 2j) = (n/2 + p + 2)x - x(x + 1)$ essential positions.

Replacing x by y gives the result for colors that appear in only quadrants 2 and 4. This yields

$$g(x, y, z) \leq n(x + y)/2 - x^2 - y^2 + (n/2 - p)z + (2p + c + 2 - n)n/2 + 2p - c.$$

Let $f(x, y, z)$ be the function given on the right-hand side of the last inequality. Notice that the partial derivative of f with respect to z is $n/2 - p > 0$, so no critical point of f can occur in the interior of the region $x + y + z \leq 3c - 2p - n$; $x, y, z \geq 0$. If we look at the boundary of our region and examine corners and the points obtained by setting partial derivatives equal to zero, we obtain the following candidates for the maximum value of f . $A = f(0, 0, 3c - 2p - n)$, $B = f(0, n/4, 0)$, $C = f(0, 3c - 2p - n, 0)$, $D = f(0, p/2, 3c - 5p/2 - n)$, $E = f(n/4, n/4, 0)$, $F = f(3c/2 - p - n/2, 3c/2 - p - n/2, 0)$, and $G = f(p/2, p/2, 3c - 3p - n)$. It is easy to show that $G > A$, $E > B$, $G > D$, $F > C$, so we need only consider E, F, G .

The total number of essential positions we must color is $N = p^2 + p + n^2/4 + n/2$. $N - E = p^2 - p(n + 1) + 5n^2/8 - n/2 + c - cn/2$. This quadratic has discriminant $-9n^2 + 4n + 2nc - 4c + 1$. This is negative when $c = \lfloor 3n/4 \rfloor - 1$ and so $N - E > 0$. Similar calculations for $N - F$ give a quadratic in p^2 with leading term $3p^2$. It has discriminant $1 - 17n^2 - 18c^2 + 2n + 36cn$, which is negative when $c = \lfloor 3n/4 \rfloor - 1$. Therefore, $N - F > 0$.

The situation for $N - G$ is different. We have $N - G = -3p^2/2 - p(1 + n - 3c) + 5n^2/4 - n/2 + c - 2nc$. When $c = \lfloor 3n/4 \rfloor - 1$, $(N - G) > 0$ if $p = n/3, n/2 - 1$, so $N - G$ is positive in the range $n/3 \leq p < n/2$. Therefore, we cannot color all of the positions in the essential triangles with $\lfloor 3n/4 \rfloor - 1$ colors. $\square$

Section 2.7: Pagenumber of $K_{m,n}$ for Large m

As mentioned earlier, Bernhart and Kainen noted that $t(K_{m,n}) = n$ for $m > n(n-1)$, by using the pigeonhole principle to obtain a large twist. Our construction in Theorem 2.5 shows that $t(K_{2n-4,n}) \leq n - 1$; note also that Theorem 2.11 shows that regular embeddings of $K_{2n-2,n}$ require n pages. We conjecture that $t(K_{2n-4,n}) = n - 1$ and $t(K_{2n-3,n}) = n$. Toward this we offer the following.

THEOREM 2.16: $t_2(K_{2n-3,n}) = n$.

PROOF: With the ordering $Y_1, \dots, Y_q, X_1, \dots, X_p, Y_{q+1}, \dots, Y_n, X_{p+1}, \dots, X_m$, we may assume $p = n - 2$. Otherwise we have a run of n X 's, yielding a twist of size n . Lemma 2.8 reduces us to coloring only D_0 and D_1 in each quadrant, as indicated in Figure 2.6, but the aim of this proof is to show we cannot do even that with $n - 1$ colors. For ease of discussion, index the quadrants clockwise from the upper left, as usual, and let D_0^s, D_1^s be the diagonals of interest in quadrant s .

In the two bottom quadrants, $D_0^4 \cup D_1^3$ and $D_1^4 \cup D_0^3$ each form a twist of size $n - 1$. Hence all $n - 1$ colors must appear in each, so the color in row i is the same in both twists. Let α be the one color that now appears in both lower quadrants. Note that in the upper quadrants α can appear only in the HOME columns. Also, no color can appear in column 1 and column n of the upper quadrants, because every color appears in a column less than n in the lower right or a column greater than 1 in the lower left.

Because α can appear only in columns $q, q + 1$ in the upper quadrants, α cannot appear in $D_1^1 \cup D_1^2$, the diagonals next to the main diagonal there. However, $D_1^1 \cup D_1^2$ is a twist of size

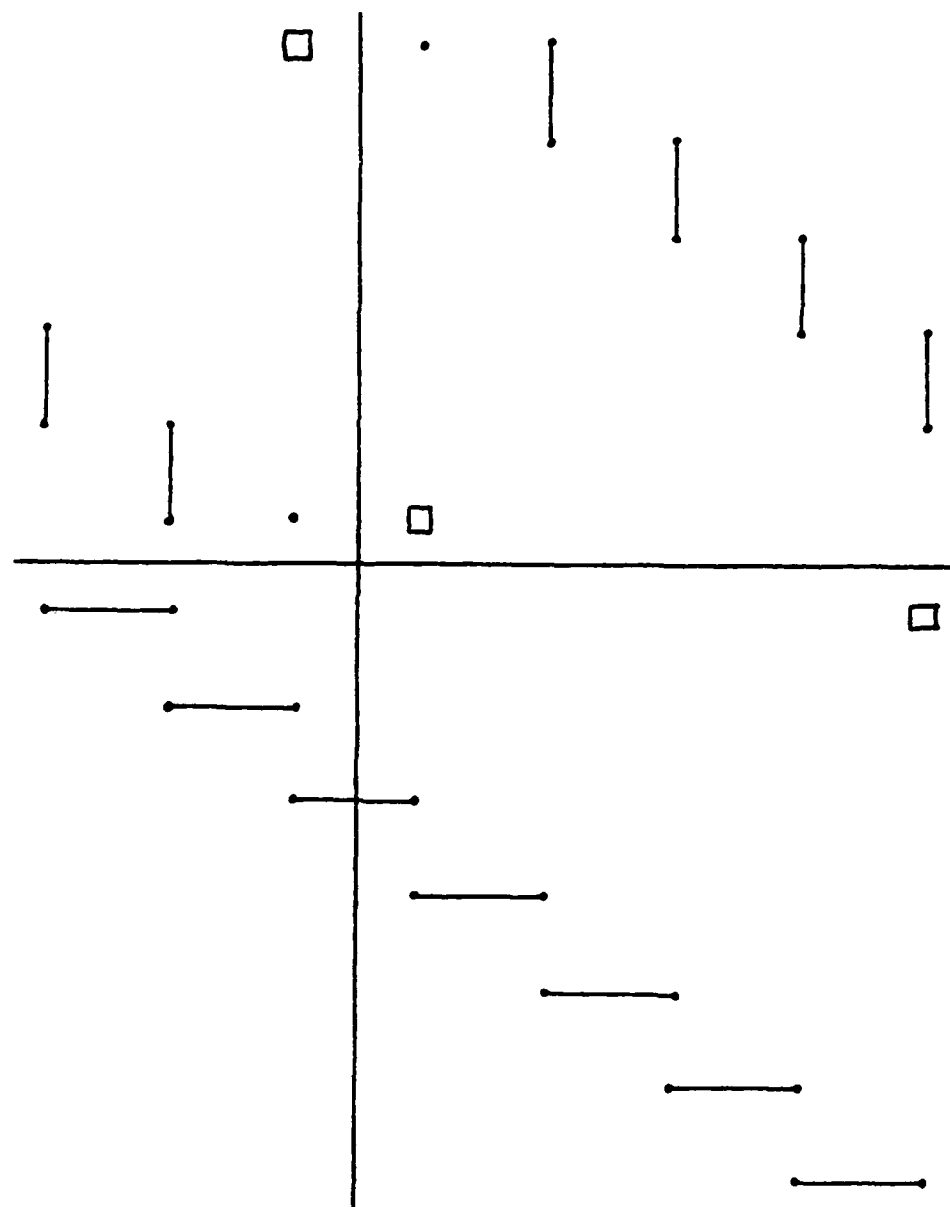


Figure 2.6: Problematic edges of $K_{2n-3,n}$.

$n - 2$, so it receives the $n - 2$ other colors. Let β, γ be the colors on D_1^1 in column 1 and D_1^2 in column n . (If $q = 1$, then D_1^1 is empty and we can set $\beta = \alpha$; similarly if $q = n - 1$, then D_1^2 is empty and we set $\gamma = \alpha$.)

Now consider the colors available for D_0^1 in column 1 and D_0^2 in column n . By considering the twist $D_1^1 \cup D_1^2$, we see that the only colors that can be used are β, γ . Since no color appears in column 1 and column n , we must put β on column 1 in D_0^1 and γ on column n in D_0^2 . Looking at successively increasing columns in D_0^1 and decreasing columns in D_0^2 , the only color available is the color in the same column of $D_1^1 \cup D_1^2$. Reaching the HOME columns, this leaves only α available to color both $(1, q + 1)$ and $(n - 2, q)$, but it cannot color both. $\square$

CHAPTER 3

GENERALIZED BANDWIDTHS OF GRAPHS

Section 3.1: Introduction

In this chapter we will study the bandwidth parameter and one of its generalizations. The bandwidth problem seems to have originated in the 1950's when structural engineers analyzed steel frameworks by manipulating their structural matrices. To make operations such as inversion of matrices and finding determinants as fast as possible, they sought an "equivalent" matrix with all non-zero entries lying within a narrow band about the main diagonal. This is where the term bandwidth originated.

In the early 1960's, the bandwidth problem for graphs was first studied at the Jet Propulsion Laboratory in Pasadena, California. L. H. Harper and A. W. Hales [19, 20] considered the following problem in coding theory. Suppose a binary channel is to be used to send one of 2^n possible integers $1, 2, \dots, 2^n$. An integer may be assigned to each of the 2^n n -tuples of zeroes and ones. If it is assumed that only single bit errors are likely in a transmitted n -tuple, how should the integers be assigned in order to minimize the maximum magnitude of error in the encoded integer? Looking at this from a graph theory standpoint we have the following problem. How can the integers $1, 2, \dots, 2^n$ be assigned to the vertices of the n -cube such that the maximum absolute difference of the numbers assigned to adjacent vertices is minimized? This was the origin of the bandwidth problem for graphs.

Chung [11] and Chinn, Chvatalova, Dewdney, and Gibbs [10] have presented surveys on the bandwidth parameter.

From the formal definition of this parameter, which we give next, it is not hard to see the relationship between the bandwidth problems for matrices and graphs. Indeed, if "equivalence" of matrices means simultaneous permutation of its rows and columns, then the bandwidth of a graph is the bandwidth of its adjacency matrix. This explains the origin of the term, though the formal definition doesn't mention matrices.

DEFINITION 3.1: Let $G = (V, E)$ be a graph. A 1-1 function $f: V \rightarrow \mathbb{Z}$, the integers, is called a *numbering* of G . The *bandwidth* of a numbering f for G , denoted by $b(G; f)$, is given by

$$b(G; f) = \max \{ |f(u) - f(v)| : (u, v) \in E(G) \}$$

The *bandwidth* of G , denoted by $b(G)$, is given by

$$b(G) = \min \{ b(G; f) : f \text{ is a numbering of } G \}$$

A numbering f of G is *optimal* if $b(G; f) = b(G)$.

We can generalize Definition 3.1 by assigning a k -tuple of integers to each vertex instead of a single integer. We have the following definition.

DEFINITION 3.2: Let $G = (V, E)$ be a graph. A 1-1 function $f: V \rightarrow \mathbb{Z}^k$ is called a *k-dimensional numbering* of G . If $x = (x_1, x_2, \dots, x_k)$ and $y = (y_1, y_2, \dots, y_k)$ are points in $\mathbb{Z}^k$, then we define

$$|x - y| = \sum_{i=1}^k |x_i - y_i|$$

The *k-dimensional bandwidth* of a numbering f for G , denoted by $b_k(G; f)$ is given by

$$b_k(G; f) = \max \{ |f(u) - f(v)| : (u, v) \in E(G) \}$$

The *k-dimensional bandwidth* of G , denoted by $b_k(G)$, is defined to be

$$b_k(G) = \min \{ b_k(G; f) : f \text{ is a } k\text{-dimensional numbering of } G \}$$

The metric used in Definition 3.2 is called the l_1 or "taxi" metric. We could have used the Euclidean distance, but the "taxi" metric is more natural as a special case of the following general problem defined by Chung [11]. Assign the vertices of a graph G to distinct vertices of a host graph H , by a function f . If $xy \in E(G)$ and $d_H(x, y) = t$, we say that edge xy has dilation t in this embedding. Bandwidth arises when the host graph is a grid graph. We seek the embedding f that minimizes maximum dilation of edges. The *k-dimensional grid graph* has vertex set $\mathbb{Z}^k$ consisting of all k -tuples of integers with x adjacent to y for $x, y \in \mathbb{Z}^k$ when $|x - y| = 1$. The k -dimensional bandwidth problem seeks an embedding of the vertices of G at distinct vertices of the k -dimensional grid graph H such that the maximum distance in H between images of adjacent

vertices in G is minimized.

Graph labeling problems with grid graphs as host graph often arise in formulating circuit layout models involved in VLSI design. In particular, the bandwidth of a layout is directly related to the performance of the circuit if the length of an edge is proportional to the propagation delay through the wire. In models with higher dimensional grid graphs as host graphs, bandwidth problems correspond to layout problems in multi-layer circuits. Propagation delay is even more relevant in distributed computing, where the host graph may model a communication network such as the hypercube.

In this chapter we will find upper and lower bounds on the k -dimensional bandwidth of graphs in terms of other parameters. We also find the bandwidth and generalized bandwidths of several classes of graphs.

Section 3.2: Bounds on the k -dimensional Bandwidth

Our first result gives an upper bound on the k -dimensional bandwidth of a graph in terms of the bandwidth of the graph.

THEOREM 3.3: For any graph G , $b_k(G) \leq k \lceil (b(G))^{1/k} \rceil - k + 1$.

PROOF: Let $b = b(G)$ and let $B = \lceil b^{1/k} \rceil$. Assume f is an optimal one-dimensional labeling of G .

Suppose v is a vertex in G . Write $f(v)$ as $f(v) = B^{k-1}q(v) + r(v)$, where $0 \leq r(v) < B^{k-1}$. Let

$a_0(v), a_1(v), \dots, a_{k-2}(v)$ be the B -ary expansion of $r(v)$. In other words, $r(v) = \sum_{i=0}^{k-2} a_i(v)B^i$,

with $0 \leq a_i(v) \leq B - 1$. Define the k -dimensional labeling

$$g(v) = (a_0(v), a_1(v), \dots, a_{k-2}(v), q(v)).$$

For any $v, w \in V(G)$ and any $0 \leq i \leq k - 2$, $|a_i(v) - a_i(w)| \leq B - 1$. If v and w are adjacent, then $|f(v) - f(w)| \leq b$, and therefore, $|q(v) - q(w)| \leq B$. Hence,

$$|g(v) - g(w)| \leq kB - k + 1. \quad \square$$

We will see in Theorem 3.22 that Theorem 3.3 is asymptotically best possible up to a constant factor in k . We next consider the k -dimensional bandwidth of the Cartesian product of graphs. We will use it to bound the k -dimensional bandwidth of the n -cube. (The reverse problem of embedding grids in hypercubes is under active study in the computational community.)

THEOREM 3.4: Let $G = G_1 \times G_2 \times \cdots \times G_t$ be a Cartesian product of t factors. Let $m_1 + m_2 + \cdots + m_t = k$ be any composition of k with t parts. Then $b_k(G) \leq \max\{b_{m_i}(G_i)\}$.

