

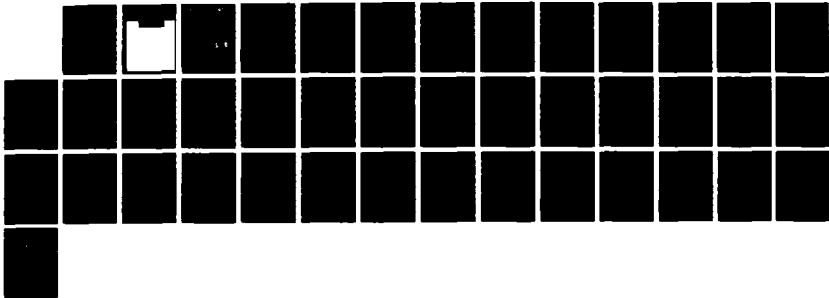
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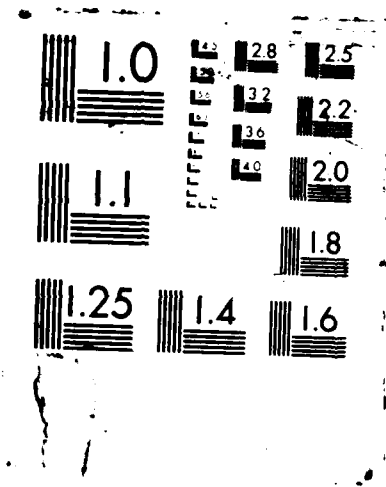
DIFFUSION APPROXIMATIONS AND NEARLY OPTIMAL MAINTENANCE
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Lefschetz Center for Dynamical Systems
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by

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Policies for System Breakdown and Repair Problems**

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Abstract

We consider the problem of service system deterioration and maintenance when there are only small statistical differences in the quality (or time required) of each produced item when the system is in the different states of deterioration, but where those marginal differences are economically important. This is somewhat analogous to the situation in the modelling of queues in heavy traffic, where the main effects which are dealt with might also be considered to be 'marginal' ones. In our case, the production of each item takes a random length of time and the deterioration during any production or sampling period can have a fairly general (and state dependent) statistical relation with this time and with the quality of the production. Due to this generality, there are several continuous parameter interpolations (of the sequence of conditional probabilities of the system states, given the observed data) which are appropriate for purposes of the weak convergence, each with its own advantages. [One can work with the 'natural time scales' of the deterioration process, or with that of the sampling process, or with something in between.] The diffusion process limit is obtained when the random sequences (time, quality) are appropriately correlated. The limit process is of the form of a filtering problem for white noise corrupted observations of a function of a Markov chain, but the limit problem is somewhat non-standard since the effective noise covariance and the signal part of the effective observation might depend on the current conditional probability, due to the nature of the 'scaling'.

For one classical case, it is shown that the optimal repair policy for the ideal limit diffusion model is nearly optimal for the actual physical process and average cost per unit time problem), when compared with the optimal policies

for the physical process. The method can be used for all the other cases also. The result is useful since the optimal policies for the physical models are not usually possible to get. It illustrates the power and potential of the diffusion approximation approach for such 'nearly' optimal control purposes.

Key Words: weak convergence, diffusion approximation, break down and repair models, nearly optimal control policies, deteriorating systems in heavy traffic.

AMS #93E11, 93E25, 90B25

Introduction

In the last decade or two, there has been a considerable effort directed towards getting (reflected) diffusion approximations of suitably scaled queues or networks of queues in heavy traffic: Lemoine [1978], Harrison [1978], Reiman [1984]. We are concerned with a somewhat non-classical problem of sampling inspection and repair which is motivated by the same basic concerns that underlie the heavy traffic work. Let $Q^\epsilon(t)$ denote the (\mathbb{R}^T -valued) content (the number either in service or waiting for service at the various servers) of the system at time t . In the heavy traffic work, loosely speaking, the arrival and service rates are such that as some parameter ϵ goes to zero, so does the difference of the service and arrival rates. For $X^\epsilon(t) \equiv \sqrt{\epsilon}Q(t/\epsilon)$, the references show that $\{X^\epsilon(\cdot)\}$ is tight in the Skorohod topology on $D^T[0, \infty)$, and that $X^\epsilon(\cdot)$ converges weakly to an appropriate reflected Wiener process or diffusion.

Typically, service rate minus arrival rate = $O(\sqrt{\epsilon})$ at each server, and the drift in the limit Wiener process or diffusion is (loosely speaking) proportional to $\lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{\epsilon}}$ [service rate - arrival rate]. Apparently, in many applications, this 'marginal drift' is economically important. For one example, in the manufacture of computer chips, the efficiency is typically very high, the processors (servers) are usually heavily loaded, and productivity differences of one percent or so can give a meaningful competitive edge.

With the scaling used in such problems, it is implicitly implied that the value or opportunity loss for an idle server is $O(\sqrt{\epsilon})$ per potential service period. Of course, one primary motivation is that the diffusion limits are usually much easier to study or to control than are the actual physical

processes, and their analytical or optimal (if controlled, (see Kushner and Ramachandran [1987])) properties can shed considerable light on the behavior of the physical processes.

Taking our motivation from the above considerations of the heavy traffic problem, it is natural to study the system deterioration, inspection and repair problem (the maintenance and repairman problem with sampling inspection) where (a) marginal returns are economically important, (b) each individual inspection or service interval has relatively little effect on the overall system, (c) the system and statistics are sufficiently complicated (or are non-Markov) so that the optimal repair policy is very difficult or impossible to obtain, (d) the appropriate diffusion limit (for the conditional distribution process of the system state given the observed data) and its associated 'repair' problem (once that is identified) is relatively simple to solve or is known, (e) one can show that applying the optimal policy for the limit process and cost to the physical system and associated cost yields a cost (average cost per unit time criterion) which is almost as good as that obtained for the actual optimal policy for the physical system in the sense that the difference in the relevant costs converge to zero as $\epsilon \rightarrow 0$. Here, ϵ is a scaling parameter which 'measures' the relative differences between production quality in the different system states. The 'limit' is always in the sense of weak convergence in the Skorohod topology on $D^r[0, \infty)$ (Billingsley [1968], Kurtz [1981]).

In our problem the production or service system deteriorates in time, samples of the production are observed, and repair undertaken when called for. As in the usual repairmen model, the repair can be either total or partial. Following the usual spirit of the heavy traffic modelling, where 'marginal'

differences in production are important, the average percentage of 'bad' items produced would vary 'marginally' with a parameter ϵ . We introduce the precise model below. Consider, for example, the case where the system is either in a good (G) or in a bad (B) state, and that the quality of the produced object is either good or bad. Owing to the importance of marginal returns, the value of a defective (bad) item might be of the *same order* as the opportunity loss attained when the server is idle. Thus, loosely speaking, $P\{\text{bad item produced} | B\} - P\{\text{bad item produced} | G\} = O(\sqrt{\epsilon})$.

The problem, as formulated here, has a number of novel features (and difficulties). The form of the limit filtering or estimation problem is not obvious a priori, and the presence of the control or repair strategy involves questions of 'admissibility' of the limit controls for the limit problem. Also, we treat correlated observations or service intervals. Additionally, the service intervals are allowed to depend (statistically) on the unknown state of the system. Also, since weak convergence on $D^*[0, \infty)$ tells us little about the 'large time' values of the processes, some additional work is needed to deal with the average cost per unit time problem.

The salient features of the problem are illustrated by the single server case, and we stick to that case until the comments at the end of the paper. Let Δt_n^ϵ denote the time required to process the n^{th} item. Until the end of the paper, we suppose that all items are inspected. This is just for the sake of notational simplicity in the main part of the development, and does not affect the final result. The sampling could be at fixed or random intervals also.

For notational convenience and w.l.o.g., assume that the server is working all the time. If the processor does not deteriorate when not in operation, then

this assumption is not restrictive at all. On the other hand if the processor does deteriorate when not in use - but at a different rate, or if there are several classes of uses, each with a different effect on the deterioration, then we can let the k_n^ϵ below depend on the class of use. If the sequence of the classes of the arrivals can be imbedded in a finite state Markov process, then the development and results are essentially the same as what follows.

The System Model, Let $S(\cdot)$ denote a continuous parameter Markov chain on $I = \{1, 2, \dots, M\}$, with infinitesimal transition probabilities $\{p_{\alpha\beta}\}$. The value of $S(\cdot)$ denotes the deterioration state of the system, but the rate at which 'time progresses' for $S(\cdot)$ depends on how the deterioration takes place as a function of use. After the n^{th} item is produced, we observe ξ_n^ϵ , a bounded random variable, one of whose components is Δt_n^ϵ , and the other component(s) concern the quality of the item. Let \mathcal{D}_n^ϵ denote the σ -algebra generated by $\{\xi_1^\epsilon, \dots, \xi_n^\epsilon\}$, and $k(\cdot)$ a bounded measurable function. Define $k_n^\epsilon = k(\xi_n^\epsilon)$ and $R_n^\epsilon = \sum_{i=1}^n k_i^\epsilon$. We could let $N = \infty$, without much additional work, but the additional notation and detail probably outweigh the advantages. $S(\cdot)$ is the underlying deterioration process. The 'effective time step' k_n^ϵ for $S(\cdot)$ per observation is the deterioration, and can depend statistically on the 'quality' observation.

