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POINTWISE STABILIZATION FOR COUPLED QUASILINEAR AND LINEAR WAVE EQUATIONS

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§I. Introduction.

A large structure is formed by the coupling of simple structural elements. In this paper we study the simplest type of such structures which is made up to two coupled strings modelled by quasilinear or linear wave equations. We install two stabilizers: one at the (left) boundary and one at an in-span point. We wish to study the exponential stability property of this coupled dynamic structure. The nonlinear or linear partial differential equations are described below:

$$(\text{nonlinear}) \quad \begin{cases} \dfrac{\partial^2}{\partial t^2} w(x,t) - \dfrac{\partial}{\partial x}\left[\sigma_1\left(\dfrac{\partial w(x,t)}{\partial x}\right)\right] = 0, & 0 < x < 1, \\[3mm] \dfrac{\partial^2}{\partial t^2} w(x,t) - \dfrac{\partial}{\partial x}\left[\sigma_2\left(\dfrac{\partial w(x,t)}{\partial x}\right)\right] = 0, & 1 < x < 2, \end{cases} \tag{1.1}$$

$$(\text{linear}) \quad \begin{cases} \dfrac{\partial^2}{\partial t^2} w(x,t) - c_1^2 \dfrac{\partial^2 w(x,t)}{\partial x^2} = 0, & 0 < x < 1, \\[3mm] \dfrac{\partial^2}{\partial t^2} w(x,t) - c_2^2 \dfrac{\partial^2 w(x,t)}{\partial x^2} = 0, & 1 < x < 2, \end{cases} \tag{1.2}$$

where

σ_1, σ_2 satisfy $\sigma_i(0) = 0$, $\sigma_i'(u) > 0$, $i = 1, 2$,

and

$c_1 = \sqrt{T_1/\rho}$, $c_2 = \sqrt{T_2/\rho}$

T_i = tension constant on string i, $i = 1, 2$,

ρ = mass density per unit length.

We have tacitly assumed that the mass densities on both strings are identical. (This information is contained in σ_1 and σ_2 in (1.1)). The length of each string has been normalized to 1.

At the left end $x = 0$, a stabilizer is installed satisfying, respectively, the following condition

$$\sigma_1(w_x(0,t)) - k_0^2 w_t(0,t) = 0 \qquad \text{(for (1.1))}, \quad k_0^2 \geq 0, \qquad (1.3)$$

$$c_1^2 w_x(0,t) - k_0^2 w_t(0,t) = 0 \qquad \text{(for (1.2))}, \quad k_0^2 \geq 0. \qquad (1.4)$$

At the intermediate node $x = 1$, another stabilizer is installed according to one of the following two sets of dissipative transmission conditions (compare [2]) hold:

$$\left. \begin{array}{l} \sigma_1(w_x(1^-,t)) = \sigma_2(w_x(1^+,t)), \\[6pt] w_t(1^-,t) - w_t(1^+,t) = -k_1^2\sigma_1(w_x(1^-,t))(= -k_1^2\sigma_2(w_x(1^+,t))), \quad k_1^2 \geq 0, \end{array} \right\} (1.5)$$

or

$$\left. \begin{array}{l} w(1^-,t)) = w(1^+,t), \\[6pt] \sigma_1(w_x(1^-,t)) - \sigma_1(w_x(1^+,t)) = -k_1^2 w_t(1^-,t)(= -k_1^2 w_t(1^+,t)), \quad k_1^2 \geq 0, \end{array} \right\} (1.5)'$$

for the nonlinear system (1.1). For the linear system, the counterparts are

$$\left. \begin{array}{l} c_1^2 w_x(1^-,t) = c_2^2 w_x(1^+,t) \\[6pt] w_t(1^-,t) - w_t(1^+,t) = -k_1^2 c_1^2 w_x(1^-,t)(= -k_1^2 c_2^2 w_x(1^+,t)), \quad k_1^2 \geq 0, \end{array} \right\} (1.6)$$

or

$$\left. \begin{array}{l} w(1^-,t) = w(1^+,t) \\[6pt] c_1^2 w_x(1^-,t) - c_2^2 w_x(1^+,t) = -k_1^2 w_t(1^-,t)(= -k_1^2 w_t(1^+,t)), \quad k_1^2 \geq 0. \end{array} \right\} (1.6)'$$

Note that if $k_1^2 = 0$ in (1.5), (1.5)', (1.6) or (1.6)', then there is no loss of energy at $x = 1$ and the joint is conservative. We call k_0^2 and k_1^2 the feedback gains at $x = 0$ and $x = 1$.

At the right end $x = 2$, we may assume that it is either fixed or free:

$$w(2,t) = 0 \qquad \text{(fixed end)} \qquad (1.7)$$

$$w_x(2,t) = 0 \qquad \text{(free end)} \qquad (1.7)'$$

We want to study the effects of stabilizers for coupled nonlinear and linear vibrating strings as described above. We will be primarily concerned with the

following question:

[Q] "When does the solution of the above coupled wave equations decay exponentially?"

The answer to the above question is well understood in the case of a single (nonlinear or linear) vibrating string. Greenberg and Li [6] proved that for (1.1), (1.3) and a conservative boundary condition at $x = 1$, the solution decays exponentially in the C^1-norm for any $k_0^2 > 0$, provided that the initial data is sufficiently small, smooth, and that appropriate compatibility conditions are satisfied. In the linear case, uniform exponential decay of energy follows as a simple exercise of the method of characteristics.

The problem of coupled vibrating strings is equivalent to that of a hyperbolic system. Some recent papers by Chen-Coleman-West [2] and Qin [9] have provided partial answers to [Q] under study here. Another recent paper [7] by F.L. Huang has also introduced a direct way of proving exponential decay of solutions for linear systems. His method is to establish a uniform bound of the resolvent operator on the imaginary axis. In this paper, we will provide answers to [Q] along the directions of [2], [7] and [9].

The organization of our paper is as follows. In §II, we first give a counterexample of Qin's theorem in [9] and state its correct version. In §III, we apply the corrected theorem of Qin to coupled nonlinear vibrating strings and state some sufficient conditions for exponential decay of solutions with small data. In §IV, we apply Huang's theorem to give a complete answer to [Q] for coupled linear strings.

§II. A Counterexample and Correction to a Theorem by T.H. Qin [9] for Quasilinear Hyperbolic Systems with Dissipative Boundary Conditions

Let

$$\frac{\partial u(x,t)}{\partial t} + A(u(x,t))\frac{\partial u(x,t)}{\partial x} = 0, \qquad 0 < x < 1, \quad t > 0 \qquad (2.1)$$

be a first order quasilinear hyperbolic system, where $u \in \mathbb{R}^N$ and $A(u)$ is an $N \times N$ sufficiently smooth matrix function of the variable u only.