PROOF: Let f_i be an optimal m_i -dimensional labeling of G_i . If u and v are adjacent in G_i , then $|f_i(v) - f_i(u)| \leq b_{m_i}(G_i)$. Let $x_{u_1, u_2, \dots, u_t}$ be the vertex of G corresponding to $u_i \in G_i$. Consider the k -dimensional labeling formed by concatenating the labelings $f_i(u_i)$.

$$f(x_{u_1, u_2, \dots, u_t}) = (f_1(u_1), f_2(u_2), \dots, f_t(u_t))$$

Suppose $x_{u_1, u_2, \dots, u_t}$ is adjacent to $x_{v_1, v_2, \dots, v_t}$ in G . Call this a type i edge if u_i is adjacent to v_i in G_i and $u_j = v_j$ for $j \neq i$. Suppose $x_{u_1, u_2, \dots, u_t}$ is adjacent to $x_{v_1, v_2, \dots, v_t}$ by a type i edge. Then

$$|f(x_{u_1, u_2, \dots, u_t}) - f(x_{v_1, v_2, \dots, v_t})| = |f_i(u_i) - f_i(v_i)| \leq b_{m_i}(G_i)$$

Since every edge in the Cartesian product is type i for some i , $b_k(G; f) \leq \max\{b_{m_i}(G_i)\}$. $\square$

Harper [20] found an exact formula for the bandwidth of the n -cube Q_n by using the following labeling of the vertices. (1) Label any vertex with 1. (2) If 1, 2, ..., l have been assigned, then assign $l + 1$ to any unlabeled neighbor of the lowest numbered vertex having unlabeled neighbors.

$$\text{THEOREM 3.5: } b(Q_n) = \sum_{i=0}^{n-1} \left\lfloor \frac{i}{2} \right\rfloor$$

Using Theorems 3.4 and 3.5, we will get an upper bound on the k -dimensional bandwidth of the n -cube.

COROLLARY 3.6: $b_k(Q_n) \leq \sum_{i=0}^{\lfloor n/k \rfloor - 1} \binom{i}{\lfloor i/2 \rfloor}.$

PROOF: Q_n is the Cartesian product of k factors of the form $Q_{\lfloor n/k \rfloor}$ or $Q_{\lfloor n/k \rfloor + 1}$. By Theorems 3.4 and

3.5, $b_k(Q_n) \leq b(Q_{\lfloor n/k \rfloor}) = \sum_{i=0}^{\lfloor n/k \rfloor - 1} \binom{i}{\lfloor i/2 \rfloor}.$ $\square$

Dewdney [14] proved the following lower bound for the bandwidth of any graph G . It is known as the *density* bound, being due to the density of integer points in a neighborhood of a given point.

THEOREM 3.7: For any graph G with maximum degree Δ , $b(G) \geq \lfloor \Delta/2 \rfloor$

We now want to generalize this result to the k -dimensional bandwidth of a graph. First, we will count the number of labels within distance t of a given point.

DEFINITION 3.8: For $x \in \mathbb{Z}^k$ and any integer $t \geq 0$, let $B_k(x, t)$ denote the k -dimensional ball of radius t around x . $B_k(x, t) = \{z \in \mathbb{Z}^k : |z - x| \leq t\}$ in our metric. The size of $B_k(x, t)$ is independent of x . Therefore, let $\rho_k(t) = |B_k(x, t)|$.

LEMMA 3.9: $\rho_k(t) = \sum_{i=0}^k \binom{k}{i} \binom{t}{i} 2^i$. In particular, $\rho_2(t) = 2t(t+1) + 1 = \frac{(2t+1)^2 + 1}{2}$ and

$$\rho_1(t) = 2t + 1.$$

PROOF: We will count the integer points within distance t of x . Let $d > 0$ and consider the points at distance d from x . Such points differ from x in some number $i > 0$ of coordinates. The total distance d is apportioned among these i nontrivial positions. The non-zero difference in a given coordinate may be positive or negative. Since the number of compositions of d with i positive parts is $\binom{d-1}{i-1}$, we thus have $\sum_{i=1}^k \binom{k}{i} \binom{d-1}{i-1} 2^i$ points at distance d from x . Summing over

$0 \leq d \leq t$ to obtain the number of points within distance t , we then interchange the order of

summation and apply the identity $\sum_{d=1}^t \binom{d-1}{i-1} = \binom{t}{i}$ to obtain $\rho_k(t) = \sum_{i=0}^k \binom{k}{i} \binom{t}{i} 2^i$ $\square$

THEOREM 3.10: For any graph G with maximum degree Δ , and any k , $b_k(G) \geq t$, where t is the least integer such that $\rho_k(t) \geq \Delta + 1$. In particular, $b_2(G) \geq \lceil (-1 + \sqrt{1 + 2\Delta})/2 \rceil$ and $b(G) \geq \lceil \Delta/2 \rceil$.

PROOF: Suppose $b_k(G) = s$. If x is the label assigned to a vertex of maximum degree in some optimal labeling, then its neighbors must have labels in $B_k(x, s)$. This implies that $\Delta \leq \rho_k(s) - 1$ and by definition of t , we have $s \geq t$.

When $k = 1, 2$, solving $2t \geq \Delta$ and $\frac{(2t+1)^2 - 1}{2} \geq \Delta$ yields the special cases. $\square$

Section 3.3: The Bandwidth of Clique-stars

Equality in Theorem 3.10 clearly holds when G is a star. We now want to consider a more general class of graphs obtained by adding edges to join all central vertices in several disjoint stars. In particular, the *double star*, denoted by $S_{m,n}$, is formed by adding an edge between the central vertices of S_{m-1} and S_{n-1} . Syslo and Zak [29] found an exact formula for the bandwidth of the double star.

THEOREM 3.11: If $m \geq n$, then

$$b(S_{m,n}) = \max \{ \lceil m/2 \rceil, \lceil (m+n-1)/3 \rceil \}$$

We now wish to generalize the idea of the double star by replacing the two-clique with a larger clique. We have the following definition.

DEFINITION 3.12: Suppose $n_1 \geq n_2 \geq \dots \geq n_c \geq c-1 \geq 0$. Define the *c-clique star*

$S_{n_1, n_2, \dots, n_c}$ as follows. Begin with a c -clique having vertices $x_1, x_2, \dots, x_c$. For $1 \leq i \leq c$, add

$n_i - c + 1$ vertices adjacent to x_i only, so that x_i has degree n_i . There are now

$n = \left(\sum_{i=1}^c n_i \right) - c(c-2)$ vertices altogether, of which $n - c$ have degree one. We will call the

vertices of degree one *leaves*.

We now wish to generalize the result of Syslo and Zak to the bandwidth of the c -clique star. We will need to use the following elementary and well-known results.

THEOREM 3.13: If H is a subgraph of G , then $b(G) \geq b(H)$.

This result holds since the difference between the labels on any two adjacent vertices in H must be considered in finding the bandwidth of G .

THEOREM 3.14: If K_n is the complete graph on n vertices, then $b(K_n) = n - 1$.

Since every vertex in K_n is adjacent to every other vertex, there are two labels differing by at least $n - 1$, such that their underlying vertices are adjacent.

THEOREM 3.15: If G is any graph containing n vertices, then $b(G) \geq \frac{n-1}{D}$ where D is the diameter of G .

In this situation, the labels 1 and n must be on vertices that are at distance at most D from one another, and the result follows.

THEOREM 3.16: Suppose $n_1 \geq n_2 \geq \dots \geq n_c \geq c - 1 \geq 0$. Then for the c -clique star $S_{n_1, n_2, \dots, n_c}$ with n vertices

$$b(S_{n_1, n_2, \dots, n_c}) = \max \{ c - 1, \lfloor n_1/2 \rfloor, \lfloor (n - 1)/3 \rfloor \}.$$

PROOF: Let $G = S_{n_1, n_2, \dots, n_c}$, and $b = b(G)$. First consider the lower bound. Since K_c is a subgraph of G , Theorems 3.13 and 3.14 imply that $b \geq c - 1$. Since n_1 is the degree of a vertex in G , Theorem 3.7 implies that $b \geq \lfloor n_1/2 \rfloor$. Since G has n vertices and diameter 3, Theorem 3.15 implies that $b \geq \lfloor (n - 1)/3 \rfloor$.

For the upper bound, we construct a labeling, which we view as placing the vertices at consecutive integers on the real line. In other words, we will specify a linear ordering of the vertices. An edge is *good* if the distance between its endpoints in this ordering is at most

$$m = \max \{ c - 1, \lfloor n_1/2 \rfloor, \lfloor (n - 1)/3 \rfloor \}.$$

For any c -clique star, there is an optimal ordering in which the order of the leaves respects the order of the clique vertices. In other words, if x_i appears to the left of x_j , then all leaf neighbors of x_i appear to the left of all leaf neighbors of x_j . To prove this, note that if two leaves were out of order, then switching their labels would decrease the maximum difference on their two edges, but change no other edge difference. Therefore, to specify a potentially optimal labeling, it suffices to specify the order and position of the clique vertices in the vertex ordering.

The simplest constructions have the clique vertices together and the leaf vertices to their left and right, which in fact is forced if $m = c - 1$. If $m > c - 1$, some leaves may creep in among the clique vertices in the ordering. At most m leaves can be placed to the left or right of the extreme clique vertices, or else we get a bad edge. Hence, we define $m_1 = \min\{m, n - c\}$, $m_2 = \min\{m, n - c - m_1\}$, and $m' = \max\{0, n - c - 2m\}$. Note that $m_1 + m_2 + m' = n - c$. We say that an ordering is *proper* if the leaf order respects the order of the clique vertices and there are m_1 leaves to the left of all clique vertices, m_2 leaves to their right, and m' leaves among them. We will construct a proper ordering with all edges good.

In every proper ordering, all clique edges are good, because $c - 1 \leq m$ and $n - 1 \leq 3m$ imply $c - 1 + m' \leq m$ for the largest difference on a clique edge. This also implies that the edges incident to the m' leaves among the clique vertices are good. Hence, we need only check the edges involving leaf vertices to the left and right. To do this we repeatedly apply one simple idea, which we call the "consecutivity lemma":

In a proper ordering, if the neighbors of the leaf vertices appearing to the left of all clique vertices appear consecutively as the leftmost clique vertices, then the edges involving these leaves are good. A similar result holds for the leaves to the right of the clique vertices.

This is easily proved by induction. Let $x_1, x_2, \dots, x_t$ be the clique neighbors of these leaves. Because the leaves respect the clique order, the neighbor of the leftmost leaf is x_1 and the distance between them is $m_1 \leq m$. Hence, all edges involving the leftward leaf neighbors of x_1 are good. For the induction step, suppose all edges involving leftward leaf vertices of $x_1, x_2, \dots, x_{j-1}$ are good. Since x_j is one to the right of x_{j-1} and the leftmost neighbor of x_j is to the right of the leftmost neighbor of x_{j-1} , the edges involving the leftward leaf neighbors of x_j are also good.

Now we are ready to complete the construction. Let r be the smallest j such that

$\sum_{i=1}^j (n_i - c + 1) \geq m_1 + m'$. Let t be the largest i such that x_i has leaf neighbors. In other words,

$n_i \geq c$, but $n_i = c - 1$ for any $i > t$. If $\sum_{i=1}^r (n_i - c + 1) = m_1 + m'$, we succeed easily by letting

$x_1, x_2, \dots, x_r$ be the leftmost clique vertices, consecutively, letting $x_{r+1}, \dots, x_t$ be the rightmost consecutively, and placing the remaining $c - t$ clique vertices between them as indicated below.

m_1 leaves $|x_1 \cdots x_r|$ m' leaves and $c - t$ clique vertices $|x_t \cdots x_{r+1}|$ m_2 leaves

This ordering satisfies the consecutivity lemma on both sides, so all edges are good.

We are left with the case $\sum_{i=1}^r (n_i - c + 1) > m_1 + m'$. Now, x_r has a neighbor to the right of

the clique vertices and the situation is more difficult. To overcome this difficulty, we must use the flexibility of shifting x_r rightward without changing the leaf ordering. The leaf ordering remains the same if x_r is placed anywhere between $X_1 = \{x_1, \dots, x_{r-1}\}$ and $X_2 = \{x_{r+1}, \dots, x_t\}$. We want to choose one of these positions so that the edges from x_r to the left and right are good.

Note that either X_1 or X_2 defined above may be empty. Let p_1 and p_2 be the total number of leaves adjacent to the vertices in X_1 and X_2 , respectively. If $p_1 \geq m_1$, we could easily correct the problem by moving x_r next to X_2 and applying the consecutivity lemma, so we may assume $p_1 < m_1$. We also have $p_2 < m_2$ by $p_1 + (n_r - c + 1) > m_1 + m'$ and similar reasoning. In other words, x_r must have leaf neighbors to the left and to the right of the clique vertices. All leaf neighbors of X_1 are to the left, and all leaf neighbors of X_2 are to the right.

If the labels start at 1, the leftward leaf neighbors of x_r will be close enough if $f(x_r) \leq p_1 + 1 + m$. To stay to the left of X_2 , we must also have $f(x_r) \leq n - m_2 - (t - r)$. The rightward leaf neighbors of x_r will be close enough if $f(x_r) \geq n - p_2 - m$, and to stay to the right of X_1 , we must have $f(x_r) \geq m_1 + r$. Thus we can successfully place x_r if $\max \{ m_1 + r, n - p_2 - m \} \leq \min \{ n - m_2 - (t - r), p_1 + 1 + m \}$.

Of these conditions, $m_1 + r \leq n - m_2 - (t - r)$ is the easiest to prove and follows from $m_1 + m_2 \leq n - c$. Next, $m_1 + r \leq m + 1 + p_1$ follows from $m_1 \leq m$ and $p_1 \geq r - 1$, the latter holding because each vertex in X_1 has a leaf neighbor to its left. Similarly, $n - p_2 - m \leq n - m_2 - (t - r)$ follows from $m_2 \leq m$ and $p_2 \geq t - r$.

Finally, consider $n - p_2 - m \leq p_1 + 1 + m$. Since $p_1 + p_2$ accounts for all leaves except those adjacent to x_r , we have $p_1 + p_2 = n - c - (n_r - c + 1) = n - 1 - n_r$. The desired inequality then follows from $2m \geq n_1 \geq n_r$. $\square$

For any graph G , let $m_i(G)$ be the maximum number of vertices in an induced subgraph of G having diameter i . Then $b(G) \geq \max \{ \lfloor (m_i - 1)/i \rfloor \}$. Call this the *subgraph-diameter bound* or *density bound* on $b(G)$. This is the common generalization of Dewdney's density bound and Theorem 3.15.

The subgraph-diameter bound is tight for cliques and stars, which are graphs of diameter one and two. Theorem 3.16 has provided us with a class of graphs of diameter three for which the subgraph-diameter bound is tight. We pose the question of characterizing the graphs of diameter two and three for which the subgraph diameter bound is tight.

Finding the largest subgraph of diameter i for each i is NP-hard, since it includes the maximum clique problem. However, if the clique size is bounded, Theorem 3.16 gives a lower bound on $b(G)$ that improves Dewdney's and can be computed in polynomial time.

COROLLARY 3.17: Given a graph G , let ν_c be the maximum number of vertices in any c -clique star contained in G . Then

$$b(G) \geq \max \{ \lfloor (\nu_1 - 1)/2 \rfloor, \omega(G) - 1, \lfloor (\nu_2 - 1)/3 \rfloor, \lfloor (\nu_3 - 1)/3 \rfloor, \dots, \lfloor (\nu_{\omega(G)} - 1)/3 \rfloor \}$$

PROOF: Since no c -clique star in G has a clique whose size exceeds $\omega(G)$ or a vertex of degree exceeding $\nu_1 - 1$, these are the best lower bounds obtainable from Theorem 3.16 for any clique-star in G . $\square$

Section 3.4: The Two-dimensional Bandwidth of Double Stars

In the remainder of this chapter, we analyze the higher-dimensional bandwidth of some special graphs. By Theorem 3.10, the two-dimensional bandwidth of the star $S_{\Delta+1}$ is $\lfloor (-1 + \sqrt{1 + 2\Delta})/2 \rfloor$. Theorem 3.16 suggests that we next consider two-clique stars. For this we will need a counting lemma. Henceforth, we let $\epsilon_r = 1$ if r is odd and $\epsilon_r = 0$ if r is even, to facilitate combining cases. In the discussion of two-dimensional balls, we will drop the subscript 2 in the notation.

