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CODING CAPACITY OF GENERALIZED ADDITIVE CHANNELS

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Introduction

The generalized additive channel was introduced in [1]. It is described by an additive noise process with sample functions inducing a measure on a linear topological vector space, and by a constraint which includes dimensionality. The coding capacity of the matched channel was analyzed in [1], with an exact value obtained for the Gaussian channel and an upper bound for a class of nonGaussian channels. Bounds on the coding capacity for the mismatched Gaussian generalized channel were obtained in [2].

In this paper, the exact coding capacity of the mismatched Gaussian generalized channel is determined, along with an upper bound for a class of nonGaussian mismatched channels. The set of admissible constraints is also greatly increased over that considered in [2]. Although the treatment here is restricted to noise measures induced on a separable Hilbert space, it can readily be seen that the results extend immediately to the class of linear topological vector spaces considered in [1]. The results of the present paper are partly based on the Hilbert space results on information capacity given in [3]; for the extension to linear topological vector spaces, one would use the corresponding results given in [4]. The focus on Hilbert space is useful for application of the results given here to the discrete-time or continuous-time additive channel.

The basic path followed here is well-known to information theorists, appearing in the analysis of much simpler channels. A generalization of Feinstein's Fundamental Lemma is used to obtain a lower bound on capacity, and Fano's inequality is used to obtain an upper bound. However, the generality of the model requires a development considerably different from that of the classical treatment; central to these results is the spectral representation of unbounded self-adjoint operators.

determining bounds on coding capacity of the continuous-time channel. These bounds will be given elsewhere.

μ_{GN} is defined as the zero-mean Gaussian cylinder set measure on H having the same covariance operator as μ_N . The entropy $H_{GN}(N)$ of μ_N with respect to μ_{GN} is defined as follows. Let H_n be any finite-dimensional subspace of H , with μ_N^n and μ_{GN}^n the measures induced on H_n by the projection operator $P_{H_n} : H \rightarrow H_n$. Let $H_{GN}(N|H_n)$ be the entropy of μ_N^n with respect to μ_{GN}^n :

$H_{GN}(N|H_n) = \infty$ if it is false that $\mu_N^n \ll \mu_{GN}^n$, while otherwise

$$H_{GN}(N|H_n) = \int_{H_n} \left[\log \frac{d\mu_N^n}{d\mu_{GN}^n} \right] d\mu_N^n. \text{ Define } H_{GN}(N) \text{ by } H_{GN}(N) = \sup_{H_n \subset H, n \geq 1} H_{GN}(N|H_n).$$

The induced measures μ_{GN}^n and μ_N^n are always countably additive for any finite-dimensional subspace H_n , while the measure μ_{GN} will be countably additive if and only if R_N is trace-class.

Since R_W^{-1} exists and $\text{range}(R_W^{\frac{1}{2}}) \subset \text{range}(R_N^{\frac{1}{2}})$, $R_N = R_W^{\frac{1}{2}}(I+S)R_W^{\frac{1}{2}}$ for a self-adjoint linear operator S , with $(I+S)^{-1}$ existing and bounded [5]. θ is the smallest limit point of the spectrum of S . A limit point of the spectrum is either the limit of a sequence of distinct eigenvalues, or an eigenvalue of infinite multiplicity, or a point of the continuous spectrum [6].

Coding Capacity

Theorem 1: (1) If $H_{GN}(N) < \infty$, then

$$C_W^\infty(P) \leq \frac{1}{2} \log \left[1 + \frac{P}{1+\theta} \right].$$

(2) If $H_{GN}(N) < \infty$ and $\dim(H) < \infty$, then $C_W^\infty(P) = 0$.

(3) If μ_N is Gaussian and $\dim(H) = \infty$, then $C_W^\infty(P) = \frac{1}{2} \log \left[1 + \frac{P}{1+\theta} \right]$.

Proof: The complete theorem will first be proved under the assumption that $\theta < \infty$.

