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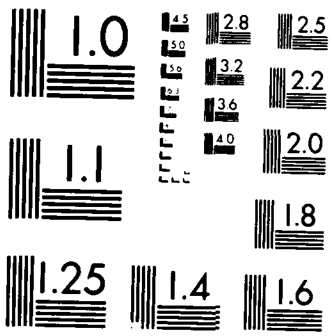
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COORDINATED SCIENCE LABORATORY
College of Engineering
Applied Computation Theory

COMPACT REPRESENTATION OF THE SEPARATING k-SETS OF A GRAPH

Arkady Kanevsky

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

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REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: Unclassified	1b. RESTRICTIVE MARKINGS: None
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION / AVAILABILITY OF REPORT: Approved for public release; distribution unlimited
2b. DECLASSIFICATION / DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S): UILU-ENG-88-2204 (ACT-88)	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION: Coordinated Science Lab, University of Illinois; 6b. OFFICE SYMBOL (If applicable): N/A	7a. NAME OF MONITORING ORGANIZATION: NSF, ONR, and Semiconductor Research Corporation
6c. ADDRESS (City, State, and ZIP Code): 1101 W. Springfield Avenue, Urbana, IL 61801	7b. ADDRESS (City, State, and ZIP Code): 800 N. Quincy, Arlington, VA 22217 (ONR); 1800 G St., Washington, DC 20550 (NSF); Research Triangle Park, NC 27709 (SRC)
8a. NAME OF FUNDING / SPONSORING ORGANIZATION: NSF, JSEP, and SRC; 8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: NSF: ECS 8404866, JSEP: N00014-84-C-0149, SRC: 87-DP-109
8c. ADDRESS (City, State, and ZIP Code): see block 7b.	10. SOURCE OF FUNDING NUMBERS

10. SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO	WORK UNIT ACCESSION NO.

11. TITLE (Include Security Classification)

Compact Representation of the Separating k-sets of a Graph

12. PERSONAL AUTHOR(S)

Kanevsky, Arkady

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
Technical	FROM ____ TO ____	1988 January	29

16. SUPPLEMENTARY NOTATION

17. COSATI CODES

FIELD	GROUP	SUB-GROUP

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)

graph theory, data structures, connectivity

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

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20. DISTRIBUTION / AVAILABILITY OF ABSTRACT: ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION: Unclassified
22a. NAME OF RESPONSIBLE INDIVIDUAL	22b. TELEPHONE (Include Area Code) / 22c. OFFICE SYMBOL

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ABSTRACT

We present an $O(n)$ space representation for the separating k-sets of an undirected k-connected graph G for fixed k, where n is the cardinality of the vertex set of G. Namely, the total space used by the representation is $O(k^2 n)$. We also improve the upper bound on the number of separating k-sets of G to $O(2^k \frac{n^2}{k})$, which has a matching lower bound.

1. Introduction

Connectivity is an important graph property and there has been a considerable amount of work on algorithms for determining connectivity of graphs [BeX,Ev2,EvTa,Ga,GiSo,LiLoWi]. An undirected graph $G=(V,E)$ is *k-connected* if for any subset V' of $k-1$ vertices of G the subgraph induced by $V-V'$ is connected [Ev]. A subset V' of k vertices is a *separating k-set* for G if the subgraph induced by $V-V'$ is not connected. For $k=1$ the set V' becomes a single vertex which is called an *articulation point*, and for $k=2,3$ the set V' is called a *separating pair* and a *separating triplet*, respectively. Efficient algorithms are available for finding all separating k-sets in k-connected undirected graphs for $k \leq 3$ [Ta,HoTa,MiRa,KaRa].

In [KaRa2,Ka] we addressed the question of the maximum number of separating pairs, triplets and k-sets in biconnected, triconnected and k-connected undirected graphs, respectively?

An undirected graph G on n vertices has a trivial upper bound of $\binom{n}{k}$ on the number of separating k-

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This research was supported by NSF under ECS 8404866, the Semiconductor Research Corporation under 87-DP-109 and the Joint Services Electronics Program under N00014-84-C-0149.

sets, $k \geq 1$. The graph that achieves this bound for all k is a graph on n vertices without any edges. For $k=1$ the maximum number of articulation points in a *connected* graph is $(n-2)$ and a graph that achieves it is a path on n vertices. For $k=2$ the maximum number of separating pairs in an undirected biconnected graph is $\frac{n(n-3)}{2}$ and a graph that achieves it is a cycle on n vertices [KaRa2]. Further, we observed that there is an $O(n)$ representation for the separating pairs in any biconnected graph (although the number of such pairs could be $\Theta(n^2)$) [KaRa2]. For $k=3$ the maximum number of separating triplets in a triconnected graph is $\frac{(n-1)(n-4)}{2}$ and we presented a graph, namely the *wheel* [Tu], that achieves it [KaRa2]. The number of separating k-sets in a k-connected graph is $O(3^k n^2)$ and we show that the bound is tight up to the constant [Ka]. The lower bound on the number of separating k-sets in a k-connected undirected graph is $\Omega(2^k \frac{n^2}{k^2})$.

In this paper we present a linear representation of separating k-sets in k-connected undirected graphs. For $k=2$ representation is different from the one presented in [KaRa2]. We also give the alternative prove of the upper bound on the number of separating k-sets, which match the previous upper bounds for $k=2$ and $k=3$, and improves the upper bound for general k to $O(2^k \frac{n^2}{k})$. We will first present representation for $k=2$ and $k=3$ and then generalized the technique for general k.

2. Graph-theoretic definitions

An *undirected graph* $G=(V,E)$ consists of a *vertex set* V and an *edge set* E containing unordered pairs of distinct elements from V. A *path* P in G is a sequence of vertices $<v_0, \cdots, v_k>$ such that $(v_{i-1}, v_i) \in E, i=1, \cdots, k$. The path P *contains* the vertices $v_0, \cdots, v_k$ and the edges $(v_0, v_1), \cdots, (v_{k-1}, v_k)$ and has *endpoints* v_0, v_k, and *internal vertices* $v_1, \cdots, v_{k-1}$.

We will sometimes specify a graph G structurally without explicitly defining its vertex and edge sets. In such cases, $V(G)$ will denote the vertex set of G and $E(G)$ will denote the edge set of G. Also, if $V' \subseteq V$ and $v \in V$ we will use the notation $V' \cup v$ to represent $V' \cup \{v\}$.

An undirected graph $G=(V,E)$ is connected if there exists a path between every pair of vertices in V. For a graph G that is not connected, a *connected component* of G is an induced subgraph of G which is maximally connected.

A vertex $v \in V$ is an *articulation point* of a connected undirected graph $G=(V,E)$ if the subgraph induced by $V-\{v\}$ is not connected. G is *biconnected* if it contains no articulation point.

Let $G=(V,E)$ be a biconnected undirected graph. A pair of vertices $v_1, v_2 \in V$ is a *separating pair* for G if the induced subgraph on $V-\{v_1,v_2\}$ is not connected. G is *triconnected* if it contains no separating pair.

A triplet (v_1,v_2,v_3) of distinct vertices in V is a *separating triplet* of a triconnected graph if the subgraph induced by $V-\{v_1,v_2,v_3\}$ is not connected. G is *four-connected* if it contains no separating triplets.

Let $G=(V,E)$ be an undirected graph and let $V' \subseteq V$. A graph $G'=(V',E')$ is a *subgraph* of G if $E' \subseteq E \cap \{(v_i,v_j) \mid v_i,v_j \in V'\}$. The *subgraph of G induced by V'* is the graph $G''=(V',E'')$ where $E''=E \cap \{(v_i,v_j) \mid v_i,v_j \in V'\}$.

3. Representation for k=2

Let $G=(V,E)$ be an undirected biconnected graph with n vertices and m edges. We denote with $g(n)$ the upper bound on the size of a compact representation of separating pairs of a graph on n vertices. Let $\{v_1,v_2\}$ be a separating pair that divides G into nonempty G_1 and G_2. Let $\{w_1,w_2\}$ be a "cross" separating pair with $w_1 \in G_1$ and $w_2 \in G_2$. It divides G_1 into G'_1 and G''_1, and divides G_2 into G'_2 and G''_2 (see Figure 1).

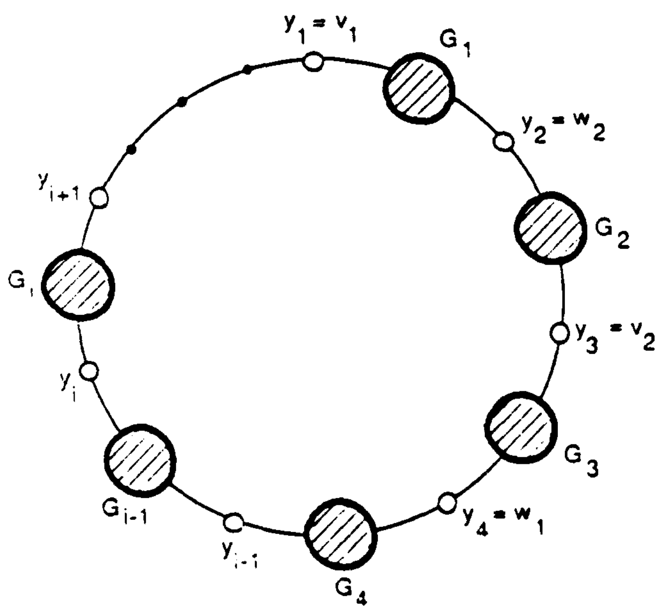

Figure 1.
Representation for k=2.

Consider a maximal set of vertices u in G_2 such that $\{w_1,u\}$ is a cross separating pair and, analogously, consider a maximal set of vertices x in G_1 such that $\{x, w_2\}$ is a cross separating pair. The set of u's is the set of articulation points in G_2. Moreover, the set of u's along with the subgraphs of G_2 between them is a path from v_1 to v_2. Analogously, the set x's is a set of articulation points of G_1 with additional condition that the x's along with the subgraphs of G_1 between them is a path from v_1 to v_2. Number the vertices v_1, u's, v_2, and x's by y_1, y_2 and so on going clockwise along the paths. We denote by G_i the subgraph of G between y_i and y_{i+1}. Note that some G_i can be empty (consists of a single edge). Thus, the graph G becomes a cycle with vertices y's and G_i's alternating on it. Every pair of vertices y's give a separating pair of G unless they are adjacent and the subgraph between them is empty. Hence, we can represent all of them by the following structure:

1) the cycle: the set of vertices y's

2) a vertex for every G_i with a flag to specify if G_i is empty. Edges between G_i and y_i, y_{i+1}.

Note that when there are no cross separating pairs then we get a trivial cycle with two vertices v_1 and v_2 and two edges connecting them. Since the sets x's and u's are maximal all other separating pairs are inside $G_i \cup y_i \cup y_{i+1}$. Note that G_i can be the union of disconnected components, but each of them is connected to y_i and y_{i+1}. Let the cardinality of set of vertices y's be l. Based upon the above observations we get the following recurrence relation

$$g(n) \le \sum_{i=1}^{l} g(n_i + 2) + 4l ,$$

where $g(n_i + 2)$ represent the upper bound for all separating pairs inside $G_i \cup y_i \cup y_{i+1}$. The cardinality of $G_i = n_i$, and $\sum_{i=1}^{l} (n_i + 1) = n$. Any $g(n)$ that satisfy the recurrence will be an upper bound on the size of representation of separating pairs of G. Clearly, linear $g(n)$ is one of them (see Appendix).