LEMMA 3.18: Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be two-dimensional integer labels with $|x_1 - y_1| = s$ and $|x_2 - y_2| = t$, where $t \leq s$ and $s + t = r \leq d$. Let $A = B(x, d) \cap B(y, d)$ and let $C = B(x, d) \cup B(y, d)$. Then

$$|A| = \frac{(2d - r)^2 - \epsilon_r}{2} + (t + 1)(2d - r + 1)$$

$$|C| \leq \frac{7d^2 + 6d + 2 + \epsilon_r}{2}$$

with equality in the latter if and only if $t = 0$ and $r = d$.

PROOF: We may assume by symmetry that $x_1 \leq y_1$ and $x_2 \leq y_2$. Consider Figure 3.1. A is the set of integer points bounded by the rectangle v_1, v_2, v_3, v_4 , where v_4 is d units to the left of y and v_2 is d units to the right of x . v_1 and v_3 are determined by extending segments of slopes ± 1 from v_2 and v_4 . Note that v_1 and v_3 have integer coordinates if and only if r is even. Thus the extreme rows of A have one point if r is even and two points if r is odd. Let A_0 be the points of A on or between the rows containing x and y and let A^+ and A^- contain the points of A above and below A_0 , respectively. Note that A^+ and A^- have the same size.

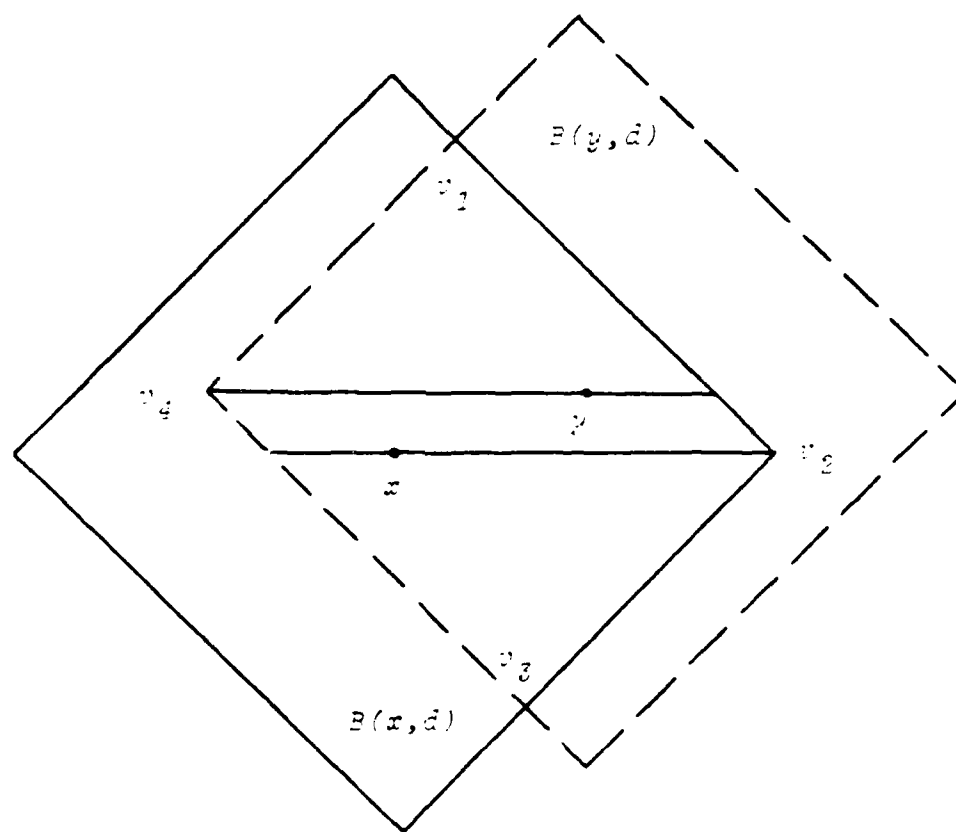


Figure 3.1: Diagram for Lemma 3.18.

Due to the relative positions of x and y , A_0 has $d - r$ points directly to the right of y .

Hence, each row of A_0 has $d + 1 + d - r$ points. A_0 has $t + 1$ rows, so

$$|A_0| = (t + 1)(2d - r + 1)$$

The largest row of A^+ has $2d - r - 1$ points. If r is odd, summing the even numbers to this point yields

$$|A^+| = 2 \sum_{i=0}^{(2d-r-1)/2} i = 2 \frac{(2d-r-1)(2d-r+1)}{8} = \frac{(2d-r)^2 - \epsilon_r}{4}$$

If r is even, we count the odd numbers up to $2d - r - 1$. It is well-known that the sum of the first j odd natural numbers is j^2 , so in this case $|A^+| = (2d - r)^2/4$. Together, we get

$$|A| = |A_0| + 2|A^+| = \frac{(2d-r)^2 - \epsilon_r}{2} + (t+1)(2d-r+1).$$

Now consider C . By Lemma 3.9, $|B(x, d)| = |B(y, d)| = 2d(d+1) + 1$. Hence,

$|C| = 2 + 4d(d+1) - |A|$. To maximize $|C|$, we must minimize $|A|$. Over fixed d and r , we can minimize $|A|$ only by setting $t = 0$ and $s = r$. Then the formula for $|A|$ depends only on r and d , and we can minimize $|A|$ only by setting $r = d$, in which case

$$|A| = \frac{d^2 + 2d + 2 - \epsilon_r}{2}$$

Hence,

$$|C| \leq 2 + 4d(d+1) - \frac{d^2 + 2d + 2 - \epsilon_r}{2} = \frac{7d^2 + 6d + 2 + \epsilon_r}{2},$$

with equality if and only if $t = 0$ and $r = d$. $\square$

THEOREM 3.19: Let d, m, n be positive integers with $m \geq n$, and let $f(d) = 2d(d+1)$ and

$$g(d) = \frac{7d^2 + 6d + 2 + \epsilon_d}{2}. \text{ Then } b_2(S_{m,n}) = d \text{ if and only if the following three conditions are}$$

satisfied:

- (1) $m \leq f(d)$
- (2) $m + n \leq g(d)$

$$(3) m > f(d-1) \text{ or } m+n > g(d-1)$$

PROOF: Suppose that the conditions are satisfied. Note that $f(d)$ is the size of the d -ball in two dimensions, and $g(d)$ is the maximum union of two such d -balls around points whose distance is at most d . These facts follow from Lemma 3.18. Since $S_{m,n}$ has adjacent vertices of degrees m and n , condition 3 implies $b_2(S_{m,n}) > d-1$.

Let u and v be the vertices of degree m and n respectively. Assign the label $x = (0, 0)$ to u and $y = (d, 0)$ to v . Assign labels in $B(x, d) - B(y, d)$ to the leaf neighbors of u , using labels in $B(x, d) \cap B(y, d)$ when and if $B(x, d) - B(y, d)$ is exhausted. Assign unused labels in $B(y, d) \cup B(x, d)$ to the leaf neighbors of v . By Lemma 3.9, Lemma 3.18, and conditions (1) and (2), there are enough labels available at each stage of this construction and $b_2(S_{m,n}) \leq d$.

Finally, note that there is exactly one value of d for which these conditions are satisfied, since both f and g are strictly increasing functions of d . This value is the least integer d such that both $f(d) \geq m$ and $g(d) \geq m+n$ are true. $\square$

Section 3.5: The k -dimensional Bandwidth of Complete Graphs

Now we shall consider the k -dimensional bandwidth of the complete graph. This will show that the bound in Theorem 3.3 is best possible up to a factor depending only on k . Several concepts and a lemma relating balls with other sets in Z^k will be helpful.

DEFINITION 3.20: Let 0_k and 1_k denote the vectors of all zeroes and all ones in Z^k . A vector e with entries ± 1 is called a *sign vector*. Given a point x , a sign vector e , and an integer $t \geq 0$, let $H(x, e, t)$ denote the set of integer points in the $t+1$ hyperplanes given by the equations $e \cdot z = e \cdot x + j$ for integers j with $-|t/2| \leq j \leq |t/2|$. A sign vector is *positive* if either (1) $e \cdot 1_k > 0$ or (2) $e \cdot 1_k = 0$ and $e_1 = 1$.

LEMMA 3.21: Let T denote the collection of positive sign vectors in Z^k . Let

$S = \bigcup \{ H(x, e, t) : e \in T \}$ and let $y = x + (1, 0, 0, \dots, 0)$. Then $S = B_k(x, t/2)$ if t is even and

$S = B_k(x, \lfloor t/2 \rfloor) \cup B_k(y, \lfloor t/2 \rfloor)$ if t is odd. Furthermore, S is a maximum-sized collection of points in Z^k whose elements have pairwise distance at most t .

PROOF: First suppose t is even. For any point z , define $\epsilon_x(z)$ to be the sign vector e such that $e_i = 1$ if $z_i \geq x_i$ and $e_i = -1$ if $z_i < x_i$. Then in our metric,

$$|z - x| = \epsilon_x(z) \cdot (z - x) = \epsilon_x(z) \cdot z - \epsilon_x(z) \cdot x.$$

If $z \in S$, then $|e \cdot z - e \cdot x| \leq t/2$ for every sign vector e , including $\epsilon_x(z)$. Hence, $|z - x| \leq t/2$ and $S \subseteq B_k(x, t/2)$.

Conversely, if $z \in B_k(x, t/2)$, then for any sign vector e we have

$$|e \cdot z - e \cdot x| = \left| \sum_{i=1}^k e_i(z_i - x_i) \right| \leq \sum_{i=1}^k |z_i - x_i| = |z - x| \leq \frac{t}{2}.$$

Hence, $B_k(x, t/2) \subseteq S$.

Now suppose t is odd. From the last case, we see that $S - B_k(x, \lfloor t/2 \rfloor)$ consists of all points z such that $\lfloor t/2 \rfloor < e \cdot z - e \cdot x \leq \lfloor t/2 \rfloor$ for all $e \in T$. Suppose $z \in B_k(y, \lfloor t/2 \rfloor) - B_k(x, \lfloor t/2 \rfloor)$.

Then $|z - x| > \lfloor t/2 \rfloor$ and

$$\begin{aligned} \lfloor t/2 \rfloor &\geq |z - y| = \sum_{i=1}^k |z_i - y_i| = |z_1 - x_1 - 1| + \sum_{i=2}^k |z_i - x_i| \\ &\geq |z_1 - x_1| - 1 + \sum_{i=2}^k |z_i - x_i| = |z - x| - 1 \end{aligned}$$

Therefore, $|z - x| \leq \lfloor t/2 \rfloor + 1 = \lfloor t/2 \rfloor$. This is the same description we have for $S - B_k(x, \lfloor t/2 \rfloor)$ and hence, $S = B_k(x, \lfloor t/2 \rfloor) \cup B_k(y, \lfloor t/2 \rfloor)$.

Finally, let U be a collection of points in Z^k such that $|u - v| \leq t$ for all $u, v \in U$. For any positive sign vector e and $u, v \in U$, we have

$$|e \cdot u - e \cdot v| \leq |u - v| \leq t$$

This means that all elements of U belong to $t + 1$ consecutive parallel hyperplanes perpendicular to e for any $e \in T$. Hence, U belongs to a translation of S . The fact that translation does not change cardinality completes the proof. $\square$

THEOREM 3.22: Let $f_k(t) = |B_k(x, t/2)| = \rho_k(t/2)$ if t is even, and if t is odd let $f_k(t) = |B_k(x, \lfloor t/2 \rfloor) \cup B_k(y, \lfloor t/2 \rfloor)|$ where $|x - y| = 1$. Then $b_k(K_n)$ is the least value of t such that $f_k(t) \geq n$. In particular, $b_2(K_n)$ is the least t such that $\frac{(t+1)^2 + 1 - \epsilon_t}{2} \geq n$, so $b_2(K_n) = \lfloor \sqrt{2n} - 1 \rfloor$, and for larger values of k , $b_k(K_n) = (k!n)^{1/k} + o(n^{1/k})$.

PROOF: By Lemma 3.21, $f_k(t)$ is the maximum size of a clique whose points can be given k -dimensional labels with pairwise distance at most t , which proves the first statement. For $k = 2$,

we have $f_k(t) = \rho_k(t/2) = \frac{(t+1)^2 + 1}{2}$ if t is even. If t is odd, recall that Lemma 3.18 says that

$$|B(x, d) \cap B(y, d)| = \frac{(2d-1)^2 - 1}{2} + 2d = 2d^2 \text{ when } x \text{ and } y \text{ are adjacent. Taking}$$

$$d = \lfloor t/2 \rfloor, \text{ we have } |S| = 2\rho_2(d) - 2d^2 = t^2 + 1 - \frac{(t-1)^2}{2} = \frac{(t+1)^2}{2}.$$

For larger, k , we need the asymptotic behavior of $\rho_k(t/2)$ as a function of t . Recall that

$$\rho_k(d) = \sum_{i=0}^k \binom{k}{i} \binom{d}{i} 2^i. \text{ When } d \text{ is much larger than } k, \text{ this sum is dominated by its last term,}$$

which itself is asymptotic to $\frac{d^k 2^k}{k!}$. Hence we have $f_k(t) \sim t^k / k!$. In other words,

$$\lim_{t \rightarrow \infty} \frac{f_k(t) k!}{t^k} = 1. \text{ Turning this around yields } b_k(K_n) = (k!n)^{1/k} + o(n^{1/k}). \quad \square$$

In Theorem 3.3, we showed $b_k(G) \leq k(B-1) + 1$ for all G , where $B = \lfloor b(G)^{1/k} \rfloor$.

Theorem 3.22 shows that Theorem 3.3 is best possible in terms of n , though it may be off by a factor depending on k . In particular, given a fixed value b , let G be the graph on vertices

$v_1, v_2, \dots, v_n$ for which v_i and v_j are adjacent if $|i - j| \leq b$. Then G has bandwidth b and contains K_{b+1} , so by Theorem 3.22, $b_k(G) \geq (k!b)^{1/k} + o(b^{1/k})$. Since $k! \sim k^k e^{-k} \sqrt{2\pi k}$ the ratio

between the upper bound on $b_k(G)$ in terms of $b(G)$ and the value of $b_k(G)$ in terms of $b(G)$ for this construction is asymptotic for fixed k to $e(2\pi k)^{-1/2k}$. In fact, since $k^{-1/2k}$ decreases to 1, the ratio is bounded by a constant. In other words, we have graphs for all n whose leading order

behavior satisfies $\sqrt{2b(G)} \leq \max\{b_2(G)\} \leq 2\sqrt{b(G)}$ and this ratio of $\sqrt{2}$ improves with increasing k .

Section 3.6: The k -dimensional Bandwidth of Integer Simplices

We end this chapter with the class of graphs that introduced us to the bandwidth problem in the following form: How can billiard balls be stacked so that the difference of the numbers on balls that are touching is minimized? West [32] has found some upper bounds for this question. This problem gives rise to the following class of graphs.

DEFINITION 3.23: The n -dimensional integer simplex with parameter j , denoted by $I(n, j)$, has

vertex set $\{(x_1, x_2, \dots, x_{n+1}) : 0 \leq x_i \leq j-1, \sum_{i=1}^{n+1} x_i = j-1\}$. Suppose

$X = (x_1, x_2, \dots, x_{n+1})$ and $Y = (y_1, y_2, \dots, y_{n+1})$. X is adjacent to Y if there exist integers p and q with $p \neq q$ such that $x_i = y_i$ if $i \neq p, q$, $x_p = y_p + 1$, and $x_q = y_q - 1$.