Suppose that μ_N is Gaussian, with $\Theta < \infty$. We will show that

$$C_W^\infty(P) \geq \frac{1}{2} \log \left[1 + \frac{P}{1+\Theta} \right].$$

Fix any $\delta > 0$. Since $1 + \Theta$ is the smallest limit point of the spectrum of the self-adjoint operator $I + S$, there exists an infinite o.n. set $\{v_n, n \geq 1\}$ in the range of the projection operator $P_{1+\Theta+\delta}$, where $\{P_t, t \in \mathbb{R}\}$ is the left-continuous resolution of the identity for $I + S$ such that $x \in \mathcal{D}(I+S)$ if and only if $\int_0^\infty \lambda^2 d\|P_\lambda x\|^2 < \infty$, and then $(I+S)x = \int_0^\infty \lambda dP_\lambda x$ where the integral exists as a limit in the strong operator topology [6].

If x is any element in $\text{span}\{v_n, n \geq 1\}$, then $P_t x = x$ for $t \geq 1+\Theta+\delta$, since then $P_t \sum_{i=1}^M \langle x, v_i \rangle v_i = \sum_{i=1}^M \langle x, v_i \rangle P_t v_i = \sum_{i=1}^M \langle x, v_i \rangle v_i$. Thus, if x is in $\text{span}\{v_1, \dots, v_n\}$, then

$$\begin{aligned} \int_0^\infty t^2 d\langle P_t x, x \rangle &= \int_0^{1+\Theta+\delta} t^2 d\langle P_t x, x \rangle = \int_0^{1+\Theta+\lambda} t^2 d\|P_t x\|^2 \\ &\leq (1+\Theta+\lambda)^2 \int_0^{1+\Theta+\lambda} d\|P_t x\|^2 \leq (1+\Theta+\lambda)^2 \|x\|^2. \end{aligned}$$

This also shows that $\text{span}\{v_n, n \geq 1\}$ is contained in $\mathcal{D}(I+S)$, and that

$$\|(I+S)x\|^2 \leq (1+\Theta+\delta)^2 \|x\|^2 \text{ for all } x \text{ in } \text{span}\{v_n, n \geq 1\}.$$

$$\|(I+S)^{\frac{1}{2}} x\|^2 \leq (1+\Theta+\delta) \|x\|^2 \text{ if } x \in \text{span}\{v_n, n \geq 1\}.$$

Let U be the unitary operator in H which satisfies $R_W^{\frac{1}{2}}(I+S)^{\frac{1}{2}} U^* = R_N^{\frac{1}{2}}$ [5].

For each v_n , define $u_n = Uv_n$, so that $(I+S)^{\frac{1}{2}} U^* u_n = (I+S)^{\frac{1}{2}} v_n$.

Choose Q in $(0, P)$. For $n \geq 1$, define μ_X^n to be the zero-mean Gaussian measure with covariance operator $\frac{Q}{1+\Theta+\delta} \sum_{i=1}^n R_N^{\frac{1}{2}} u_i \otimes R_N^{\frac{1}{2}} u_i$. Let

$$H_n = \text{span}\{R_N^{\frac{1}{2}} u_1, \dots, R_N^{\frac{1}{2}} u_n\}. \text{ Note that } H_n \subset \text{range}(R_W^{\frac{1}{2}}), \text{ because } R_N^{\frac{1}{2}} u_i =$$

$$R_W^{\frac{1}{2}}(I+S)^{\frac{1}{2}} U^* u_i = R_W^{\frac{1}{2}}(I+S)^{\frac{1}{2}} v_i; \text{ since } \mu_X^n[H_n] = 1, \text{ this shows that}$$

$$\mu_X^n[\text{range}(R_W^{\frac{1}{2}})] = 1. \text{ Let } \mu_{XY}^n \text{ and } \mu_X^n \otimes \mu_Y^n \text{ be the joint cylinder set measures}$$