4. Representation for k=3

The *wheel* W_n [Tu] is C_{n-1} together with a vertex v and an edge between v and every vertex on C_{n-1}. It is easy to see that W_n is triconnected and has $\frac{(n-1)(n-4)}{2}$ separating triplets.

Assume there exists a separating triplet $\{v_1, v_2, v_3\}$ in G, which separates G into nonempty G_1 and G_2 (see Figure 2).

Lemma 1: Only one of these three vertices has type 3 separating triplets $\{w_1, v_i, w_2\}$ such that $w_1 \in G_1$ and $w_2 \in G_2$ [KaRa2].

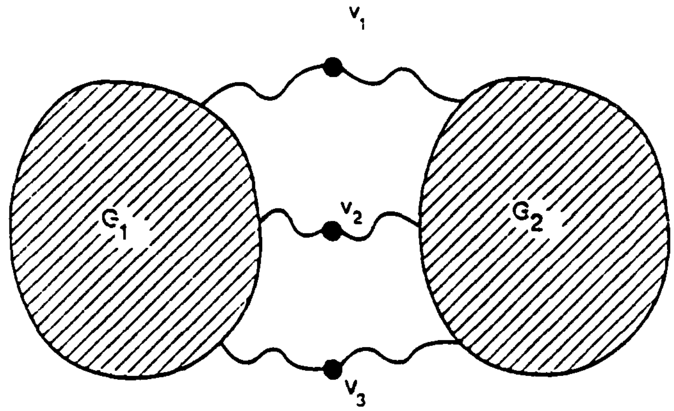

Figure 2.
Separating G into G_1 and G_2 by separating triplet $\{v_1,v_2,v_3\}$

Proof: Assume there is separating triplet $\{w_1,v_2,w_2\}$ of the third type in G, where $w_1 \in G_1$ and $w_2 \in G_2$. It separates G_1 into K_1 and K_2, and separates G_2 into K_3 and K_4. Vertices v_1 and v_3 must belong to the different components with respect to separating triplet $\{w_1,v_2,w_2\}$, otherwise either $\{w_1,v_2\}$ is a separating pair, or $\{w_2,v_2\}$ is a separating pair, or both.

Claim 1 Vertex v_2 has a direct edge to every nonempty subgraph K_1,K_2,K_3,K_4.

W.L.O.G. assume that K_1 is not empty and $\forall x \in K_1$, $(x,v_2) \notin E$. Then $\{v_1,w_1\}$ is a separating pair of G, which separates K_1 from the rest of the graph.

$\square$

Now, we will prove that there are no separating triplets of the third type which use v_1 or v_3. We will prove this by contradiction. W.L.O.G. assume there is a separating triplet $\{u_1,v_1,u_2\}$, where $u_1 \in G_1$ and $u_2 \in G_2$ (u_1 may be equal to w_1 and u_2 may be equal to w_2).

Case 1: $u_1 \in K_2$, if K_2 is not empty (see Figure 3).

By Claim 1 for v_1 and the existence of separating triplet $\{u_1,v_1,u_2\}$, K_1, w_1, $K_2 - u_1$ belong to the same connected component with respect to separating triplet $\{u_1,v_1,u_2\}$. If v_2 belongs to the same component then $\{v_2,u_2\}$ is a separating pair which separates $K_3 \cup w_2 \cup K_4 \cup v_3$ from the rest of the graph. If v_2 does not belong to the same component then $\{v_1,u_1\}$ is a separating pair which separates $K_1 \cup w_1 \cup K_2 - u_1$ from the rest of the graph.

Analogously, $u_2 \notin K_4$.

Case 2: $u_1 = w_1$.

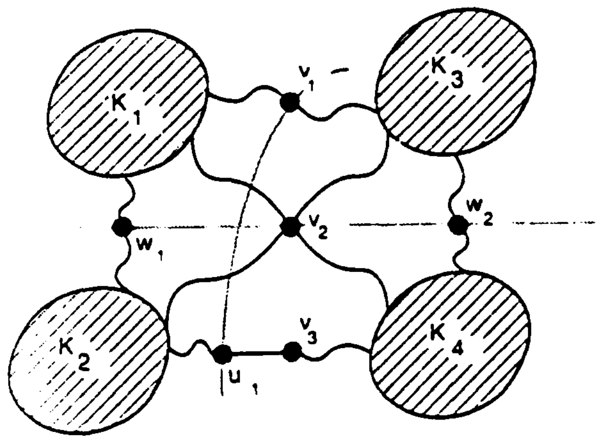

Figure 3.
Illustrating Case 1 in the proof of Lemma 1.

Since $\{u_1, v_1, u_2\}$ is a separating triplet then v_2 does not have any edges to K_1 and hence, K_1 is empty by Claim 1. But then $\{v_1, u_2\}$ is a separating pair, if $\{u_1, v_1, u_2\}$ is a separating triplet.

Analogously, $u_2 \neq w_2$.

Case 3: $u_1 \in K_1$ and $u_2 \in K_3$.

If $\{u_1, v_1, u_2\}$ is a separating triplet then either $\{u_1, u_2\}$, or $\{u_1, v_1\}$, or $\{v_1, u_2\}$ is a separating pair.

That means that if there is a separating triplet of the third type which uses one of the $v_i, i=1,2,3$ then there are no separating triplets of the third type that use the other $v_j, j=1,2,3,\ j \neq i$.

□

Let $\{v_1, v_0, v_2\}$ be a separating triplet of a graph G on n vertices, and v_0 be the only one of the three vertices of this separating triplet which might participate in a separating triplets of the third type with respect to $\{v_1, v_0, v_2\}$. Consider all separating triplets of the third type $\{w_1, v_0, w_2\}$ such that $w_1 \in G_1$ and $w_2 \in G_2$, together with $\{v_1, v_0, v_2\}$. All such separating triplets use v_0 as the "central" vertex. Rename the vertices w_1's, w_2's, v_1 and v_2 into $\{v_1, v_2, \cdots, v_l\}$ going clockwise, such that they form the wheel with v_0 in a center, where any two nonadjacent vertices form a separating triplet with v_0. The subgraphs between v_i and v_{i+1} are denoted with G_i, and some of them may be empty. Now, the graph G looks like a wheel with v_0 in a center v_i, and G_i $(i=1, \cdots, l)$ on a cycle.

Every pair of vertices on the cycle of the wheel form a separating triplet with v_0 unless they are adjacent (v_i and v_{i+1}) and the subgraph (G_i) between them is empty. Hence, we can represent these separating triplets by the following structure:

1) the wheel: $\{v_0, v_1, \cdots, v_k\}$ with edges of G

2) a vertex for every G_i with a flag to specify if G_i is empty. The edges between G_i and v_i, v_{i+1} and between v_0 and v_i, G_i with flags to specify if the edge is real.

Let us see where the rest of separating triplets of G lie.

Observation The remaining separating triplets belong to $G_i \cup v_0 \cup v_i \cup v_{i+1} \cup$ the neighbor of v_i in G_{i-1} if such a neighbor is unique $\cup$ the neighbor of v_{i+1} in G_{i+1} if such a neighbor is unique.

Let $\{w_1, w_2, w_3\}$ be a separating triplet with $w_1 \in G_1$ and $w_2, w_3 \in G_2$. The separating triplet $\{w_1, w_2, w_3\}$ separates G_1 into L_1 and L_2, and separates G_2 into L_3 and L_4 (Figure 4).

Let us see how the original separating triplet $\{v_1, v_2, v_3\}$ is separated by the separating triplet $\{w_1, w_2, w_3\}$.

The vertices $\{v_1, v_2, v_3$ cannot belong to the same connected component of G with respect to the separating triplet $\{w_1, w_2, w_3\}$, otherwise either w_1 would be an articulation point, or $\{w_2, w_3\}$ would be a separating pair, or both. W.L.O.G. assume that v_1 belongs to one connected component and v_2, v_3 to the other.

Subgraph L_1 must be empty, otherwise $\{w_1, v_1\}$ becomes a separating pair. Since the graph is triconnected, we have

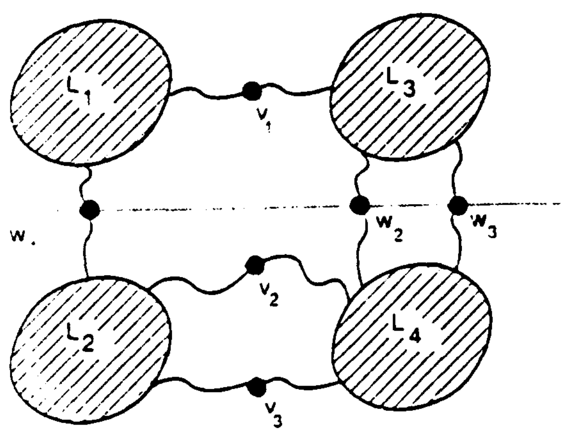

Figure 4.
Illustrating the proof of the Observation.

1) $(w_1, v_1) \in E$,

2) $\exists\, x,y \in L_3 \cup w_2 \cup w_3$: $(x,v_1) \in E$, $(y,v_1) \in E$ and

3) $\forall\, z \in L_2 \cup L_4 \cup v_2 \cup v_3$: $(z,v_1) \notin E$.

Hence, vertex w_1 is the unique neighbor of vertex v_1 in G_1. Moreover, if there are any separating triplets with one vertex in G_1 and two in G_2 which separate v_1 from v_0 and v_2, then w_1 is one of the vertices of the triplet.

A separating triplet cannot have all its three vertices in three different G_i's otherwise two of these vertices would form a separating pair. From the proof of the Lemma 1 and the fact that the set $\{v_1, v_2, \cdots, v_k\}$ is maximal, we know that if there is a separating triplet which involves a vertex from G_i, then the other two vertices belong to $\{v_i\} \cup \{v_{i+1}\} \cup \{v_0\} \cup G_i$ and the neighbor of v_i in G_{i-1}, if such a neighbor is unique, and symmetrically a 'unique' neighbor of v_{i+1} in G_{i+2}. This proves the Observation.

□

Let $g(n)$ be the size of a compact representation of the separating triplets in a graph on n vertices, and let the number of vertices in G_i be n_i. Then $\sum_{i=1}^{k}(n_i + 1) + 1 = n$, and we can write the following recurrence relation

$$g(n) = \sum_{i=1}^{l} g(n_i + 5) + (6l + 1)\,,$$

where $(6l + 1)$ stands for the space used to store the wheel information including multiple edges. The solution to this recurrence is clearly linear (see Appendix). This proves that there is a succinct $O(n)$ size representation of the separating triplets.