The graph $I(2, 7)$ is shown in figure 3.2. We have the following upper bounds on the bandwidth of $I(2, j)$ and on the two-dimensional bandwidth of $I(3, j)$.

THEOREM 3.24: $b(I(2, j)) \leq j$.

PROOF: We can think of $I(2, j)$ as consisting of j rows of vertices such that the r th row contains

r vertices. Let $N_r = \frac{r(r+1)}{2}$. N_r is the total number of vertices in the first r rows of $I(2, j)$.

Label the vertices in the r th row of $I(2, j)$ from left to right with the labels

$1 + N_{r-1}, 2 + N_{r-1}, \dots, N_r$. The maximum difference of the labels in rows r and $r-1$ is

$N_r - N_{r-1} = r$. Therefore, $b(I(2, j)) \leq j$ $\square$

THEOREM 3.25: $b_2(I(3, j)) \leq j-1$ for all $j \geq 3$.

PROOF: We can think of $I(3, j)$ as $I(2, 1)$ stacked on top of $I(2, 2)$, ..., stacked on top of $I(2, j)$.

Each vertex in $I(2, r-1)$ determines a triangle in $I(2, r)$ and is adjacent to each vertex in that

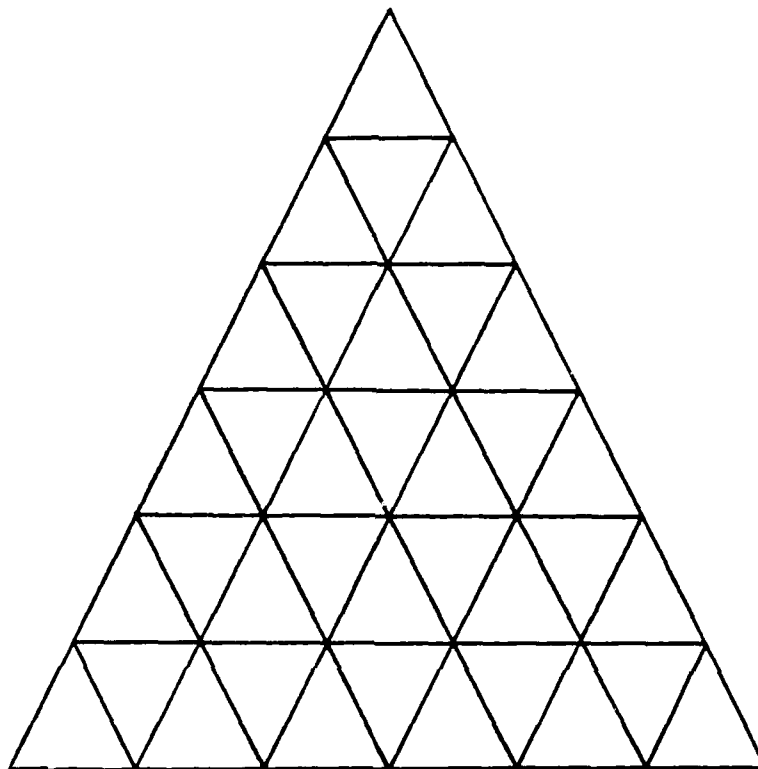


Figure 3.2: The two-dimensional integer simplex $I(2, 7)$.

triangle. We will label $I(3, j)$ keeping these facts in mind.

For $j = 3$, label the vertex in $I(2, 1)$ with the label $(0, 2)$. Label the vertices in $I(2, 2)$ with the labels $(2, 2)$, $(1, 1)$, and $(1, 2)$, starting in the first row and labeling each row from left to right. Similarly, label the vertices in $I(2, 3)$ with the labels $(4, 2)$, $(3, 1)$, $(3, 2)$, $(2, 0)$, $(2, 1)$, and $(2, 3)$. It is clear that no two adjacent vertices have labels that differ by more than 2.

Suppose $j \geq 4$. For $1 \leq i \leq j$ and $1 \leq r \leq j - 2$, label the r th row of $I(2, i)$ from left to right using the following labels $((i - 1)(j - 2) - r + 1, 0)$, $((i - 1)(j - 2) - r + 1, 1)$, ..., $((i - 1)(j - 2) - r + 1, r - 1)$. For $i = j - 1, j$, label the $(j - 1)$ st row of $I(2, i)$ from left to right with the following labels $((i - 1)(j - 2) - j + 2, 1)$, $((i - 1)(j - 2) - j + 2, 2)$, ..., $((i - 1)(j - 2) - j + 2, j - 1)$. Label the j th row of $I(2, j)$ from left to right with the labels $((j - 1)(j - 2) - j + 1, 2)$, $((j - 1)(j - 2) - j + 1, 3)$, ..., $((j - 1)(j - 2) - j + 1, j + 1)$.

For $1 \leq i \leq j - 2$, there is clearly no overlap in the first coordinates of the labels on the two-dimensional simplices since each contains at most $j - 2$ rows. The first row of $I(2, j - 2)$ and the last row of $I(2, j - 1)$ have labels with the same first coordinate, but the second coordinates differ. The first two rows of $I(2, j - 1)$ and the last two rows of $I(2, j)$ have labels that share the same first coordinates, but the second coordinates are different. Therefore, no label is repeated in our construction.

Within $I(2, i)$ it is clear that the maximum difference between adjacent labels is three. For edges between $I(2, i)$ and $I(2, i + 1)$, the first coordinates of the labels differ by $j - 2$ or $j - 3$. If the first coordinates differ by $j - 2$, then the second coordinates differ by at most one. If the first coordinates differ by $j - 3$, then the second coordinates differ by at most 2. Therefore,

$$b_2(I(3, j)) \leq j - 1. \quad \square$$

We end this chapter with the following question. Is $b_k(I(k + 1, j)) \leq j - k + 1$ for $j \geq k + 1$?

CHAPTER 4

GENERALIZED CHROMATIC NUMBERS OF GRAPHS

Section 4.1: Introduction

One of the most intensively studied parameters associated with a graph is its chromatic number, which is defined as follows.

DEFINITION 4.1: A k -coloring of a graph G is an assignment of k colors to the vertices of G . A k -coloring is *proper* if no two adjacent vertices receive the same color. The *chromatic number* of a graph G , denoted by $\chi(G)$, is the smallest k such that there is a proper k -coloring of G . A *color class* for a color c is the set of vertices that receive color c .

The chromatic number is such a natural and important concept that many generalizations of it have been considered.

Stahl [28] has defined a proper n -tuple coloring of G as an assignment of n distinct colors to each vertex of G such that no two adjacent vertices share a color. The n th chromatic number of G , denoted by $\chi_n(G)$, is the smallest number of colors needed to give G a proper n -tuple coloring. He has derived a lower bound on χ_n in terms of χ_{n-1} and has determined χ_n for several classes of graphs.

Albertson and Berman [2] and Erdos, Rubin and Taylor [15] have considered list colorings, which can be described as follows. For every vertex v in G , let $L(v)$ be a list of colors. A proper list coloring of G is a function f that assigns to v one of the colors in $L(v)$ such that no two adjacent vertices receive the same color. The list chromatic number of G is the smallest l such that there is a proper list coloring of G from any collection of lists having $|L(v)| = l$ for all vertices.

Matula [23] has yet another generalization which we will discuss later.

Akiyama [1] and Johns and Saba [22] have independently considered two related generalizations which we study further in this chapter.

The definition of chromatic number requires that each color class induce the empty graph. We can think of this in two ways: (1) every component of the graph induced by a color class must be a single vertex or (2) the two-path, P_2 , is forbidden as a subgraph of the graph induced by a color class. Using these two ideas, we can generalize the concept of chromatic number of a graph in the following two ways.

DEFINITION 4.2: Let G and H be graphs. An H -required k -coloring of G is an assignment of k colors to the vertices of G such that every component of the graph induced by a color class is a subgraph of H . The H -required chromatic number of a graph G , denoted by $\chi_R(G; H)$, is the smallest k such that an H -required k -coloring of G exists.

DEFINITION 4.3: Let G and H be graphs. An H -forbidden k -coloring of G is an assignment of k colors to the vertices of G such that H is not a subgraph of the graph induced by a color class. The H -forbidden chromatic number of G , denoted by $\chi_F(G; H)$, is the smallest k such that an H -forbidden k -coloring of G exists.

We can generalize these definitions to families of graphs as follows.

DEFINITION 4.4: Let $\mathbf{H}$ be a family of graphs. An $\mathbf{H}$ -required k -coloring of G is an assignment of k colors to the vertices of G such that every component of the graph induced by a color class is a member of $\mathbf{H}$. The $\mathbf{H}$ -required chromatic number of G is the smallest k such that an $\mathbf{H}$ -required k -coloring of G exists.

DEFINITION 4.5: Let $\mathbf{H}$ be a family of graphs. An $\mathbf{H}$ -forbidden k -coloring of G is an assignment of k colors to the vertices of G such that no member of $\mathbf{H}$ is a subgraph of the graph induced by any color class. The $\mathbf{H}$ -forbidden chromatic number of G is the smallest k such that an $\mathbf{H}$ -forbidden k -coloring of G exists.

Akiyama [1] has considered path and star-required chromatic numbers and has obtained upper bounds for outerplanar graphs. He also derived an upper bound for the P_2 -required chromatic number of any graph. We will restrict our attention to the classes of paths and stars as our forbidden or required configurations.

Notice the following facts. If H_1 does not contain H_2 , then every H_1 -required coloring is an H_2 -forbidden coloring and $\chi_F(G : H_2) \leq \chi_R(G : H_1)$ for all G . If every connected graph not containing H_2 is contained in H_1 , then every H_2 -forbidden coloring is an H_1 -required coloring and $\chi_R(G : H_1) \leq \chi_F(G : H_2)$ for all G . In particular, $\chi_R(G : P_2) = \chi_F(G : P_3)$ and $\chi(G) = \chi_F(G : P_2)$, but in general $\chi_R(G : P_3) \geq \chi_F(G : P_4)$. Also note $\chi_F(G : H) \leq \chi(G)$ and $\chi_R(G : H) \leq \chi(G)$, the former assuming that H has at least two vertices. Furthermore, since $P_2 = S_2$ and $P_3 = S_3$, results about small paths and stars coincide. Finally, $\chi_R(G : H_1) \geq \chi_R(G : H_2)$ and $\chi_F(G : H_1) \geq \chi_F(G : H_2)$ if H_1 is a subgraph of H_2 .

In this chapter we will consider several famous theorems on the chromatic number of a graph and obtain extensions to generalized chromatic numbers. In particular, we seek analogues for generalized chromatic numbers of bounds on $\chi(G)$ in terms of other graph parameters. Then we will find the generalized chromatic numbers for several classes of graphs. This will include consideration of the behavior of generalized chromatic numbers under Cartesian product, which is analogous to that of chromatic number, but more complicated.

Section 4.2: Triangle-free Graphs with Large Chromatic Number

In any ordinary proper coloring of a graph the vertices of a clique must all be assigned different colors. Thus, a graph with a large clique must have a large chromatic number. It is well-known that there are graphs with no large clique but arbitrarily large chromatic number.

THEOREM 4.6 (Mycielski): For any positive integer k , there exists a triangle-free graph G_k with $\chi(G_k) = k$.

Mycielski [7, 25] proved this theorem by using the following construction. Let $G_1 = K_1$ and $G_2 = K_2$. Suppose G_k has been constructed and contains m vertices $v_1, v_2, \dots, v_m$. Form G_{k+1} from G_k by adding $m + 1$ vertices $u_1, u_2, \dots, u_m$ and v and joining u_i to v and to the neighbors of v_i .

Our first result shows that this theorem can be extended to the S_n -required chromatic number, using a different construction.

THEOREM 4.7: Let n and k be positive integers. Then there exists a triangle-free graph $G_{n,k}$ such that $\chi_R(G_{n,k}; S_n) = k$.

PROOF: Let n be a fixed positive integer. Let $G_{n,1} = K_1$. Clearly, $G_{n,1}$ contains no triangles and $\chi_R(G_{n,1}; S_n) = 1$. We provide a construction for $G_{n,k}$ by induction on k .

Suppose $G_{n,1}, G_{n,2}, \dots, G_{n,k-1}$ have been constructed. Let v_i be the number of vertices in $G_{n,i}$. Let $c = (v_1 v_2 \dots v_{k-1})^n$. Let V be a set of c vertices. We will construct $G_{n,k}$ as follows: Take n copies of each of $G_{n,1}, G_{n,2}, \dots, G_{n,k-1}$. Each element of V corresponds to a selection of one vertex from each of the $n(k-1)$ graphs above. $G_{n,k}$ is obtained from V and the n copies of $G_{n,1}, G_{n,2}, \dots, G_{n,k-1}$ by joining each vertex in V to the $n(k-1)$ vertices to which it corresponds. For example, note that $G_{n,2} = S_{n+1}$, but beyond that the graphs are very complicated.

By induction hypothesis, none of $G_{n,1}, G_{n,2}, \dots, G_{n,k-1}$ contains a triangle. There are no edges between copies of $G_{n,1}, G_{n,2}, \dots, G_{n,k-1}$, and each vertex in V has only one edge to each copy of $G_{n,1}, G_{n,2}, \dots, G_{n,k-1}$; hence $G_{n,k}$ is also triangle-free.

$G_{n,k}$ is k -colorable in the following manner. Color the copies of $G_{n,1}, G_{n,2}, \dots, G_{n,k-1}$ with $k-1$ colors. Color each vertex in V with color k . Since none of $G_{n,1}, G_{n,2}, \dots, G_{n,k-1}$ contains a monochromatic S_n , neither does $G_{n,k}$.

It remains only to show that $G_{n,k}$ is not $(k-1)$ -colorable. For $1 \leq i \leq k-1$, designate the copies of $G_{n,i}$ in $G_{n,k}$ as $G_{n,i}^1, G_{n,i}^2, \dots, G_{n,i}^n$. Consider a legal coloring; for each $j = 1, 2, \dots, n$, we will select vertices $x'_1, x'_2, \dots, x'_{k-1}$ from $G_{n,1}^j, G_{n,2}^j, \dots, G_{n,k-1}^j$ having distinct colors. There is

a vertex x'_1 in $G'_{n,1}$ of some color i_1 . Since $G'_{n,2}$ requires two colors, there is a vertex x'_2 in $G'_{n,2}$ of some color $i_2 \neq i_1$. Having selected $x'_1, x'_2, \dots, x'_{l-1}$ in this manner, the fact that $G'_{n,l}$ requires l colors implies that there is a vertex x'_l in $G'_{n,l}$ having some color $i_l \neq i_1, i_2, \dots, i_{l-1}$. Continuing this process we get vertices $x'_1, x'_2, \dots, x'_l$ such that all of the colors $1, 2, \dots, k-1$ are used on them. Having selected vertices for $j = 1, 2, \dots, n$, there is a vertex y in V such that y is adjacent to all these x'_j . In particular, y is adjacent to n vertices of color i , for $1 \leq i \leq k-1$. Therefore, y can get none of the colors $1, 2, \dots, k-1$. This gives $\chi_R(G_{n,k}; S_n) \geq k$. $\square$

If we let $n = 1$ in the construction given in the last theorem, we get a different construction from that used by Mycielski. Our resulting graph $G_{1,k}$ has more vertices than his k -chromatic triangle-free graph.

Section 4.3: Degree Bounds on Chromatic Number

We now want to consider upper bounds on chromatic numbers. Introductory courses in graph theory often begin discussion of chromatic number by bounding it in terms of maximum degree. By coloring $v_1, v_2, \dots, v_n$ "greedily" (i. e. assigning each vertex the least color not used on earlier neighbors), we obtain $\chi(G) \leq \Delta + 1$.