We use the following system deterioration model. The ' ϵ ' scaling is consistent with the usual heavy traffic assumptions - here, the deterioration is slow on the 'real' time scale - but goes at the 'normal' rate on the interpolated time scale. The system state at the completion of the n^{th} production is defined to be $S_n^\epsilon = S(\epsilon R_n^\epsilon)$. We use

$$(1.1) \quad P\{S_{n+1}^\epsilon = \beta | S_n^\epsilon = \alpha, \xi_{n+1}^\epsilon\} = P\{S_{n+1}^\epsilon = \beta | S_n^\epsilon = \alpha, \mathcal{D}_{n+1}^\epsilon\} \\ = \epsilon k(\xi_{n+1}^\epsilon) p_{\alpha\beta} + o(\epsilon).$$

One special case is where $k(\xi_n^\epsilon) = k_0$, a constant. Here, the deterioration process is just a function of the *number* of items produced. If $k(\xi_{n+1}^\epsilon) = k_0 \Delta t_{n+1}^\epsilon$, then the deterioration rate is a function of the *actual processing time*. The form (1.1) allows the deterioration which takes place during the n^{th} processing interval to be correlated with the quality of the item produced in that interval. More generally, we could let $k(\cdot)$ depend on $\alpha = S_n^\epsilon$ also.

Define $P_n^\epsilon(\alpha) = P\{S_n^\epsilon = \alpha | \mathcal{D}_n^\epsilon\}$. To get P_{n+1}^ϵ from P_n^ϵ , ξ_{n+1}^ϵ , \mathcal{D}_n^ϵ , we need to take the new observation ξ_{n+1}^ϵ as well as the possibility of deterioration on $[R_n^\epsilon, R_{n+1}^\epsilon)$ into account.

We will prove the weak convergence of a continuous parameter interpolation of $\{P_n^\epsilon(\alpha)\}$. It will be shown that the limit process is of the form of an optimal filter for certain white noise corrupted observations on a jump Markov process model. The exact form of the limit process is far from obvious, however. Ultimately, our interest is in the control - repair problem. We show that the limit filter is the filter for a continuous parameter sampling and repair problem for which the optimal solution is known for some important special cases, and can be computed as the solution to a control problem for a diffusion process in others. Furthermore, the optimal policy for the limit is 'nearly optimal' for the actual physical process, when compared to the optimal policy for the (fixed and small ϵ) physical process.

Continuous parameter interpolations. Owing to the two ways of measuring the passage of time which are implicit in our problem (the scale which increases as do the number of observations n , and the scale which increases as does the

deterioration R_n^ϵ), there are several continuous parameter interpolations which make sense for the physical process each with its own advantages from the point of view of applications (see below). We now describe three basic interpolations. The analysis is virtually the same for all of them. Define

$$R_\epsilon(t) = \epsilon \sum_1^n k_i^\epsilon \quad \text{on } [n\epsilon, n\epsilon + \epsilon), \quad T_\epsilon(t) = \epsilon \sum_1^n \Delta t_i^\epsilon \quad \text{on } [n\epsilon, n\epsilon + \epsilon),$$

$$R_\epsilon^{-1}(t) = \max_{i=1}^n \{ \epsilon n : \epsilon \sum_{i=1}^n k_i^\epsilon \leq t \}, \quad T_\epsilon^{-1}(t) = \max_{i=1}^n \{ n : \epsilon \sum_{i=1}^n \Delta t_i^\epsilon \leq t \}.$$

Interpolation 1. (*Deterioration process time*): Define $P_\alpha^\epsilon(t) = P_{R_\epsilon^{-1}(t)/\epsilon}^\epsilon(\alpha)$ (i.e., it equals P_n^ϵ on the interval $[R_n^\epsilon, R_{n+1}^\epsilon)$, and set $S^\epsilon(t) = S(t)$. Here the progress of time is marked by the rate of system deterioration, and $R_\epsilon^{-1}(t)/\epsilon$ is the number of items produced by interpolated time t .

Interpolation 2. (*Real time*). Define

$$P_\alpha^\epsilon(t) = P_n^\epsilon \quad \text{on } [T_n^\epsilon, T_{n+1}^\epsilon) \quad \text{and} \quad S^\epsilon(t) = S(R_\epsilon(T_\epsilon^{-1}(t))).$$

Our $S^\epsilon(t)$ here is the actual state of the system when $[t/\epsilon]$ samples are taken.

Interpolation 3. (*Sampling time*). Define $P_\alpha^\epsilon(t) = P_n^\epsilon(\alpha)$ on $[n\epsilon, n\epsilon + \epsilon)$, $S^\epsilon(t) = S(R^\epsilon(t))$. This is the 'usual' interpolation for $\{P_n^\epsilon(\alpha)\}$, since each 'step' or 'sample' corresponds to ϵ in the interpolated time scale.

We work with the interpolation 1 - in the deterioration or system state time. Later, (Section 4) we (trivially) convert the result for that case to the result for interpolation 3. The other case is similar. Interpolation 1 perhaps is most natural for the pure filtering problem - since the time scale is that of the signal $S(\cdot)$ (hence the evolution operator involved in the drift term of the limit process is that of $S(\cdot)$), even though the observations (samples) are taken at random intervals. The interpolation 3 seems to be most natural for the control

problem in the sense that (loosely speaking) the mean costs associated with 'bad production' are proportional to the (interpolated) time that the system is in a 'bad' state. But, the frequency of repairs (and the associated repair costs) is most meaningful in interpolation 2. In fact, simple rescalings take us from one case to another.

Section 2 contains the basic assumptions and the definition of the terms in which the limit filter will be defined. Section 3 deals with the weak convergence of the true physical filters to the limit filter. Various extensions are dealt with in Section 4: intermittent sampling, other interpolations and 'non-anticipative' properties of limits of certain functionals. In Section 5, we show that the limit filter can be used to get a nearly optimal repair and maintenance policy for the actual physical system for a cost function of the average cost per unit time type.

2. Assumptions and Definitions

For motivation of our definition of the observation model (ξ_n^ϵ) given below, first consider the classical case where the state space is (G, B) . Let (ξ_n^ϵ) be i.i.d. under each G and B (the most common assumption in inspection - repair models). The discussion in the last section on 'marginal' differences can be quantified by writing, for example,

$$(2.1) \quad P_G^\epsilon(\xi) = P\{\xi_n^\epsilon \in [\xi, \xi + d\xi] | S_n^\epsilon = G\} = [P_B(\xi) + \sqrt{\epsilon} \beta(\xi)] \lambda(d\xi),$$

where $P_G^\epsilon, P_B^\epsilon$ are densities with respect to the given measure $\lambda(\cdot)$. [We could replace $\beta(\cdot)$ by $\beta^\epsilon(\cdot)$ where $\beta^\epsilon(\cdot)/\beta(\cdot) \xrightarrow{\epsilon \rightarrow 0} 1$.] Let $\beta(\xi)/P_B^\epsilon(\xi)$ be bounded and $\int \beta(\xi) \lambda(d\xi) = 0$ (required for the probability to integrate to unity). If $\sqrt{\epsilon} \beta(\cdot)$ is replaced by a term of smaller order, then as $\epsilon \rightarrow 0$, B and G are asymptotically indistinguishable by the observations. If the order is greater than $\sqrt{\epsilon}$, then as $\epsilon \rightarrow 0$, one can distinguish B and G in essentially 'zero interpolated time'.

If the (ξ_n^ϵ) are correlated, then a more complex model is needed. The next most complex model involves a first order Markov dependence, where we use ($\alpha = G$ or B here)

$$(2.2) \quad P\{\xi_{n+1}^\epsilon \in A | \xi_n^\epsilon = \xi, S_n^\epsilon = \alpha\} = \int_A P_\alpha(\xi, \xi') \lambda(d\xi')$$

and

$$(2.3) \quad P_G^\epsilon(\xi, \xi') = P_B(\xi, \xi') + \sqrt{\epsilon} \beta(\xi, \xi'), \quad \int \beta(\xi, \xi') \lambda(d\xi') = 0,$$

where $P_G(\xi, \cdot), P_B(\xi, \cdot)$ are densities relative to a measure $\lambda(\cdot)$ for each ξ , and $\beta(\xi, \xi')/P_B(\xi, \xi')$ is bounded.

We prefer to work with the more complex Markov dependence model in this paper, since it illustrates the possible directions of further generalizations

and allows (state dependent) correlation in the service time or 'quality' sequence - even though the notation is more complicated than that for the i.i.d. case.

For the state space $I = \{1, 2, \dots, M\}$ case, we extend (2.2), (2.3) as follows, where $P_0(\xi, \cdot)$, $P_i(\xi, \cdot)$ are densities with respect to a measure $\lambda(\cdot)$: There is P_0 :

$$(2.4) \quad P_i^\epsilon(\xi, \xi') = P_0(\xi, \xi') + \sqrt{\epsilon} \beta_i(\xi, \xi').$$

[Again $\beta^\epsilon : \beta^\epsilon/\beta \rightarrow 1$ could be used.] Although our observations ξ_n^ϵ are finite dimensional, it is not necessarily true that the dimension of the observational process which appears in the limit filtering problem is finite dimensional (a surprise!). It will be finite dimensional only if (2.4) takes the special factored form

$$(2.5) \quad P_i^\epsilon(\xi, \xi') = P_0(\xi, \xi') + \sqrt{\epsilon} \sum_{j=1}^N \beta_j(i) P_j(\xi, \xi'),$$

for some $N < \infty$. We will use this factored form. It seems that any problem can be approximated arbitrarily well by such a form.

Assumptions.

A2.1. In (2.4), (2.5), the $P_i(\xi, \xi')/P_0(\xi, \xi')$ are bounded uniformly in ξ, ξ' and $\int P_i(\xi, \xi') \lambda(d\xi') = 0$, all i, ξ . The range of ξ is compact.