5. Representation for general k

Let $G=(V,E)$ be an undirected k-connected graph with n vertices and m edges. We denote with $g(n)$ and $f(n)$ the upper bounds on the size of representation and the number of separating k-sets for k-connected graph on n vertices. Let $V' = \{v_1, v_2, \cdots, v_k\}$ be a separating k-set, whose removal separates G into nonempty G_1 and G_2 (see Figure 5). A separating k-set $\{w_1, w_2, \cdots, w_k\}$ of G is a *cross* separating k-set with respect to V' if $\exists\, i,j$: $w_i \in G_1$ and $w_j \in G_2$. Let the cardinalities of G_1 and G_2 be l and $n-l-k$, respectively. Let the upper bound on the size of the representation of the cross separating k-sets be $h(l,n-l)$, and the maximum number of cross separating k-sets be $r(l,n-l)$. Then any $g(n)$ and $f(n)$ that satisfy the recurrences

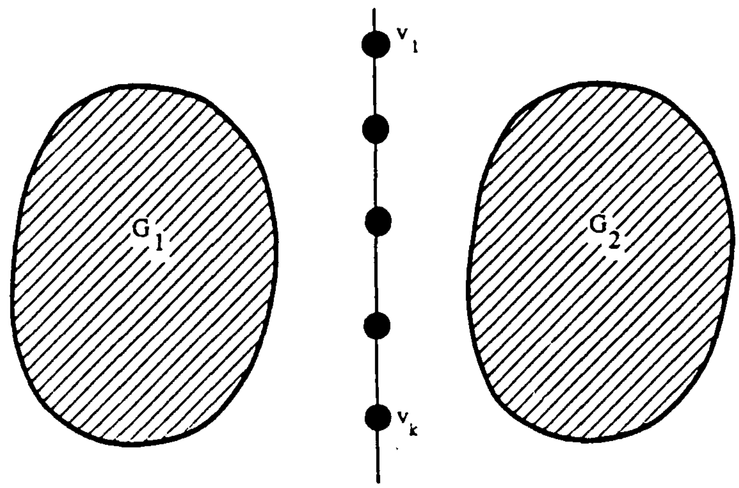

Figure 5.
Dividing G into G_1 and G_2 by separating k-set $\{v_1, \cdots, v_k\}$

$$g(n) = \left[g(l+k) + g(n-l) + h(l, n-l) \right] ,$$

$$f(n) = \left[f(l+k) + f(n-l) + r(l, n-l) + 1 \right] ,$$

are upper bounds on the size of representation and the number of separating k-sets in G. Now we will derive upper bounds for the functions h and r and tune up the recurrences.

Let $\{w_1, w_2, \cdots, w_k\}$ be a cross separating k-set with $\{w_1, \cdots, w_s\} \subset G_1$, $\{w_{s+t+1}, \cdots, w_k\} \subset G_2$ and $\{w_{s+1}, \cdots, w_{s+t}\} \subset \{v_1, \cdots, v_k\}$. The separating k-set $\{w_1, w_2, \cdots, w_k\}$ separates G_1 into G_3 and G_4, separates G_2 into G_5 and G_6, and divides $\{v_1, \cdots, v_k\}$ into $\{v_1, \cdots, v_r\}$, $\{v_{r+t+1}, \cdots, v_k\}$ and $v_{r+i} = w_{s+i}$, $i = 1, \ldots, t$. (see Figure 6)

Case 1 None of G_i, $i = 3,4,5,6$ are empty. (see Figure 6)

The sets $\{w_1, w_2, \cdots, w_{s+t}, v_1, \cdots, v_r\}$, $\{w_1, w_2, \cdots, w_{s+t}, v_{r+t+1}, \cdots, v_k\}$, $\{v_1, \cdots, v_{r+t}, w_{s+t+1}, \cdots, w_k\}$ and $\{v_{r+1}, \cdots, v_k, w_{s+t+1}, \cdots, w_k\}$ are separating sets of G that separate G_3, G_4, G_5 and G_6 respectively, so their cardinalities are greater than or equal to k. Then,

$$\begin{cases} s + t + r \geq k \\ r + t + k - s - t \geq k \\ s + t + k - r - t \geq k \\ k - r + k - s - t \geq k \end{cases} \Rightarrow \begin{cases} r + s + t \geq k \\ r \geq s \\ s \geq r \\ k \geq r + s + t \end{cases} \Rightarrow \begin{cases} r = s \\ r + s + t = k \end{cases}$$

From now on we replace the subscript r by s. Let $A = \{v_1, \cdots, v_s\}$, $B = \{v_{s+t+1}, \cdots, v_k\}$, $C = \{w_1, \cdots, w_s\}$, $D = \{w_{s+t+1}, \cdots, w_k\}$, and $T = \{v_{s+1}, \cdots, v_{s+t}\} = \{w_{s+1}, \cdots w_{s+t}\}$. For Case 1 $|A| = |B| = |C| = |D| = \frac{k-t}{2}$.

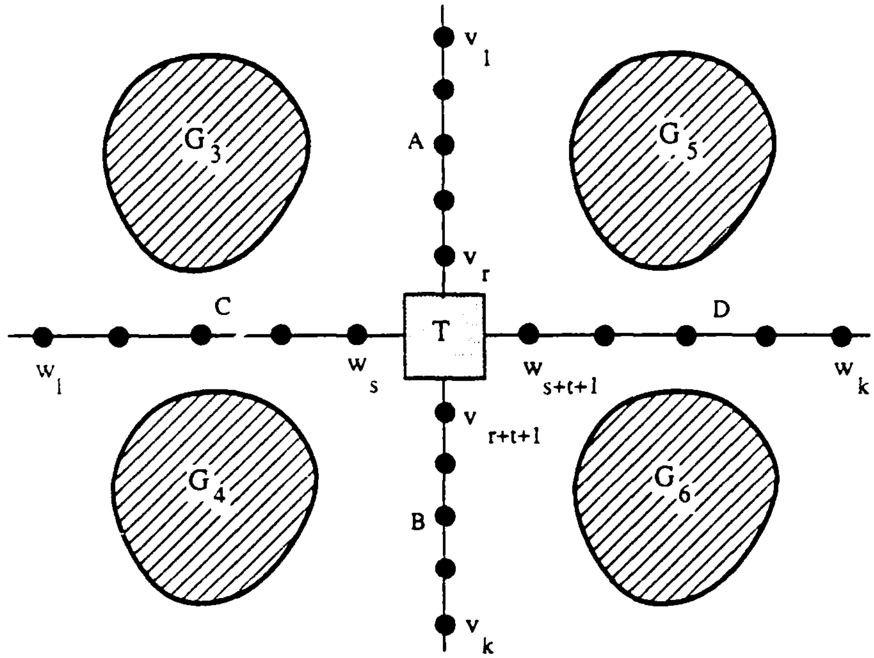

Figure 6.
Dividing G into nonempty components by separating k-sets
$\{v_1, \cdots, v_k\}$ and $\{w_1, \cdots, w_k\}$.

Claim 2 $\forall i \;\; i = s+1,\ldots,t \;\exists\, x_j \in G_j,\; j = 3,4,5,6$: $(v_i, x_j) \in E$.

Proof: W.L.O.G. assume $\exists v_i$: $\forall x \in G_3$: $(x, v_i) \notin E$. Then $\{v_1, \cdots, v_{s+t}, w_1, \cdots, w_s\} - \{v_i\}$ is a separating (k-1)-set.

□

Claim 3 For every $x \in A$ there are $y \in G_3$ and $z \in G_5$, such that $(x,y) \in E$ and $(x,z) \in E$. Analogously, for every vertex x of B, C and D there are vertices y and z in appropriate neighboring G_i, i=3,4,5,6, which are adjacent to x.

Proof: W.L.O.G. assume there is $x \in A$ such that for every $y \in G_3$ $(x,y) \notin E$. Then $A \cup C \cup T$-$\{x\}$ is a separating (k-1)-set.

□

Lemma 2 All cross separating k-sets containing $C \cup T$ and at least one fixed vertex of D can be represented in $O((\frac{k-t}{2})^2)$ space, and their number is $O(2^{\frac{k-t}{2}})$.

Proof: Assume we have a separating k-set $\{w_1, \cdots, w_{s+t+a}, x_{s+t+a+1}, \cdots, x_{s+t+a+b}, y_{s+t+a+b+1}, \cdots, y_k\}$, where x's$\in G_5$, y's$\in G_6$, $a \geq 1$, and either b or $k-s-t-a-b$ is greater or equal to 1 (the new cross separating k-set is different from the old one) (see Figure 7).

Let $H = \{x_{s+t+a+1}, \cdots, x_{s+t+a+b}\}$ (x's) and $I = \{y_{s+t+a+b+1}, \cdots, y_k\}$ (y's), and let D be divided into $D' = \{w_{s+t+1}, \cdots, w_{s+t+a}\}$, E which is in the same connected component as G_3, A, and part of G_5, and F which is in the

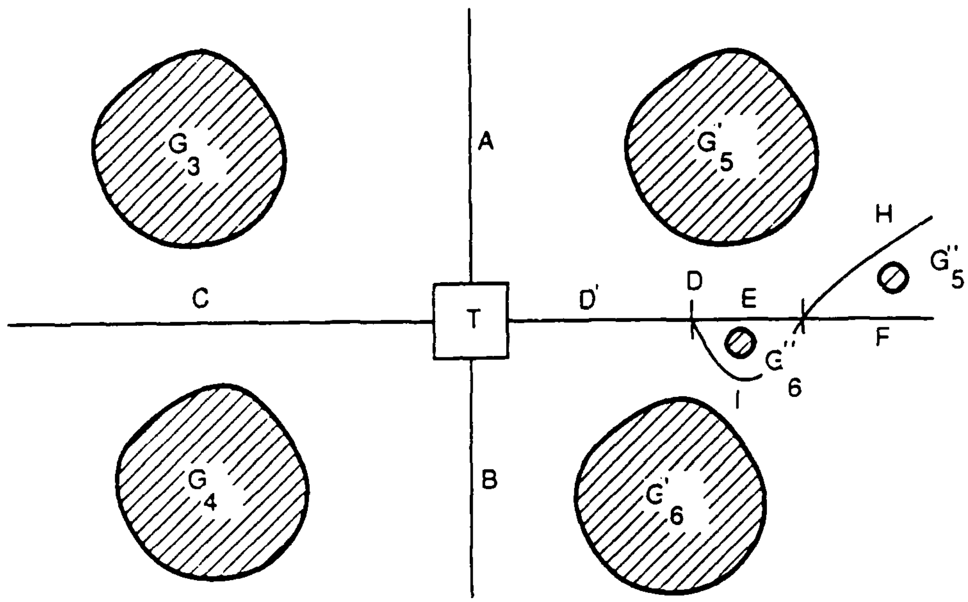

Figure 7.
Illustrating the proof of Lemma 2.

same connected component as G_4, B and part of G_6. Also let H divide G_5 into G'_5 and G''_5, and let I divide G_6 into G'_6 and G''_6 (see Figure 7).