As in [9], assume that

(A1) System (2.1) is hyperbolic for sufficiently small $|u|$ in the following sense:

i) The matrix $A(u)$ has N smooth real eigenvalues $\lambda_1(u), \ldots, \lambda_N(u)$ and

$$\lambda_1(0) \leq \ldots \leq \lambda_m(0) < 0 < \lambda_{m+1}(0) \leq \ldots \leq \lambda_N(0). \qquad (2.2)$$

ii) $A(u)$ has N linearly independent left eigenvectors

$$\xi_j(u) = (\xi_{j1}(u),\ldots,\xi_{jN}(u)), \qquad 1 \leq j \leq N \qquad (2.3)$$

corresponding to each real eigenvalue $\lambda_j(u)$. □

Without loss of generality, we assume that the matrix

$$\xi(u) = \begin{bmatrix} \xi_{11}(u)\ldots\ldots\xi_{1N}(u) \\ \ldots\ldots\ldots\ldots \\ \xi_{N1}(u)\ldots\ldots\xi_{NN}(u) \end{bmatrix} \qquad (2.4)$$

is identity when $u = 0$, i.e.,

$$\xi(0) = \begin{bmatrix} 1 & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}_{N \times N}. \qquad (2.5)$$

Let

$$u^I = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}, \qquad u^{II} = \begin{bmatrix} u_{m+1} \\ \vdots \\ u_N \end{bmatrix}. \qquad (2.6)$$

Due to (2.2) and (2.5), general boundary conditions for system (2.1) should have the following forms:

$$\left. \begin{array}{ll} u^{II} = F(u^I) & \text{at} \quad x = 0, \\ u^I = G(u^{II}) & \text{at} \quad x = 1, \end{array} \right\} \qquad (2.7)$$

where

$$F : \mathbb{R}^m \to \mathbb{R}^{N-m}, \qquad G : \mathbb{R}^{N-m} \to \mathbb{R}^m$$

$$F = \begin{bmatrix} F_{m+1} \\ \vdots \\ F_N \end{bmatrix}, \qquad G = \begin{bmatrix} G_1 \\ \vdots \\ G_m \end{bmatrix}$$

are C^1-smooth vector-valued functions. We define

$$B_1 = \frac{\partial F}{\partial u^I}(0) \cdot \frac{\partial G}{\partial u^{II}}(0) : \text{an } (N-m) \times (N-m) \text{ matrix,}$$

$$B_2 = \frac{\partial G}{\partial u^{II}}(0) \cdot \frac{\partial F}{\partial u^I}(0) : \text{an } m \times m \text{ matrix,}$$

where

$$\frac{\partial F}{\partial u^I} = \frac{\partial(F_{m+1}, \ldots, F_N)}{\partial(u_1, \ldots, u_m)} , \qquad \frac{\partial G}{\partial u^{II}} = \frac{\partial(G_1, \ldots, G_m)}{\partial(u_{m+1}, \ldots, u_N)} .$$

Let the initial condition of (2.1) be

$$u(x,0) = \phi(x), \qquad 0 \leqslant x \leqslant 1. \tag{2.8}$$

In addition to (A1), we assume

(A2) The initial and boundary conditions (2.8) and (2.7) satisfy the following:

i) $F(0) = 0, \quad G(0) = 0.$

ii) At $x = 0$ and $x = 1$, the following compatibility conditions are satisfied:

$$\phi^{II}(0) = F(\phi^I(0)), \quad \phi^I(1) = G(\phi^{II}(1)) \tag{2.9}$$

$$[A^{21}(\phi(0)) - \frac{\partial F}{\partial u^I}(\phi^I(0))A^{11}(\phi(0))]\frac{\partial \phi^I(0)}{\partial x} +$$
$$\tag{2.10}$$
$$[A^{22}(\phi(0)) - \frac{\partial F}{\partial u^I}(\phi^I(0))A^{12}(\phi(0))]\frac{\partial \phi^{II}(0)}{\partial x} = 0.$$

$$[A^{11}(\phi(1)) - \frac{\partial G}{\partial u^{II}}(\phi^{II}(1))A^{21}(\phi(1))]\frac{\partial \phi^I(1)}{\partial x} +$$
$$\tag{2.11}$$
$$[A^{12}(\phi(0)) - \frac{\partial G}{\partial u^{II}}(\phi^{II}(1))A^{22}(\phi(1))]\frac{\partial \phi^{II}(1)}{\partial x} = 0,$$

where A^{11}, A^{12}, A^{21} and A^{22} are blocks of sub-matrices of A corresponding to the decomposition $u = (u^I, u^{II})$:

$$A(u) = \begin{bmatrix} A^{11}(u) & A^{12}(u) \\ A^{21}(u) & A^{22}(u) \end{bmatrix}. \qquad\qquad \square$$

For any $n \times n$ square matrix $M = (m_{ij})$, we define its absolute value matrix $\tilde{M}$ by

$$\bar{M} = (|m_{ij}|)_{n \times n}.$$

This matrix $\bar{M}$ satisfies the property that

$$|\bar{M}v|_\infty \leq \left[\max_{1 \leq i \leq n} \sum_{j=1}^{n} |m_{ij}| \right] |v|_\infty, \quad \text{for all} \quad v \in \mathbb{R}^n,$$

where

$$|v|_\infty = \max_{1 \leq i \leq n} |v_i| \quad \text{for} \quad v = (v_1, \ldots, v_n) \in \mathbb{R}^n,$$

We thus define a norm $|\cdot|$ for M by

$$|M| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |m_{ij}| \quad (= |\bar{M}|).$$

We now state the "theorem" of T.H. Qin in [9].

Theorem 0. Assume (A1) and (A2). Assume further that the spectral radii of B_1 and B_2 are less than 1. Then there exists $\delta > 0$ such that the mixed initial-boundary value problem (2.1), (2.7), (2.8) admits a unique global smooth solution $u(t,x)$ for $t \geq 0$ and $|u(t,\cdot)|_{C^1(0,1)}$ decays exponentially

$$|u(t,\cdot)|_{C^1(0,1)} \leq Ke^{-\alpha t}, \quad K, \alpha > 0, \quad \text{for all} \quad t \geq 0, \tag{2.12}$$

provided that $|\phi|_{C^0(0,1)} + |\phi'|_{C^0(0,1)} < \delta$, where in (2.12), K depends on ϕ and the rate of decay α is at least

$$\alpha = -\frac{\lambda_{min} \cdot \ln \sigma}{2p} \tag{2.13}$$

with

$$\lambda_{min} = \min_{1 \leq i \leq N} |\lambda_i|_{C^0(-\delta', \delta')}, \quad \text{(for} \quad \delta' > 0 \quad \text{some small number)} \tag{2.14}$$

p is the smallest positive integer making

$$\max(|B_1^p|, |B_2^p|) = \sigma_0 < 1 \tag{2.15}$$

and σ is any real number in $(\sigma_0, 1)$. $\qquad \qquad \square$

It is well-known that solutions to quasilinear hyperbolic systems with smooth

data develop shocks within finite duration. Therefore it is required in the conditions of Qin's theorem that the C^1-norm of ϕ be sufficiently small. Thus normally exponential stability (2.12) can only be expected for small data.

The conditions on the spectral radii of B_1 and B_2 in the statement of "Theorem 0" seem so intuitive and natural, as we anticipate that waves would lose strength exponentially after repeated reflections on energy absorbing boundaries. Unfortunately, the conclusion is erroneous due to the fact that in a system there are several waves travelling with different speeds, thus their superpositions may form a wave which is undamped. The following is a counterexample.