THEOREM 4.8: For any graph G with maximum degree Δ , $\chi(G) \leq \Delta + 1$.

Akiyama [1] gave a nice extension of this to $\chi_R(G; P_2)$. Recall that this equals $\chi_F(G; S_3)$; we extend his proof to obtain the analogous bound for $\chi_F(G; S_{k+1})$.

THEOREM 4.9: If G is an r -regular graph, then $\chi_F(G; S_{k+1}) \leq \lceil (r+1)/k \rceil$.

PROOF: Given any $\lceil (r+1)/k \rceil$ -coloring of G , let m_i be the number of edges in the graph induced by the i th color class. Let $m = \sum_{i=1}^{\lceil (r+1)/k \rceil} m_i$. Choose a coloring that minimizes m . If this coloring is

not S_{k+1} -forbidden, we may assume v is a vertex in the i th color class with at least k neighbors in

the i th color class. Then v has at most $r - k$ other neighbors among the $\lfloor (r + 1)/k \rfloor - 1$ other color classes. Since $\frac{r - k}{\lfloor (r - k + 1)/k \rfloor} < k$, there is some color j whose color class contains at most $k - 1$ neighbors of v . If we recolor v with color j , then we get an assignment of colors to the vertices that reduces m by at least one. This contradicts the choice of the coloring. Therefore, a coloring minimizing m is an S_{k+1} -forbidden $\lfloor (r + 1)/k \rfloor$ -coloring.

The next result follows immediately from Akiyama's result, because every graph with maximum degree Δ has a Δ -regular supergraph. $\square$

COROLLARY 4.10: If G is any graph with maximum degree Δ , then $\chi_F(G : S_{k+1}) \leq \lfloor (\Delta + 1)/k \rfloor$.

Brooks [7, 8] characterized the graphs that achieve this upper bound for $k = 1$.

THEOREM 4.11 (Brooks): If G is a connected, simple graph with maximum degree Δ and G is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta$.

For all k , the bound of Corollary 4.10 is achieved for complete graphs and cycles. Indeed since all edges are available, $\chi_F(K_{\Delta+1} : H) \geq \lfloor (\Delta + 1)/k \rfloor$ for any H with $k + 1$ vertices. We want to know whether there are other graphs that achieve the upper bound. The answer to this question is yes. Theorem 4.13 will prove this for $k = 2$. Before presenting this, we note that we do not have a characterization of the extremal graphs. However, there is a concept of generalized chromatic number for which an analogue of Brooks' Theorem has been found.

Let A be a subset of the vertex set of G . A is an i -independent set of G if all subgraphs of the subgraph induced by A have edge-connectivity at most i . $\chi_i(G)$ is the size of a minimum cover of $V(G)$ by i -independent sets. $\chi_0(G)$ is the chromatic number of G . Matula [23] has shown the following extension of Brooks' Theorem.

THEOREM 4.12 (Matula): Let $\sigma(G)$ be the largest edge-connectivity of the subgraphs of G . Let $i \geq 0$ and let G be any connected graph with maximum degree Δ , other than a complete graph K_n ,

with $n = k(i + 1) + 1$, an $(i + 1)$ -regular graph with $\sigma(G) = i + 1$, or an odd cycle. Then $\chi_i(G) \leq \lfloor \Delta/(i + 1) \rfloor$.

THEOREM 4.13: Suppose $\Delta \geq 2$. Then there exists a graph G with maximum degree Δ such that

$$\chi_R(G : P_2) = \lfloor (\Delta + 1)/2 \rfloor = \lfloor \Delta/2 \rfloor + 1.$$

PROOF: Suppose Δ is even. Let $G = K_{\Delta+2} - M$, where M is any complete matching on these $\Delta + 2$ vertices. Note that G is Δ -regular.

Suppose we can color G in $\Delta/2$ colors. Then some color class must contain at least three vertices. Suppose x_i, x_j , and x_l are vertices of G that receive the same color. Since we removed only one edge from each vertex in K_n , these three vertices must induce P_3 or C_3 . In either case, we get a forbidden monochromatic configuration. Therefore, we cannot color G in $\Delta/2$ colors.

If Δ is odd, let G_Δ be any graph with maximum degree Δ that contains $K_{\Delta+1} - M$. Then $\lfloor \Delta/2 \rfloor + 1 \geq \chi_R(G_\Delta : P_2) \geq \chi_R(G_{\Delta-1} : P_2) = \lfloor (\Delta - 1)/2 \rfloor + 1 = \lfloor \Delta/2 \rfloor + 1. \quad \square$

It is not immediately obvious how to extend this construction to $\chi_F(G : S_{k+1})$. For example, in seeking a six-regular graph with S_4 -forbidden chromatic number $\lfloor (6 + 1)/3 \rfloor = 3$, one might consider $K_8 - M$. However, $\chi_F(K_8 - M : S_4) = 2$, by two classes of equal size that induce four-cycles.

Section 4.4: Towards an Analogue of Brooks' Theorem

Since the class of graphs described in Theorem 4.13 is easy to describe, we want to know if we can obtain a result similar to Brooks' Theorem for the P_2 -required chromatic number. In other words, can we find a class of graphs C such that if G is not a member of C , then

$\chi_R(G : P_2) \leq \lfloor \Delta/2 \rfloor$? The next theorem shows that this class must be quite large. If $\Delta = 2$ or 3 , then it is clear that any graph with maximum degree Δ requires 2 colors and the upper bound is met. For $\Delta = 4$, the following theorem shows that there are other classes of graphs that require $\lfloor \Delta/2 \rfloor + 1$ colors.

To prove the following theorem, we will give two different constructions.

THEOREM 4.14: For every $n \geq 5$, there exists a connected 4-regular graph G_n such that G_n contains n vertices and $\chi_R(G_n; P_2) = 3$.

PROOF: We know that for $n = 5$, the complete graph K_5 requires three colors. For $n = 6$, we know that K_6 with a matching removed requires 3 colors. Therefore, we will assume that $n \geq 7$.

Suppose $n = 2j + 1$, where $j \geq 3$. The vertices of G_n will be called

$$A, B, C, D, E, x_1, x_2, \dots, x_{j-2}, y_1, y_2, \dots, y_{j-2}.$$

We define the following edges, where $2 \leq i \leq j - 3$.

VERTEX	ADJACENT VERTICES
--------	-------------------

A	B, C, x_1, y_1
B	A, C, D, x_1
C	A, B, E, y_1
D	B, E, x_{j-2}, y_{j-2}
E	C, D, x_{j-2}, y_{j-2}
x_1	A, B, x_2, y_2
y_1	A, C, x_2, y_2
x_{j-2}	D, E, x_{j-3}, y_{j-3}
y_{j-2}	D, E, x_{j-3}, y_{j-3}
x_i	$x_{i-1}, x_{i+1}, y_{i-1}, y_{i+1}$
y_i	$x_{i-1}, x_{i+1}, y_{i-1}, y_{i+1}$

The graph G_{13} is shown in Figure 4.1.

Suppose we can color G_n with two colors. Consider the vertices A, x_1 , and y_1 . They cannot all receive the same color or we would have a monochromatic P_3 . Suppose that A and x_1 both receive color 1. Then B, C , and y_1 all must receive color 2. This gives a monochromatic P_3 . By symmetry, we must color A with color 1 and x_1 and y_1 with color 2. Now both x_2 and y_2 must receive color 1, since x_1 and y_1 are adjacent to both of them. Continuing this process, it follows that x_i and y_i both receive color 1 when i is even and color 2 when i is odd. In particular, x_{j-2} and y_{j-2} both receive color 1 when j is even and color 2 when j is odd. If j is odd, then A, D , and E all receive color 1. Then both B and C receive color 2 and we have a monochromatic P_3 induced by B, C , and y_1 . If j is even, then D and E both receive color 2. Therefore, both B and C are

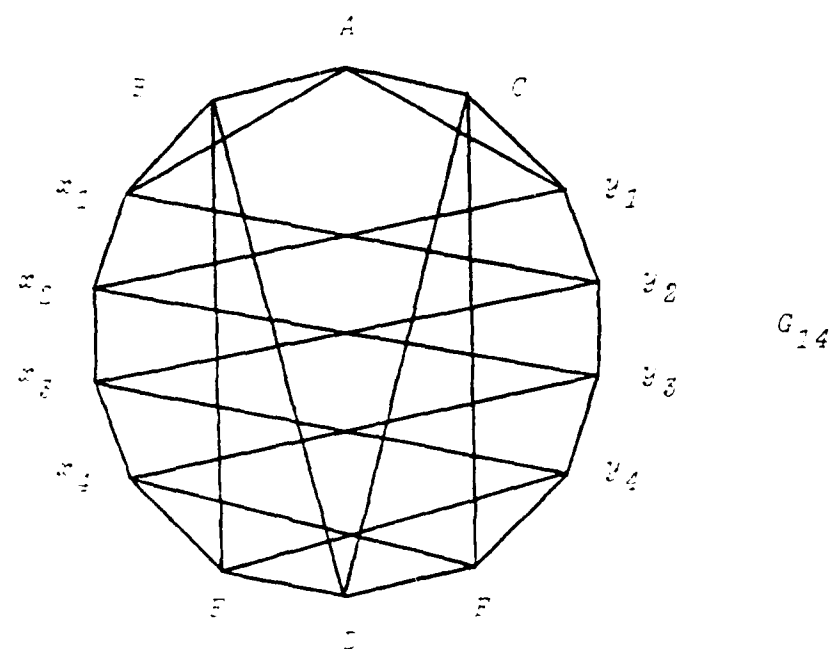
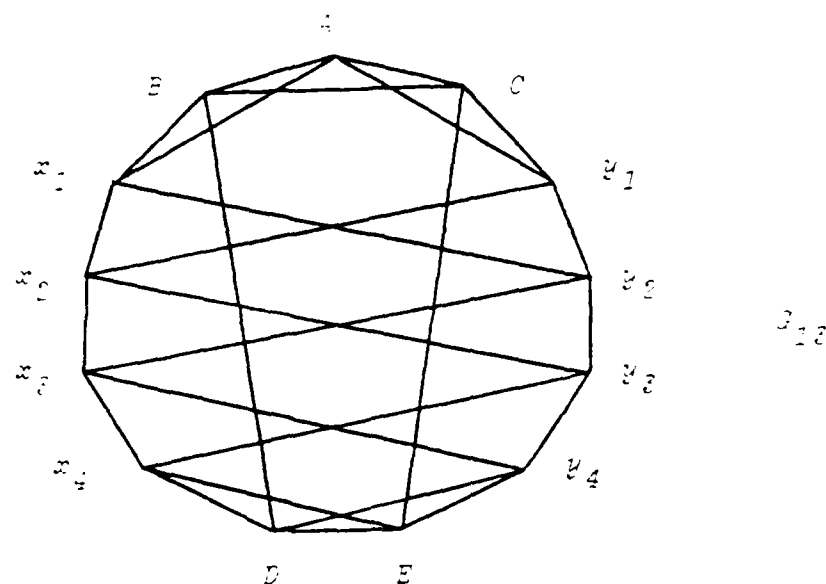


Figure 4.1: The graphs G_{13} and G_{14} of Theorem 4.14.

colored 1 and we have a monochromatic C_3 induced by A , B , and C . This implies that

$\chi_R(G_n; P_2) \geq 3$. We know $\chi_R(G_n; P_2) \leq 3$ by Theorem 4.9.

Suppose $n = 2j$ where $j \geq 4$. The vertices of G_n will be called

$$A, B, C, D, E, F, x_1, x_2, \dots, x_{j-3}, y_1, y_2, \dots, y_{j-3}.$$

We define the following edges, where $2 \leq i \leq j-4$.

VERTEX	ADJACENT VERTICES
A	B, C, x_1, y_1
B	A, D, E, x_1
C	A, D, F, y_1
D	B, C, E, F
E	B, D, x_{j-3}, y_{j-3}
F	C, D, x_{j-3}, y_{j-3}
x_1	A, B, x_2, y_2
y_1	A, C, x_2, y_2
x_{j-3}, y_{j-3}	E, F, x_{j-4}, y_{j-4}
x_i, y_i	$x_{i-1}, x_{i+1}, y_{i-1}, y_{i+1}$

The graph G_{14} is shown in Figure 4.1.

Suppose we can color G_n with two colors. As before, not all of A , x_1 , and y_1 can receive the same color. If A and x_1 both receive the same color, then C , y_1 , and y_2 must get the other color, yielding a monochromatic P_3 . If x_1 and y_1 both are colored with color 2, then x_i and y_i both receive color 1 when i is even and color 2 when i is odd. Then E and F both receive color 1 when j is even and color 2 when j is odd. This implies that D is colored 2 when j is even and 1 when j is odd. When j is even A , E , and F all receive color 1, which implies that B and C both are colored 2. But D is colored 2 in this case and we get a monochromatic P_3 induced by B , C , and D . If j is odd, E and F both receive color 2 and A and D both receive color 1. Then both B and C receive color 2, which gives the monochromatic P_3 induced by F , C , and y_1 . Therefore, we cannot color G_n with 2 colors. We know that $\chi_R(G_n; P_2) \leq 3$ by Theorem 4.9. $\square$

We present an additional construction, due to its simplicity.

ALTERNATE PROOF: When n is not a multiple of four the following construction also requires three colors. The vertices of G_n will be called $x_0, x_1, \dots, x_{n-1}$. The adjacencies are as

follows: x_i is adjacent to x_{i+1} and x_{i+2} , where the addition is performed modulo n . G_n consists of an n -cycle with chords between vertices that are at distance two along the cycle. Call this graph C_n^2 .

Suppose that we can color C_n^2 with two colors. If we alternate the colors on the outer cycle, then x_0, x_2 , and x_4 all receive the same color and this induces a monochromatic P_3 . Therefore, we can assume that x_0 and x_1 both receive color 1. This implies that x_2 and x_3 both receive color 2. This can be continued indefinitely around the cycle in the pattern 11221122... if and only if n is a multiple of four. Hence, C_n^2 requires three colors if n is not a multiple of four, but has a P_2 -required two-coloring if n is a multiple of four.

When n is a multiple of four, but $n \neq 8$, define G as an n -cycle on $x_0, x_1, \dots, x_{n-1}$ as before, together with the edges x_i, x_{i+2} when i is even, but x_i, x_{i+4} when i is odd. ($n \neq 8$ is required for four-regularity.) The chords of length two force the coloring 11221122... as before, but then every fourth vertex receives the same color and the chords x_i, x_{i+4} yield a large monochromatic subgraph.

Finally, when $n = 8$, we may choose from several four-regular graphs with $\chi_R(G; P_2) = 3$, such as G_3 from the previous construction. $\square$

We know of three four-regular graphs G_1, G_2 , and G_3 on eight vertices with $\chi_R(G_n; P_2) = 3$. These are shown in Figure 4.2. We conjecture these are the only such graphs. Note that G_3 is the graph presented earlier, and that G_1, G_2 , and G_3 are distinct graphs.