Separating sets $T+D'+E+H$ and $T+D'+F+I$ separate G''_5 and G''_6, respectively. The cardinalities of these separating sets are less than k. Hence, G''_5 and G''_6 are empty. Moreover, since $C+T+D'+H+F$ and $C+T+D'+E+I$ are separating sets and $C+T+D$ and $C+T+D'+H+I$ are separating k-sets, $|E| = |H|$, and $|I| = |F|$. Note that the argument still holds if either H or I are empty.

Next, we will show that if we replace part of E and/or part of F we will necessarily use only vertices of H and/or I for it, regardless of whether we replace part of D' or not. In other words, H and I are unique for E and F. The proof is by contradiction.

Assume that there exist $I_1+H_1 \neq I+H$, such that $C+T+D'+H_1+I_1$ is a separating k-set. Let $H_1 \subseteq G_5$ and $I_1 \subseteq G_6$. Also, let I_1+H_1 divide E into E_1 and E_2, and divide F into F_1 and F_2 (see Figure 8).

Let H_1 be separated into two parts, H'_1 adjacent to E and E''_1 adjacent to F. By the above arguments H'_1 is adjacent to E_1, H''_1 is adjacent to F_2, and I_1 is adjacent to E_2+F_1. Since all neighbors of E in G_6 are also in I, and all neighbors of F in G_5 are also in H, $H''_1 \subset H$ and I_1 is divided into $I'_1 = I \cup I_1$ and $I''_1 = I_1 - I'_1$. Let $H' = H - H''_1$ and let $I' = I - I'_1$.

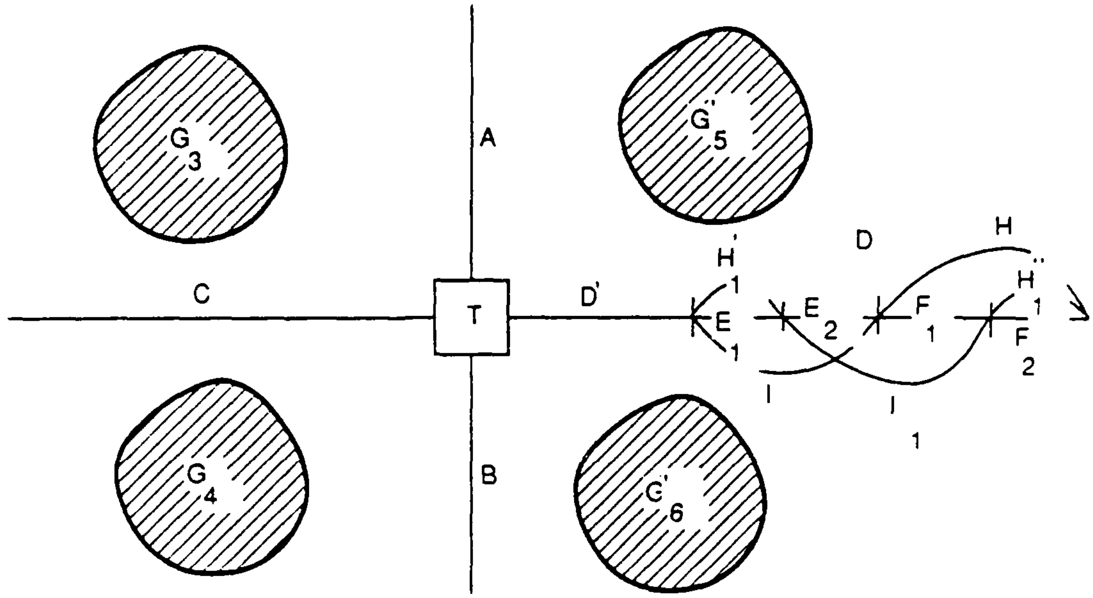

Figure 8.
Illustrating the uniqueness of a replacement for a part of cross separating k-set.

The separating set $T+D'+H'_1+H$ separates E_1 from the rest of the graph and has cardinality is less than k. Hence, E_1 is empty and we have $I = I'_1$, $E = E_2$ and $H_1 = H''_1$. Analogously, the separating set $T+D'+I_1+H$ separates F_1 from the rest of the graph and has cardinality is less than k. Hence, F_1 is empty and we have $F = F_2$, $E = E_1$, $H = H_1$ and $I = I_1$. This contradict the assumptions.

Note that the arguments still hold if either H or I are empty, or if we replace only parts of E and F. If part of D' is replaced as well, then we will not replace it, so that we will look only at the replacements for E and F. Also, if there exists a separating k-set that replaces F by H, then there is no $I_1 \subseteq G_6$ that replaces any part of F for any cross separating k-set described in Lemma 2.

Thus, any replacement of any part of F for any cross separating k-set specified by Lemma 2 lies in H. The set of vertices which is used for all possible replacement of any part of D for a cross separating k-sets specified by Lemma 2 will be called the *fringe* of D, where H is the fringe of F and I is the fringe of E. Note that there could be parts of D which do not have any replacements. The cardinality of the fringe of D is less than $\frac{k-t}{2} = |D|$. Hence, the representation of all cross separating k-sets with $C+T$ fixed along with at least one vertex from D takes $O((\frac{k-t}{2})^2)$ space, where $O((\frac{k-t}{2})^2)$ space is needed to specify all edges between D and its fringe. This proves the space complexity for the representation.

The number of different subsets of D is $2^{|D|}$. Since for every subset $E+F$ of D there is a unique replacement, (if it exists) that a separating k-set specified by Lemma 2, the number of separating k-sets with $C+T$ fixed along with at least one vertex from D is upper bounded by $O(2^{\frac{k-t}{2}})$. This proves the second part of the Lemma.

□

Corollary All cross separating k-sets containing $T+D$ and at least one vertex from C can be represented in $O((\frac{k-t}{2})^2)$ space, and their number is $O(2^{\frac{k-t}{2}})$.

Take the maximal set X of disjoint $C \in G_1$ such that C_i+T+D is a separating k-set. Analogously, take the maximal set Y of disjoint $D \in G_2$ such that $C+T+D_i$ is a separating k-set. For T fixed, all cross separating k-sets are upper bounded by $O(2^{\frac{k-t}{2}} |X| 2^{\frac{k-t}{2}} |Y|) = O(2^{k-t} |X| |Y|)$, and are represented in $O((\frac{k-t}{2})^2 (|X| + |Y|))$ space. Next we will see how many different T's we need to consider.

Take the smallest $T = T_1$ such that a cross separating k-set will have nonempty G_i i=3,4,5,6, if it exist. If there exist a separating k-set with different $T = T_2, T_1 \neq T_2$, then it can be of four different types:

Type 1). $T_2 \cap A \neq \emptyset$ and $T_2 \cap B \neq \emptyset$,

Type 2). $\left[T_2 \cap A = \emptyset \text{ or } T_2 \cap B = \emptyset\right]$ and $T_1 \cap T_2 \neq \emptyset$,

Type 3). $\left[T_2 \cap A = \emptyset \text{ or } T_2 \cap B = \emptyset\right]$ and $T_1 \cap T_2 = \emptyset$,

Type 4). $T_2 \cap A = \emptyset$ and $T_2 \cap B = \emptyset$.

Let us first consider type 4 cross separating k-sets. Since T_2 must lie completely inside T_1 and T_1 has the smallest cardinality, then $T_2 = T_1$. Let the cardinality of X, the maximal disjoint set of C's, be l_1, and let the cardinality of the maximal disjoint set Y be l_2, where $l_1 + l_2 = l$. Let us number A, the set X, B and the set Y. So A becomes A_1, the "nearest" D from Y becomes A_2, and so on going clockwise. The cardinality of this set is $l+2$. From the proof of the Lemma 2 we know that all cross separating k-sets of type 4 consist of three parts: T_1, C which is inside G_1 and is inside some C's from set X and its fringe, and D which is inside G_2 and is inside some D's from set Y and its fringe. Note that $T \cup$ any two $A_i, i=1, \cdots, l+2$ are also separating k-sets if the parts of the graph between them are nonempty. We can also replace parts of A_i by its fringe as long the above condition will be true. Let the part of the graph G between A_i and $A_{i+1}, i=1, \cdots, l+2$ be $G_i, i=1, \cdots, l+2$ (i in this case taken mod $l+2$). Let G_i − the fringe of A_i in G_i − the fringe of A_{i+1} in G_i be $G'_i, i=1, \cdots, l+2$. The only case when $T \cup A_i \cup A_j$ (or parts of the fringe of A_i and A_{i+1}) $i<j$ is not a separating k-set when $i=j-1$ and $G'_i=\varnothing$.

Based upon above observations the structure (structure 1) which covers all cross separating k-sets of type 4 will be the following:

1) A_i with its fringes for all $i=1,\cdots,l+2$,

2) For every nonempty $G'_i, i=1,\cdots,l+2$ we fill all nonexistent edges of the complete graph on the neighbors of G'_i as real edges. If $G'_i, i=1,\cdots,l+2$ is empty for some i then we fill these edges as virtual edges. All of the edges of G between A_i and $G_{i+1}, i=1,\cdots,l+2$ are in the structure as real edges.

Let us see where the rest of the separating k-sets lie assuming there are no cross separating k-sets of type 1 and type 2. Note that we allow separating k-sets of type 3. Let us first the definition of the exceptional separating k-sets. The separating k-set is ***exceptional*** if it separates only part of A_i an nothing else for $i=1,\cdots,l+2$.

Lemma 3: All separating k-sets which are not covered by the structure 2 and not of type 1 and 2 and not exceptions are inside $G_i \cup A_i$ and its fringes inside $G_{i-1} \cup A_{i+1}$ and its fringes inside G_{i+1}.

Proof: Since there are no type 1 and type 2 and no exceptions in separating k-sets, no separating k-set is using T. There are also no cross separating k-set which are not covered by the structure 1. Let us see what happens if a separating k-set crosses some $A_i, i=1,\cdots,l+2$ (see Figure 9).

W.L.O.G. let $E\cup F\cup H$ is this separating k-set, which crosses A_i, where $E\subset G_5$, $F\subset G_6$ and $H\subset A_i$. It divides A_i into A'_i, A''_i and H. It also divides G_5 into G'_i and G''_i, and it divides G_6 into G'_6 and G''_6. Both A''_i and A'_i are nonempty, otherwise the set Y is not maximal, or there is no cross separating k-sets. If G''_5 and G''_6 are nonempty then $E\cup H\cup A''_i$ and $F\cup H\cup A''_6$ are separating sets with cardinalities bigger or equal to k. But both of them can not have cardinality bigger or equal to k, hence, one of G''_5 or G''_6 must be empty. W.L.O.G. let G''_6 be empty. Since $A_{l+1}\cup T\cup A_i$ and $A_{l+1}\cup T\cup A'_i\cup H\cup F$ are separating k-set and separating set, respectively, $|F|\geq|A''_i|$. Since $E\cup H\cup A''_i$ is a separating set, since both G''_5 and G''_6 can not be empty (exception), $|A''_i|\geq|F|$. Hence, $A''_i|=|F|$, and F is part of the fringe of A_i.