We consider a linear system formed by the coupled wave equations (cf. (1.2))

$$\frac{\partial^2 w(x,t)}{\partial t^2} - c_1^2 \frac{\partial^2 w(x,t)}{\partial x^2} = 0, \qquad 0 < x < 1, \quad t > 0$$

$$\frac{\partial^2 w(x,t)}{\partial t^2} - c_2^2 \frac{\partial^2 w(x,t)}{\partial x^2} = 0, \qquad 1 < x < 2, \quad t > 0 \qquad (2.16)$$

$$(c_2^2 > c_1^2)$$

with boundary and transmission conditions

$$\begin{cases} w(0,t) = 0 & \text{at left end } x = 0, \\ w_x(2,t) = 0 & \text{at right end } x = 2, \\ w_t(1^-,t) - w_t(1^+,t) = -k_1^2 c_1^2 w_x(1^-,t) \\ c_1^2 w_x(1^-,t) = c_2^2 w_x(1^+,t) \end{cases} \quad \text{at } x = 1, \text{ cf. (1.6).} \qquad (2.17)$$

Let us transform them into a hyperbolic system by letting, for $x \in (0,1)$,

$$w_1(x,t) = \frac{1}{2}\left[-c_2 \frac{\partial}{\partial x} w(2-x,t) + \frac{\partial}{\partial t} w(2-x,t)\right],$$

$$w_2(x,t) = \frac{1}{2}\left[c_1 \frac{\partial}{\partial x} w(x,t) + \frac{\partial}{\partial t} w(x,t)\right],$$

$$w_3(x,t) = \frac{1}{2}\left[-c_1 \frac{\partial}{\partial x} w(x,t) + \frac{\partial}{\partial t} w(x,t)\right],$$

$$w_4(x,t) = \frac{1}{2}\left[c_2 \frac{\partial}{\partial x} w(2-x,t) + \frac{\partial}{\partial t} w(2-x,t)\right].$$

Then $w = (w_1, w_2, w_3, w_4) = (w^I, w^{II})$ $(w^I = (w_1, w_2), w^{II} = (w_3, w_4))$ satisfies

$$\frac{\partial}{\partial t}w(x,t) + \begin{bmatrix} -c_2 & & 0 \\ & -c_1 & \\ & c_1 & \\ 0 & & c_2 \end{bmatrix} \frac{\partial}{\partial x}w(x,t) = 0$$

$$w^{II}(0,t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} w^I(0,t) = D_0 w^I(0,t) \qquad\qquad (2.18)$$

$$w^I(1,t) = (c_1+c_2+k_1^2 c_1 c_2)^{-1} \begin{bmatrix} 2c_1 & -c_1+c_2+k_1^2 c_1 c_2 \\ c_1-c_2+k_1^2 c_1 c_2 & 2c_2 \end{bmatrix} w^{II}(1,t)$$
$$= D_1 w^{II}(1,t).$$

It is easy to see that (A1) and (A2)(i) are satisfied. (A2)(ii) is satisfied if the initial condition is chosen to satisfy the compatibility conditions. (2.5) is trivially satisfied. Now, $D_0 D_1$ is computed to be

$$B_1 = D_0 D_1 = -(c_1+c_2+k_1^2 c_1 c_2)^{-1} \begin{bmatrix} c_1-c_2+k_1^2 c_1 c_2 & +2c_2 \\ -2c_1 & c_1-c_2-k_1^2 c_1 c_2 \end{bmatrix}, \qquad (2.19)$$

which has eigenvalues

$$\mu_{1,2} = -(c_1+c_2+k_1^2 c_1 c_2)^{-1}(c_1-c_2 \pm [c_1 c_2(k_1^4 c_1 c_2 - 4)]^{1/2}). \qquad (2.20)$$

$B_2 = D_1 D_0$ has identical eigenvalues as $B_1 = D_0 D_1$ as given above. It is easy to see that

$$|\mu_{1,2}| < 1, \quad \text{for any } k_1^2 > 0.$$

Hence all of the hypotheses of "Theorem 0" are satisfied.

We now prove that solutions to (2.18) do not decay exponentially by showing that the coupled wave system (2.15), (2.17) has at least some eigenvalues located on the imaginary axis. We write an eigensolution of (2.16), (2.17) as

$$w(x,t) = \begin{cases} e^{\lambda t}\phi_1(x), & 0 \le x \le 1 \\ e^{\lambda t}\phi_2(x), & 1 \le x \le 2. \end{cases} \qquad (2.21)$$

Then

$$\left. \begin{array}{l} \phi_1''(x) - (\lambda/c_1)^2 \phi_1(x) = 0, \quad 0 \le x \le 1, \\ \phi_2''(x) - (\lambda/c_2)^2 \phi_2(x) = 0, \quad 1 \le x \le 2. \end{array} \right\} \qquad (2.22)$$

Due to the first two boundary conditions in (2.17), we have

$$\phi_1(x) = A_1[\exp(\lambda x/c_1) - \exp(-\lambda x/c_1)], \qquad 0 \leqslant x \leqslant 1, \left.\vphantom{\phi_2}\right\}$$
$$\phi_2(x) = A_2[\exp(\lambda(x-2)/c_2 + \exp(-\lambda(x-2)/c_2)], \; 1 \leqslant x \leqslant 2. \tag{2.23}$$

The second two transmission conditions in (2.17) impose that

$$c_1 A_1 \lambda[\exp(\lambda/c_1) + \exp(-\lambda/c_1)] = c_2 A_2 \lambda[\exp(-\lambda/c_2) - \exp(\lambda/c_2)]$$
$$\lambda A_1[\exp(\lambda/c_1) - \exp(-\lambda/c_1)] - \lambda A_2[\exp(\lambda/c_2) + \exp(-\lambda/c_2)]$$
$$+ \; k_1^2 c_1 A_1 \lambda[\exp(\lambda/c_1) + \exp(-\lambda/c_1)] = 0. \tag{2.24}$$

Therefore all eigenfrequencies λ are determined by

$$0 = \Delta = \det \begin{bmatrix} (e^{\lambda/c_1} - e^{-\lambda/c_1}) + k_1^2 c_1 (e^{\lambda/c_1} + e^{-\lambda/c_1}) & -(e^{\lambda/c_2} + e^{-\lambda/c_2}) \\ c_1(e^{\lambda/c_1} + e^{-\lambda/c_1}) & c_2(e^{\lambda/c_2} - e^{-\lambda/c_2}) \end{bmatrix} \tag{2.25}$$

$$= c_1[e^{(1/c_1 + 1/c_2)\lambda} + e^{-(1/c_1 - 1/c_2)\lambda} + e^{(1/c_1 - 1/c_2)\lambda} + e^{-(1/c_1 + 1/c_2)\lambda}]$$

$$+ \; c_2[e^{(1/c_1 + 1/c_2)\lambda} - e^{-(1/c_1 - 1/c_2)\lambda} - e^{(1/c_1 - 1/c_2)\lambda} + e^{-(1/c_1 + 1/c_2)\lambda}]$$

$$+ \; k_1^2 c_1 c_2[e^{(1/c_1 + 1/c_2)\lambda} + e^{-(1/c_1 - 1/c_2)\lambda} - e^{(1/c_1 - 1/c_2)\lambda} - e^{-(1/c_1 + 1/c_2)\lambda}].$$

Let the wave speeds on two strings be commensurable:

$$\frac{c_1}{c_2} = \frac{L}{M}, \quad L, M \text{ are relatively prime positive integers}, \quad L < M. \tag{2.26}$$

