

AD-A190 326

ADMISSIBLE BAYES TESTS FOR STRUCTURAL RELATIONSHIP(U)

1/1

PITTSBURGH UNIV PA CENTER FOR MULTIVARIATE ANALYSIS

W SMITH ET AL SEP 87 TA-87-31 AFOSR-TA-87-1939

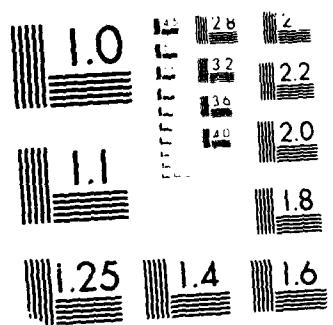
UNCLASSIFIED

F49620-85-C-0008

F/G 12/3

NL

END
DATE
48



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

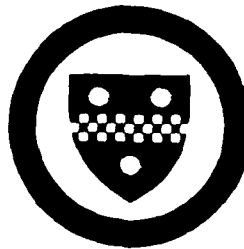
APOSR-TR. 87-1959

ADMISSIBLE BAYES TESTS FOR STRUCTURAL
RELATIONSHIP

Wei-Hsiung Shen and Bimal K. Sinha*

Center for Multivariate Analysis
University of Pittsburgh
Pittsburgh PA 15260

Center for Multivariate Analysis
University of Pittsburgh



87 12 29 212

ADMISSIBLE BAYES TESTS FOR STRUCTURAL
RELATIONSHIP

Wei-Hsiung Shen and Bimal K. Sinha*

Center for Multivariate Analysis
University of Pittsburgh
Pittsburgh PA 15260

September 1987

Technical Report Number 87-31

Center for Multivariate Analysis
University of Pittsburgh
Pittsburgh PA 15260

Accession For

NTIS

DTIC

Unann

Unann

Unann

Unann

Unann

Unann

Unann

Unann

Unann

Unann

Unann

Unann

Unann

Unann

Unann

Unann

Unann

Unann

Unann

Unann

Unann

Unann

Unann

Unann

Unann

Unann

Unann

*Research supported by the Air Force Office of Scientific Research (AFOSR),
under Contract F49620-85-C-0008. The United States Government is
authorized to reproduce and distribute reprints for governmental purposes
notwithstanding any copyright notation hereon.

ADMISSIBLE BAYES TESTS FOR STRUCTURAL RELATIONSHIP

BY WEI-HSIUNG SHEN

University of Maryland Baltimore County

AND

BIMAL K. SINHA*

University of Maryland Baltimore County

and

Center for Multivariate Analysis, University of Pittsburgh

Abstract

It is an open problem to construct a test for structural relationship among the mean vectors of several multivariate normal populations with known but unequal covariance matrices. In this paper, a class of admissible Bayes tests for the above problem is derived. As a byproduct, in the special case of known and equal covariance matrices, the likelihood ratio test of Rao(1973) is shown to be admissible Bayes.

*Research supported by the Air Force Office of Scientific Research (AFOSR) under contract F49620-85-C-008. The United States Government is authorized to reproduce and distribute reprints for governmental purposes.

AMS 1980 subject classification. PRIMARY 62H15; SECONDARY 62C15, 62F15.

Key words and phrases. Structural relationship, admissible Bayes test, likelihood ratio test, prior distribution.

1. Introduction. Fisher(1939) pointed out the importance of tests for structural relationship among the mean vectors of several multivariate normal populations. To be specific, consider k independent p -variate multinormal populations, $N_p(\mu_i, \Sigma_i)$, $i = 1, \dots, k$ and the problem of testing

$$(1.1) \quad H_0 : H\mu_i = \xi, i = 1, \dots, k \quad \text{versus}$$

$$H_1 : \mu_1, \dots, \mu_k \text{ arbitrary}$$

where $H : s \times p$ is an unknown matrix of known rank $s < p$ and ξ is an unknown s -vector. The hypothesis H_0 , if true, implies that the k mean vectors $\mu_1, \dots, \mu_k$ lie in a $(p-s)$ -dimensional subspace of the Euclidean p -space E^p rather than in E^p itself, and provides a structural relationship among the mean vectors. As Fisher(1939) and Rao(1973) rightly emphasized, it is essential that in dealing with several multinormal populations a hypothesis of the form (1.1) be tested prior to using the models for prediction etc.

It is an open problem in the literature to construct a suitable test for the hypotheses in (1.1) based on independent samples from the k populations when $\Sigma_1, \dots, \Sigma_k$ are unequal. In the case when $\Sigma_1, \dots, \Sigma_k$ are equal (and $= \Sigma$, say), Rao(1973) derived the likelihood ratio test (LRT) of H_0 against H_1

which rejects the null hypothesis for large values of $T = \sum_{i=1}^s \lambda_i (B\Sigma^{-1})$. Here B is the matrix of the sums of squares and products due to the hypotheses (between groups), $\lambda_1 < \lambda_2 < \dots < \lambda_p$ are the ordered roots of $B\Sigma^{-1}$, and the matrix Σ is assumed known. If Σ is unknown, one replaces Σ in the definition of T by W , the within matrix sum of squares and products due to error (within groups). That the resultant test is the LRT is proved in Fujikoshi(1974) and also in Rao(1985).