The construction of small graphs with large chromatic number suggests another question, that of "critical" graphs for values of relaxed chromatic numbers. G is k -critical if G is connected, $\chi(G) = k$, and deletion of any edge reduces $\chi(G)$. The only two-critical graph is P_2 , and the three-critical graphs are the odd cycles. The k -clique is k -critical. Similarly, we say that G is H -required k -critical (or H -forbidden k -critical) if $\chi_R(G; H) = k$ ($\chi_F(G; H) = k$) and deletion of any edge reduces $\chi_R(G; H)$ ($\chi_F(G; H)$). Although we have not found P_2 -required k -critical graphs for all k , we can say the following:

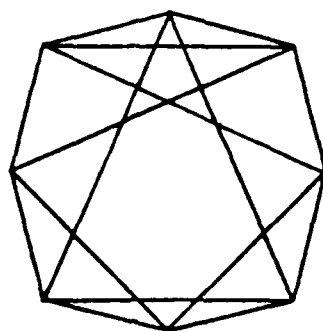
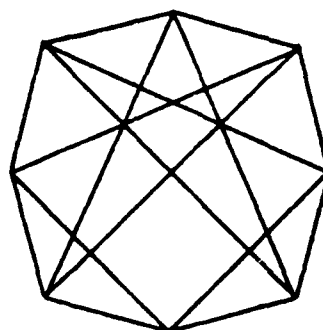
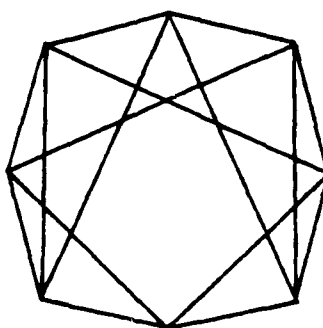
 G_1  G_2  G_3

Figure 4.2: The graphs G_1 , G_2 , and G_3 .

THEOREM 4.15: The only P_2 -required two-critical graph is P_3 . If n is an odd multiple of two, let G_n be the graph on vertices $x_0, x_1, \dots, x_{n-1}$ with edges x_i, x_{i+1} for all i and x_i, x_{i+2} for even values of i . Then G_n is P_2 -required three-critical. However, these are not the only P_2 -required three-critical graphs.

PROOF: $\chi_R(G : P_2) \geq 2$ if and only if G contains P_3 , but $\chi_R(P_3 - e : P_2) = 1$ for any edge e in P_3 . Now consider G_n . If G_n has a P_2 -required two-coloring the colors cannot alternate, because this yields a monochromatic $C_{n/2}$. If two consecutive vertices x_i and x_{i+1} get the same color, we may assume x_1 and x_2 get color 1. Then avoiding P_3 implies x_3 and x_4 get color 2, etc. Since n is an odd multiple of two, this pattern yields a contradiction of x_{n-1} and x_0 getting color 1. Hence, $\chi_R(G_n : P_2) \geq 3$. We can achieve this by two-coloring the cycle on the even vertices, and using the third color on the odd vertices.

Now, there are two types of edge-deletions to consider to show criticality: (1) deletion of an edge on the n -cycle and (2) deletion of an edge on the $n/2$ -cycle generated by the vertices with even index. For (1), we can assume that we remove the edge x_{n-1}, x_0 . Starting with x_0 , color the vertices 1212211221122...221. For (2), we can assume we remove the edge x_0, x_2 . Starting with x_0 , color the vertices 21211221122...112. These are both legal two-colorings, so G_n is P_2 -required three-critical.

Every P_2 -required three-chromatic graph must contain a P_2 -required three-critical graph. The graph G_1 , of Figure 4.2 contains the six-vertex critical graph just described. However, neither G_2 nor G_3 contains this graph, nor is either critical. Hence, they contain some other as-yet undetermined P_2 -required three-critical subgraph. $\square$

Interesting problems that remain here are the characterization of P_2 -required three-critical graphs, the discovery of P_2 -required k -critical graphs, and the discovery of H -required or H -forbidden three-critical graphs for other H .

For $\Delta = 5$ any graph containing one of the graphs described in Theorem 4.14 requires 3 colors. We end this section with one more graph that requires $\lfloor \Delta/2 \rfloor + 1$ colors. Consider the graph H_{10} shown in Figure 4.3. H_{10} is six-regular and our next theorem will show that it requires four colors.

THEOREM 4.16: $\chi_R(H_{10}; P_2) = 4$

PROOF: Suppose that we can color H_{10} with three colors. Then some color class must contain at least four vertices. We will show there can be no such class.

Suppose A is contained in a color class with three other vertices. At most one of E, F, G, H, I, and J can receive the same color as A. If one of them is the same color as A, then at most one of B, C, and D can receive that same color. Hence, if A is to appear in a color class with three other vertices, then the class must contain B, C, and D, but this produces a monochromatic C_3 . Therefore, A cannot appear in a color class with three other vertices.

Suppose that B is contained in a color class with three other vertices. Then we know that A cannot be in that class and at most one of C and D can receive the same color as B. At most one of E, F, G, H, and I can receive the same color as B. Therefore, J must be in the same color class as B. Both J and B are adjacent to C; therefore, C cannot be in the color class with them. Therefore, our color class must contain B, D, J, and one of $\{E, F, G, H, I\}$. But each of $\{E, F, G, H, I\}$ is adjacent to a vertex of the edge DJ, so none of them can get this color, and we see that B cannot appear in the same color class as three other vertices. Similarly, D cannot appear in a color class containing four vertices.

If C is contained in a color class containing three other vertices, then those three vertices must be among E, F, G, H, I, and J. This implies that at least two of E, G, H, and J must receive the same color as C. But C is adjacent to all four of these vertices and this would produce a monochromatic P_3 . Therefore, C cannot belong to a color class of size four.

Using the facts established above we see that our color class of size four must contain four of the vertices E, F, G, H, I, and J. Suppose that F appears in the class. Then at most one of G, I, and

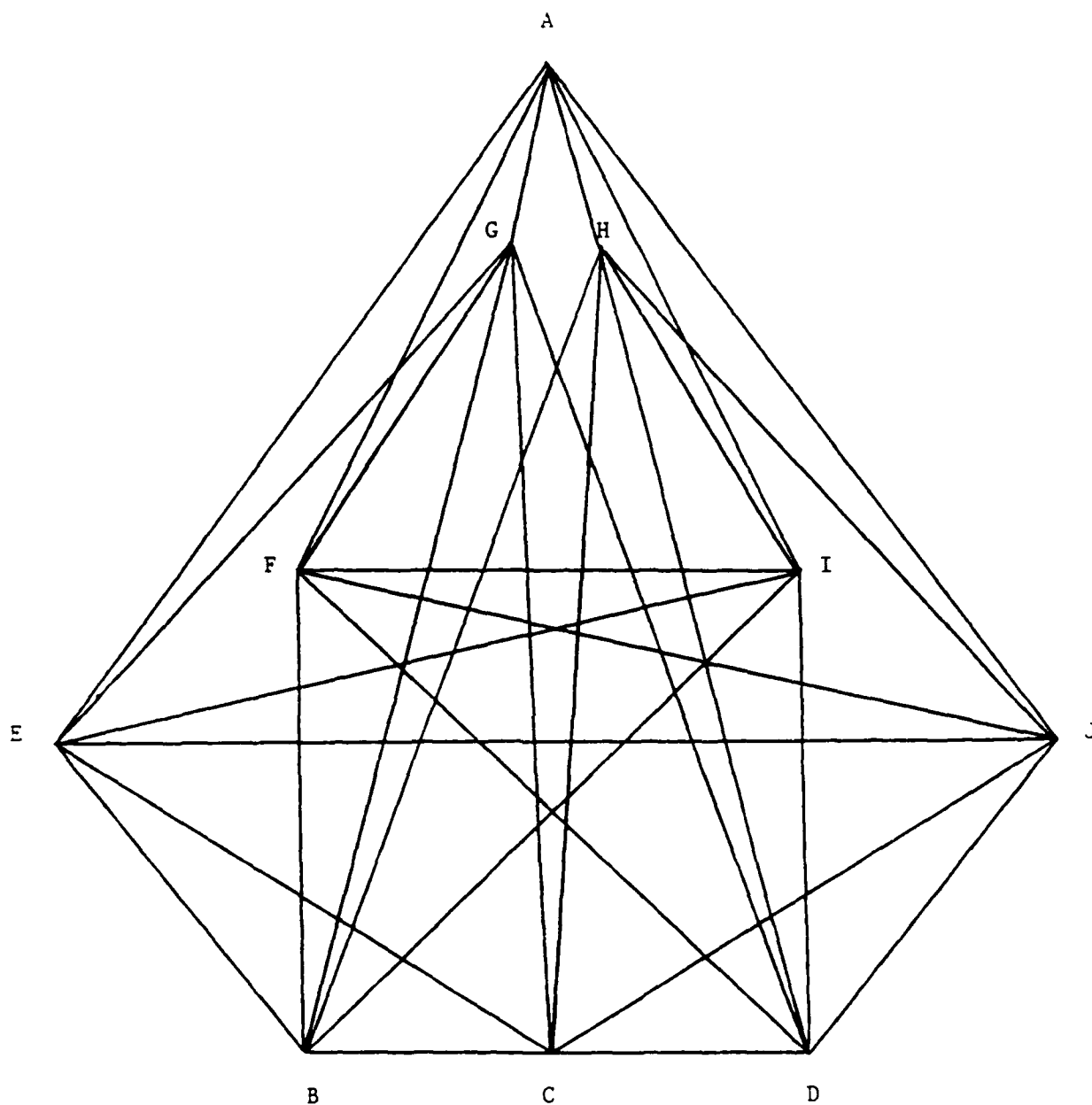


Figure 4.3: The graph H_{10} of Theorem 4.16.

J can receive the same color since F is adjacent to all three of them. This implies that E and H must be the same color as F. But G, I, and J are all common neighbors of E and F, so we cannot add any of them. Therefore, F cannot be the same color as three other vertices. By symmetry I cannot be in a color class containing three other vertices. This implies that our color class of size four must contain the vertices E, G, H, and J. But this produces a monochromatic P_4 . Therefore, no color class can contain four vertices and $\chi_R(H_{10}; P_2) = 4$. $\square$

In view of Theorems 4.14 and 4.16 it seems unlikely that a Brooks'-like theorem exists for the P_2 -required chromatic number.

Section 4.5: Colorings of Complete Multipartite Graphs

We will now find the generalized chromatic numbers for several classes of graphs. The first graphs we wish to consider are the complete multipartite graphs. Throughout this discussion, we will write a complete multipartite graph as $K_{n_1, n_2, \dots, n_p}$ and assume that $n_1 \leq n_2 \leq \dots \leq n_p$.

Define $N_i = \sum_{j=1}^i n_j$. It is easy to show that $\chi(K_{n_1, n_2, \dots, n_p}) = p$. The lower bound holds since K_p is a subgraph of $K_{n_1, n_2, \dots, n_p}$. The upper bound is obtained by assigning color i to every vertex in the i th part.

We now want to consider the generalized chromatic numbers of the complete multipartite graphs. We begin by finding the star-required chromatic number.

THEOREM 4.17: Let $G = K_{n_1, n_2, \dots, n_p}$. Let s be the largest integer such that $n_s \leq m - 1$. $s = 0$ if $n_1 \geq m$. Let t be the smallest integer such that $N_t > s - t$. Then

$$\chi_R(G; S_m) = p - t + 1$$

PROOF: Case 1: $s = 0$

If $s = 0$ then $t = 1$ and $n_i \geq m$ for all i . In this case, we must show $\chi_R(G; S_m) = p$. The upper bound follows by giving the vertices of each part a single color.

For the lower bound, consider $H = K_{l_1, l_2, \dots, l_p}$, where $l_i = m$ for $1 \leq i \leq p$. Since $n_1 \geq m$, H is a subgraph of G . The number of vertices in H is $n = pm = m(p-1) + m$. If we color H with $p-1$ colors then some color class must contain $m+1$ vertices. Such a class must induce either S_{m+1} or a cycle. Since these configurations are forbidden,

$$\chi_R(G; S_m) \geq \chi_R(H; S_m) \geq p.$$

Case 2: $s > 0$

Since $s > 0$, then $N_s > 0 = s - s$ and therefore, $t \leq s$. Consider the following coloring scheme. Give each of the parts of sizes $n_1, n_{t+1}, \dots, n_p$ a single distinct color, using $p-t+1$ colors. Give each vertex in parts $1, 2, \dots, t-1$ a distinct color from the colors assigned to parts $t, t+1, \dots, s$. By definition of t we know that $N_{t-1} \leq s-t+1$, so this completes a coloring. For $1 \leq i \leq s-t+1$, we know that $n_{t-1+i} + 1 \leq m$. Since $n_s \leq m-1$, these non-trivial color classes induce stars with at most m vertices. Hence, $\chi_R(G; S_m) \leq p-t+1$.

$$\text{For the lower bound, let } H = K_{l_1, l_2, \dots, l_p}, \text{ where } l_i = \begin{cases} n_i & \text{if } i \leq t \\ n_t & \text{if } t \leq i \leq s \\ m & \text{if } i \geq s+1 \end{cases}$$

The number of vertices in H is $n = N_t + (s-t)n_t + (p-s)m$. No color class may contain more than m vertices or a monochromatic cycle or S_{m+1} will result. At most $p-s$ classes can contain m vertices. Suppose we can color H with $p-t$ colors. Then $s-t$ of these classes must account for $N_t + (s-t)n_t \geq s-t+1 + (s-t)n_t$ vertices. But then some color class must contain at least $n_t + 2$ vertices. This will produce a monochromatic cycle. Since H is a subgraph of G , $\chi_R(G; S_m) \geq p-t+1$. $\square$

Theorem 4.17 gives us the P_2 and P_3 -required chromatic numbers for the complete multipartite graph. We also have the next result immediately.

COROLLARY 4.18: If $m \geq 4$ and $G = K_{n_1, n_2, \dots, n_p}$, then $\chi_R(G; P_m) = \chi_R(G; S_3)$.

PROOF: Any set of four vertices in G must induce one of the following: the empty graph,

$K_{1,3}$, $K_{2,2}$, $K_{1,1,2}$, or K_4 . Therefore requiring P_m is the same as requiring S_3 . $\square$

We now want to consider the forbidden chromatic numbers of the complete multipartite graphs. Johns and Saba [22] proved the following theorem.

THEOREM 4.19: Let $G = K_{n_1, n_2, \dots, n_p}$. If $n_1 = n_2 = \dots = n_r = 1$, then $\chi_F(G : P_3) = p - \lfloor r/2 \rfloor$.

We will generalize this theorem to all complete multipartite graphs and to paths of any length. First, we need the following lemma.

LEMMA 4.20: Let $G = K_{n_1, n_2, \dots, n_p}$. Let $k = \max\{m, n_p + \lfloor m/2 \rfloor\}$. Then any induced subgraph on k vertices contains a copy of P_m .

PROOF: Let A be a set of k vertices and let A_i be its intersection with the i th part. If $|A_i| < \lfloor m/2 \rfloor$ for every i , number the vertices of A so that those from each A_i appear consecutively. Consider the vertices numbered $1, \lfloor m/2 \rfloor + 1, 2, \lfloor m/2 \rfloor + 2, \dots$ ending at m if m is even and $\lfloor m/2 \rfloor$ if m is odd. Consecutive vertices listed above have labels differing by at least $\lfloor m/2 \rfloor - 1$, which guarantees that they belong to different A_i 's. Therefore, they form a path of m vertices.

If there is an r such that $|A_r| \geq \lfloor m/2 \rfloor$, then we must have $n_p \geq \lfloor m/2 \rfloor$ and $k = n_p + \lfloor m/2 \rfloor$. Since $|A_r| \leq n_p$, then $|A - A_r| \geq \lfloor m/2 \rfloor$ and the subgraph of G induced by A contains a copy of P_m alternating between A_r and $A - A_r$. $\square$

Consider the following recursive algorithm, which we will show produces an optimal P_m -forbidden coloring of $G = K_{n_1, n_2, \dots, n_p}$.

COLMULT

- (1) If $n_p < \lfloor m/2 \rfloor$, choose $m - 1$ arbitrary vertices (if available) for one color class C , and color $G - C$ recursively. If G has fewer than $m - 1$ vertices, assign them all to the single class C .

- (2) If $n_p \geq \lfloor m/2 \rfloor$, choose $\lfloor m/2 \rfloor - 1$ vertices (if available) in order from the smallest parts.

Combine these with the largest part to obtain one color class C , and color $G - C$ recursively.