Write $\lambda = iMc_1 \beta$, $\beta \in \mathbb{C}$. Then (2.25) becomes

$$\Delta = c_1[e^{i(M+L)\beta} + e^{-i(M+L)\beta} + e^{i(M-L)\beta} + e^{-i(M-L)\beta}] \tag{2.27}$$

$$+ \; c_2[e^{i(M+L)\beta} + e^{-i(M+L)\beta} - e^{i(M-L)\beta} - e^{-i(M-L)\beta}]$$

$$+ \; k_1^2 c_1 c_2[e^{i(M+L)\beta} - e^{-i(M+L)\beta} - e^{i(M-L)\beta} + e^{-i(M-L)\beta}]$$

$$= 2c_1[\cos(M+L)\beta + \cos(M-L)\beta] + 2c_2[\cos(M+L)\beta - \cos(M-L)\beta]$$

$$+ \; i \, 2k_1^2 c_1 c_2[\sin(M+L)\beta - \sin(M-L)\beta].$$

Then $\Delta = 0$ if and only if

$$(c_1 + c_2)\cos(M+L)\beta + (c_1 - c_2)\cos(M-L)\beta = 0 \left.\vphantom{\sin}\right\}$$
$$\sin(M+L)\beta - \sin(M-L)\beta = 0. \tag{2.28}$$

If

$$\frac{c_1}{c_2} = \frac{L}{M} = \frac{2n}{2n+1}, \quad (L = 2n, \ M = 2n+1)$$

then $M+L = 4n+1$, $M-L = 1$, $c_1+c_2 = \frac{4n+1}{2n+1}c_2$, $c_1-c_2 = \frac{-1}{2n+1}c_2$, and (2.28) gives

$$\left.\begin{array}{l} (4n+1)\cos(4n+1)\beta - \cos\beta = 0 \\[2mm] \sin(4n+1)\beta - \sin\beta = 0. \end{array}\right\} \tag{2.29}$$

It is obvious that $\beta = \pi/2$ is a solution of (2.29). Hence

$$\lambda = iMc_1\beta = i\left(n + \frac{1}{2}\right)\pi c_1, \quad \text{any positive integer} \quad n,$$

is an eigenfrequency of (2.16), (2.17) which is undamped as $\mathrm{Re}\ \lambda = 0$. This is a counterexample to "Theorem 0".

The fallacy of "Theorem 0" is due to the fact that in [9] the author has forgotten to take certain absolute values between steps (52) – (54) in [9, pp. 295–296]. The mistakes can be easily corrected by changing B_1, B_2 in the statement of "Theorem 0" to $\bar{\bar{B}}_1$, $\bar{\bar{B}}_2$. We state the corrected version of Qin's theorem below:

<u>Theorem 1.</u> Assume (A1) and (A2). Assume further that the spectral radii of $\bar{\bar{B}}_1$ and $\bar{\bar{B}}_2$ are less than 1. Then there exists $\delta > 0$ such that the mixed initial-boundary value problem (2.1), (2.7), (2.8) admits a unique global smooth solution $u(t,x)$ for $t \geq 0$ and $\|u(t,\cdot)\|_{C^1(0,1)}$ decays exponentially

$$\|u(t,\cdot)\|_{C^1(0,1)} \leq Ke^{-\alpha t}, \quad K,\alpha > 0 \quad \text{for all} \quad t \geq 0, \tag{2.30}$$

provided that $\|\phi\|_{C^0(0,1)} + \|\phi'\|_{C^0(0,1)} < \delta$, where in (2.30), k depends on ϕ and the rate of decay α is at least

$$\alpha = -\frac{\lambda_{min}\cdot\ln\sigma}{2p}, \tag{2.31}$$

with

$$\lambda_{min} = \min_{1\leq i\leq n}\|\lambda_i\|_{C^0(-\delta',\delta')}, \qquad \text{(for some small } \delta' > 0)$$

p is the smallest positive integer making

$$\max(\|\bar{\bar{B}}_1^p\|\cdot\|\bar{\bar{B}}_2^p\|) = \sigma_0 < 1 \tag{2.32}$$

and σ is any real number in $(\sigma_0, 1)$. $\qquad\qquad\qquad\qquad$ □

Note that the above correction imposes very severe restrictions on matrices B_1 and B_2. It does not apply to many cases studied in [2] e.g. as one realizes that in [2] the spectral radii of B_1 and B_2 are less than 1, but those of $\bar{\bar{B}}_1$ or $\bar{\bar{B}}_2$ are usually not less than 1.

§III. Coupled Nonlinear Vibrating Strings with Point Stabilizers.

Let us first consider the system (1.1), (1.3), (1.5) and (1.7). We will transform the system into a form which allows the application of Theorem 1.

Without loss of generality, we assume that

$$\sigma_1'(0) \geq \sigma_2'(0). \qquad\qquad\qquad (3.1)$$

Define

$$y_1(x,t) = \frac{\partial w(x,t)}{\partial x}, \quad y_2(x,t) = -\frac{\partial}{\partial x}w(2-x,t), \\ z_1(x,t) = \frac{\partial w(x,t)}{\partial x}, \quad z_2(x,t) = \frac{\partial}{\partial x}w(2-x,t), \qquad (3.2)$$

for $x \in (0,1)$, $t \geq 0$. Use the following Riemann invariants to diagonalize the system:

$$u_i = \frac{1}{2}(z_i + \int_0^{y_i} \sqrt{\sigma_i'(\eta)}d\eta), \\ u_{3-i} = \frac{1}{2}(z_i - \int_0^{y_i} \sqrt{\sigma_i'(\eta)}d\eta), \qquad i = 1,2. \qquad (3.3)$$

Then $u = (u_1, u_2, u_3, u_4) = (u^I, u^{II})(u^I = (u_1, u_2), u^{II} = (u_3, u_4))$ satisfies

$$\frac{\partial}{\partial t}u + \begin{bmatrix} -c_1(u_1-u_4) & 0 & 0 & 0 \\ 0 & -c_2(u_2-u_3) & 0 & 0 \\ 0 & 0 & c_2(u_2-u_3) & 0 \\ 0 & 0 & 0 & c_1(u_1-u_4) \end{bmatrix}\frac{\partial}{\partial x}u = 0, \\ 0 < x < 1, \quad t > 0. \qquad (3.4)$$

where

$$c_i(\gamma) = \sqrt{\sigma_i'(U(\gamma))}, \qquad i = 1,2, \qquad (3.5)$$

denotes the wave speed on string i, and $U(\gamma)$ is definded implicitly by

$$\gamma = \int_0^{u(\gamma)} \sqrt{\sigma_i(\eta)} \; d\eta. \tag{3.6}$$

We now transform boundary and transmission conditions (1.3), (1.7) and (1.5) in terms of u:

$$u^{II}(0,t) = \begin{bmatrix} u_3(0,t) \\ u_4(0,t) \end{bmatrix} = F(u^I(0,t)) = \begin{bmatrix} F_3(u^I(0,t) \\ F_4(u^I(0,t)) \end{bmatrix}$$

$$u^I(0,t) = \begin{bmatrix} u_1(1,t) \\ u_2(1,t) \end{bmatrix} = G(u^{II}(0,t)) = \begin{bmatrix} G_1(u^{II}(1,t) \\ G_2(u^{II}(1,t)) \end{bmatrix}$$

where $F(0) = G(0) = 0$, and

$$F_3(u^I) = F_3(u_2) = -u_2$$

$$\frac{\partial F_3}{\partial u_1} = 0, \qquad \frac{\partial F_3}{\partial u_2} = -1$$