defined by μ_X^n and μ_N . Since μ_X^n gives full measure to H_n , we can replace μ_N by the measure $\mu_N \circ P_n^{-1}$, where P_n is the projection operator with range equal to H_n . Thus the joint measure of interest is concentrated on $H_n \times H_n$, and if B_1 and B_2 are Borel sets in H_n , then $\mu_{XY}^n[B_1 \times B_2] = \mu_X^n \otimes \mu_N\{(x, y): (x, x + P_n y) \in B_1 \times B_2\}$. Similarly, $\mu_Y^n[B_2] = \mu_X^n \otimes \mu_N\{(x, y): x + P_n y \in B_2\}$. Since both μ_{XY}^n and $\mu_X^n \otimes \mu_Y^n$ are countably additive measures on $H_n \times H_n$, the results of [3] can be applied. Set $F_n = \{x \in \text{range}(R_W^{\frac{1}{2}}): \|x\|_W^2 \leq nP\}$.

It will now be shown that $\mu_X^n[F_n^c] \rightarrow 0$ as $n \rightarrow \infty$. Note that $\mu_X^n = \mu_{T^n} \circ (R_N^{\frac{1}{2}})^{-1}$, where μ_{T^n} is the zero-mean Gaussian measure with covariance operator

$$\begin{aligned} \frac{Q}{1+\theta+\delta} \sum_{i=1}^n u_i \otimes u_i, \text{ so that } x &= \sum_{i=1}^n \langle x, u_i \rangle u_i \text{ a.e. } d\mu_{T^n}(x). \text{ Thus} \\ \mu_X^n[F_n^c] &= \mu_{T^n}\{x: \|R_W^{-\frac{1}{2}} R_N^{\frac{1}{2}} x\|^2 > nP\} = \mu_{T^n}\{x: \|(I+S)^{\frac{1}{2}} U^* x\|^2 > nP\} \\ &= \mu_{T^n}\{x: \|(I+S)^{\frac{1}{2}} U^* \sum_{i=1}^n \langle u_i, x \rangle u_i\|^2 > nP\} \\ &\leq \mu_{T^n}\{x: (1+\theta+\delta) \sum_{i=1}^n \langle u_i, x \rangle^2 > nP\}. \end{aligned}$$

The random variables $\{\langle u_i, \cdot \rangle, i \leq n\}$ are i.i.d. Gaussian random variables with respect to μ_{T^n} , mean zero and variance $Q/[1+\theta+\delta]$. Applying Chebyshev's

inequality, one has $\mu_X^n[F_n^c] \leq \frac{2nQ^2}{[nP-nQ]^2}$, so that $\mu_X^n[F_n^c] \rightarrow 0$ as $n \rightarrow \infty$.

From the proof of Prop. 2 of [7],

$$\frac{d\mu_{XY}^n}{d\mu_X^n \otimes \mu_Y^n}(x, y) = \frac{1}{2} \sum_{i=1}^n (a_i^2(x, y) - b_i^2(x, y)) + \frac{1}{2} n \log(1 + \frac{Q}{1+\theta+\delta})$$

where $\{a_1, \dots, a_n, b_1, \dots, b_n\}$ is a family of i.i.d. Gaussian random variables with respect to μ_{XY}^n , each having zero mean and variance

$$\left[\frac{Q/(1+\theta+\delta)}{1+Q/(1+\theta+\delta)} \right]^{\frac{1}{2}} = \left[\frac{Q}{1+\theta+\delta+Q} \right]^{\frac{1}{2}}. \text{ Take } \gamma > 0, \text{ and define}$$

$$\alpha_n = \frac{1}{2} n \log \left[1 + \frac{Q}{1+\theta+\delta} \right] - n\gamma,$$

$$A_n = \{(x,y): \log \frac{d\mu_{XY}^n}{d\mu_X^n d\mu_Y^n}(x,y) > \alpha_n\},$$

so that $A_n^c = \{(x,y): \frac{1}{2} \sum_{i=1}^n (a_i^2 - b_i^2) \leq -n\gamma\}$. Since the sequence of random variables $(a_i^2 - b_i^2)$ are independent and have zero mean w.r.t. μ_{XY} , Chebyshev's inequality gives $\mu_{XY}^n[A_n^c] \leq \frac{1}{n^2\gamma} 4n \left[\frac{Q}{1+\theta+\delta+Q} \right]^2 \rightarrow 0$.