Let us see if a cross separating k-set crosses two adjacent A_i's. W.L.O.G. $E\cup H_1\cup F\cup H_2\cup I$ is a separating k-set, which divides A_i into A'_i, H_1, and A''_i, and divides A_{i+1} into A'_{i+1}, H_2, and A''_{i+1}. It separates G_{i-1} into G'_{i-1} and G''_{i-1}, it separates G_i into G'_i and G''_i, it separates G_{i+1} into G'_{i+1} and G''_{i+1}. By the above argument,

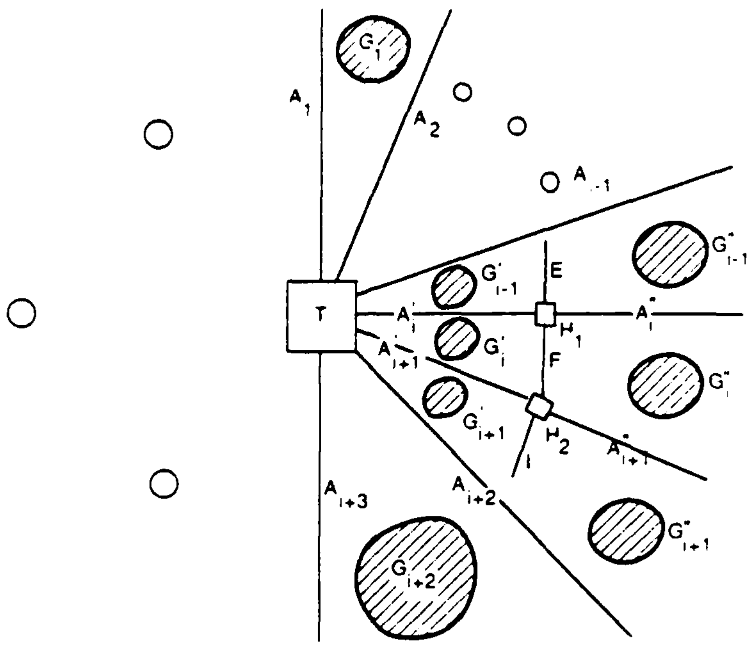

Figure 9.
Illustrating the proof of Lemma 3.

G''_{i-1} and G''_{i+1} are empty, and E belongs to the fringe of A_i, and I belongs to the fringe of $A_{i=1}$. Note that we don't need to use the assumption that there are no exceptions. A cross separating k-set can not cross three adjacent A_i's, since with respect to the middle A_i non of G''_5 and G''_6 can not be empty. Hence, all other separating k-set, except exceptions, belong to $G_i \cup A_i \cup$ its fringes in $G_{i-1} \cup A_{i+1} \cup$ its fringes in G_{i+1}.

□

Let us now consider exceptions. W.L.O.G. let there exist an exceptional separating k-set, which separates part of A_i. In other words, there is a separating k-set which separates part of A_i (A'_i), such that all of the vertices not in $A_i \cup T$ are neighbors of A'_i. The number of the neighbors of A'_i in $G_{i-1} \cup A_{i-1} \cup G_i \cup A_{i+1}$ is less than k. Consider the minimal set of subsets of A_i that covers all vertices of A_i which can be separated by some exceptional separating k-set. The number of subsets in this set is less than or equal to the cardinality of A_i, whence is at most $\frac{k-t}{2}$. The number of neighbors of A_i that are used for separating these subsets is less than or equal to k vertices per subsets, so their total is at most $\frac{k^2}{2}$. Note that $\frac{k^2}{2}-k$ such vertices can be inside either $G_{i-1} \cup A_{i-1}$ or $G_i \cup A_{i+1}$. Moreover, if $v \in A_i$ participates in some subset of A_i, that can be separated by an exceptional separating k-set, then v has less than k vertices in $G_{i-1} \cup A_{i-1} \cup G_i \cup A_{i+1}$. Hence, if we take the union of the following sets

1) $G_i \cup A_i \cup A_{i+1}$

2) the neighbors of A_i in $G_{i-1} \cup A_{i-1}$, that are used for exceptional separating k-sets

3) the fringe of A_i

4) the neighbors of A_{i+1} in $G_{i+1} \cup A_{i+2}$, that are used for exceptional separating k-sets

5) the fringe of A_{i+1} for all i's,

will contain all separating k-sets which are not covered by the structure.

The number of exceptional separating k-set for A_i is bounded by the number of different subsets of A_i. Hence, it is less than or equal to $2^{\frac{k-t}{2}}$. Thus, the number of exceptional separating k-sets is at most $(l+2)2^{\frac{k-t}{2}}$.

Based upon this Lemma and the above observation about exceptions, and using structure 1, we can write the following recurrence, which is valid if there are no type 1 or type 2 separating k-sets:

$$g(n) = \sum_{i=1}^{l+2} g(n_i + k(k-t)+t) + (l+2)(\frac{k-t}{2})k + t ,$$

where every term inside the sum covers one of the G_i's, and $(l+2)(\frac{k-t}{2}) + t$ is the upper bound on the size of the structure 1. Note that $\sum_{i=1}^{l+2} n_i + \frac{(l+2)(k-t)}{2} + t = n$. The solution to this recurrence is $O(kn + k^3)$ (see Appendix). Note that each $(n_i + k(k-t)+t)$ is less than n itself.

Analogously, the recurrence on the upper bound on the number of separating k-sets become

$$f(n) = \sum_{i=1}^{l+2} f(n_i + k(k-t)+t) + 2^{k-t} l \frac{l+2}{2} + 2^{\frac{k-t}{2}} (l+2) .$$

The solution to this recurrence is $O(2^k \frac{n^2}{k})$. Note that all cross separating k-set of type 3 are covered by these recurrences.

Now we will look at type 1. Let $T_2 \cap A = T'_2$, $T_2 \cap B = T''_2$, and $T_1 \cap T_2 = \bar{T}_2$. With respect to a new cross separating k-set which uses T_2 some G_i i=3,4,5,6 could be empty. Let us first look at a harder case when none of G_i i=3,4,5,6 are empty with respect to a new cross separating k-set.

A new cross separating k-set must cross C and D of the old cross separating k-set which uses T_1, otherwise the Claim 2 with respect to the new cross separating k-set will be violated (see Figure 10).

Second, $\bar{T}_2 = T'_2$, otherwise Claim 2 will be contradicted for the old cross separating k-set.

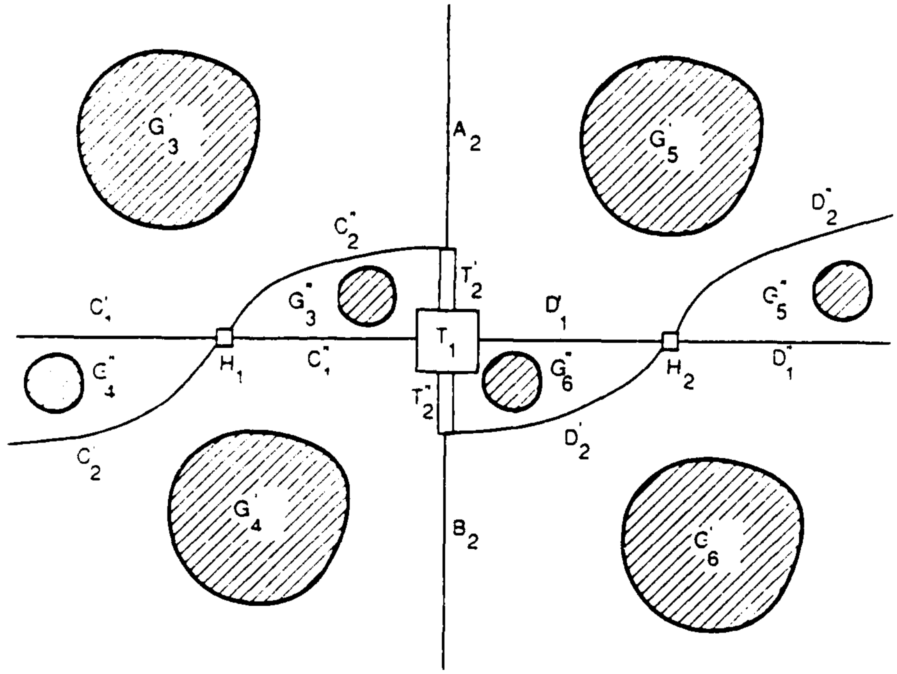

Figure 10.
Illustrating the configuration between two cross separating k-sets which use different T's.

Third, $C'_1+C'_2+H_1+T_1+T''_2$, $C''_1+C''_2+H_1+T_1+T'_2$, $D'_1+D'_2+H_2+T_1+T''_2$, and $D''_1+D''_2+H_2+T_1+T''_2$ are separating sets with cardinalities less than k, which separate G''_4, G''_3, G''_6, and G''_5, respectively. Hence, G''_3, G''_4, G''_5, and G''_6 are empty.

Fourth, $C'_1+H_1+C''_2+T_2+D'_2+H_2+D''_2$, $C'_2+H_1+C''_2+T_2+D'_2+H_2+D''_1$, $C'_2+H_1+C''_1+T_2+D'_2+H_2+D''_2$, and $C'_2+H_1+T_2+D'_1+H_2+D''_2$ are separating sets. Hence, $|C'_1| \geq |C'_2|$, $|D'_1| \geq |D'_2|$, $|C''_1| \geq |C''_2|$, and $|D''_1| \geq |D''_2|$. Also, $C'_1+H_1+C''_2+T'_2+T_1+D'_1+H_2+D''_1$, $C'_2+T''_2+H_1+C''_1+T_1+D'_1+H_2+D''_1$, $C'_1+H_1+C''_1+T_1+T''_2+D'_2+H_2+D''_1$, and $C'_1+H_1+C''_1+T_1+T'_2+D'_1+H_2+D''_2$ are separating sets. Hence,

$$\begin{cases} |C'_2| + |T''_2| \geq |C'_1| \geq |C'_2| > 0 \\ |C''_2| + |T'_2| \geq |C''_1| \geq |C''_2| > 0 \\ |D'_2| + |T''_2| \geq |D'_1| \geq |D'_2| > 0 \\ |D''_2| + |T'_2| \geq |D''_1| \geq |D''_2| > 0 \end{cases}$$

Also since we are still in a Case 1 with respect to both old and new cross separating k-sets, we have the following equalities

$$\begin{cases} |T'_2| = |T''_2| \\ |A_2| = |B_2| = |D'_2| + |H_2| + |D''_2| = |C'_2| + |H_1| + |C''_2| \end{cases}$$

Note that the set T'_2 has edges to the set D''_1, the set T''_2 has edges to the set D'_1, the set T''_2 has edges to the set C'_1, and the set T'_2 has edges to the set C''_1, because of the Claim 2 with respect to the new cross separating k-set. Hence, the maximal disjoint sets for C's and D's (X and Y) will have cardinalities equal to 1.

Let us take a maximal T_2, and let us take the fringes of A_2, B_2, C and D (see Figure 11).