If G has fewer than $\lfloor m/2 \rfloor - 1$ vertices outside the largest part, color all of G with the single class C .

LEMMA 4.21: COLMULT gives a legal P_m -forbidden coloring of $G = K_{n_1, n_2, \dots, n_p}$. Furthermore, in applying COLMULT all applications of case (2) precede all applications of case (1).

PROOF: Any collection of at most $m - 1$ vertices contains no copy of P_m . Any collection of vertices with at most $\lfloor m/2 \rfloor - 1$ vertices outside a single part contains no copy of P_m . Finally, if $n_p < \lfloor m/2 \rfloor$, any deletion of vertices preserves this property, so no application of case (2) can follow an application of case (1). $\square$

THEOREM 4.22: Suppose $G = K_{n_1, n_2, \dots, n_p}$. Let $N = \sum_{i=1}^p n_i$, and $a_j = (\lfloor m/2 \rfloor - 1)(p - j)$. Let t

be the smallest integer such that $N_t > a_t$ and let s be the largest integer such that $n_s < \lfloor m/2 \rfloor$.

Then COLMULT colors G optimally with

$$\chi_F(G : P_m) = \begin{cases} \lfloor N/(m-1) \rfloor & \text{if } n_p < \lfloor m/2 \rfloor \\ p - t + 1 & \text{if } n_p \geq \lfloor m/2 \rfloor \text{ and } n_t \geq \lfloor m/2 \rfloor \\ p - s + \lfloor (N_s - a_s)/(m-1) \rfloor & \text{if } n_p \geq \lfloor m/2 \rfloor \text{ and } n_t < \lfloor m/2 \rfloor \end{cases}$$

PROOF: Case 1: $n_p < \lfloor m/2 \rfloor$

In COLMULT, classes of Type 1 are always chosen, yielding $\chi_F(G : P_m) \leq \lfloor N/(m-1) \rfloor$.

With a smaller number of classes in a vertex partition, some class must contain m vertices. By Lemma 4.20, such a class contains a copy of P_m .

Case 2: $n_p \geq \lfloor m/2 \rfloor$ and $n_t \geq \lfloor m/2 \rfloor$

The facts $n_t \geq \lfloor m/2 \rfloor$, $N_t > a_t$ and $N_{t-1} \leq a_{t-1}$ guarantee that the first $p - t$ applications of COLMULT are of Type 2, and that after this at most $\lfloor m/2 \rfloor - 1$ vertices remain uncolored outside

the t th part. Hence one additional application completes the coloring.

For the lower bound, let $H = K_{l_1, l_2, \dots, l_p}$, where $l_i = \begin{cases} n_i & \text{if } i \leq t \\ n_t & \text{if } i \geq t \end{cases}$.

The number of vertices in H is $n = N_t + (p - t)n_t \geq (\lfloor m/2 \rfloor - 1 + n_t)(p - t) + 1$. If we color H with $p - t$ colors then some color class must contain at least $\lfloor m/2 \rfloor + n_t$ vertices. By Lemma 4.20, the subgraph of H induced by such a class would contain a copy of P_m . Hence H requires at least $p - t + 1$ colors. Since H is a subgraph of G , the bound also holds for G .

Case 3: $n_p \geq \lfloor m/2 \rfloor$ and $n_t < \lfloor m/2 \rfloor$

By the definition of s , $n_{s+1} \geq \lfloor m/2 \rfloor > n_s$. Also, the condition on t implies that $s \geq t$, so $N_s \geq N_t > a_t \geq a_s$. Let $c = N_s - a_s$.

Together, $n_{s+1} \geq \lfloor m/2 \rfloor$ and $N_s > a_s$ imply that the first $p - s$ applications of COLMULT are of Type 2, coloring all vertices in parts $s + 1, s + 2, \dots, p$ and a_s vertices in parts $1, 2, \dots, s$. We have c vertices left to color, and $n_s < \lfloor m/2 \rfloor$ implies that COLMULT finishes the job with $\lfloor c/(m - 1) \rfloor$ additional colors.

For the lower bound, let $H = K_{l_1, l_2, \dots, l_p}$, where $l_i = \begin{cases} n_i & \text{if } i \leq s \\ \lfloor m/2 \rfloor & \text{if } i > s \end{cases}$.

Because $l_p = \lfloor m/2 \rfloor$, Lemma 4.20 implies that every set of m vertices induces a graph containing P_m . Hence we need only count the number of vertices in H and divide by $m - 1$ to get a lower bound. The number of vertices in H is $n = N_s + \lfloor m/2 \rfloor(p - s) = N_s - a_s + (m - 1)(p - s)$. Hence, H requires at least $p - s + \lfloor (N_s - a_s)/(m - 1) \rfloor$ colors. Since $n_{s+1} \geq \lfloor m/2 \rfloor$, H is a subgraph of G , and the lower bound applies for G also. $\square$

The question of finding the star-forbidden chromatic numbers of the complete multipartite graphs appears to be a much more difficult problem. For example, if we wish to forbid the six-star S_6 , then there are configurations other than an independent set which contain at least six vertices, but do not induce S_6 . Two such configurations in the complete multipartite graph are $K_{4,4}$ and

$K_{2,2,2}$. As the size of the star grows the number of these configurations grows also.

Section 4.6: Colorings of the Petersen Graph and its Generalizations

One of the most famous graphs in graph theory is the Petersen graph. It contains ten vertices $x_0, x_1, \dots, x_4$ and $y_0, y_1, \dots, y_4$. x_i is adjacent to x_{i+1} and y_i . y_i is adjacent to y_{i+2} where the addition is performed modulo five. Our next two classes of graphs will generalize the Petersen graph.

DEFINITION 4.23: Let n and k be integers satisfying $n \geq 3$, $1 \leq k \leq n-1$, and $2k \neq n$. The *generalized Petersen graph*, denoted by $P(n, k)$, has vertex set $x_0, x_1, \dots, x_{n-1}, y_0, y_1, \dots, y_{n-1}$. The edges are defined as follows: x_i is adjacent to x_{i+1} and y_i , and y_i is adjacent to y_{i+k} , where the addition is performed modulo n .

$P(5, 2)$ is the Petersen graph. For general n and k , let g be the greatest common divisor of n and k . $P(n, k)$ is a three-regular graph consisting of an n -cycle formed by the x 's and g cycles of length n/g formed by the y 's. By a reflection, $P(n, k)$ is isomorphic to $P(n, n-k)$. Therefore, we will assume that $1 \leq k < n/2$.

We will now find the chromatic number and P_2 -required chromatic numbers of the generalized Petersen graphs. Although the correct bounds follow immediately from Theorems 4.9 and 4.11, we will provide explicit constructive colorings. Since we will exhibit P_2 -required two-colorings, there is no need to consider other generalized chromatic numbers.

THEOREM 4.24: $\chi(P(n, k)) = \begin{cases} 2 & \text{if } n \text{ is even and } k \text{ is odd} \\ 3 & \text{otherwise} \end{cases}$

PROOF: If n is even and k is odd then the following assignment is a proper two-coloring.

$$f(x_i) = \begin{cases} 0 & \text{if } i \text{ is even} \\ 1 & \text{if } i \text{ is odd} \end{cases}$$

$$f(y_i) = 1 - f(x_i)$$

By construction, x_i and y_i receive different colors. Also for any edge x_i, x_j or y_i, y_j , i and j have different parity and therefore different colors.

For each other case, $P(n, k)$ is not two-colorable, because $x_0, x_1, \dots, x_k, y_k, y_0$ is an odd cycle if k is even and $x_0, x_1, \dots, x_{n-1}$ is an odd cycle if k is odd. It remains only to present a proper three-coloring. Consider the following three-coloring.

$$f(y_l) = \begin{cases} 2 & \text{if } n - k \leq l \leq n - 1 \\ 0 & \text{if } jk \leq l < \min \{ (j+1)k, n - k \} \text{ and } j \text{ is even} \\ 1 & \text{if } jk \leq l < \min \{ (j+1)k, n - k \} \text{ and } j \text{ is odd} \end{cases}$$

$$f(x_l) = \begin{cases} 2 & \text{if } 0 \leq l < n - k \text{ and } l \text{ is even} \\ 1 - f(y_l) & \text{if } 0 \leq l < n - k \text{ and } l \text{ is odd} \end{cases}$$

$$f(x_{n-k}) = \max \{ 0, 1 - f(x_{n-k-1}) \}$$

$$f(x_l) = 1 - f(x_{l-1}) \text{ if } n - k < l \leq n - 1$$

Since the color on x_l is restricted to $\{0, 1\}$ when $n - k \leq l \leq n - 1$, we have constructed the coloring so that x_l and y_l have different colors for all l . Among the y_l 's, vertices are adjacent when the indices differ by $k \bmod n$. Each y_l belongs to a string of at most k consecutive y 's of the same color, with the k y 's to either side of the block having a different color. Finally, for the edges x_l, x_{l+1} , the colors start with 2 at x_0 and alternate between 2 and non-2 until x_{n-k} is reached. For color 0, 1, 2, respectively on x_{n-k-1}, x_{n-k} gets the distinct color 1, 0, 0, and thereafter the colors alternate between 0 and 1 until returning to 2 at x_0 . $\square$

Our next result will find the P_2 -required chromatic number of the generalized Petersen graph. This theorem follows directly from Akiyama's Theorem since it is three-regular, but we will give explicit constructions that achieve that bound.

THEOREM 4.25: $\chi_P(P(n, k); P_2) = 2$

PROOF: Clearly, we cannot color $P(n, k)$ with one color or we would induce a monochromatic cycle. Therefore, we need only show that the upper bound is two.

Case 1: Either n is not congruent to 1 modulo $2k$ or k is odd

Consider the following coloring of the vertices. For $0 \leq i \leq k-1$, define

$$f(y_i) = \begin{cases} 0 & \text{if } i \text{ is even} \\ 1 & \text{if } i \text{ is odd} \end{cases}$$

For $k \leq i \leq n-1$, let $f(y_i) = 1 - f(y_{i-k})$. For $0 \leq i \leq n-1$, let $f(x_i) = 1 - f(y_i)$.

By construction, no three adjacent y 's receive the same color and x_i and y_i receive different colors for all i . Therefore, the only way a monochromatic P_3 could occur would be among three adjacent x 's. Let $n = qk + r$. Starting with x_0 , we can think of the x 's as falling into q blocks of size k and one block of size r . Note that within any block the colors alternate. Therefore, if three adjacent x 's are to receive the same color, they must fall into three different blocks. This can happen only if $r = 1$ and $i = n-1$. In other words, $n \equiv 1 \pmod{k}$. If q is even, then by assumption k is odd and x_0 and x_{n-1} receive different colors. If q is odd, then x_0 and x_{n-2} receive different colors. In either case, no three adjacent x 's receive the same color.

Case 2: $n \equiv 1 \pmod{2k}$ and k is even

Let f be defined as in case 1, except let $f(x_0) = 0$ and $f(y_0) = 1$. We have eliminated the possibility of x_0 , x_{n-1} , and x_{n-2} receiving the same color, but we now must be concerned with y_{n-k} , y_0 , and y_k receiving the same color. We know that $n = a(2k) + 1$. Therefore, $n - k = (2a - 1)k + 1$. Since $2a - 1$ is odd, $f(y_{n-k}) = 1 - f(y_1) = 1 - f(y_0)$. Therefore, no three adjacent y 's receive the same color. $\square$

The next graphs that we will consider are also generalizations of the Petersen graph.

DEFINITION 4.26: Let k be a positive integer. Consider the set $S = \{1, 2, 3, \dots, 2k+1\}$. The *odd graph of order k* , denoted by O_k , contains n vertices, where $n = \binom{2k+1}{k}$. The vertices in O_k correspond to the subsets of size k of S . Two vertices are adjacent if their underlying subsets are disjoint.

O_2 is the Petersen graph and O_1 is the three-cycle. In general, O_k is a $(k + 1)$ -regular graph. We will now give the chromatic number of the odd graphs. This appears as a problem in Biggs [5].

THEOREM 4.27: $\chi(O_k) = 3$

PROOF: For the lower bound, we show that O_k has the odd cycle C_{2k+1} as a subgraph. View the elements of S modulo $(2k + 1)$, that is $2k + 1 = 0$. Place these elements consecutively around a cycle. Let A_1 consist of the first k , A_2 the next k and so on. A_3 overlaps A_1 , but each consecutive pair of sets are disjoint and therefore correspond to adjacent vertices in O_k . Set A_{2k+1} begins $2k(k)$ vertices after set A_1 , and ends $2k^2 + k - 1 = (2k + 1)k - 1$ vertices after A_1 began. Since $(2k + 1)k$ is divisible by $2k + 1$, this means that A_{2k+1} consists of the last k elements and A_{2k+1} and A_1 correspond to adjacent vertices in O_k , completing the cycle. Since the chromatic number of any odd cycle is three and the odd graphs contain odd cycles, we know that the chromatic number of the odd graph is at least three.

For the upper bound, we construct a three-coloring of O_k . Let x be a vertex in O_k , and let X be the set that corresponds to x . Consider the following coloring of O_k :

$$l(x) = \begin{cases} 1 & \text{if } 1 \in X \\ 2 & \text{if } 2 \in X \text{ and } 1 \notin X \\ 3 & \text{if } 1, 2 \notin X \end{cases}$$

Suppose x and y are adjacent vertices and let X and Y be the corresponding sets. Then $X \cap Y = \emptyset$. If $l(x) = 1$, then $1 \in X$. Therefore, $1 \notin Y$ and $l(y) \neq 1$. If $l(x) = 2$, then $2 \in X$. This implies that $2 \notin Y$ and $l(y) \neq 2$. If $l(x) = 3$, then $1, 2 \notin X$. Since $|X \cup Y| = 2k$ and $X \cup Y \subset \{1, 2, \dots, 2k + 1\}$, it follows that $1 \in Y$ or $2 \in Y$. Therefore, $l(y) \neq 3$, so no two adjacent vertices receive the same color and we have $\chi(O_k) \leq 3$. $\square$

For the odd graphs, a slight relaxation yields a savings in colors.

THEOREM 4.28: $\chi_R(O_k : P_2) = 2$.

PROOF: Since O_k contains P_3 , one color will not suffice. We construct a two coloring. If x is a vertex in O_k , let X be the set to which it corresponds. Consider the following coloring scheme:

$$l(x) = \begin{cases} 1 & \text{if } 1 \in X \\ 2 & \text{if } 1 \notin X \end{cases}$$

Let x and y be adjacent vertices and let X and Y be the corresponding sets. Then $X \cap Y = \emptyset$. Suppose $l(x) = 1$. Then $1 \in X$, which implies that $1 \notin Y$ and $l(y) \neq 1$. Hence all neighbors of a vertex labeled 1 have label 2 and we get no monochromatic P_2 in color 1.

Now suppose $l(x) = 2$ and $l(y) = 2$. Let z be a vertex adjacent to x with $z \neq y$. Let Z be the set that corresponds to z . Then we have $1 \notin X$, $1 \notin Y$, $X \cap Z = \emptyset$, and $Z \neq Y$. We know $|X| = |Y| = k$. Therefore, $X \cup Y = \{2, 3, \dots, 2k + 1\}$. This gives $1 \in Z$ and $l(z) = 1$. Hence, there can be a monochromatic P_2 in color 2, but not a monochromatic P_3 in color 2. Therefore, $\chi_R(O_k : P_2) \leq 2$. $\square$

Section 4.7: Colorings of Cartesian Products

The last graphs that we will consider are the products of cycles. We will first consider the product of two cycles $C_m \times C_n$, where $n, m \geq 3$. Label the vertices as $x_{i,j}$ with $0 \leq i \leq m - 1$ and $0 \leq j \leq n - 1$. $x_{i,j}$ is adjacent to $x_{i,j-1}$, $x_{i,j+1}$, $x_{i-1,j}$, and $x_{i+1,j}$, where the addition is modulo m in the first coordinate and modulo n in the second coordinate. We will lay out the vertices in a grid with m rows labeled $0, 1, \dots, m - 1$ and n columns labeled $0, 1, \dots, n - 1$. From left to right, row r will read $x_{r,0}, x_{r,1}, \dots, x_{r,n-1}$.