Let $R < \frac{1}{2} \log \left[1 + \frac{Q}{1+\theta+\delta} \right]$ and set $k_n = \lfloor e^{nR} \rfloor$. Then, $k_n e^{-\alpha_n} \leq e^{nR+n\gamma-\frac{1}{2}n \log \left[1 + \frac{Q}{1+\theta+\delta} \right]}$. By the Thomasian-Kadota generalization of Feinstein's Fundamental Lemma (see, e.g., [1, p. 165]), there exists a code (k_n, F_n, ϵ_n) with $\epsilon_n \leq k_n e^{-\alpha_n} + \mu_{XY}^n(A_n^c) + \mu_X^n(F_n^c)$. From above, both $\mu_{XY}^n(A_n^c)$ and $\mu_X^n[F_n^c]$ tend to zero as $n \rightarrow \infty$. Considering $k_n e^{-\alpha_n}$, choose γ so that $R + \gamma < \frac{1}{2} \log \left[1 + \frac{Q}{1+\theta+\delta} \right]$. Then $k_n e^{-\alpha_n} \rightarrow 0$ also.

This shows that any rate less than $\frac{1}{2} \log \left[1 + \frac{Q}{1+\theta+\delta} \right]$ is admissible, for all $Q < P$ and for all $\delta > 0$. Hence, the supremum over all admissible rates must be at least $\frac{1}{2} \log \left[1 + \frac{P}{1+\theta} \right]$, so that $C_W^\infty(P) \geq \frac{1}{2} \log \left[1 + \frac{P}{1+\theta} \right]$ when μ_N is Gaussian.

Now consider the case of a possibly nonGaussian μ_N , not necessarily countably additive, with $\theta < \infty$ and $H_{GN}(N) < \infty$. Proceeding exactly as in the proof of this result for the matched channel [1, pp. 167-168], it is found that any admissible R must satisfy $R \leq \limsup_n \frac{1}{n} C_W^n(P)$. $C_W^n(P)$ is the information capacity of the additive Gaussian channel with noise covariance operator R_N , subject to the constraints that $\text{support}(\mu_X)$ has linear dimension $\leq n$ and $\int_H \|x\|_W^2 d\mu_X(x) \leq nP$.

It now remains only to verify that $\lim_{n \rightarrow \infty} \frac{C_W^n(P)}{n} = \frac{1}{2} \log \left[1 + \frac{P}{1+\theta} \right]$. To show this, one can apply Theorem 2 of [3]. If the operator S has no eigenvalues less than θ , then $C_W^n(nP) = \frac{n}{2} \log \left[1 + \frac{nP}{n[1+\theta]} \right]$ for all $n \geq 1$, so $\lim_{n \rightarrow \infty} \frac{1}{n} C_W^n(nP)$ exists and equals $\frac{1}{2} \log \left[1 + \frac{P}{1+\theta} \right]$.

If the operator S has a finite set of eigenvalues less than θ , $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_K < \theta$, then $\sum_{i=1}^K \lambda_i + nP > K\theta$ for sufficiently large n , so that applying Theorem 2(c) of [3],

$$\frac{1}{n} C_W^n(nP) = \frac{1}{2n} \sum_{i=1}^K \log \left[\frac{1+\theta}{1+\lambda_i} \right] + \frac{1}{2} \log \left[1 + \frac{nP + \sum_{i=1}^K (\lambda_i - \theta)}{n(1+\theta)} \right]$$

and this again converges to the limit $\frac{1}{2} \log \left[1 + \frac{P}{1+\theta} \right]$.