C'_1 does not have the fringe in G_4, otherwise part of C'_1 which has a fringe becomes a part of I'_1. If C'_1 has the fringe in G_3 then the part of C'_1 which has the fringe can be separated from the rest of the graph by a separating set $C'_2+T''_2+T_1+$ the fringe of C'_1 in G_3, whose cardinality is less than k. Hence, C'_1 does not have the fringe. Analogously, C''_1, D'_1, and D''_1 do not have the fringes. Symmetrically, T'_2 and T''_2 do not have the fringes.

Let $\hat{T}_2$ be the union of vertices which are used for all possible T_2 which create a cross separating k-sets with nonempty G_i i=3,4,5,6. Let $\hat{D}'_1$ be the union of all possible D'_1, $\hat{D}''_1$ be the union of all possible D''_1, $\hat{C}'_1$ be the union of all possible C'_1, $\hat{C}''_1$ be the union of all possible C''_1, $\hat{C}'_2$ be the union of all possible C'_2, $\hat{C}''_2$ be the union of all possible C''_2, $\hat{D}'_2$ be the union of all possible D'_2, and $\hat{D}''_2$ be the union of all possible D''_2. Let us show that all of these sets are disjoint.

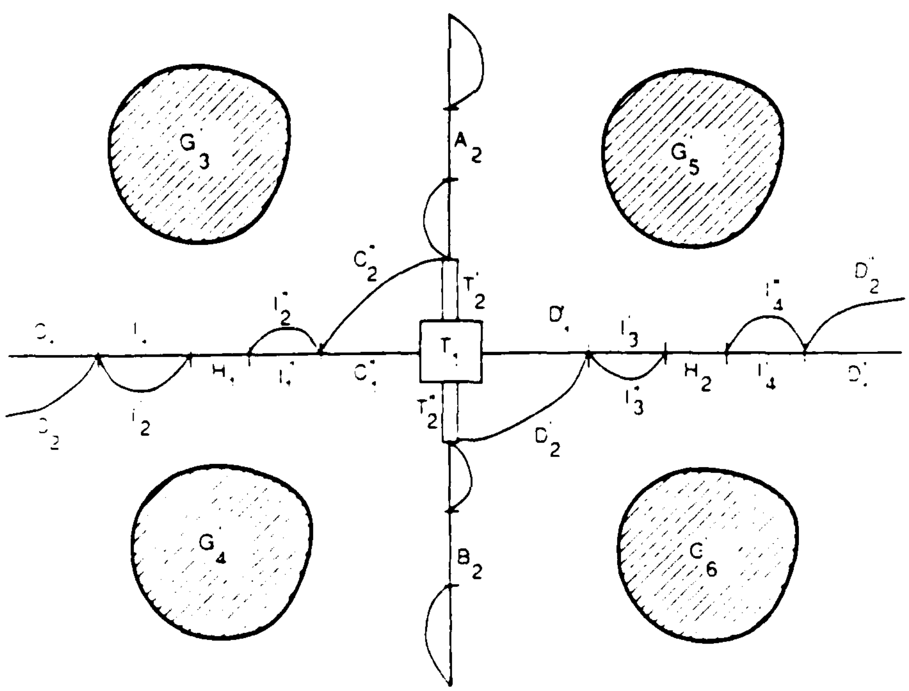

Figure 11.
Illustrating the representation of separating k-sets of Case 1
if two or more different intersecting T's exist.
(Structure 2).

Since all of them are symmetric we will prove it only for $\hat{C}'_1$ and $\hat{C}''_1$. Assume there are T_3 and T_4 such that C''_1 for T_3 is not disjoint from C'_1 for T_4. Then nonempty intersection of C''_1 for T_3 and C'_1 for T_4 is separated from the rest of the graph by a separating set C''_2 for $T_3 \cup T'_3 \cup T_1 \cup T''_4 \cup C'_2$ for T_4, whose cardinality is less than k. This contradiction proves the statement.

The cardinality of the union $\hat{D}''_2 \cup \hat{D}'_2 \cup I''_4 \cup I'_4$ is less than $\frac{k-t}{2}$, and analogously, the cardinality of $\hat{C}''_2 \cup \hat{C}'_2 \cup I'_1 \cup I''_2$ is less than $\frac{k-t}{2}$. Let us call $\hat{C}'_2$, $\hat{C}''_2$, $\hat{D}'_2$, and $\hat{D}''_2$ the *pseudofringe*. Note that A and B might have fringes , but by the symmetry $\hat{T}_2 - T_1$ does not have any fringes.

The structure which represent all separating k-sets for all possible T's will the following (structure 2):

1) the original separating k-set with its fringes,

2) the cross separating k-set with minimum cardinality T_1 with its fringes and pseudofringes,

3) for every nonempty G'_i i=3,4,5,6 we will fill all nonexistent edges of the complete graph on the neighbors of G'_i, if G'_i is empty for any i=3,4,5,6 we will fill these nonexistent edges of this complete graph by the virtual edges. (For G'_3 we fill the edges between the vertices of the fringe of A in G_3, T_1, $\hat{T}'_2$, part of A_2 which does not have any fringes, $\hat{C}'_1$, I'_1, H_1, I''_2 and $\hat{C}''_2$).

From the construction of the structure it is easy to see that this structure cavers all cross separating k-sets for all possible T's, of type 1. Let us see now where the rest of the separating k-sets lie, if we have separating k-sets of type 1.

If there exists T_2 with at least one of the G_i empty i=3,4,5,6, assuming it is not exception, such that there is another T_2 with $T_2 \cap T_1$ is nonempty along with nonempty $T_2 \cap B$ and $T_2 \cap A$, then all cross separating k-sets of this T_2 are covered by the above structure. (They belong to the fringes of A and/or B in G_1 or G_2 and the rest belong to the original cross separating k-set with its fringes or pseudofringes). So all cross separating k-sets are covered by this structure, assuming there are no exceptions, hence, all separating k-sets are either inside $G_1 \cup A \cup B \cup T_1 \cup$ the fringes of A and B in G_2, or $G_2 \cup A \cup B \cup T_1 \cup$ the fringes of A and B in G_1, or cross separating k-sets covered by the structure. Since the structure is symmetric, we can look at the cross separating k-sets where the original separating k-set is $C \cup D \cup T_1$. Then the pseudofringes of C and D become the pseudofringes of A and B. With respect to this separation of G all separating k-sets are either inside $G_3 \cup G_4 \cup C \cup D \cup T_1 \cup$ the fringe of C in G_3 and the fringe of

D in G_6, or inside $G_4 \cup G_6 \cup C \cup D \cup T_1 \cup$ the fringe of C in G_3 and the fringe of D in G_5, or separating k-sets covered by the structure. But since in both cases they are the same separating k-sets, all separating k-sets are either inside $G_3 \cup A \cup T_1 \cup C \cup$ the fringe of C in $G_4 \cup$ the fringe of A in G_5, or inside $G_4 \cup B \cup C \cup T_1 \cup$ the fringe of B in G_6, or inside $G_5 \cup A \cup D \cup T_1 \cup$ the fringe of A in $G_3 \cup$ the fringe of D in G_6, or inside $G_6 \cup B \cup D \cup T_1 \cup$ the fringe of B in $G_4 \cup$ the fringe of D in G_5, or the separating k-sets covered by the structure. To cover all exceptions we will do what we did for types 3 and 4 separating k-sets, we will add $k(k-t)$ neighbors of A, B, C and D to each of G_3, G_4, G_5 and of G_6 which can participate in exceptional separating k-sets. Hence, the size of representation is

$$g(n) = \sum_{i=1}^{4} g(n_i + k(k-t)+t) + 8\frac{(k-t)}{2}k + t ,$$

where every term inside the sum covers one of G_i i=3,4,5,6 along with its appropriate neighbors and fringes, and $8\frac{(k-t)}{2}k + t$ is the upper bound on the size of the structure. Note that $\sum_{i=1}^{4} n_i + 2k - t = n$, hence the solution to the above recurrence is $O(nk + k^3)$ (see Appendix). The number of exceptional separating k-sets is upper bounded by $4 \cdot 2^{\frac{k-t}{2}}$. The upper bound on the number of separating k-sets become

$$f(n) = \sum_{i=1}^{4} f(n_i + k(k-t)+t) + \binom{4}{2} \cdot 2^{k-t} + 4 \cdot 2^{\frac{k-t}{2}} .$$

The solution to it is $O(2^k n + 2^k k^2)$ (see Appendix).

Let us now see what happens if we are in type 2 and no separating k-sets of type 1 exist. W.L.O.G. assume there is a separating k-set which uses $T_2 = T'_2 \cup \overline{T}_2$, where $T'_2 \in A$ and $\overline{T}_2 \in T_1$, and no separating k-set of type 1 exist (see Figure 12).

If G_i's i=3,4,5,6 are nonempty with respect to a new cross separating k-set then we become in the Case 1 with respect to a new cross separating k-set, hence $|A_2| = |B|$ which is impossible. Hence, one of the G_i i=3,4,5,6 with respect to a new cross separating k-set must be empty. W.L.O.G. let the empty G_i be either G_3 or G_4 with respect to the new cross separating k-set. If G_4 is empty then G_5 with respect to the new cross separating k-set must be empty, otherwise $T_1 \cup T'_2 \cup A_2 \cup D_2$ of the new cross separating k-set becomes a separating set with cardinality less than k. Hence, if G_4 is empty then all cross separating k-set of type 2 belong to the original separating k-set with its fringes. Then all separating k-set are either inside $G_3 \cup A \cup B \cup T_1 \cup$ the fringe of A in $G_5 \cup$ the fringe of B in G_6, or $G_5 \cup A \cup B \cup T_1 \cup$ the fringe of A in $G_3 \cup$ the fringe of B in G_4, or they belong to the union of $A \cup B \cup T_1 \cup$ the fringes of A and B. Note that the latter separating k sets are covered by the structure 2. We can write the recurrences

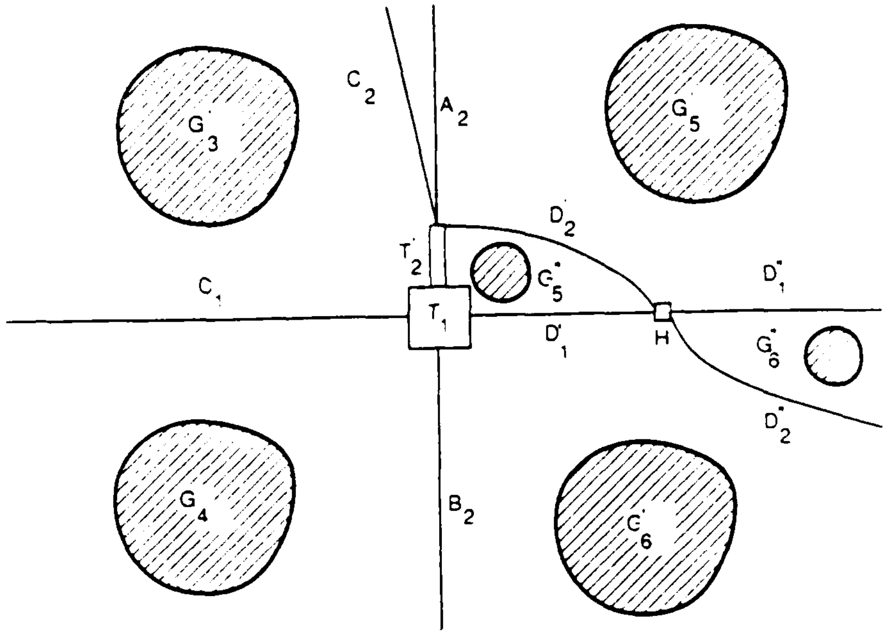

Figure 12.
Illustrating type 2 separating k-set when no type 1 separating k-set exist.