$$F_4(u^I) = F_4(u_1)$$

$$\frac{\partial F_4}{\partial u_1} = \frac{c_1(u_1 - F_4(u_1)) - k_0^2}{c_1(u_1 - F_4(u_1)) + k_0^2}, \quad \left(\left| \frac{\partial F_4}{\partial u_1} \right| < 1 \right),$$

$$\frac{\partial F_4}{\partial u_2} = 0.$$

$$\left. \begin{aligned} \frac{\partial G_1}{\partial u_1} &= \frac{2c_2}{c_1 + c_2 + k_1^2 c_1 c_2} \\[2ex] \frac{\partial G_1}{\partial u_4} &= \frac{c_1 - c_2 + k_1^2 c_1 c_2}{c_1 + c_2 + k_1^2 c_1 c_2} \\[2ex] \frac{\partial G_2}{\partial u_3} &= \frac{-c_1 + c_2 + k_1^2 c_1 c_2}{c_1 + c_2 + k_1^2 c_1 c_2} \\[2ex] \frac{\partial G_2}{\partial u_4} &= \frac{2c_1}{c_1 + c_2 + k_1^2 c_1 c_2} \end{aligned} \right\} \qquad \begin{aligned} c_1 &= c_1(G_1(u^{II}) - u_4) \\ c_2 &= c_2(G_2(u^{II}) - u_3). \end{aligned}$$

Therefore

$$\frac{\partial F}{\partial u^{\mathrm{I}}}(0) = \begin{bmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} \\ \frac{\partial F_4}{\partial u_1} & \frac{\partial F_4}{\partial u_2} \end{bmatrix}(0) = \begin{bmatrix} 0 & -1 \\ \frac{c_1(0)-k_0^2}{c_1(0)+k_0^2} & 0 \end{bmatrix}$$

$$\frac{\partial G}{\partial u^{\mathrm{II}}}(0) = \begin{bmatrix} \frac{\partial G_1}{\partial u_3} & \frac{\partial G_1}{\partial u_4} \\ \frac{\partial G_2}{\partial u_3} & \frac{\partial G_2}{\partial u_4} \end{bmatrix}(0) = [c_1(0) + c_2(0) + k_1^2 c_1(0)c_2(0)]^{-1} \cdot$$

$$\begin{bmatrix} 2c_2(0) & c_1(0)-c_2(0)+k_1^2 c_1(0)c_2(0) \\ -c_1(0)+c_2(0)+k_1^2 c_1(0)c_2(0) & 2c_1(0) \end{bmatrix}.$$

The reader can see that the matrix D_1 in (2.18) looks almost like the transpose matrix of the matrix above.

So

$$B_1 = [c_1+c_2+k_1^2 c_1 c_2]^{-1} \begin{bmatrix} c_1-c_2-k_1^2 c_1 c_2 & -2c_1 \\ 2c_2 \cdot \frac{c_1-k_0^2}{c_1+k_0^2} & (c_1-c_2+k_1^2 c_1 c_2)\frac{c_1-k_0^2}{c_1+k_0^2} \end{bmatrix}$$

$$B_2 = [c_1+c_2+k_1^2 c_1 c_2]^{-1} \begin{bmatrix} \frac{c_1-k_0^2}{c_1+k_0^2}(c_1-c_2+k_1^2 c_1 c_2) & -2c_2 \\ 2c_1 \cdot \frac{c_1-k_0^2}{c_1+k_0^2} & c_1-c_2-k_1^2 c_1 c_2 \end{bmatrix}$$

where in the above, we have abbreviated $c_1(0)$ and $c_2(0)$ as c_1 and c_2, respectively. Under the previous assumption that $c_1 \succ c_2$ (i.e., (3.1)), we have

$$\bar{\bar{U}}_1 = [c_1+c_2+k_1^2 c_1 c_2]^{-1} \begin{bmatrix} |c_1-c_2-k_1^2 c_1 c_2| & 2c_1 \\ \frac{2c_2}{c_1+k_0^2}|c_1-k_0^2| & \frac{(c_1-c_2+k_1^2 c_1 c_2)}{c_1+k_0^2}|c_1-k_0^2| \end{bmatrix}$$

$$= \begin{bmatrix} K_2 & 2c_1 \Delta^{-1} \\ 2c_1 \Delta^{-1} K_3 & K_1 K_3 \end{bmatrix}$$

(3.7)

$$\bar{\bar{B}}_2 = [c_1+c_2+k_1^2c_1c_2]^{-1} \begin{vmatrix} \dfrac{(c_1+c_2+k_1^2c_1c_2)|c_1-k_0^2|}{c_1+k_0^2} & 2c_2 \\ \dfrac{2c_1}{c_1+k_0^2}|c_1-k_0^2| & |c_1-c_2-k_1^2c_1c_2| \end{vmatrix}$$

$$= \begin{bmatrix} K_1K_3 & 2c_2\Delta^{-1} \\ 2c_1\Delta^{-1}K_3 & K_2 \end{bmatrix}. \tag{3.8}$$

where in the above

$$\Delta = c_1+c_2+k_1^2c_1c_2$$

$$K_1 = \Delta^{-1}(c_1-c_2+k_1^2c_1c_2)$$

$$K_2 = \Delta^{-1}|c_1-c_2-k_1^2c_1c_2| \tag{3.9}$$

$$K_3 = |c_1-k_0^2|/(c_1+k_0^2).$$

The spectral radii of $\bar{\bar{B}}_1$ and $\bar{\bar{B}}_2$ are determined from $|\lambda|$ of

$$\det(\lambda-\bar{\bar{B}}_1) = \det(\lambda-\bar{\bar{B}}_2) = \lambda^2-(K_1K_3+K_2)\lambda+K_1K_2K_3-4c_1c_2\Delta^{-2}K_3 = 0. \tag{3.10}$$

According to Theorem 1, a sufficient condition for exponential decay of solutions with small data is that the roots λ_1, λ_2 of (3.10) satisfy

$$\max(|\lambda_1|, |\lambda_2|) < 1. \tag{3.11}$$

The above can be determined by the standard Routh stability criterion in automatic control as follows. Let us use the following Möbius transformation mapping the interior of the unit circle into the left half plane:

$$\lambda = \frac{z+1}{-z+1} \tag{3.12}$$
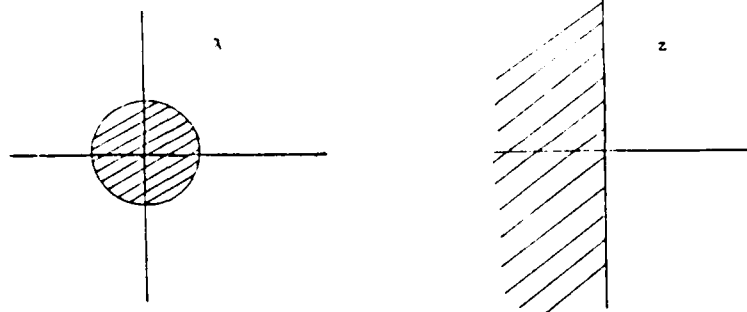
Substituting (3.12) into (3.10), we get

$$\tilde{A}_0 z^2 + \tilde{A}_1 z + \tilde{A}_2 = 0 \tag{3.13}$$

where

$$
\begin{aligned}
\tilde{A}_0 &= 1+K_1 K_3+K_2+K_1 K_2 K_3-4c_1 c_2 \Delta^{-2} K_3 \ (> 0) \\
\tilde{A}_1 &= 2(1-K_1 K_2 K_3+4c_1 c_2 \Delta^{-2} K_3) \\
\tilde{A}_2 &= 1-K_1 K_3-K_2+K_1 K_2 K_3-4c_1 c_2 \Delta^{-2} K_3.
\end{aligned}
\right\} \tag{3.14}
$$