Finally, suppose that S has an infinite sequence of eigenvalues (λ_n) strictly less than θ . Since θ is the smallest limit point of the spectrum, $\lambda_n \uparrow \theta$. This means that for any fixed P , $KP + \sum_{i=1}^K \lambda_i > K\lambda_K$ for all sufficiently large K . To see this, one notes that for any $\Delta > 0$, there exists M_0 such that $\theta - \lambda_i < \Delta$ for $i > M_0$. Thus, for $K > M_0$,

$$K\lambda_K - \sum_{i=1}^K \lambda_i \leq \sum_{i=1}^{M_0} (\lambda_K - \lambda_i) + (K-M_0)\Delta \leq \sum_{i=1}^{M_0} (\theta - \lambda_i) + (K-M_0)\Delta,$$

so that

$$\frac{1}{K} \left[K\lambda_K - \sum_{i=1}^K \lambda_i \right] \leq \frac{1}{K} \left[\sum_{i=1}^{M_0} (\theta - \lambda_i) + (K-M_0)\Delta \right],$$

with the right side converging to Δ as $K \rightarrow \infty$. Thus, choosing $\Delta < P$,

$KP + \sum_{i=1}^K \lambda_i > K\lambda_K$ for K sufficiently large. One can thus apply part (c) of

Theorem 2 of [3], giving

$$C_W^n(nP) = \frac{1}{2} \sum_{i=1}^n \log \left[\frac{1+\theta}{1+\lambda_i} \right] + \frac{n}{2} \log \left[1 + \frac{nP + \sum_{j=1}^n (\lambda_j - \theta)}{n(1+\theta)} \right].$$

Since $\log \frac{1+\theta}{1+\lambda_n} \rightarrow 0$, $\frac{1}{n} \sum_{i=1}^n \log \left[\frac{1+\theta}{1+\lambda_i} \right] \rightarrow 0$. Similarly, $\frac{1}{n} \sum_{i=1}^n (\lambda_i - \theta) \rightarrow 0$.

Thus, one again has $\lim_n \frac{1}{n} C_W^n(nP) = \frac{1}{2} \log \left[1 + \frac{P}{1+\theta} \right]$; part (1) is proved, and this also completes the proof of part (3).

If $\dim \text{range}(R_N) = M < \infty$, then in the immediately preceding result one has for n sufficiently large,

$$C_W^n(nP) = \frac{1}{2} \sum_{i=1}^M \log \left[\frac{M + nP + \sum_{j=1}^M \beta_j}{M(1+\beta_i)} \right]$$

where $\beta_1 \leq \beta_2 \leq \dots \leq \beta_M$ are the eigenvalues of S . In this case,

$\lim_n \frac{1}{n} C_W^n(P) = 0$, so that $R > 0$ is not permissible.

The theorem is now proved when $\theta < \infty$. If $\theta = \infty$, then obviously $C_W^\infty(P) \geq \frac{1}{2} \log \left[1 + \frac{P}{1+\theta} \right] = 0$. Part (2) of the theorem can be ignored, since $\theta = \infty$ cannot occur unless $\text{range}(R_N)$ is infinite-dimensional. Thus, it only remains to prove part (1), and this is equivalent to showing that

$\overline{\lim}_n \frac{1}{n} C_W^n(nP) = 0$ when $\theta = \infty$. If there exists an integer M such that

$\lambda_{n+1} > P + \frac{1}{n} \sum_{i=1}^n \lambda_i$ for all $n \geq M$, then

$$\overline{\lim}_n \frac{1}{n} C_W^n(P) = \lim_n \frac{1}{2n} \sum_{j=1}^M \log \left[\frac{P + \frac{1}{M} \sum_{i=1}^M \lambda_i + 1}{1 + \lambda_j} \right] = 0.$$