A2.2. Define the one-step transition function $P(\xi, 1, d\xi') = P_0(\xi, \xi') \lambda(d\xi')$, and let the associated Markov chain have a unique invariant measure $\mu^0(\cdot)$.

We use E^0 to denote the expectation with respect to $\mu^0(\cdot)$. The assumption seems unrestricted and trivially holds in the i.i.d. case, where we

have $P_0(\xi, \xi') = P_0(\xi')$, etc. and $\mu^0(d\xi) = P_0(d\xi)\lambda(d\xi)$. Note that the 'noise' which appears is not that of the observations directly, but rather how the observations 'affect' the marginal differences in the distributions.

A2.3. The matrix

$$\Sigma_1 = \left\{ E^0 \int \frac{P_i(\xi, \xi') P_j(\xi, \xi')}{P_0(\xi, \xi')} \lambda(d\xi'), i, j \in N \right\}$$

is nonsingular. (This is rather unrestrictive - if it were singular, we would use a representation (2.5) of lower dimension. In the i.i.d. case, the E^0 is redundant.)

A2.4. $k(\cdot)$ and Δt^ϵ are bounded, measurable and $\inf_{\xi} k(\xi) > 0$, $\inf \Delta t^\epsilon > 0$. There are measurable functions $\bar{k}(\cdot)$, $\bar{\Delta t}(\cdot)$ such that

$$E[k(\xi_n^\epsilon) | S_{n-1}^\epsilon = i, \xi_{n-1}^\epsilon = \xi] = E[k(\xi_n^\epsilon) | \mathcal{D}_{n-1}^\epsilon, S_{n-1}^\epsilon = i] = \bar{k}(i, \xi),$$

$$E[\Delta t_n^\epsilon | S_{n-1}^\epsilon = i, \xi_{n-1}^\epsilon = \xi] = \bar{\Delta t}(i, \xi),$$

and $\bar{k}(i, \cdot)$ and $\bar{\Delta t}(i, \xi)$ are continuous for each i .

A2.5. $\int \bar{k}(s, \xi') P_0(\xi, \xi') \lambda(d\xi')$, $\int \bar{\Delta t}(s, \xi') P_0(\xi, \xi') \lambda(d\xi')$,

and

$$\int \frac{\rho_i(\xi, \xi') \rho_j(\xi, \xi')}{P_0(\xi, \xi')} \lambda(d\xi')$$

are continuous in ξ , for each s .

For any function $h(\cdot)$ on I and $P = \{P_i\}$, define $h(P) = \sum_i h(i) P_i$. The conditions on Δt in (A2.4, A2.5) are needed only for interpolation 2.

The general filtering equation for a Markov chain signal and white noise corrupted observations. For future reference and for purposes of identifying the weak limit of $\{P^\epsilon(\cdot), S^\epsilon(\cdot)\}$, we write the general equation for the evolution of the conditional probability distribution for this classical filtering problem. Let $H_i^0(\cdot)$, $i \in M$, be bounded continuous real valued functions and $w_i^0(\cdot)$, $i \in N$, stationary Wiener processes, independent of $S(\cdot)$, and with correlation matrix $\Sigma_0 = (E w_i^0(1) w_j^0(1))$, $i, j \in N$. Write $H^0 = (H_1^0, \dots, H_N^0)'$, $y_i^0(t) = \int_0^t H_i^0(S(\tau)) d\tau + w_i^0(t)$, $y^0 = (y_1^0, \dots, y_N^0)'$. Define the conditional probabilities $P_s(t) = P\{S(t) = s | y^0(\tau), \tau \leq t\}$. Write $P(t) = (P_i(t), i \in M)$. Then (Liptser and Shiryaev [1977], Elliot [1982])

$$(2.6) \quad dP_s(t) = \left[\sum_{i \neq s} p_{is} P_i(t) - \sum_{i \neq s} p_{si} P_s(t) \right] dt + [H^0(s) - H^0(P(t))] \Sigma_0^{-1} [dy^0 - H^0(P(t)) dt].$$

We use the following definitions in the next section.

Define $K(s) = \int \bar{k}(s, \xi) \mu^0(d\xi) = E^0 \bar{k}(s, \xi)$. Define

$$h_i(s, \xi) = \int \sum_{j=1}^N \frac{\rho_j(\xi, \xi') \rho_j(\xi, \xi') \beta_j(s)}{P_0(\xi, \xi')} \lambda(d\xi'),$$

and $H_i(s) = E^0 h_i(s, \xi)$. Then $H(s) = \Sigma_1 \beta(s) = (H_i(s), i \in N)$, where we write $\beta(s) = (\beta_1(s), \dots, \beta_N(s))'$.

3. Weak Convergence of $\{P_n^\epsilon(\cdot)\}$.

Theorem 3.1 shows that the $\{P_n^\epsilon(\cdot), i \in M\}$ converge weakly to a filter of the type (2.6). The form of the filter is somewhat non-standard, and is discussed after Theorem 3.1. Define $b_s^1(P) = \sum_{i \neq s} p_{is} P_i$, $b_s^2(P) = \sum_{i \neq s} p_{si} P_s$, for $P = (P_i, i \in M)$, and set $b_s(P) = b_s^1(P) - b_s^2(P)$, $b(P) = \{b_s(P), s \in I\}$. Recall that interpolation 1 is used in Theorem 3.1. If $k(\xi_n^\epsilon) = \Delta t_n^\epsilon$, then interpolations 1 and 2 are the same.

Theorem 3.1. *Assume (1.1) and (A2.1) to (A2.5). There is a Wiener process $w(\cdot) = (w_1(\cdot), \dots, w_N(\cdot))'$, with covariance matrix $\Sigma_1 t$ and independent of $S(\cdot)$ such that $\{P_n^\epsilon(\cdot), S(\cdot), s \in M\} \Rightarrow \{P_s(\cdot), S(\cdot), s \in M\}$ in the Skorohod topology on $D^{M+1}[0, \infty)$, where*

$$(3.1) \quad dP_s(t) = b_s(P(t))dt + P_s(t) \left[\frac{H(s)}{K(P(t))} - \frac{H(P(t))}{K(P(t))} \right] \left[\frac{\Sigma_1}{K(P(t))} \right]^{-1} \left[dy - \frac{H(P(t))}{K(P(t))} dt \right]$$

and

$$(3.2) \quad y(t) = \int_0^t \frac{H(S(\tau))}{K(P(\tau))} d\tau + \int_0^t \frac{dw(\tau)}{K^{1/2}(P(\tau))}.$$

Remark. Comparing (3.1) with (2.6) yields that (3.1) is a filtering equation for the effective observation process

$$(3.3) \quad dy = dy^0 = \frac{dw}{K^{1/2}(P(t))} + \frac{H(S(t))}{K(P(t))} dt \\ \equiv dw^0 + H^0(S(t), P(t))dt.$$

This is somewhat non-standard since the noise and signal parts of the observation process depend on the estimates $\{P_i(t)\}$. Since these are just

functions of the past observed data, a straightforward extension of the proof of (2.6) in Liptser and Shiryaev [1977] or Elliot [1982] yields that the solution of (3.1) satisfies $P_s(t) = P\{S(t) = s | y(\tau), \tau \leq t\}$. The 'weighing' of the observation term by $K^{-1}(P(t))$ arises owing to the fact that the 'natural' scale for the *observation process* is our *interpolation 3*, the sample time scale, but the interpolation which is used to get (3.1) is our *interpolation 1*, the deterioration process time. The drift term in (3.1) is not 'weighed', since it is just the operator of the deterioration process $S(\cdot)$ in its 'own' time scale. The term $\int_a^{a+\delta a} dt/K^{-1}(P(t))$ is (loosely speaking) the average number (conditioned on the 'data' up to t) of samples taken in the time that $S(\cdot)$ evolves from $S(a)$ to $S(a + \delta a)$. See the proof below. In the scale of *interpolation 2*, both the drift and diffusion terms would be scaled by functions of $P(\cdot)$, unless $k(\xi) = \Delta t$. With *interpolation 3*, the drift only would be scaled by such a function, (see Section 4), since there the evolution of $S^\epsilon(\cdot)$ depends on the evolution of $R^\epsilon(\cdot)$.

It is necessary that we treat $\{P_\alpha^\epsilon(\cdot), \alpha \in I\}$ and $S(\cdot)$ together, in order that the limit process be interpretable as a filtering process with observation process (3.2) or (3.3). Note that the dimension of the observation process in N and the underlying signal $S(\cdot)$ appears only via $H(\cdot)$. Since the limit conditional probability process (3.1) (or the analogs for the other interpolations) is much simpler than the evolution equation of Bayes' rule for our semi Markov process model, it has the usual advantage of diffusion approximations. It can be used to evaluate the effects of different deterioration or sampling or repair processes. Furthermore (Section 5) it is the correct process to use to get the 'approximately' optimal repair and maintenance policy.

Proof. 1⁰. The proof starts with Bayes' rule and via successive rearrangements of the terms, puts the expression in a form where essentially standard weak convergence methods yield both tightness and the desired expressions for the limit, as well as the independence of $S(\cdot)$ and $w(\cdot)$ and the non-anticipativeness of the $\{P_s(\cdot), s \in M\}$ with respect to $w(\cdot)$.