similar to the above ones except for the sum which will be up to 2 instead of up to 4. The solution will be still of the same order. If G_3 is empty then $|C_2| \geq |A_2|$, otherwise $C_2 \cup T'_2 \cup T_1 \cup B$ is a separating set with cardinality less than k. If D_2 crosses D_1 (see Figure 12) then $A_2 \cup T'_2 \cup T_1 \cup D_2$ is a separating set, so $|C_2| = |A_2|$. $C \cup T_1 \cup D'_1 \cup H \cup D''_2$ is a separating set, so $|D''_2| \geq |D''_1|$. Also $C_2 \cup T_2 \cup D'_2 \cup H \cup D''_1$ is a separating set, so $|D''_1| \geq |D''_2|$. Combining these two we get $|D''_1| = |D''_2|$. Since, $C \cup T_1 \cup T'_2 \cup D'_2 \cup H \cup D''_1$ and $C_2 \cup T'_2 \cup T_1 \cup D'_1 \cup H \cup D''_2$ are separating sets, so $|T'_2 \cup D'_2| \geq |D'_1| \geq |D'_2|$. Since $T_1 \cup D''_1 \cup H \cup D''_2$ separates G''_6 from the rest of the graph, and since the cardinality of this separating set is less than k, G''_6 is empty. Hence, D''_2 belongs to the fringe of D in G_6. $\bar{T}_2 = T_1$ in order for the Claim 2 with respect to the old cross separating k-set to be true. And since $|C_2| + |T'_2| = |A|$ and since the cardinality of the new cross separating k-set is k, $|D'_2| = |D'_1|$. So, all cross separating k-sets of this type belong to $G_5 \cup A \cup D \cup T_1 \cup$ the fringe of A in $G_3 \cup$ the fringe of D in G_6, if there are no exceptional separating k-sets. Also in the maximal set of disjoint D's (Y) all of D's except D_2 belong to G_6. If G_5 with respect to the new cross separating k-set is nonempty, then by the above argument C_2 will belong to the fringe of A. Hence, all cross separating k-sets belong to the set mentioned above, namely, $G_5 \cup A \cup T_1 \cup D_2 \cup$ the fringe of A in $G_1 \cup$ the fringes of D_2 in G_6.

Let us take the maximal set of C's and D's (X and Y). We know that all cross separating k-sets of type 2 with nonempty G_5 belong to $G_5 \cup A \cup D \cup T_1 \cup$ the fringe of A in $G_3 \cup$ the fringe of D in G_6. Since we need to consider all symmetric cases, and since we don't have any cross separating k-sets of type 1, all cross separating k-sets of the type 2 belong to $G_3 \cup A \cup C \cup T_1 \cup$ the fringe of A in $G_5 \cup$ the fringe of C in G_4, or $G_4 \cup B \cup C \cup T_1 \cup$ the fringe of B in $G_6 \cup$ the fringe of C in G_3, or $G_5 \cup A \cup D \cup T_1 \cup$ the fringe of A in $G_3 \cup$ the fringe of D in G_6, or $G_6 \cup B \cup D \cup T_1 \cup$ the fringe of B in $G_4 \cup$ the fringe of D in G_5. Note that C's and D's are not the same in these sets. In case of G_3 C is "nearest" to A, in case of G_4 C is "nearest" to B, in case of G_5 D is "nearest" to A, and in case of G_6 D is "nearest" to B. Let us see where the rest of separating k-sets must lie. First, if there are no cross separating k-sets with G_5 nonempty (or same other appropriate symmetric G_i i=3,4,5,6) then it is still possible to have a cross separating k-sets.

All cross separating k-sets consist of three parts: part one is in G_1, part two is in G_2 and part three is T_1. Part one belongs to some C from the set X or its fringe or the fringe of A in G_3 or the fringe of B in G_4. Part two belongs to some D from the set Y or its fringe or the fringe of A in G_5 or the fringe of B in G_6. That covers all cross separating k-sets which use T_1, otherwise either set X or set Y is not maximal. We don't have any cross separating k-sets of type 1. All cross separating k-sets of type 2 with nonempty appropriate G_i with respect to them belong to the part of the graph between A and the nearest D in G_2 along with A and its fringe and D and its fringe. Hence, all other separating k-sets belong to $G_1 \cup A \cup B \cup T_1$ with its fringes, or $G_2 \cup A \cup B \cup T_1$ with its fringes.

Hence, all cross separating k-sets of type 2, except exceptions are covered by the structure 2 or inside the the subgraphs associated by G_1, G_{l_1+1}, G_{l_1+2} and G_{l+2}. As for the exceptions the upper bounds we got for types 3 and 4 still hold, since no part of T_1 can be separated by them (otherwise Claim 2 is contradicted). So, the recurrence which were written for the type 3 and 4 separating k-sets covers type 2 cross separating k-sets also, including exceptions. That conclude Case 1.

$\square$

Case 2 For any separating k-set every cross separating k-set will have one of the G_i i=3,4,5,6 empty. Not every vertex in both G_1 and G_2 can be used for cross separating k-sets.

W.L.O.G. let G_3 will be empty (see Figure 13).

Since G_4 is nonempty by assumption, and G_5 is nonempty since there are no exception, $C \cup T \cup B$ and $A \cup T \cup D$ are separating sets. So their cardinalities are bigger or equal to k, hence, $|C| = |A|$ and $|B| = |D|$. So, C is part of the fringe of A in G_1. Since this true for every T, all cross separating k-sets belong to $G_1 \cup A \cup T \cup B \cup$ the fringes of

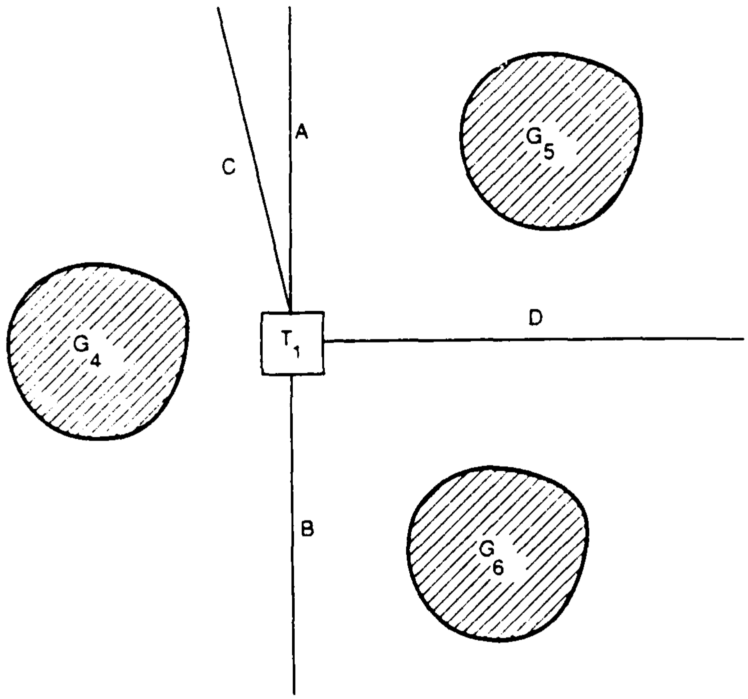

Figure 13.
Illustrating Cases 2 and 3.

A and B in G_2, or $G_2 \cup A \cup T \cup B \cup$ the fringes of A and B in G_1, except for exceptions. So all separating k-sets including the exceptions are either inside $G_1 \cup A \cup B \cup T \cup$ appropriate at most k^2 neighbors of $A \cup T \cup B$ in G_2 or inside $G_2 \cup A \cup B \cup T \cup$ appropriate at most k^2 neighbors of $A \cup T \cup B$ in G_1 which are used in exceptional separating k-sets. Hence,

$$g(n) = g(n_1 + k(k-1)) + g(n_2 + k(k-1)) + 4k^2,$$

where n_1 and n_2 are the cardinalities of G_1 and G_2. We still have that $n_1 + n_2 + k = n$, and the solution to this recurrence is $O(k^2 + n)$ (see Appendix). Note that $n_i + k(k-1) < n$ for $i=1,2$.

For the upper bound on the number of separating k-sets we get the following equality

$$f(n) = f(n_1 + 2k) + f(n_2 + 2k) + 2^k,$$

where 2^k covers all exceptional separating k-sets. And its solution is clearly smaller than $O(2^k \frac{n^2}{k})$ (see Appendix). That conclude Case 2.

□

Case 3 For every separating k-set all cross separating k-sets are lopsided (one of the G_i i=3,4,5,6 will be empty). And either G_1 or G_2 are such that every vertex of them is used for some cross separating k-set.

W.L.O.G. let G_3 be empty and the smallest G_1 every vertex of G_1 is used for some cross separating k-set (see Figure 13). There are two subcases: either G_5 or G_6 are empty, otherwise we will be in Case 2. Take C as large as possible.

If G_6 is empty then $A \cup B \cup C \cup D \cup T$ with all edges between them and filling real edges for nonempty G_5 and G_4 and virtual otherwise (analogous to the structure 1) will specify all cross separating k-sets. If G_5 is empty then $C \cup T \cup D$ separate A from the rest of the graph. Hence, $C \cup T \cup D$ is an exceptional separating k-set. So the third structure will be the following:

1) A, B and T - the original separating k-set,

2) All the neighbors of $A \cup B \cup T$ that are used for a cross separating k-sets with edges between them and the original separating k-set.

since the remaining separating k-sets are inside $G_2 \cup A \cup B \cup T$, we derive the following recurrence relation:

$$g(n) = g(n-1) + k^2,$$

whose solution is $f(n) = O(k^2 n)$. Analogously, we have the following recurrence relation for the upper bound on the number of separating k-sets

$$f(n) = f(n-1) + 2^k,$$

whose solution is $O(2^k n)$.

□

That conclude the proof of all cases. Our final result is that all separating k-sets have $O(k^2 n)$ space representation, and their number is $O(2^k \frac{n^2}{k})$.

Acknowledgement

I am indebted to Prof. Vijaya Ramachandran for her helpful comments, suggestions, discussions and references.