Suppose that there exists a subsequence (n_k) of the integers such that for all $k \geq 1$, $\lambda_{n_k+1} - \frac{1}{n_k} \sum_{i=1}^{n_k} \lambda_i \leq P$. This gives

$$\overline{\lim}_n \frac{1}{n} C_W^n(nP) = \overline{\lim}_k \frac{1}{2n_k} \sum_{i=1}^{n_k} \log \left[\frac{P - \left[\lambda_{n_k+1} - \frac{1}{n_k} \sum_{j=1}^{n_k} \lambda_j \right] + 1 + \lambda_{n_k+1}}{1 + \lambda_i} \right] \quad (\gamma)$$

$$\leq \overline{\lim}_k \frac{1}{2n_k} \sum_{i=1}^M \log \left[\frac{P + 1 + \lambda_{n_k+1}}{1 + \lambda_i} \right] + \overline{\lim}_k \frac{1}{2n_k} \sum_{i=M+1}^{n_k} \log \left[\frac{P + 1 + \lambda_{n_k+1}}{1 + \lambda_i} \right]$$

for any fixed integer M . Now, since $\frac{1}{n_k} \sum_{i=1}^{n_k} \left[\lambda_{n_k+1} - \lambda_i \right] \leq P$, and since

$\frac{1}{n_k} \sum_{i=1}^{n_k} \frac{\lambda_i}{1+\lambda_i} \rightarrow 1$ as $k \rightarrow \infty$, we must have that $\frac{1}{n_k} \sum_{i=1}^{n_k} \frac{\lambda_{n_k+1}}{1+\lambda_i}$ is bounded, so that

$\frac{\lambda_{n_k+1}}{n_k} \leq C_0$ for some $C_0 < \infty$ and all $k \geq 1$. The first term on RHS(γ) above is

then

$$\leq \overline{\lim}_k \frac{M}{2n_k} \log \left[\frac{P + 1 + C_0 n_k}{1 + \lambda_1} \right] = 0.$$

We now have, for any $M \geq 1$,

$$\begin{aligned} \overline{\lim}_n \frac{1}{n} C_W^n(nP) &\leq \overline{\lim}_k \frac{1}{2n_k} \sum_{i=M+1}^{n_k} \log \left[\frac{P + 1 + \lambda_{n_k+1}}{1 + \lambda_i} \right] \\ &\leq \overline{\lim}_k \frac{1}{2n_k} \sum_{i=M+1}^{n_k} \left[\frac{P + \lambda_{n_k+1} - \lambda_i}{1 + \lambda_i} \right] \\ &\leq \overline{\lim}_k \frac{1}{2n_k} \sum_{i=M+1}^{n_k} \left[\frac{\lambda_{n_k+1} - \lambda_i}{1 + \lambda_{M+1}} \right] + \frac{P}{2(1+\lambda_{M+1})} \\ &\leq \frac{P}{1 + \lambda_{M+1}}. \end{aligned}$$

Since M is arbitrary and $\lambda_n \rightarrow \infty$, $\overline{\lim}_n \frac{1}{n} C_W^n(nP) = 0$, and thus $C_W^\infty(P) = 0$ when

$\theta = \infty$.

□

Bounds on Coding Capacity of the Discrete-Time Gaussian Channel

We now consider the following situation. A zero-mean Gaussian stochastic process $\{N_t, t = 1, 2, \dots\}$ is represented by a bounded, non-negative, self-adjoint operator R_N in ℓ_2 ; R_N is an infinite matrix with $R_N(i, j) = EN_i N_j$. The constraint is given in terms of a second such operator R_W in ℓ_2 . The basic assumption to be made is that $\text{range}(R_N^{\frac{1}{2}})$ contains $\text{range}(R_W^{\frac{1}{2}})$.

A simple example of such a channel and constraint is the memoryless Gaussian channel with $R_W = I$ (leading to an average power constraint) and R_N given by $R_N(i, j) = \alpha_j^2 \delta_{ij}$, with $\alpha_j^2 \geq \epsilon$ for all $j \geq 1$, some $\epsilon > 0$.