2⁰. By Bayes' rule, (where the $P(\xi_n^\epsilon | \cdot)$ are the obvious densities with respect to $\lambda(\cdot)$)

$$\begin{aligned}
 (3.4) \quad P(S_{n+1}^\epsilon = s | \mathcal{D}_{n+1}^\epsilon) &= \sum_i P(S_{n+1}^\epsilon = i | \mathcal{D}_n^\epsilon, \xi_{n+1}^\epsilon) \cdot P(S_{n+1}^\epsilon = s | S_n^\epsilon = i, \xi_{n+1}^\epsilon) \\
 &= \frac{\sum_i P(\xi_{n+1}^\epsilon | S_n^\epsilon = i, \mathcal{D}_n^\epsilon) \cdot P(S_n^\epsilon = i | \mathcal{D}_n^\epsilon) P(S_{n+1}^\epsilon = s | S_n^\epsilon = i, \xi_{n+1}^\epsilon)}{\sum_i P(\xi_{n+1}^\epsilon | S_n^\epsilon = i, \mathcal{D}_n^\epsilon) P(S_n^\epsilon = i | \mathcal{D}_n^\epsilon)} \\
 &= \frac{[P(\xi_{n+1}^\epsilon | S_n^\epsilon = s, \mathcal{D}_n^\epsilon) P_n^\epsilon(s) (1 - \epsilon \sum_{i \neq s} p_{si} k_{n+1}^\epsilon + o(\epsilon)) \\
 &\quad + \sum_{i \neq s} P(\xi_{n+1}^\epsilon | S_n^\epsilon = i, \mathcal{D}_n^\epsilon) P_n^\epsilon(i) (\epsilon k_{n+1}^\epsilon p_{is} + o(\epsilon))]}{\sum_i P_n^\epsilon(i) P(\xi_{n+1}^\epsilon | S_n^\epsilon = i, \mathcal{D}_n^\epsilon)}.
 \end{aligned}$$

Now, use (2.5) in (3.4) in the form

$$\begin{aligned}
 P(\xi_{n+1}^\epsilon | S_n^\epsilon = i, \mathcal{D}_n^\epsilon) &= \\
 &P_0(\xi_n^\epsilon, \xi_{n+1}^\epsilon) + \sqrt{\epsilon} \sum_{j=1}^N \beta_j(i) \rho_j(\xi_n^\epsilon, \xi_{n+1}^\epsilon).
 \end{aligned}$$

Expand the resulting expression in powers of $\sqrt{\epsilon}$ and divide all terms by $P_0(\xi_n^\epsilon, \xi_{n+1}^\epsilon)$. Define $\rho_j^n = \rho_j(\xi_n^\epsilon, \xi_{n+1}^\epsilon)$ and $P_0^n = P_0(\xi_n^\epsilon, \xi_{n+1}^\epsilon)$. Then

$$\begin{aligned}
 (3.5) \quad P_{n+1}^\epsilon(s) &= P_n^\epsilon(s) + o(\epsilon) - \epsilon b_s^2(P_n^\epsilon) k_{n+1}^\epsilon + \epsilon b_s^1(P_n^\epsilon) k_{n+1}^\epsilon \\
 &+ \sqrt{\epsilon} \sum_{j=1}^N \beta_j(s) \frac{\rho_j^n}{P_0^n} P_n^\epsilon(s) - \sqrt{\epsilon} \left[\sum_i P_n^\epsilon(i) \sum_{j=1}^N \beta_j(i) \frac{\rho_j^n}{P_0^n} \right] P_n^\epsilon(s) \\
 &- \epsilon \left[\sum_i P_n^\epsilon(i) \sum_{j=1}^N \beta_j(i) \frac{\rho_j^n}{P_0^n} \right] \left[\sum_{j=1}^N \beta_j(s) \frac{\rho_j^n}{P_0^n} \right] P_n^\epsilon(s) \\
 &+ \epsilon \left[\sum_i P_n^\epsilon(i) \sum_{j=1}^N \beta_j(i) \frac{\rho_j^n}{P_0^n} \right]^2 P_n^\epsilon(s) \\
 &\equiv P_n^\epsilon(s) + o(\epsilon) + \epsilon \left[b_s^1(P_n^\epsilon) - b_s^2(P_n^\epsilon) \right] k_{n+1}^\epsilon + T_{1,n}^\epsilon + \dots + T_{4,n}^\epsilon,
 \end{aligned}$$

where the $T_{i,n}^\epsilon$ are the last four terms on the right of (3.5). Rearranging (3.5) and using the definitions of $b_s(P)$ and $\beta_j(P)$ yields (where we write $P_n^\epsilon = (P_n^\epsilon(i), i \in M)$)

$$\begin{aligned}
 (3.6) \quad P_{n+1}^\epsilon(s) &= P_n^\epsilon(s) + o(\epsilon) + \epsilon k_{n+1}^\epsilon b_s(P_n^\epsilon) \\
 &+ \sqrt{\epsilon} P_n^\epsilon(s) \sum_{j=1}^N \frac{\rho_j^n}{P_0^n} (\beta_j(s) - \beta_j(P_n^\epsilon)) \\
 &- \epsilon P_n^\epsilon(s) \left(\sum_{j=1}^N \beta_j(P_n^\epsilon) \frac{\rho_j^n}{P_0^n} \right) \left[\sum_{j=1}^N \frac{\rho_j^n}{P_0^n} (\beta_j(s) - \beta_j(P_n^\epsilon)) \right].
 \end{aligned}$$

3⁰. To arrange the proper centering of the terms in (3.6), we next evaluate the conditional expectation (given \mathcal{D}_n^ϵ) of the last two terms in (3.6), which we call $\sqrt{\epsilon} P_n^\epsilon(s) \hat{T}_1(s, \xi_n^\epsilon, \xi_{n+1}^\epsilon, P_n^\epsilon)$ and $-\epsilon P_n^\epsilon(s) \hat{T}_2(s, \xi_n^\epsilon, \xi_{n+1}^\epsilon, P_n^\epsilon)$, resp. For notational simplicity, we sometimes use subscripts to denote the data being conditioned on.

We have, via the definition $h_j(P, \xi) = \sum h_j(i, \xi) P_i$ and (A2.1),

$$\begin{aligned}
 \sqrt{\epsilon} E_{\xi_n^\epsilon, P_n^\epsilon} [\rho_j^n / P_0^n] &= \sqrt{\epsilon} E_{D_n^\epsilon} [\rho_j^n / P_0^n] \\
 &= \sqrt{\epsilon} \sum_i \int \frac{\rho_j(\xi_n^\epsilon, \xi)}{P_0(\xi_n^\epsilon, \xi)} \left[P_0(\xi_n^\epsilon, \xi) + \sqrt{\epsilon} \sum_{k=1}^N \rho_k(\xi_n^\epsilon, \xi) \beta_k(i) \right] P_n^{\epsilon(i)\lambda}(d\xi) \\
 &= \epsilon \sum_i \int \frac{\rho_j(\xi_n^\epsilon, \xi)}{P_0(\xi_n^\epsilon, \xi)} \sum_{k=1}^N \rho_k(\xi_n^\epsilon, \xi) \beta_k(i) P_n^{\epsilon(i)\lambda}(d\xi) = \epsilon h_j(P_n^\epsilon, \xi_n^\epsilon).
 \end{aligned}$$

From this, we get

$$\begin{aligned}
 (3.7) \quad E_{D_n^\epsilon} \sqrt{\epsilon} P_n^\epsilon(s) \hat{T}_1(s, \xi_n^\epsilon, \xi_{n+1}^\epsilon, P_n^\epsilon) \\
 = \epsilon P_n^\epsilon(s) \sum_{j=1}^N [\beta_j(s) - \beta_j(P_n^\epsilon)] h_j(P_n^\epsilon, \xi_n^\epsilon).
 \end{aligned}$$

By a similar calculation we have (modulo $o(\epsilon)$) that the expression in (3.7) equals $\epsilon P_n^\epsilon(s) E_{D_n^\epsilon} \hat{T}_2(s, \xi_n^\epsilon, \xi_{n+1}^\epsilon, P_n^\epsilon)$.

Thus, we can center both terms of the \hat{T}_i -terms about their conditional expectations and change (3.6) by only $o(\epsilon)$. Note that

$$(3.8) \quad \epsilon P_n^\epsilon(s) \left[\hat{T}_2(s, \xi_n^\epsilon, \xi_{n+1}^\epsilon, P_n^\epsilon) - E_{D_n^\epsilon} \hat{T}_2(s, \xi_n^\epsilon, \xi_{n+1}^\epsilon, P_n^\epsilon) \right] \equiv q_n^\epsilon$$

is a martingale difference term with variance bounded by $\epsilon^2 K$ for some constant K . Consequently, the continuous parameter interpolation (with either interpolation 1,2 or 3) of $\sum_i q_i^\epsilon$ will converge weakly to the zero process as $\epsilon \rightarrow 0$. Equivalently, the continuous parameter interpolation of the sums of the centered last term in (3.6) converges to the zero process. W.l.o.g., we henceforth include this term in the $o(\epsilon)$, but center the \hat{T}_1 term about its conditional expectation.