Condition (3.11) is now equivalent to that (3.13) has no roots on the open right half plane. This can be determined by the Routh criterion as follows:

$$
\begin{array}{ll}
\tilde{A}_0 & \tilde{A}_2 \\
\tilde{A}_1 & 0 \\
\tilde{B}_1 & 0
\end{array}
\qquad
\tilde{B}_1 = \frac{\tilde{A}_1 \cdot \tilde{A}_2 - \tilde{A}_0 \cdot 0}{\tilde{A}_1} = \tilde{A}_2 \tag{3.15}
$$

Therefore (3.11) is satisfied if and only if the following condition is satisfied:

> The first column $\tilde{A}_0$, $\tilde{A}_1$, $\tilde{B}_1$ ($= \tilde{A}_2$) in (3.15) is of one sign
> and $\tilde{A}_1 \neq 0$, $\tilde{A}_2 \neq 0$, i.e., all of coefficients $\tilde{A}_0$, $\tilde{A}_1$ and $\tilde{A}_2$ (3.16)
> in (3.13) are nonzero and positive.

We summarize the above in

Theorem 2. Assume (A1) and (A2) as in §II. If $k_0^2 > 0$ and $k_1^2 > 0$ in (1.3) and (1.5) are chosen such that (3.16) is satisfied, then the exponential decay property (2.30) holds for (1.1), (1.3), (1.5) and (1.7) provided that the initial data $\phi(x)$ satisfies compatibility conditions and has sufficiently small C^1-norm □

From condition (3.16) it is not difficult to check that if $k_1^2 > 0$, we can always find $k_0^2 > 0$ such that condition (3.16) is satisfied. Therefore exponential stability (for small data) can always be achieved by using a single stabilizer at the left end $x = 0$. In particular, we examine the case $k_1^2 = 0$:

$$\tilde{A}_0 > 0$$

$$\tilde{A}_1 = 2\left[1 - \frac{(c_1-c_2)^2}{c_1+c_2} \frac{|c_1-k_0^2|}{c_1+k_0^2} + \frac{4c_1 c_2}{(c_1+c_2)^2} \frac{|c_1-k_0^2|}{c_1+k_0^2}\right]$$

$$\tilde{A}_2 = 1 - \frac{c_1 + c_2}{c_1 + c_2} \frac{|c_1 - k_0^2|}{c_1 + k_0^2} - \left(\frac{c_1 - c_2}{c_1 + c_2}\right) + \left(\frac{c_1 - c_2}{c_1 + c_2}\right)^2 \frac{|c_1 - k_0^2|}{c_1 + k_0^2} - \frac{4c_1 c_2}{(c_1 + c_2)^2} \frac{|c_1 - k_0^2|}{c_1 + k_0^2} .$$

If k_0^2 is chosen equal (or very close) to c_1, we will always get $\tilde{A}_1 > 0$, $\tilde{A}_2 > 0$. This is well known in wave propagation theory as $k_0^2 = c_1$ in (1.3) corresponds to the characteristic impedence boundary condition which causes maximum energy loss of waves at $x = 0$. Indeed, the closer k_0^2 to c_1 is, the smaller the power p in (2.32) becomes. Therefore the rate of decay α in (2.31) will become larger. On the other hand, if k_0^2 is not close to c_1, then $\tilde{A}_1$ and $\tilde{A}_2$ may easily become negative, therefore Theorem 2 will no longer be applicable. This is in sharp contrast to the linear case (cf. Theorem 7 later) where $k_0^2 > 0$ can be arbitrary when $k_1^2 = 0$.

What happens if $k_0^2 = 0$? Can we get exponential stability by using only one stabilizer in the middle ($x = 1$)? The reader can easily check that condition (3.16) is never satisfied no matter what value $k_1^2 > 0$ is chosen. In fact, in general the solution will not decay exponentially as the linear counterexample in §II has already shown. One might wonder whether nonlinearity would work any differently. The answer is still no as the following theorem suggests.

<u>Theorem 3.</u> Assume that $k_0^2 = 0$ in (1.3). Then there are C^2-solutions to (1.1), (1.3), (1.5), (1.7) which are undamped for any $k_1^2 > 0$, for some nonlinear σ_1, σ_2.

<u>Proof.</u> We first note that when $k_0^2 = 0$, (1.3) becomes

$$w_x(0,t) = 0. \tag{1.3'}$$

We first assume that the following boundary value problem

$$
\begin{aligned}
&\frac{d^2}{dt^2} \tilde{w}(x,t) - \frac{d}{dx} \sigma\left[\frac{d}{dx} \tilde{w}(x,t)\right] = 0, \quad 0 < x < 1 \\
&\tilde{w}(0,t) = 0 \\
&\tilde{w}_x(1,t) = 0
\end{aligned}
\tag{3.17}
$$

has an undamped C^2 solution for certain σ. Then we can construct an undamped solution for (1.1), (1.3)', (1.5), (1.7) as follows. Define

$$
y(x,t) = \begin{cases}
\tilde{w}(-x,t), & -1 < x < 0, \\
\tilde{w}(x,t), & 0 < x < 1, \\
\tilde{w}(2-x,t), & 1 < x < 2,
\end{cases}
$$

$$w(x,t) = \begin{cases} y(2x-1,t), & 0 < x < 1, \\ y(x,t), & 1 < x < 2. \end{cases}$$

At $x = 0$,

$$w(0^-,t) = w(0^+,t) = 0, \quad w_x(0^-,t) = w_x(0^+,t),$$

$$w_t(0^-,t) = w_t(0^+,t) = 0, \quad w_{xx}(0^-,t) = w_{xx}(0^+;t),$$

$$w_{tt}(0^-,t) = w_{tt}(0^+,t) = 0,$$

so $y \in C^2(-1,1)$ and $w \in C^2(0,1)$. w satisfies

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\left[\frac{1}{2}\,\sigma\left(\frac{1}{2}\,\frac{\partial w}{\partial x}\right)\right] = 0, \qquad 0 < x < 1, \quad t > 0$$

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\left[-\sigma\left(-\,\frac{\partial w}{\partial x}\right)\right] = 0, \qquad 1 < x < 2, \quad t > 0.$$

Set

$$\sigma_1(\eta) = \frac{1}{2}\,\sigma\left(\frac{1}{2}\,\eta\right),$$

$$\sigma_2(\eta) = -\sigma(-\eta).$$

Then $\sigma_i(0) = 0$ and $\sigma_i' > 0$, $i = 1,2$, and w satisfies (1.1). At $x = 0$ and 2,

$$w_x(0,t) = 2y_x(-1,t) = 2\tilde{w}_x(1,t) = 0,$$

$$w(2,1) = y(0,t) = \tilde{w}(0,t) = 0.$$

At $x = 1$,

$$w_x(1^-,t) = w_x(1^+,t) = 0$$

$$w_t(1^-,t) = w_t(1^+,t),$$

so (1.5) is satisfied. Hence w is a C^2-soluton of (1.1), (1.3)', (1.5), (1.7) which is undamped.