We start this section with a general theorem on the generalized chromatic number of the product of any two graphs.

THEOREM 4.29: Let F , G , and H be any graphs. Then

$$\max\{\chi_R(G : F), \chi_R(H : F)\} \leq \chi_R(G \times H : F)$$

and

$$\chi_R(G \times H : F) \leq \min\{\max\{\chi_R(G : F), \chi(H)\}, \max\{\chi(G), \chi_R(H : F)\}\}.$$

The same result is true when χ_R is replaced by χ_F .

PROOF: The lower bound holds since G and H are subgraphs of $G \times H$.

Let $g : V(G) \rightarrow \{0, 1, \dots, a-1\}$ be an optimal F -required (F -forbidden) coloring of G . Let $h : V(H) \rightarrow \{0, 1, \dots, b-1\}$ be an optimal coloring of H . Let $m = \max\{a, b\}$. If (u, v) is a vertex in $G \times H$, define $f(u, v) = (g(u) + h(v)) \bmod m$. Under f , no vertices adjacent along H edges get the same color, so monochromatic subgraphs in G cannot get any longer and the upper bound follows. $\square$

We get several useful corollaries from this theorem.

COROLLARY 4.30: $\chi(G \times H) = \max\{\chi(G), \chi(H)\}$

PROOF: Since $\chi(G) = \chi_R(G : P_1)$, the upper and lower bounds in Theorem 4.29 are the same. $\square$

COROLLARY 4.31: $\chi(C_m \times C_n) = \begin{cases} 2 & \text{if both } m \text{ and } n \text{ are even} \\ 3 & \text{if } m \text{ or } n \text{ is odd} \end{cases}$

PROOF: Since $\chi(C_j) = 2$ if j is even and $\chi(C_j) = 3$ if j is odd, this result follows immediately from Corollary 4.30. $\square$

COROLLARY 4.32: If m or n is even and $k \geq 2$, then $\chi_R(C_m \times C_n : P_k) = 2$.

PROOF: For every p , $\chi_R(C_p : P_k) = \chi_R(C_p : P_2) = 2$, and $\chi(C_p) = 2$ if p is even. Thus, in Theorem 4.29, the upper and lower bounds are both 2. $\square$

By applying Corollary 4.32 recursively, we obtain the following result.

COROLLARY 4.33: If at most one of $n_1, n_2, \dots, n_d$ is odd, and $k \geq 2$, then

$$\chi_R(C_{n_1} \times C_{n_2} \times \dots \times C_{n_d} : P_k) = 2.$$

COROLLARY 4.34: $2 \leq \chi_R(C_{n_1} \times C_{n_2} \times \cdots \times C_{n_d}; P_k) \leq 3$.

PROOF: The lower bound is obtained as in Corollary 4.32. The upper bound follows since $\chi(C_m) \leq 3$ for all m . $\square$

We already know that the P_k -required chromatic number of $C_m \times C_n$ is two if either m or n is even. We now want to find the P_k -required chromatic number of $C_m \times C_n$ when both m and n are odd. To do this we will need the following lemmas.

LEMMA 4.35: If $\chi_R(C_{n_1} \times C_{n_2} \times \cdots \times C_{n_d}; F) = 2$, then

$$\chi_R(C_{n_1} \times C_{n_2} \times \cdots \times C_{n_{d-1}} \times C_{n_d+2}; F) = 2.$$

PROOF: Let A and B be two adjacent "slabs" perpendicular to the d th direction. Insert two new "slabs" X and Y between A and B such that X is adjacent to A and Y is adjacent to B . On Y use the same coloring that was used on A . On X use the complement of the coloring used on A . No adjacent vertices between "slabs" A and X or between X and Y receive the same color. Between "slabs" B and Y , we have the same configuration we had previously between A and B . Therefore, this is still a legal two-coloring. $\square$

The purpose of Lemma 4.35 is to reduce the problem of finding the required chromatic numbers of $C_m \times C_n$ to finding the minimum odd value such that the required chromatic number is two.

Our next lemma will help prove lower bounds for $\chi_R(C_3 \times C_n; P_k)$.

LEMMA 4.36: Suppose n is odd. Then two-coloring $C_3 \times C_n$ yields a monochromatic copy of P_4 or C_4 .

PROOF: If two adjacent columns of $C_3 \times C_n$ contain four vertices that have the same color, then

we have one of the following configurations: $\begin{smallmatrix} & c & cc \\ cc & & \\ cc & & \\ & c & c \end{smallmatrix}$ or cc or c . The first configuration contains a mono-

chromatic C_4 , the second contains a monochromatic P_4 and the third contains a monochromatic P_4 . Therefore, we can assume that column j contains two vertices with color 1 if j is even and column j contains two vertices with color 2 if j is odd. But then column 0 and column $n - 1$ both contain two vertices with color 1 and we have a monochromatic P_4 or C_4 . $\square$

We are now ready to give the P_k -required chromatic number of $C_m \times C_n$.

THEOREM 4.37: $\chi_R(C_m \times C_n; P_k) = 2$, except when one of the following conditions holds:

- (1) $k = 2$ and m and n are odd
- (2) $k = 3$, $m = 3$ and n is odd
- (3) $k = 4$ and $m = n = 3$

PROOF: By Corollary 4.34, Lemma 4.35, and Lemma 4.36 it suffices to show

$\chi_R(C_3 \times C_3; P_5) = \chi_R(C_3 \times C_5; P_4) = \chi_R(C_5 \times C_5; P_3) = 2$, and that $\chi_R(C_3 \times C_3; P_4) > 2$ and $\chi_R(C_m \times C_n; P_2) > 2$ when m and n are odd.

If we require P_5 , a legal two-coloring of $C_3 \times C_3$ is

1	1	2
2	1	1
2	2	1

If we require P_4 , a legal two-coloring of $C_3 \times C_5$ is

1	2	1
2	1	2
1	2	1
2	1	1
1	2	2

If we require P_3 , a legal two-coloring of $C_5 \times C_5$ is

1	2	2	1	2
1	2	1	2	2
2	1	2	1	1
1	2	2	1	2
2	1	1	2	1

Suppose $\chi_R(C_3 \times C_3; P_4) = 2$. Then without loss of generality, we can assume that color 1 appears at least five times. No row or column can be all the same color for this would give us a monochromatic cycle, which is forbidden. Therefore, we can assume that we have color 1 appearing twice in row 1, twice in row 2, and at least once in row 3. Hence, we can assume that row 1 looks like:

1	1	2
---	---	---

If row 2 is the same as row 1, then our graph contains a monochromatic cycle.

We can assume that we have the following:

1	1	2
2	1	1
x	y	z

where at least one of x, y, z is 1. If $y = 1$ then we have a monochromatic cycle. If $x = 1$ or $z = 1$ then we have a monochromatic P_5 , which is not allowed. Therefore, $\chi_R(C_3 \times C_3; P_4) = 3$.

Suppose m and n are odd and that $\chi_R(C_m \times C_n; P_2) = 2$. If $x_{i,j}$ and $x_{i,j+1}$ both receive the same color, then $x_{i+1,j}, x_{i+1,j+1}, x_{i-1,j}, x_{i-1,j+1}$ all receive the other color, where the arithmetic is done modulo m on the first index and modulo n on the second index. Without loss of generality, this implies that $x_{k,j}$ and $x_{k,j+1}$ both receive color 1 when k is even and color 2 when k is odd. But this means that $x_{1,j}, x_{1,j+1}, x_{m,j}, x_{m,j+1}$ all receive color 2. This implies that we have a monochromatic cycle, which is forbidden. Therefore, colors must alternate in the columns and the rows. But then $x_{1,1}, x_{1,n}, x_{m,1}, x_{m,n}$ all receive the same color and we have a monochromatic cycle, which

is not allowed. Therefore, $\chi_R(C_m \times C_n; P_2) = 3$. $\square$

We can use Theorem 4.37 to give us the next result on the product of more than two cycles.

COROLLARY 4.38: $\chi_R(C_{n_1} \times C_{n_2} \times \cdots \times C_{n_d}; P_k) = 3$ when

- (1) $k = 2$ and two of the n_i are odd.
- (2) $k = 3$, $\min \{ n_i \} = 3$, and one of the remaining n_i is odd.
- (3) $k = 4$ and two of the n_i equal 3.

PROOF: The upper bound follows from Corollary 4.34. The lower bound follows since the configurations of Theorem 4.37 are subgraphs of $C_{n_1} \times C_{n_2} \times \cdots \times C_{n_d}$. $\square$

Since $C_m \times C_n$ has no vertices of degree exceeding four, and since $S_2 = P_2$ and $S_3 = P_3$, not much work remains to determine the star-required chromatic number of $C_m \times C_n$.

THEOREM 4.39: If $k \geq 4$, then

$$\chi_R(C_m \times C_n; S_k) = \begin{cases} 3 & \text{if } m = 3 \text{ and } n \text{ is odd} \\ 2 & \text{otherwise} \end{cases}$$

PROOF: If $k \geq 6$ then S_k is not an induced subgraph of any color class since the maximum degree of a vertex in $C_m \times C_n$ is four. Therefore, $\chi_R(C_m \times C_n; S_k) = \chi_R(C_m \times C_n; S_4)$. Hence we may assume k is either 4 or 5.

Since P_3 is a subgraph of S_k , it follows that $\chi_R(C_m \times C_n; S_k) \leq \chi_R(C_m \times C_n; P_3)$, and the upper bound follows immediately from Corollary 4.32 and Theorem 4.37. Since $C_m \times C_n$ is not a star we need at least two colors. Finally, Lemma 4.36 implies that $\chi_R(C_3 \times C_n; S_k) \geq 3$ if n is odd. $\square$

When we allow other small graphs and merely forbid a long path or star, we may get colorings with fewer colors. The next two theorems consider this for $C_m \times C_n$. Recall that $\chi_F(G; S_3) = \chi_F(G; P_3) = \chi_R(G; P_2) = \chi_R(G; S_2)$ and $\chi_F(G; P_2) = \chi_F(G; S_2) = \chi(G)$.

THEOREM 4.40: If $k \geq 4$, then

$$\chi_F(C_m \times C_n; P_k) = \begin{cases} 1 & \text{if } mn \leq k - 1 \\ 3 & \text{if } m = 3, n \text{ is odd and } k = 4 \\ 2 & \text{otherwise} \end{cases}$$

PROOF: If $mn \leq k - 1$, then $C_m \times C_n$ contains no copy of P_k and we can color it with one color.

Suppose $m = 3, n$ is odd, and $k = 4$. Clearly, $\chi_F(C_3 \times C_n; P_4) \leq \chi_R(C_3 \times C_n; P_3) = 3$.

Lemma 4.36 implies that two colors do not suffice since P_4 is a subgraph of C_4 .

Suppose neither of the above cases holds. Since $mn \geq k$, then clearly $\chi_F(C_m \times C_n; P_k) \geq 2$.

The upper bounds for all cases except $m = n = 3$ and $k = 5$ follow from Theorem 4.37. The following is a legal two-coloring of $C_3 \times C_3$ if we forbid a monochromatic P_5 .

1 2 1

2 1 2

1 2 1

Therefore, we get an upper bound of 2 as claimed. $\square$

THEOREM 4.41: If $k \geq 4$, then

$$\chi_F(C_m \times C_n; S_k) = \begin{cases} 1 & \text{if } k \geq 6 \\ 2 & \text{if } k = 4, 5 \end{cases}$$

PROOF: If $k \geq 6$, then S_k is not a subgraph of $C_m \times C_n$ and therefore, we need only one color.

If $k = 4, 5$ and we color $C_m \times C_n$ with one color then we get a monochromatic S_5 , which is forbidden. Therefore, we have a lower bound of two. Consider the following assignment of colors to $C_m \times C_n$.

1 2 1 2 1 2...

2 1 2 1 2 1...

1 2 1 2 1 2...

2 1 2 1 2 1...

...

...

...

Each vertex explicitly has at least two neighbors with the other color, so no vertex can have three neighbors with its own color and we have no monochromatic S_4 . $\square$

The results on $C_m \times C_n$ suggest the following questions:

- (1) Can $\chi_R(G \times H : P_2)$ be arbitrarily large if $\chi_R(G : P_2)$ and $\chi_R(H : P_2)$ are bounded?
- (2) Consider $\chi_R(C_{n_1} \times C_{n_2} \times \cdots \times C_{n_d} : P_k)$ where all of the n_i are odd. What is the largest value of k such that three colors are always required no matter how large $\min \{ n_i \}$ is? Does it remain at two?
- (3) For the same situation as described in (2), what is the smallest value of k such that two colors always suffice no matter how small $\max \{ n_i \}$ is?
- (4) For what value of k does a single even n_i allow a legal P_k -required two-coloring?

As a partial answer to (2), we offer the following.

THEOREM 4.42: If m , n , and p are odd, then $\chi_R(C_m \times C_n \times C_p : P_3) = 3$.

PROOF: The upper bound follows from Corollary 4.34. If we two-color $C_m \times C_n \times C_p$ then Theorem 4.37 implies that we must obtain a monochromatic P_3 . Without loss of generality, we may assume that the monochromatic P_3 occurs in some copy of $C_m \times C_n$. Then copies of $C_m \times C_n$ adjacent to the first copy must contain monochromatic P_3 in the other color in corresponding positions. Continuing this reasoning and using the fact that p is odd, we get two monochromatic P_3 's

adjacent to one another in adjacent copies of $C_m \times C_n$. But this gives a monochromatic cycle.

Therefore, we must use three colors. $\square$

In view of this result, we conjecture that the answer to (2) is at least d .

We now consider question (3). For $d = 3$, the next result will show that there is no k that satisfies (3).

THEOREM 4.43: If n is odd, then for all k , $\chi_R(C_3 \times C_3 \times C_n; P_k) = 3$.

PROOF: Suppose we two-color $C_3 \times C_3 \times C_n$. Then by Lemma 4.36 each copy of $C_3 \times C_n$ contains a monochromatic P_4 , and we are done for $k = 2, 3$. Suppose $k \geq 4$. Let a, b, c, d be the monochromatic P_4 in one of the copies of $C_3 \times C_n$. Then in the other two copies of $C_3 \times C_n$, the vertices adjacent to b, c must receive the other color. But this gives a monochromatic C_4 . $\square$

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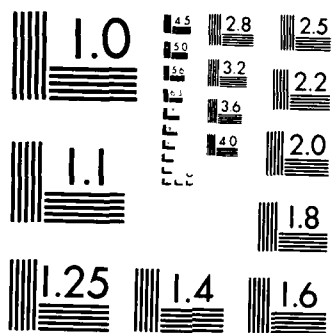
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VITA

Margaret Lefevre Weaver was born in Charleston, West Virginia on November 30, 1953. She entered Virginia Polytechnic Institute and State University in 1971 and received a B.S. in Mathematics and Chemistry in June, 1975. While enrolled at the University of Illinois from 1975 to 1977 she was employed as a teaching assistant in the Mathematics Department. She received a M.S. in Statistics in 1977. From 1977 to 1980 she was employed as an Instructor in the Mathematics Department at the University of North Alabama in Florence, Alabama. In 1980, she was hired by Eastern Illinois University as a Mathematics instructor. At that time she returned to the University of Illinois as a graduate student. From 1985 to 1987 she was on leave from Eastern Illinois University and held the position of research assistant with the Coordinated Sciences Laboratory at the University of Illinois. She is currently an Assistant Professor in the Mathematics Department at Eastern Illinois University in Charleston, Illinois.

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