Now, centering and rewriting (3.6) yields (recall the definitions in the last paragraph of Section 2)

$$\begin{aligned}
 P_{n+1}^\epsilon(s) &= P_n^\epsilon(s) + o(\epsilon) + \epsilon k_{n+1}^\epsilon b_s(P_n^\epsilon) \\
 &\quad + P_n^\epsilon(s) \sum_{i=1}^N [\beta_i(s) - \beta_i(P_n^\epsilon)] \left[\sqrt{\epsilon} \frac{\rho_i^\epsilon}{P_0^n} - \epsilon h_i(P_n^\epsilon, \xi_n^\epsilon) \right] \\
 &= P_n^\epsilon(s) + o(\epsilon) + \epsilon k_{n+1}^\epsilon b_s(P_n^\epsilon) \\
 (3.9) \quad &\quad + P_n^\epsilon(s) [H(s) - H(P_n^\epsilon)]' \Sigma_1^{-1} [\delta Y_n^\epsilon - \epsilon h(P_n^\epsilon, \xi_n^\epsilon)] \\
 &\equiv P_n^\epsilon(s) + o(\epsilon) + \epsilon k_{n+1}^\epsilon b_s(P_n^\epsilon) + P_n^\epsilon(s) [H(s) - H(P_n^\epsilon)]' \Sigma_1^{-1} \delta V_n^\epsilon
 \end{aligned}$$

where

$$\delta Y_n^\epsilon = \left\{ \sqrt{\epsilon} \frac{\rho_i^\epsilon}{P_0^n}, i \in N \right\}, \quad h(P, \xi) = \{h_i(P, \xi), i \in N\}.$$

and δV_n^ϵ is defined in the obvious way. By the definition of $h(P, \xi)$, and the results above (3.7) the term in the right hand bracket of (3.9) (namely $\{\delta V_n^\epsilon\}$, the 'approximate innovations' sequence) is a martingale difference with respect to the σ -algebra $\{\mathcal{D}_n^\epsilon\}$. Eqn. (3.9) is now 'close' to the desired limit (3.1), and we are ready for the weak convergence argument.

4⁰. **Tightness.** The sums of the b_s^i terms on the r.h.s. of (3.5) interpolate to (interpolation 1), resp.,

$$(3.10a) \quad - \int_0^t b_s^2(P^\epsilon(\tau)) d\tau + o(\epsilon),$$

$$(3.10b) \quad \int_0^t b_s^1(P^\epsilon(\tau)) d\tau + o(\epsilon).$$

The processes given by the integrals in (3.10) are tight, their limits are absolutely continuous with uniformly bounded derivatives. The last term on right (to be denoted by \hat{q}_n^ϵ) of (3.9) is a martingale difference with (conditional on \mathcal{D}_n^ϵ) variance bounded by ϵK , for some $K < \infty$. Consequently, the continuous parameter interpolation of $\sum_1^n \hat{q}_i^\epsilon$ is tight in $D[0, \infty)$ also (with

any interpolation 1,2 or 3).

Henceforth with a slight abuse of notation, let ϵ index a weakly convergent subsequence of $\{P_i^\epsilon(\cdot), S(\cdot), R_\epsilon(\cdot), R_\epsilon^{-1}(\cdot), i \in M\}$, with limit denoted by $\{P_i(\cdot), S(\cdot), R(\cdot), R^{-1}(\cdot), i \in M\}$. (It will turn out that the limit does not depend on the subsequence.) Thus, the terms of (3.10) converge weakly to

$$-\int_0^t b_s^2(P(\tau))d\tau, \int_0^t b_s^4(P(\tau))d\tau, \text{ resp.}$$

Note that $E_{S_n^\epsilon, \mathcal{D}_n^\epsilon} \delta Y_n^\epsilon = \epsilon h(S_n^\epsilon, \xi_n^\epsilon)$ and define $\delta w_n^\epsilon = [\delta Y_n^\epsilon - \epsilon h(S_n^\epsilon, \xi_n^\epsilon)]$ and $r_n^\epsilon = \epsilon P_n^\epsilon(s)[H(s) - H(P_n^\epsilon)]' \Sigma_1^{-1} \{ (h(S_n^\epsilon, \xi_n^\epsilon) - H(S_n^\epsilon)) - (h(P_n^\epsilon, \xi_n^\epsilon) - H(P_n^\epsilon)) \}$. We can rewrite the last term on the right of (3.9) as

$$(3.11) \quad P_n^\epsilon [H(s) - H(P_n^\epsilon)]' \Sigma_1^{-1} [\sqrt{\epsilon} \delta w_n^\epsilon + \epsilon H(S_n^\epsilon) - \epsilon H(P_n^\epsilon)] \\ + r_n^\epsilon \equiv \delta v_n^\epsilon + r_n^\epsilon.$$

It will turn out that the continuous parameter interpolation of $\sum_{i=1}^n r_i^\epsilon$ converges to the zero process. The rearrangement in (3.11) was done in order to extract what will be the signal component of the observation (namely $H(S_n^\epsilon)$) and to replace the centering $h(P_n^\epsilon, \xi_n^\epsilon)$ by its conditional mean $H(P_n^\epsilon)$. The ξ_i^ϵ is averaged out via the invariant measure $\mu^0(\cdot)$. $H(P_n^\epsilon)$ is the conditional expectation of $H(S_n^\epsilon)$ given \mathcal{D}_n^ϵ , and the resulting form (modulo the 'error' r_n^ϵ) is closer to that of the limit process.

The term $\sqrt{\epsilon} P_n^\epsilon [H(s) - H(P_n^\epsilon)]' \Sigma_1^{-1} \delta w_n^\epsilon \equiv \tilde{q}_n^\epsilon$ is a martingale difference (with respect to the σ -algebras measuring $\{\xi_i^\epsilon, i \in n, S_n^\epsilon\}$), with third moment being $O(\epsilon^{3/2})$. Thus, the weak limits of the continuous parameter interpolation of $\sum_{i=1}^n \tilde{q}_i^\epsilon$ (with interpolation intervals being either ϵ or (ϵk_n^ϵ)) are continuous processes. Thus, the limits $P_i(\cdot)$ of $P_i^\epsilon(\cdot)$ will be continuous.

5⁰. **Weak Convergence.** It will sometimes be convenient to use the

constant interpolation intervals ϵ rather than $\{\epsilon k_n^\epsilon\}$, obtain the limit and then rescale. We will state what the interpolation interval is, whenever it is important. Owing to the random number of samples taken on any (interpolated) time interval $[0, t]$, we proceed indirectly. Define $\tilde{P}_i(\cdot) = P_i(R^{-1}(\cdot))$ and $\tilde{S}(\cdot) = S(R^{-1}(\cdot))$. The limits $R(\cdot)$ and $R^{-1}(\cdot)$ are continuous and strictly monotonically increasing, and the limit of $R_\epsilon^{-1}(\cdot)$ is just $R^{-1}(\cdot)$, the inverse of the limit of $R_\epsilon(\cdot)$, so there is no ambiguity in the notation. Define $\tilde{S}^\epsilon(\cdot) = S(R_\epsilon^{-1}(\cdot))$ and $\tilde{P}_i^\epsilon(\cdot) = P_i^\epsilon(R_\epsilon^{-1}(\cdot))$. We work with \tilde{S} and \tilde{P} and then rescale. We have $(\tilde{S}(\cdot), \tilde{P}_i(\cdot), i \in M) \Rightarrow (\tilde{S}^\epsilon(\cdot), \tilde{P}_i^\epsilon(\cdot), i \in M)$.

By the definition of $\bar{k}(\cdot)$,

$$\epsilon \sum_1^n k_i^\epsilon = \epsilon \sum_1^n [k_i^\epsilon - E_{\mathcal{D}_{i-1}^\epsilon} k_i^\epsilon] + \epsilon \sum_1^n \bar{k}(P_{i-1}^\epsilon, \xi_{i-1}^\epsilon),$$

Since the first sum on the right is a martingale with quadratic variation $\leq K\epsilon^2 n$ for some $K < \infty$, the desired weak limit $R(\cdot)$ of $\{R_\epsilon(\cdot)\}$ is that of the continuous parameter interpolation (interval ϵ) of the last sum on the right.

The averaging method which will be used to get the limit process is the 'invariant measure' method used in Kushner and Shwartz [1984] for processes with limits satisfying ordinary differential equations and in Kushner [1984, Chapter 5] for limits which are general diffusion processes. However, the development here is self-contained.

Let $\Delta > 0$ such that ϵ is an integral multiple of Δ and let $\tau < t$, where t and τ are integral multiples of Δ . (The integrality is for notational convenience only; it is actually only required that the ratios go to infinity as $\epsilon \rightarrow 0$ or $\Delta \rightarrow 0$.) Write $\mathcal{D}^\epsilon(i\Delta) = \mathcal{D}_{i\Delta/\epsilon}^\epsilon$. For use below, we evaluate the weak limit of

$$(3.12) \quad \epsilon E_{\mathcal{D}^\epsilon(\tau)} \sum_{n=\tau}^t \bar{k}(P_n^\epsilon, \xi_n^\epsilon).$$

Owing to the continuity of $\bar{k}(i, \cdot)$ and the fact that $\{P^\epsilon(\cdot)\}$ is tight and has continuous limits, the weak limit of (3.12) is the same as the weak limit of (as $\epsilon \rightarrow 0$, then $\Delta \rightarrow 0$)

$$(3.13) \quad E_{\mathcal{D}^\epsilon(\tau)} \sum_{i\Delta=\tau}^t \Delta E_{\mathcal{D}^\epsilon(i\Delta)} \left[\frac{\epsilon}{\Delta} \right]_{n=m_i}^{m_i+1-1} \bar{k}(\tilde{P}^\epsilon(i\Delta), \xi_n^\epsilon),$$

where $m_i = i\Delta/\epsilon$.