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APPENDIX

$$\sum_{i=1}^{l}(n_i + 1) = n \qquad 2 \le l \le n \qquad n_i \ge 0$$

$$g(n) \le \max_l\left(\sum_{i=1}^{l} g(n_i + 2) + 4l\right)$$

Let $g(n) = 4n - 16$,

$$g(n) \le \max_l\left(\sum_{i=1}^{l} g(n_i + 2) + 4l\right) = \max_l\left(\sum_{i=1}^{l}(4(n_i + 2) - 16) + 4l\right) =$$

$$\max_l\left(4\sum_{i=1}^{l}(n_i + 1) + 4l - 16l + 4l\right) = \max_l(4n - 8l) \le 4n - 16$$

$$\sum_{i=1}^{l}(n_i + 1) + 1 = n \qquad 2 \le l \le n-1 \qquad n_i \ge 0$$

$$g(n) \le \max_l\left(\sum_{i}^{l} g(n_i + 5) + 6l + 1\right)$$

Let $g(n) = 6n - 55$,

$$g(n) \le \max_l\left(\sum_{i=1}^{l} g(n_i + 5) + 6l + 1\right) = \max_l\left(\sum_{i=1}^{l}(6(n_i - 55) + 6l + 1\right) =$$

$$\max_l\left(6\left(\sum_{i=1}^{l}(n_i + 1) + 1\right) - 31l + 6l + 1\right) = \max_l(6n - 25l - 5) \le 6n - 55$$

$$\sum_{i=1}^{l}\left(n_i + \frac{k-t}{2}\right) + t = n \qquad 0 \le t \le k-2 \qquad 2 \le l \le 2\frac{n-t}{k-t} \qquad n_i \ge 0$$

$$g(n) \le \max_l\left(\sum_{i=1}^{l} g(n_i + (k-t)k + t) + lk\frac{(k-t)}{2} + t\right.$$

Let $g(n) = 2nk - 4k^3 - 2k^2t + \frac{1}{2}k^2 - 3kt - t$,

$$g(n) \le \max_l \Big(\sum_{i=1}^{l} g(n_i + (k-t)k + t) + lk\frac{k-t}{2} + t\Big) \le$$

$$\max_l \Big(\sum_{i=1}^{l} 2k(n_i + k(k-t) + t) - 4k^3 l + 2k^2 tl + \frac{1}{2}k^2 l - ktl - tl + lk\frac{k-t}{2} + t\Big) =$$

$$\max_l \Big(2k\Big(\sum_{i=1}^{l}\Big(n_i + \frac{k-t}{2}\Big) + t\Big) - 2kl\frac{k-t}{2} - 2kt + 2k^2 l(k-t) + 2ktl - 4k^3 l + 2k^2 tl + \frac{1}{2}k^2 l - 3ktl - tl + lk\frac{k-t}{2} + t\Big) =$$

$$\max_l \Big(2kn + 2k^3(l - 2l) + 2k^2 t(-l + l) + k^2\Big(\frac{1}{2}l + \frac{l}{2} - l\Big) + kt\Big(l - 2 + 2l - \frac{l}{2} - 3l\Big) + t(-l + 1)\Big) \le$$

$$2kn - 4k^3 - 3kt + t \le 2kn - 4k^3 + 2k^2 t + \frac{1}{2}k^2 - 3kt - t$$

Hence, $g(n) = O(nk + k^3)$.

$$\sum_{i=1}^{l}\Big(n_i + \frac{k-t}{2}\Big) + t = n \qquad 2 \le l \le 2\frac{k-t}{k-t} \qquad 0 \le t \le n-2$$

$$f(n) \le \max_l \Big(\sum_{i=1}^{l} f(n_i + k(k-t) + t) + 2^{k-t}\frac{l(l-2)}{2} + 2^{\frac{k-t}{2}}l\Big)$$

Let

$$f(n) = 2^{k-t}nl - 2^{k-t}k^2 l + 2^{k-t}ktl + \frac{1}{2}2^{k-t}kl - \frac{3}{2}2^{k-t}tl + 2^{k-t}kt + \frac{1}{2}2^{k-t}k - 2\,2^{k-t}k^2 - 2^{k-t}t - \frac{1}{2}2^{k-t}l - 2\,2^{\frac{k-t}{2}},$$

$$f(n) \le \max_l \Big(\sum_{i=1}^{l}(n_i k(k-t) + t)2^{k-t}l - 2^{k-t}k^2 l^2 + 2^{k-t}ktl^2 + \frac{1}{2}2^{k-t}kl^2 - \frac{3}{2}2^{k-t}tl^2 + 2^{k-t}ktl +$$

$$\frac{1}{2}2^{k-t}kl - 2\,2^{k-t}k^2 l - 2^{k-t}tl - \frac{1}{2}2^{k-t}l^2 - 2\,2^{\frac{k-t}{2}} + \frac{1}{2}2^{k-t}l^2 - \frac{1}{2}2^{k-t}l + 2^{\frac{k-t}{2}}l\Big) = \max_l \Big(2^{k-t}\ln -$$

$$\frac{1}{2}2^{k-t}kl^2 + \frac{1}{2}2^{k-t}tl^2 - 2^{k-t}tl + 2^{k-t}k^2 l^2 - 2^{k-t}ktl^2 + 2^{k-t}tl^2 - 2^{k-t}k^2 l^2 + 2^{k-t}ktl^2 + \frac{1}{2}2^{k-t}kl^2 -$$

$$\frac{3}{2}2^{k-t}tl^2 + 2^{k-t}ktl + \frac{1}{2}2^{k-t}kl - 2\,2^{k-t}k^2 l - 2^{k-t}tl - \frac{1}{2}2^{k-t}l^2 - 2\,2^{\frac{k-t}{2}}l + \frac{1}{2}2^{k-t}l^2 - \frac{1}{2}2^{k-t}l + 2^{\frac{k-t}{2}}l\Big) =$$

$$\max_l \Big(2^{k-t}\ln - 2\,2^{k-t}k^2 l + 2^{k-t}ktl + \frac{1}{2}2^{k-t}kl - 2\,2^{k-t}kl - 2\,2^{k-t}tl - \frac{1}{2}2^{k-t}l - 2^{\frac{k-t}{2}}l\Big) \le$$

$$\max\Big(2^{k-t}\ln - 2^{k-t}k^2 l + 2^{k-t}ktl + \frac{1}{2}2^{k-t}kl - \frac{3}{2}2^{k-t}tl + 2^{k-t}kt + \frac{1}{2}2^{k-t}k - 2\,2^{k-t}k^2 - 2^{k-t}t - \frac{1}{2}2^{k-t}l - 2\,2^{\frac{k-t}{2}}\Big)$$

Hence, $f(n) = O(2^k \frac{n^2}{k} + 2^k nk)$.

$$\sum_{i=1}^{4} n_i + 2k - t = n \qquad 0 \le t \le k-2$$

$$g(n) \le \sum_{i=1}^{4} g(n_i + k(k-t) + t) + 8k\frac{k-t}{2} + t$$

Let $g(n) = 4nk - \frac{16}{3}k^3 + \frac{16}{3}k^2 t + \frac{4}{3}k^2 - \frac{16}{3}kt - \frac{1}{3}t$,

$$g(n) \le \sum_{i=1}^{4} g(n_i + k(k-t) + t) + 4(k-t)k + t \le$$

$$\sum_{i=1}^{4}(4(n_i + k(k-t) + t)k - \frac{16}{3}k^3 + \frac{16}{3}k^2 t + \frac{4}{3}k^2 - \frac{16}{3}kt - \frac{1}{3}t) + 4(k-t)k + t =$$

$$4k(\sum_{i=1}^{4} n_i + 2k - t) - 8k^2 + 4kt + 16k^3 - 16k^2 t + 16kt - \frac{64}{3}k^3 + \frac{64}{3}k^2 t + \frac{16}{3}k^2 - \frac{64}{3}kt - \frac{4}{3}t + 4k^2 - 4kt + t =$$

$$4kn + k^3(16 - \frac{64}{3}) + k^2 t(\frac{64}{3} - 16) + k^2(\frac{16}{3} - 8 + 4) + kt(4 + 16 - \frac{64}{3} - 4) + t(1 - \frac{4}{3}) =$$

$$4kn - \frac{16}{3}k^3 + \frac{16}{3}k^2 t + \frac{4}{3}k^2 - \frac{16}{3}kt - \frac{1}{3}t$$

Hence, $g(n) = O(nk + k^3)$.

$$\sum_{i=1}^{4}(n_i + \frac{k-t}{2}) + t = n \qquad 0 \le t \le n-2$$

$$f(n) \le \sum_{i=1}^{4} f(n_i + k(k-t) + t) + 6\,2^{k-t} + 4\,2^{\frac{k-t}{2}}$$

Let $f(n) = 2^{k-t}n - \frac{4}{3}2^{k-t}k^2 + \frac{4}{3}2^{k-t}kt - \frac{5}{3}2^{k-t}t + \frac{2}{3}2^{k-t}k - 2\,2^{k-t} - \frac{4}{3}2^{\frac{k-t}{2}}$,

$$f(n) \le \sum_{i=1}^{4} f(n_i + k(k-t) + t) + 6\,2^{k-t} + 4\,2^{\frac{k-t}{2}} \le \sum_{i=1}^{4}(2^{k-t}(n_i + k(k-t) + t) - \frac{4}{3}2^{k-t}k^2 + \frac{4}{3}2^{k-t}kt -$$

$$\frac{5}{3}2^{k-t}t + \frac{2}{3}2^{k-t}k - 2\,2^{k-t} - \frac{4}{3}2^{\frac{k-t}{2}}) + 6\,2^{k-t} + 4\,2^{\frac{k-t}{2}} = 2^{k-t}n - 2^{k-t}k + 2\,2^{k-t}t - 2^{k-t}t +$$

$$4\,2^{k-t}k^2 - 4\,2^{k-t}kt + 4\,2^{k-t}t - \frac{16}{3}2^{k-t}k^2 + \frac{16}{3}2^{k-t}kt - \frac{20}{3}2^{k-t}t + \frac{8}{3}2^{k-t} - \frac{16}{3}2^{\frac{k-t}{2}} + 6\,2^{k-t} + 4\,2^{\frac{k-t}{2}} =$$

$$2^{k-t}n - \frac{4}{3}2^{k-t}k^2 + \frac{4}{3}2^{k-t}kt - \frac{5}{3}2^{k-t}t + \frac{2}{3}2^{k-t}k - 2\,2^{k-t} - \frac{4}{3}2^{\frac{k-t}{2}}$$

$$n_1 + n_2 + k = n \qquad n_1, n_2 \geq 0$$

$$g(n) \leq g(n_1 + k(k-1)) + g(n_2 + k(k-1)) + 4k^2$$

Let $g(n) = n - 6k^2 + 3k$,

$$g(n) \leq n_1 + k^2 - k - 6k^2 + 3k + n_2 + k^2 - k - 6k^2 + 3k + 4k^2 = n - 6k^2 + 3k$$

$$n_1 + n_2 + k = n \qquad n_1, n_2 \geq 0$$

$$f(n) \leq f(n_1 + 2k) + f(n_2 + 2k) + 2^k$$

Let $f(n) = 2^k n - 3\,2^k k - 2^k$,

$$f(n) \leq 2^k n_1 + 2k2^k - 3\,2^k k - 2^k + 2^k n_2 + 2k2^k - 3\,2^k k - 2^k + 2^k = 2^k n - 3\,2^k k - 2^k$$

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