But Greenberg [5] has proved that with

$$\sigma(\eta) = 3c_0^2\left[\lambda - \frac{\lambda^{4/3}}{(\lambda+\eta)^{1/3}}\right], \quad \eta > -\lambda, \quad c_0 > 0, \quad \lambda > 0,$$

(3.17) has a C^2 solution which is undamped.

The proof of Theorem 3 is complete. □

Therefore we can safely state that in general exponential stability cannot be attained by using only one stabilizer at the middle of the span for coupled (linear or nonlinear) wave equations.

Although we have used the Riemann invariants u in (3.3) to give proofs and deduce stability results, the plain fact is that here as in [6] and [9] and in many other similar papers the approach is essentially based on linearization about the zero solution u = 0. Theorem 2 is but one of the many examples of the principle of linearized stability which is valid for a large class of general nonlinear distributed systems.

The case when the intermediate conditions and boundary condition are changed to (1.5)' and/or (1.7)' can be studied similarly without any difficulty.

§IV. Coupled Linear Vibrating Strings with Point Stabilizers.

In this section we study the exponential stability problem for the coupled linear system (1.2), (1.4), (1.6) and (1.7). Other transmission condition (1.6)' and boundary condition (1.7)' can be treated in a similar way.

Previously, several researchers [1], [2], [3], [8] have studied this type of problem using energy identities or the method of characteristics. We have tried both methods for system (1.2), (1.4), (1.6) and (1.7), yet we could not succeed in establishing an affirmative answer to [Q] for our problem for any wave speeds c_1, c_2 and any gain constants $k_0^2 \geq 0$, $k_1^2 \geq 0$. The difficulty probably can be interpreted as follows:

i) At the coupling point $x = 1$, there are reflections and transmissions of waves which the energy multipliers are not sophisticated enough to handle.

ii) For coupled linear strings the method of characteristics works best when the wave speeds are identical, i.e., $c_1 = c_2$, cf. [2], yielding sharp decay estimates for a (cf. (2.12)). Otherwise, this method is not convenient for coupled strings.

Of course, one could also apply the nonlinear Theorem 2 to coupled linear strings. But such a result would not be sharp as the gain coefficients k_0^2 and k_1^2 become severely restricted due to the conditions on the spectral radii of $\bar{\bar{B}}_0$ and $\bar{\bar{B}}_1$.

A recent theorem of F.L. Huang offers an extremly useful way to prove exponential stability for linear systems:

Theorem 4 [7]. Let A be the infinitesimal generator of a C_0-semigroup $\exp(tA)$, which satisfies

$$\|\exp(tA)\| \leq B_0, \quad t \geq 0, \quad \text{for some } B_0 > 0. \tag{4.1}$$

Then $\exp(tA)$ is exponentially stable (i.e, $\|\exp(tA)\| \leq Ke^{-at}$, $K, a > 0$, $t \geq 0$) if and only if

$$(i\omega | \omega \in R) \subseteq \rho(A), \quad \text{the resolvent set of } A; \qquad (4.2)$$

and

$$B_1 = \sup(\|(i\omega - A)^{-1}\| \,|\, \omega \in R) < \infty \qquad (4.3)$$

are satisfied. □

This theorem has recently been applied in [4], e.g., to establish an exponential stability result for an Euler-Bernoulli beam with bending moment proportional control.

Now we are in a position to apply Huang's Theorem 4 to (1.2), (1.4), (1.6), (1.7). Let us recast the problem into an equivalent hyperbolic system: For $x \in (0,1)$, define

$$u_1(x,t) = \frac{1}{2}\left[-c_1 \frac{\partial}{\partial x}w(x,t) + \frac{\partial}{\partial t}w(x,t)\right],$$

$$u_2(x,t) = \frac{1}{2}\left[c_2 \frac{\partial}{\partial x}w(2-x,t) + \frac{\partial}{\partial t}w(2-x,t)\right],$$

$$u_3(x,t) = \frac{1}{2}\left[-c_2 \frac{\partial}{\partial x}w(2-x,t) + \frac{\partial}{\partial t}w(2-x,t)\right],$$

$$u_4(x,t) = \frac{1}{2}\left[c_1 \frac{\partial}{\partial x}w(x,t) + \frac{\partial}{\partial t}w(x,t)\right].$$

Then (1.2) becomes

$$\frac{\partial}{\partial t}\begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix} = \begin{bmatrix}-c_1 & 0 & 0 & 0\\0 & -c_2 & 0 & 0\\0 & 0 & c_2 & 0\\0 & 0 & 0 & c_1\end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix} = Au. \qquad (4.4)$$

After straightforward calculations, we get from (1.4), (1.6), (1.7) the following boundary conditions:

At $x = 0$,

$$u^I(0,t) = \begin{bmatrix} 0 & \frac{c_1-k_0^2}{c_1+k_0^2} \\ -1 & 0 \end{bmatrix} u^{II}(0,t) = D_0 u^{II}(0,t). \qquad (4.5)$$

At $x = 1$,

$$u^{II}(1,t) = \Delta^{-1}\begin{bmatrix} 2c_1 & -c_1+c_2+k_1^2 c_1 c_2 \\ c_1-c_2+k_1^2 c_1 c_2 & 2c_2 \end{bmatrix} \cdot u^I(1,t) = D_1 u^I(1,t), \qquad (4.6)$$

where $\Delta = c_1+c_2+k_1^2 c_1 c_2$, $u^I = (u_1,u_2)$ and $u^{II} = (u_3,u_4)$.

Obviously, the underlying Hilbert space is $X = [L^2(0,1)]^4$ and

$$D(A) = \{u \in X \mid u \in [H^1(0,1)]^4, \quad u \text{ satisfies (4.5) and (4.6)}\}. \quad (4.7)$$

(The space $H^1(0,1)$ is the Sobolev space of order 1).

We solve the resolvent equation

$$\left.\begin{array}{l} (A - i\omega)u = f, \quad \omega \in \mathbb{R}, \ f \in X \text{ is arbitrary,} \\[2mm] u \in D(A), \quad \text{satisfying (4.5), (4.6)} \end{array}\right\} \qquad (4.8)$$

Writing out the above componentwise:

$$(-c_1 \frac{d}{dx} - i\omega)u_1 = f_1$$

$$(-c_2 \frac{d}{dx} - i\omega)u_2 = f_2$$

$$(c_2 \frac{d}{dx} - i\omega)u_3 = f_3$$

$$(c_1 \frac{d}{dx} - i\omega)u_4 = f_4,$$

we get

$$\left.\begin{array}{l} u_1(x) = \gamma_1 e^{-\frac{i\omega}{c_1}x} - \frac{1}{c_1}\int_0^x e^{-\frac{i\omega}{c_1}(x-\xi)} f_1(\xi)d\xi, \\[4mm] u_2(x) = \gamma_2 e^{-\frac{i\omega}{c_2}x} - \frac{1}{c_2}\int_0^x e^{-\frac{i\omega}{c_2}(x-\xi)} f_2(\xi)d\xi, \\[4mm] u_3(x) = \gamma_3 e^{\frac{i\omega}{c_2}x} + \frac{1}{c_2}\int_0^x e^{\frac{i\omega}{c_2}(x-\xi)} f_3(\xi)d\xi, \\[4mm] u_4(x) = \gamma_4 e^{\frac{i\omega}{c_1}x} + \frac{1}{c_1}\int_0^x e^{\frac{i\omega}{c_1}(x-\xi)} f_4(\xi)d\xi, \end{array}\right\} \qquad (4.9)$$