For each ω , define the measure $v_i^{\epsilon, \Delta}(\omega, \cdot)$ by

$$v_i^{\epsilon, \Delta}(\omega, \cdot) = \frac{\epsilon}{\Delta} \sum_{n=m_i}^{m_i+1-1} P\{\xi_n^\epsilon \in \cdot | \mathcal{D}^\epsilon(i\Delta)\}(\omega).$$

The inner sum of (3.13) equals

$$(3.14) \quad \int \bar{k}(\tilde{P}^\epsilon(i\Delta), \xi) v_i^{\epsilon, \Delta}(\omega, d\xi).$$

The set of measures $\{v_i^{\epsilon, \Delta}(\omega, \cdot), \omega, i, \Delta, \epsilon\}$ is tight, since the range of ξ is compact. Fix ω . With an obvious abuse of notation, let ϵ index a weakly convergent subsequence (of the originally selected weakly convergent subsequence of ϵ) such that $v_i^{\epsilon, \Delta}(\omega, \cdot) \Rightarrow v_i^\Delta(\omega, \cdot)$, for some set of measures $v_i^\Delta(\omega, \cdot)$, and all $i \leq t/\Delta$. It will turn out that this limit measure does not depend on ω, Δ or i . We next characterize $v_i^\Delta(\omega, \cdot)$.

By (2.5) and the definition of $v^{\epsilon, \Delta}(\cdot)$, the weak limit of (3.15) (as $\epsilon \rightarrow 0$) is the same as the weak limit of (3.14), mod terms of value $o(\Delta)/\Delta$ (recall that ω is fixed here).

$$\begin{aligned}
 & \frac{\epsilon}{\Delta} \sum_{n=m_i}^{m_{i+1}-1} \int \bar{k}(\bar{P}^\epsilon(i\Delta), \xi_n^\epsilon) P(d\xi_n^\epsilon | \xi_{n-1}^\epsilon, S_{n-1}^\epsilon) \cdot P(d\xi_{n-1}^\epsilon dS_{n-1}^\epsilon | \mathcal{D}^\epsilon(i\Delta)) \\
 (3.15) \quad & = \frac{\epsilon}{\Delta} \sum_{n=m_i}^{m_{i+1}-1} \int \bar{k}(\bar{P}^\epsilon(i\Delta), \xi) P_0(\xi_{n-1}^\epsilon, \xi) \lambda(d\xi) \cdot P(d\xi_{n-1}^\epsilon | \mathcal{D}^\epsilon(i\Delta)) + o(\Delta)/\Delta \\
 & = \frac{\epsilon}{\Delta} \sum_{n=m_i}^{m_{i+1}-1} \int \bar{k}(\bar{P}^\epsilon(i\Delta), \xi) P_0(\xi_{n-1}^\epsilon, \xi) \lambda(d\xi) \cdot v_i^{\epsilon, \Delta}(\omega, d\xi_{n-1}^\epsilon) + o(\Delta)/\Delta
 \end{aligned}$$

Using the first continuity condition in (A2.5) and taking limits in both (3.14) and in the last term of (3.15) yields that

$$(3.16) \quad \int \bar{k}(\bar{P}(i\Delta), \xi) v_i^\Delta(\omega, d\xi) = \int \bar{k}(\bar{P}(i\Delta), \xi') P_0(\xi, \xi') \lambda(d\xi') v_i^\Delta(\omega, d\xi).$$

By letting $\bar{k}(\cdot, \cdot)$ be replaced by an arbitrary bounded and continuous function of ξ in (3.15) and (3.16) yields that $v_i^\Delta(\omega, \cdot)$ is an *invariant measure for the Markov chain with one step transition density* $P_0(\xi, \xi') \lambda(d\xi')$. By (A2.2), this invariant measure $\mu^0(\cdot)$ is unique. Hence $v_i^\Delta(\omega, \cdot) = \mu^0(\cdot)$, and does not depend on Δ, ω or i . Thus, the weak limit of (3.10) is

$$\int_{\mathcal{T}} E^0 \bar{k}(\bar{P}(u), \xi) du = \int_{\mathcal{T}} K(\bar{P}(u)) du,$$

where (recall) $E^0 q(\xi) = \int q(\xi) \mu^0(d\xi)$ for any function $q(\cdot)$.

Let $g(\cdot)$ be bounded, real valued and continuous, By the above arguments of this section, for $\tau_i \leq \tau$, $i \leq r$,

$$\lim_{\epsilon} E g(\bar{P}^\epsilon(\tau_i), i \leq r) \left[R^\epsilon(t) - R^\epsilon(\tau) - \int_{\mathcal{T}} K(\bar{P}^\epsilon(u)) du \right] = 0.$$

Owing to the arbitrariness of $g(\cdot)$, τ_i, r, t, τ , we have that $R(t) - \int_0^t K(\bar{P}(u)) du$ is a martingale with zero quadratic variation. Hence it is constant and we have

$R(t) = \int_0^t K(\tilde{P}(u))du$. It follows that $R^{-1}(t) = K^{-1}(P(t))$; hence $R^{-1}(t) = \int_0^t \frac{du}{K(P(u))}$

An argument very similar to the above 'averaging' argument but using the last continuity condition in (A2.5), shows that the interpolation (interval ϵ) of $\epsilon \sum_1^n r_i^\epsilon$ (see above (3.11) for the definition of r_n^ϵ) converges weakly to the zero process. The limits of the interpolations (interval ϵ) of the sums in (3.11) which do not involve δw_n^ϵ are also obvious. I.e.,

$$(3.17) \quad \epsilon \sum_1^{t/\epsilon} P_n^\epsilon(s) H(P_n^\epsilon)' \Sigma_1^{-1} H(S_n^\epsilon)$$

converges weakly to

$$(3.18) \quad \int_0^t P_\bullet R^{-1}(u) H(P(R^{-1}(u)))' \Sigma_1^{-1} H(S(R^{-1}(u))) du.$$

It is not difficult to show that there is a Wiener process $\tilde{W}(\cdot)$, with covariance $\Sigma_1 t$ such that $\tilde{S}(\cdot)$ and $\tilde{P}(\cdot)$ are non-anticipative with respect to $\tilde{W}(\cdot)$ and that the process (see above (3.11) for the definition), $\sum_1^{t/\epsilon} \delta v_n^\epsilon$ weakly to

$$(3.19) \quad \int_0^t \tilde{P}_\bullet(u) [H(s) - H(\tilde{P}(u))] \Sigma_1^{-1} [d\tilde{W}(u) + H(\tilde{S}(u))du - H(\tilde{P}(u))du] \\ \equiv \int_0^t dv(u).$$

To get the proper limits with the original interpolation intervals $\{\epsilon k_n^\epsilon\}$, we actually need the limit of

$$\sum_{n=1}^{R_\epsilon^{-1}(t)/\epsilon} v_n^\epsilon.$$

But this is $\int_0^{R^{-1}(t)} dv(u) = v(R^{-1}(t))$.

Changing scale yields the existence of a Wiener process $w(\cdot)$, with

covariance $\Sigma_1 t$ such that $S(\cdot)$ and $P(\cdot)$ are non-anticipative with respect to $w(\cdot)$ and

$$(3.20) \quad v(R^{-1}(t)) = \int_0^t P_s(u) \left[\frac{H(s)}{K(P(u))} - \frac{H(P(u))}{K(P(u))} \right]' \left[\frac{\Sigma_1}{K(P(u))} \right]^{-1} \cdot \left[\frac{dw(t)}{K^{1/2}(P(u))} + \frac{H(S(u))}{K(P(u))} du - \frac{H(P(u))}{K(P(u))} du \right].$$

We omit the details concerning the mutual independence of $S(\cdot)$ and $w(\cdot)$. It is basically a consequence of the fact that as $\epsilon \rightarrow 0$, the distribution of ξ_n^ϵ depends on S_n^ϵ only via the $\sqrt{\epsilon}$ term in (2.5). The proof of the independence is accomplished by calculating the operator of the joint process $(S(\cdot), w(\cdot))$.

For the 'non-stochastic integral' terms such as (3.17), which arise from (3.11), we need the limit with t replaced by $R_\epsilon^{-1}(t)$ (for interpolation 1). Thus, the desired limit is (3.19) with t replaced by $R^{-1}(t)$. But this is just (3.18) with $R^{-1}(u)$ replaced by u and with the integrand divided by $K(P(u))$, as above. Adding these terms to (3.20) and (3.10) (with $\epsilon = 0$) yields (3.1).

Finally we note that, given $S(\cdot)$ and any $w(\cdot)$ independent of $S(\cdot)$, the solution to (1.1) is unique since $\sum P_i(s) \equiv 1$ and the coefficients are uniformly Lipschitz continuous in $(P_i, i \leq n)$. Thus the chosen weakly convergent subsequences are irrelevant, and the weak convergence holds as stated in the theorem.

Q.E.D.

4. Extensions

4.1. **Skipping samples.** Instead of sampling each item, let us sample only items $\beta_1^\epsilon, \beta_2^\epsilon, \dots$, where $\beta_n^\epsilon \uparrow \infty$, and $\beta_{n+1}^\epsilon - \beta_n^\epsilon$ is bounded. Then

$$\epsilon \sum_{i=\beta_n^\epsilon+1}^{\beta_{n+1}^\epsilon} k_i^\epsilon = \bar{K}_{n+1}^\epsilon$$

replaces the k_{n+1}^ϵ in the work of the last two sections. Let ξ_n^ϵ now denote the observation which is taken at the n^{th} actual sample, and let it include \bar{K}_n^ϵ as one component. Consider two particular cases.

Case 1: $k(\xi) = \Delta t$. Under the assumptions of Theorem 1 with \bar{K}_n^ϵ replacing k_n^ϵ , the conclusions of the theorem hold.

Case 2: Let $k(\xi) = k_0$, a constant. Then with $k_0[\beta_{n+1}^\epsilon - \beta_n^\epsilon] = \bar{K}_{n+1}^\epsilon$ replacing the k_n^ϵ in Theorem 1, the conclusions of that theorem still hold.