In the discrete-time channel, a code (k, n, ϵ) is a set of k code words $x_1, \dots, x_k$ and corresponding decoding sets $C_1, \dots, C_k$, satisfying the constraints given below, with the requirement that each x_i belong to $\mathbb{R}^n$. The decoding sets are thus Borel sets in $\mathbb{R}^n$. The constraints on the code words are that

$\|x_i\|_{W, n}^2 \leq nP$, where $\|x\|_{W, n}^2 = \|R_{W, n}^{-\frac{1}{2}} x\|_n^2$; $\|\cdot\|_n$ is the n -dimensional Euclidean norm, and $R_{W, n}$ is the restriction of R_W to $\{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$. As before, we require that $\mu_N^n\{y: y + x_i \in C_i\} \geq 1 - \epsilon$ for $i \leq k$, where μ_N^n is the measure on $\mathbb{B}[\mathbb{R}^n]$ induced from μ_N by the map $q_n: \underline{x} \rightarrow (x_1, x_2, \dots, x_n)$. $R \geq 0$ is an admissible rate if there exists a sequence of codes $(([e^{\frac{nR}{n_i}}], n_i, \epsilon_{n_i}))$ with $\epsilon_{n_i} \rightarrow 0$ as $n_i \rightarrow \infty$. The capacity $C_W^\infty(P)$ is the supremum over the set of admissible rates.

An exact expression for the coding capacity of the discrete-time Gaussian channel is given in [8]. In some applications, the value of the coding capacity will be difficult to determine, as it involves rather detailed knowledge of the spectrum of the operator S , defined above. In such cases it is useful to have bounds on coding capacity. For example, a lower bound enables one to strive toward communicating at a rate that is certain to be admissible. We give here upper and lower bounds on coding capacity.

Theorem 2: Suppose that N is Gaussian. Let θ_1 be the smallest and θ_K the largest limit point of the spectrum of the operator S . Then

$$\log \left[1 + \frac{P}{(1+\theta_K)} \right] \leq C_W^\infty(P) \leq \frac{1}{2} \log \left[1 + \frac{P}{(1+\theta_1)} \right].$$

If N is not Gaussian, and $H_{GN}(N) < \infty$, then

$$C_W^\infty(P) \leq \frac{1}{2} \log \left[1 + \frac{P}{(1+\theta_1)} \right].$$

Proof: The upper bound can be obtained from part (1) of Theorem 1. That is, we can identify $\mathbb{R}^n$ with H_n , the subspace of ℓ_2 consisting of all elements x such that $(x)_i = 0$ for $i > n$. The constraint that any admissible code word belong to H_n thus imposes an additional constraint beyond those imposed in proving the theorem; this gives $C_W^\infty(P) \leq \frac{1}{2} \log \left[1 + \frac{P}{(1+\theta_1)} \right]$.

To prove the lower bound, we can of course assume that $\theta_K < \infty$. We then simply mimic the proof of part (3) of Theorem 1, but now defining μ_X^n to be the Gaussian measure with zero mean and covariance matrix

$$R_X^n = \frac{Q}{1+\theta_K+\delta} \sum_{i=1}^{M_n^\delta} R_{N_i}^{\frac{1}{2}u_i^n} \otimes R_{N_i}^{\frac{1}{2}u_i^n}$$

where the $\{u_i^n, i \leq M_n^\delta\}$ are determined as follows. $\{v_i, i \leq M_n^\delta\}$ are o.n. elements in $\mathbb{R}^n$ such that $\|(I_n + S_n)^{\frac{1}{2}} v_i\|_n^2 \leq 1 + \theta_K + \delta$; such elements always exist [8]. $\{u_i^n, i \leq M_n^\delta\}$ are then defined by $u_i^n = U_n v_i$, where U_n is the unitary operator in $\mathbb{R}^n$ satisfying $R_{N,n}^{\frac{1}{2}} = R_{W,n}^{\frac{1}{2}} (I_n + S_n)^{\frac{1}{2}} U_n^*$. □

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