where γ_i, $1 \leq i \leq 4$, are determined from (4.5), (4.6):

$$\left.\begin{array}{l} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = D_0 \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \\[6mm] \begin{bmatrix} \gamma_3 e^{\frac{i\omega}{c_2}} + F_3 \\ \gamma_3 e^{\frac{i\omega}{c_1}} + F_4 \end{bmatrix} = D_1 \begin{bmatrix} \gamma_1 e^{-\frac{i\omega}{c_1}} - F_1 \\ \gamma_2 e^{-\frac{i\omega}{c_1}} - F_2 \end{bmatrix}, \end{array}\right\} \qquad (4.10)$$

$$\left\{ \begin{array}{l} F_j = \dfrac{1}{c_j} \displaystyle\int_0^1 e^{-\frac{i\omega}{c_j}(1-\xi)} f_j(\xi)d\xi, \quad j = 1,2 \\[4mm] F_j = \dfrac{1}{c_{s-j}} \displaystyle\int_0^1 e^{\frac{i\omega}{c_{s-j}}(1-\xi)} f_{s-j}(\xi)d\xi, \quad j = 3,4. \end{array} \right. \tag{4.11}$$

Therefore

$$(D-I)\begin{bmatrix} \gamma_3 \\ \gamma_4 \end{bmatrix} = D_1 \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} + \begin{bmatrix} F_3 \\ F_4 \end{bmatrix}, \tag{4.12}$$

$$D = \begin{bmatrix} e^{-\frac{i\omega}{c_2}} & 0 \\ 0 & e^{-\frac{i\omega}{c_1}} \end{bmatrix} D_1 \begin{bmatrix} e^{-\frac{i\omega}{c_1}} & 0 \\ 0 & e^{-\frac{i\omega}{c_2}} \end{bmatrix} D_0$$

$$= \Delta^{-1} \begin{bmatrix} (c_1-c_2-k_1^2 c_1 c_2)e^{-2i\omega/c_2} & \dfrac{c_1-k_0^2}{c_1+k_0^2} 2c_1 e^{-i\omega\left(\frac{1}{c_1}+\frac{1}{c_2}\right)} \\[4mm] -2c_2 e^{-i\omega\left(\frac{1}{c_1}+\frac{1}{c_2}\right)} & \dfrac{c_1-k_0^2}{c_1+k_0^2}(c_1-c_2+k_1^2 c_1 c_2)e^{-2i\omega/c_1} \end{bmatrix}.$$

<u>Proposition 5.</u> Let $\omega \ge R$ and $k_0^2 > 0$, $k_1^2 \ge 0$. Then

$$\|\det(D-I)\| \ge \delta, \quad \forall \omega \in R,$$

for some $\delta > 0$.

<u>Proof.</u> We have

$$\det(D-I) = \left[1 - \frac{(c_1-c_2-k_1^2 c_1 c_2)}{\Delta} e^{-\frac{2i\omega}{c_2}} \right] + \frac{c_1-k_0^2}{c_1+k_0^2}\left[\frac{c_1+c_2-k_1^2 c_1 c_2}{\Delta} e^{-\frac{2i\omega}{c_2}} \right.$$

$$\left. - \frac{c_1-c_2+k_1^2 c_1 c_2}{\Delta} \right] e^{-\frac{2i\omega}{c_1}} \tag{4.13}$$

$$= [z_1] + \frac{c_1-k_0^2}{c_1+k_0^2} [z_2]e^{-\frac{2i\omega}{c_1}}.$$

$$|\det(D-I)| \geq |z_1| - \frac{|c_1-k_0^2|}{c_1+k_0^2} |z_2|.$$

It suffices to show that

$$|z_1| \geq |z_2|, \quad \text{for all} \quad \omega \in R \tag{4.14}$$

and

$$|z_1| \geq \bar{\delta} > 0, \quad \text{for some} \quad \bar{\delta} > 0, \quad \text{for all} \quad \omega \in R. \tag{4.15}$$

But by direct calculations,

$$|z_1|^2 - |z_2|^2 = \left[1 + \left(\frac{c_1-c_2+k_1^2 c_1 c_2}{c_1+c_2+k_1^2 c_1 c_2}\right)^2 - 2 \frac{(c_1-c_2-k_1^2 c_1 c_2)}{c_1+c_2+k_1^2 c_1 c_2} \cos \frac{2\omega}{c_2}\right]$$

$$- \left[\left(\frac{c_1+c_2-k_1^2 c_1 c_2}{c_1+c_2+k_1^2 c_1 c_2}\right)^2 + \left(\frac{c_1-c_2+k_1^2 c_1 c_2}{c_1+c_2+k_1^2 c_1 c_2}\right)^2\right.$$

$$\left. - 2 \frac{(c_1+c_2-k_1^2 c_1 c_2)(c_1-c_2+k_1^2 c_1 c_2)}{(c_1+c_2+k_1^2 c_1 c_2)^2} \cos \frac{2\omega}{c_2}\right]$$

$$= \frac{8k_1^2 c_1 c_2^2}{(c_1+c_2+k_1^2 c_1 c_2)^2} \left(1 - \cos \frac{2\omega}{c_2}\right) \geq 0, \quad \text{for all} \quad \omega \in R.$$

So (4.14) is verified.

Inequality (4.15) is obvious from the definition of z_1 in (4.13), because

$$\left|\frac{c_1-c_2-k_1^2 c_1 c_2}{\Delta}\right| = \left|\frac{c_1-c_2-k_1^2 c_1 c_2}{c_1+c_2+k_1^2 c_1 c_2}\right| < 1, \quad \text{for any} \quad k_1^2 \geq 0.$$

Therefore the proof is complete. □

Lemma 6. The operator A defined in (4.4), (4.7) satisfies conditions (4.1) – (4.3), provided that $k_0^2 > 0$, $k_1^2 \geq 0$.

Proof. (4.1) is trivially satisfied because A is a dissipative operator. (4.2) follows from Proposition 5 because equations (4.12), (4.10) and (4.9) are all solvable for any given $f \in X$.

Finally, (4.3) is satisfied because in (4.11)

$$|F_j| \leq K|f_j|_{L^2(0,1)}, \qquad j = 1,2,3,4$$

are satisfied and because of Proposition 5, (4.12), (4.16), (4.10) and (4.9).

□

Hence we conclude

Theorem 7. Under the assumptions $k_o^2 > 0$, $k_1^2 \geq 0$, the energy of the coupled linear vibrating strings (1.2), (1.4), (1.6) and (1.7) decays uniformly exponentially.

□

Therefore, for coupled linear strings, one stabilizer ($k_o^2 > 0$, $k_1^2 = 0$) is sufficient to cause exponential decay of energy. The feedback gain k_o^2 can be chosen arbitrarily.

By straightforward calculations, one can easily see from (4.12) that Proposition 5 in general does not hold if $k_o^2 = 0$, $k_1^2 > 0$ (i.e., only one stabilizer is installed in the middle of the span), unless c_1 and c_2 satisfy certain special relations, such as $c_1/c_2 = 2$, e.g. More importantly, this exponential stability is not robust with respect to c_1/c_2 in the sense that if c_1/c_2 differs just slightly from 2 (or certain given number), then exponential stability no longer holds.

Therefore we see that a point stabilizer installed at the boundary is robust with respect to exponential stability for coupled nonlinear and linear vibrating strings.

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