4.2. **A Tandem Network.** Consider a simple tandem system with two processors in series and let $S(\cdot)$ still denote the deterioration process. Let each item be observed as it leaves each processor, and let $\xi_n^{i,\epsilon}$ denote the n^{th} observation at processor i . Let $\xi_n^{i,\epsilon}$ include $k_n^{i,\epsilon}$, the 'deterioration measure' between the $(n-1)^{\text{st}}$ and n^{th} observed sample at processor i . Let $S_n^{i,\epsilon}$ denote the system state at processor i at the time of the n^{th} sample. Then $S_n^{i,\epsilon} = S(\epsilon \sum_{j=1}^{n-1} k_j^{i,\epsilon})$. Assume that

$$\begin{aligned} P\{\xi_n^{i,\epsilon} \in A \mid \xi_k^{i,\epsilon}, k < n, S_{n-1}^{i,\epsilon} = s\} \\ = \int_A \left[P_0^i(\xi) + \sqrt{\epsilon} \sum_{j=1}^N \beta_j^i(s) \rho_j^i(\xi) \right] \lambda(d\xi). \end{aligned}$$

Here under each fixed system state, we let $\{\xi_n^{i,\epsilon}, n < \infty\}$ be i.i.d., and let

$\{\xi_n^{1,\epsilon}, n < \infty\}$ be independent of $\{\xi_n^{2,\epsilon}, n < \infty\}$.

Under assumptions analogous to those used in Theorem 1, we get (4.1), where the H^i, Σ_1^i, K^i are defined in the obvious way. The $w^1(\cdot)$ and $w^2(\cdot)$ are mutually independent.

$$(4.1) \quad dP_s(t) = \left[\sum_{i \neq s} p_{is} P_i(t) - \sum_{i \neq s} p_{si} P_s(t) \right] dt + P_s(t) \sum_{i=1}^2 \left[\frac{H^i(s)}{K^i(P(t))} - \frac{H^i(P(t))}{K^i(P(t))} \right] \left[\frac{\Sigma_1^i}{K^i(P(t))} \right]^{-1} \left[dy^i - \frac{H^i(P(t))}{K^i(P(t))} dt \right],$$

where

$$(4.2) \quad dy^i = \frac{H^i(S(t))}{K^i(P(t))} dt + \frac{dw^i(t)}{[K^i(P(t))]} \frac{1}{2};$$

and $w^i(\cdot)$ is a Wiener process with covariance $\Sigma_1^i t$.

The interpretation as a filtering equation is the same as for the result of Theorem 1.

The result can be extended to more general networks, to the correlated observation case, and with the intermittent sampling extensions of subsection 4.1 above.

4.3. Interpolation 3. By the definition of interpolation 3, the limit processes for that interpolation can be obtained from the result in Theorem 1 by rescaling time by $R(t)$; in particular, in interpolation 3 we have

$$(4.3) \quad dP_s(t) = K(P(t)) b_s(P(t)) dt + P_s(t) [H(s) - H(P(t))] \Sigma^{-1} [dy - H(P(t)) dt],$$

where

$$(4.4) \quad dy = H(S(R(t))) dt + dw, \text{ cov } w(t) = \Sigma_1 t.$$

To understand (4.3), (4.4), note that

$$P_s^\epsilon(t) \text{ (in scaling 3)} = P_s^\epsilon(R_\epsilon(t)) \text{ (in scaling 1)}.$$

Equivalently, there is no $K(P(t))$ 'normalization' in the the observation part of the equation since the (discrete parameter) time scale is measured simply by the number of observations taken. The $K(P(t))$ reflects the mean instantaneous 'rate of deterioration' per sample, given the observed data. The result for interpolation 2 is obtained in a similar manner.

4.4. An alternative representation for $P(\cdot)$ and non-anticipative stopping times. To prepare for the discussion in Section 5, we use interpolation 3, with $k_n^\epsilon = \Delta t_n^\epsilon$ but keep the other conditions of Theorem 1. Let $T_\epsilon \geq 0$ denote a tight and $[0, \infty)$ valued sequence of random variables, which together with $\{P^\epsilon(\cdot), R_\epsilon(\cdot), S(\cdot)\}$ converges weakly to $\{T, P(\cdot), R(\cdot), S(\cdot)\}$ as $\epsilon \rightarrow 0$. Let $I_{\{T_\epsilon \leq n\epsilon\}}$ be \mathcal{D}_n^ϵ measurable for each ϵ, n . We show that there is a N -dimensional Wiener process $v(\cdot)$ with covariance $\Sigma_1 t$ such that $I_{\{T \leq \cdot\}}, S(R(\cdot))$ and $P(\cdot)$ are non-anticipative with respect to $v(\cdot)$ and

$$(4.5) \quad dP_s(t) = K(P(t))b_s(P(t))dt + P_s(t) [H(s) - H(P(t))] \Sigma_1^{-1} dv(t), \quad s \in M.$$

Of course $v(\cdot)$ has the representation $dv = dy - H(P(t))dt$.

Let $g(\cdot)$ and $f(\cdot)$ be smooth real valued functions and let $t_i \leq t < t_i + \tau$, $i \leq q$. Choose any t_i such that $P(T = t_i) = 0$. Let A denote the operator of the system (4.5) (with $v(\cdot)$ a Wiener process with covariance $\Sigma_1 t$). Let $Q_s(P)$ denote the row vector $P_s [H(s) - H(P)] \Sigma_1^{-1}$, and let $Q(P)$ denote the matrix with s^{th} row $Q_s(P)$.

We have (under the conditions of the first three sentences of this subsection)

$$(4.6) \quad \text{Eg}(P^\epsilon(t_i), I_{\{T_\epsilon \leq t_i\}}, S^\epsilon(R_\epsilon(t_i)), i \leq q) \\ \left[f(P^\epsilon(t+\tau)) - f(P^\epsilon(t)) - \epsilon \sum_{n \leq t}^{t+\tau} (f(P_{n+1}^\epsilon) - f(P_n^\epsilon)) / \epsilon \right] = 0$$

also, the left side of (4.6) equals zero if $E_{\mathcal{D}^\epsilon(t)}$ is put before the sum and $E_{\mathcal{D}_n^\epsilon}$ before the n^{th} summand (recall that $E_{\mathcal{D}^\epsilon(t)} = E_{\mathcal{D}_{t/\epsilon}^\epsilon}$). Doing this and expanding the $E_{\mathcal{D}_n^\epsilon} f(P_{n+1}^\epsilon) - f(P_n^\epsilon)$ and using $\text{cov}_{\mathcal{D}_n^\epsilon} \delta V_n^\epsilon = \Sigma_1 \epsilon + o(\epsilon)$ (see (3.9) for the definition of δV_n^ϵ) yields

$$\epsilon E_{\mathcal{D}_n^\epsilon} [f(P_{n+1}^\epsilon) - f(P_n^\epsilon)] = \epsilon f'_p(P_n^\epsilon) b(P_n^\epsilon) E_{\mathcal{D}_n^\epsilon} \Delta t_n^\epsilon \\ + \frac{\epsilon}{2} \text{trace } f_{pp}(P_n^\epsilon) [Q(P_n^\epsilon) \Sigma_1 Q'(P_n^\epsilon)] + o(\epsilon).$$

Here, the subscripts denote the obvious gradients or Hessians.

Putting this into (4.6), averaging the sums of the $E_{\mathcal{D}_n^\epsilon} \Delta t_n^\epsilon$ as done in Theorem 1, and letting $\epsilon \rightarrow 0$ yields

$$(4.7) \quad \text{Eg}(P(t_i), I_{\{T \leq t_i\}}, S(R(t_i)), i \leq q) \left[f(P(t+\tau)) - f(P(t)) - \int_t^{t+\tau} A f(P(u)) du \right] = 0.$$

Owing to the arbitrariness of $g(\cdot)$, $f(\cdot)$, $\{t_i\}$ and q , $P(\cdot)$ solves the martingale problem with respect to the σ -algebras \mathcal{F}_t measuring $\{P(u), I_{\{T \leq u\}}, S(R(u)), u \leq t\}$. The assertions concerning $v(\cdot)$ and the non-anticipativeness property all follow from this.

This result will be used in the next section, where it will be necessary to show that the limits of the stopping (repair) times T_ϵ are 'admissible' stopping times (i.e., non-anticipative with respect to the appropriate Wiener process) for the limit filtering system.

5. The Optimal and Nearly Optimal Maintenance Policy.

The optimal stopping and repair policy for the physical process with observations $\{\xi_n^\epsilon\}$ is usually difficult to compute even in the simplest cases. It is often substantially simpler to compute a policy for the limit process. It is natural to try to use a policy which is optimal or nearly optimal for the limit process on the actual physical process. The question then arises concerning how good is this 'limit' policy when compared to the optimal policies for the physical processes. We treat one simple and classical case and show that the 'limit' policy is indeed 'nearly' optimal for small ϵ . We choose a classical model for which the calculations are simple, but the idea also works for the general case.

Problem description. The state space = $I = (G, B) = (\text{Good}, \text{Bad})$, with Bad absorbing and $p_{GB} = p > 0$. Interpolation 3 will be used, let $k_n^\epsilon = \Delta t_n^\epsilon$, $\xi_n^\epsilon = (\tilde{\xi}_n^\epsilon, \Delta t_n^\epsilon)$, where $\{\tilde{\xi}_n^\epsilon, n < \infty\}$ are the 'quality' variables and are independent of $\{\Delta t_n^\epsilon, n < \infty\}$, and the Δt_n^ϵ do not depend on the state or on ϵ . We suppose that $\{\Delta t_n^\epsilon\}$ is a stationary first order Markov process with a unique invariant measure (although the results will be true for any stationary process $\{\Delta t_n^\epsilon\}$ which is sufficiently mixing. We now only need a model of the form (2.3) for the $\{\tilde{\xi}_n^\epsilon\}$. We use

$$P_G^\epsilon(\tilde{\xi}, \tilde{\xi}') = P_B(\tilde{\xi}, \tilde{\xi}') + \sqrt{\epsilon} \rho(\tilde{\xi}, \tilde{\xi}') + o(\sqrt{\epsilon})$$

$$(5.1) \quad \int \rho(\tilde{\xi}, \tilde{\xi}') \lambda(d\tilde{\xi}') = 0.$$

Here, we can always take $N = 1$, since there are only two states. By (5.1), $B(G) = 1$, $B(B) = 0$, $P_0(\tilde{\xi}, \tilde{\xi}') = P_B(\tilde{\xi}, \tilde{\xi}')$. We use P_B as the reference measure - but it could be P_G or any 'intermediate' measure - with the same results. We

use E^0 to denote expectation with respect to any of the stationary measures either that for $\{\Delta t_n^\epsilon\}$ or that for $\{\tilde{\xi}_n^\epsilon\}$ which corresponds to the one step transition function (see (A2.2)) $P(\tilde{\xi}, 1, d\tilde{\xi}') = P_B(\tilde{\xi}, \tilde{\xi}')\lambda(d\tilde{\xi}')$. Also

$$\begin{aligned}
 h(B, \tilde{\xi}) &= 0 \\
 h(G, \tilde{\xi}) &= \int \frac{\rho^2(\tilde{\xi}, \tilde{\xi}')}{P_B(\tilde{\xi}, \tilde{\xi}')} \lambda(d\tilde{\xi}') \\
 (5.2) \quad \Sigma_1 &= E^0 h(G, \tilde{\xi}) \\
 H(B) &= 0, \quad H(G) = E^0 h(G, \tilde{\xi}) = \Sigma_1 \\
 H(P) &= H(G)(1 - P_B).
 \end{aligned}$$

We assume the appropriate forms of (A2.1) - (A2.5) throughout this section.

By the above definitions and (4.3), the limit filter is (interpolation 3)

$$\begin{aligned}
 (5.3) \quad dP_B &= (E^0 \Delta t) p_B (1 - P_B) dt \\
 &\quad - P_B (1 - P_B) H(G) \Sigma_1^{-1} [dy - (1 - P_B) H(G) dt], \\
 dy &= dw + H(S(t)) dt, \quad \text{covar } w(1) = \Sigma_1.
 \end{aligned}$$

Write $dv = dy - (1 - P_B) H(G) dt$. $v(\cdot)$ is the innovations process as discussed at the end of Section 4, and it is a Wiener process with covariance $\Sigma_1 t$. Also $P_B(\cdot)$ is non-anticipative with respect to $v(\cdot)$. Rewrite (5.3) as

$$(5.4) \quad dP_B = (E^0 \Delta t) p (1 - P_B) dt - P_B (1 - P_B) H(G) \Sigma_1^{-1} dv.$$

If $v(\cdot)$ is replaced by an arbitrary Wiener process with covariance $\Sigma_1 t$, then (5.4) has a unique non-anticipative solution $P_B(\cdot)$, and its probability law is the same as that of the solution to (5.3) or (5.4). Thus, for purposes of optimal control, if the cost criteria are in terms of $P_B(\cdot)$ only, and do not involve

$S(\cdot)$, then (5.3) and (5.4) are equivalent models. This is known as the 'separated' problem (Hausman [1985]) in stochastic control theory. This replacement or separation will be used below.

The cost criterion. If the processor is inspected and found to be G , then we charge c_1 . If it is found to be B on inspection, then we charge $c_2 > c_1$. After inspection (and repair, if B) the state is G . Furthermore, we wish to penalize the quality loss due to a bad machine. Under B or G , the distributions of the 'quality' are close. In line with our view of 'marginal' costs, the penalization should be only on the mean difference in the quality for B and G . In heavy traffic modelling of queues it is common to value lost production at $0(\sqrt{\epsilon})$ per item. With this in mind, and using the $0(\sqrt{\epsilon})$ difference in the distributions of quality, a reasonable total cost over a time interval $[0,t]$ is, for some $c_0 > 0$,

$$(5.5) \quad E \sum_{n=1}^{t/\epsilon} \epsilon c_0 P_n^\epsilon(B) I_{\{\text{complete a production at time } n\}} \\ + E \sum_{n=1}^{t/\epsilon} c_1 I_{\{\text{repair at time } n, G\}} \\ + E \sum_{n=1}^{t/\epsilon} c_2 I_{\{\text{repair at time } n, B\}}.$$

Normalizing and taking limits yields

$$(5.6) \quad \lim_t \frac{E}{t} \left[\int_0^t c_0 P_B^\epsilon(u) du + c_1 [\# \text{Repairs by } t \text{ when } G] \right. \\ \left. + c_2 [\# \text{Repairs by } t \text{ when } B] \right]$$

The observations after the repair are assumed to be independent of those before the repair. We consider only stationary admissible policies: The decision

to repair can depend only on the observed data taken since the last repair, and the decision function does not depend on the number of past repairs. Let T_ϵ denote the length of the renewal interval between repairs. Then if $ET_\epsilon < \infty$, (5.6) equals

$$(5.7) \quad \frac{E \left[\int_0^{T_\epsilon} c_0 P_B^\epsilon(u) du + c_1(1 - P_B^\epsilon(T_\epsilon)) + c_2 P_B^\epsilon(T_\epsilon) \right]}{ET_\epsilon} \equiv V^\epsilon(T_\epsilon).$$

In order to avoid some peripheral details below, we make the unrestrictive assumption that there is some small $\Delta_0 \in (0,1)$ such that repair takes place on or before the first time that $P_B^\epsilon(t) \geq 1 - \Delta_0$.

The cost and optimal policy for (5.4). Define the cost for (5.4), where T is any stopping time which is non-anticipative with respect to $v(\cdot)$.

$$V(T) = \frac{E \left[\int_0^T c_0 P_B(u) du + c_1(1 - P_B(T)) + c_2 P_B(T) \right]}{ET}$$

Theorem 2 (Shiryayev [1978]). *Let $P_B(0) = 0$. There is $\Delta_1 \in (0,1)$ such that $\bar{T} = \min \{t : P_B(t) = 1 - \Delta_1\}$ is optimal for $V(T)$, with respect to all non-anticipative T . Also $E\bar{T} < \infty$.*

We choose Δ_0 such that $\Delta_1 > \Delta_0$.

The limits of the costs and optimal stopping times for $P^\epsilon(\cdot)$. Let T_ϵ

denote the optimal stopping time for $P_B^\epsilon(\cdot)$.

Theorem 3. *Let $P_B^\epsilon(0) = 0$. Then $\{T_\epsilon\}$ is uniformly integrable.*

Proof: The theorem is a consequence of the Markov property of $S(\cdot)$ and the fact that P_n^ϵ is a conditional probability. There are $\delta_i > 0$ such that for all t and all $s \geq \delta_1$

$$P_{D^\epsilon(t)} \{P_B^\epsilon(t+s) \geq 1 - \Delta_0\} \geq \delta_2 \text{ w.p.1.}$$

This implies that all moments of the entry time into the set $[1 - \Delta_0, 1]$ are bounded.

Q.E.D.

Theorem 4. *Let $\{P_B^\epsilon(\cdot), T_\epsilon\} \Rightarrow \{P_B(\cdot), \hat{T}\}$, where T_ϵ is optimal for $P_B^\epsilon(\cdot)$ and cost (5.7). Then $P_B(\cdot)$ satisfies (5.4), \hat{T} is non-anticipative with respect to $v(\cdot)$ and $V^\epsilon(T_\epsilon) \rightarrow V(\hat{T})$.*

Proof: The weak convergence is just Theorem 1 (see also Section 4.4). The non-anticipativity was proved in Section 4.4. The convergence $V^\epsilon(T_\epsilon) \rightarrow V(\hat{T})$ is a consequence of the weak convergence and Theorem 3.

Q.E.D.

The near optimality of \bar{T} for $P_B^\epsilon(\cdot)$. Define the stopping rule \bar{T}_ϵ by $\bar{T}_\epsilon = \min \{t : P_B^\epsilon(t) \geq 1 - \Delta_1\}$. Theorem 5 says that \bar{T}_ϵ is nearly optimal for $P_B^\epsilon(\cdot)$ - and justifies using the limit model (3.1) or (5.4) for control purposes.

Theorem 5.

$$\lim_{\epsilon} [V^{\epsilon}(T_{\epsilon}) - V^{\epsilon}(\bar{T}_{\epsilon})] \geq 0.$$

Proof: Select a weakly convergent subsequence of $\{P_B^{\epsilon}(\cdot), \bar{T}_{\epsilon}\}$, $\{P_B^{\epsilon}(\cdot), T_{\epsilon}\}$, also indexed by ϵ . Then $\{P_B^{\epsilon}(\cdot), \bar{T}_{\epsilon}\} \Rightarrow \{P_B(\cdot), \bar{T}\}$ and $\{P_B^{\epsilon}(\cdot), T_{\epsilon}\} \Rightarrow \{P_B(\cdot), \hat{T}\}$, where \hat{T} is non-anticipative with respect to $v(\cdot)$ in (5.4). By Theorem 4 and the weak convergence $V^{\epsilon}(T_{\epsilon}) \rightarrow V(\hat{T})$, $V^{\epsilon}(\bar{T}_{\epsilon}) \rightarrow V(\bar{T})$. But, since \bar{T} is optimal, $V(\bar{T}) \leq V(\hat{T})$.

Q.E.D.

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