It is the object of this paper to provide a simple solution to the above open problem when $\Sigma_1, \dots, \Sigma_k$ are assumed known. We have derived in Section 2 a class of admissible Bayes tests for the hypothesis H_0 versus H_1 . Interestingly enough, inspite of the somewhat complicated structures of the model and the problem, the derived tests are of extremely simple form. In the special case of equal and known covariance matrices, the LRT of Rao(1973) is shown to be admissible Bayes.

We may recall that a Bayes critical region (for 0-1 loss function) is of the form

$$(1.2) \quad \{ \mathbf{x} : \int f(\mathbf{x}; \theta) \pi_1(d\theta) \geq \int f(\mathbf{x}; \theta) \pi_0(d\theta) + c \}$$

for some positive constant c , where $f(\mathbf{x}; \theta)$ is the underlying joint density.

θ is the vector of parameters, π_1 and π_0 are the prior probability measures over the alternative parameter space Θ_1 and the null parameter space Θ_0 respectively, and $\int$ is over the respective parameter spaces (Kiefer and Schwartz, 1965). Assuming that we have available a random sample of size n_i from the i th population, denoted as $x_{i1}, \dots, x_{in_i}$, $i = 1, \dots, k$, we can write

$$(1.3) \quad f(x; \theta) = (2\pi)^{-np/2} \prod_{i=1}^k |\Sigma_i|^{-n_i/2} \exp\left\{-1/2 \sum_{i=1}^k n_i \Sigma_i^{-1} (\bar{x}_i - \mu_i) \cdot (\bar{x}_i - \mu_i)'\right\} \exp\left\{-1/2 \sum_{i=1}^k \Sigma_i^{-1} S_i\right\}$$

where $\bar{x}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij}$, $S_i = \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)(x_{ij} - \bar{x}_i)'$, $i = 1, \dots, k$,

$n = \sum_{i=1}^k n_i$, $\theta = (\mu_1, \dots, \mu_k)$, $\Theta = \{(\mu_1, \dots, \mu_k) : \mu_i \in E^p, i = 1, \dots, k\}$,

$\Theta_0 = \{(\mu_1, \dots, \mu_k) : H\mu_i = \xi, i = 1, \dots, k, H : s \times p \text{ unknown},$

$\text{rank}(H) = s < p, s \text{ known}, \xi : s \times 1 \in E^p \text{ unknown}\}$, $\Theta_1 = \Theta - \Theta_0$.

In what follows the factor $(2\pi)^{-np/2} \prod_{i=1}^k |\Sigma_i|^{-n_i/2} \exp\left\{-1/2 \sum_{i=1}^k \Sigma_i^{-1} S_i\right\}$ appearing in $f(x; \theta)$ is ignored because it is independent of θ and has no influence on Bayes tests of the form (1.2). Thus one can write (1.2) in the form

$$(1.4) \quad \{x : \int f^*(x; \theta) \pi_1(d\theta) / \int f^*(x; \theta) \pi_0(d\theta) > c\}$$

where

$$(1.5) \quad f^*(\mathbf{x}; \theta) = \text{etr}\left\{-1/2 \sum_{i=1}^k n_i \Sigma_i^{-1} (\bar{\mathbf{x}}_i - \mu_i)(\bar{\mathbf{x}}_i - \mu_i)'\right\}.$$

Appropriate choices of π_1 and π_0 are made in Section 2 and the resultant Bayes tests are derived.

2. Admissible Bayes tests of (1.1). Under the alternative H_1 , we choose π_1 as the absolutely continuous measure on the space E^{pk} of the μ_i 's given by

$$(2.1) \quad d\pi_1(\mu_1, \dots, \mu_k) / d\mu_1 \cdots d\mu_k = \\ (2\pi)^{-kp/2} \prod_{i=1}^k |A_i|^{-1/2} \left(\prod_{i=1}^k n_i \right)^{p/2} \text{etr}\left\{-1/2 \sum_{i=1}^k n_i A_i^{-1} (\mu_i - \zeta_i)(\mu_i - \zeta_i)'\right\}$$

where A_i : $p \times p$, $p.d.$ matrix, and ζ_i : $p \times 1$ vector, $i = 1, \dots, k$. Clearly π_1 corresponds to choosing independent normal priors for each μ_i . The matrices A_i and the vectors ζ_i , $i = 1, \dots, k$, will be chosen later. This immediately results in the numerator of the expression in (1.4) as

$$(2.2) \quad \int f^*(\mathbf{x}; \theta) \pi_1(d\theta) = \\ (2\pi)^{-pk/2} \left(\prod_{i=1}^k n_i \right)^{p/2} \prod_{i=1}^k |A_i|^{-1/2} \text{etr}\left\{-1/2 \sum_{i=1}^k n_i \Sigma_i^{-1} \bar{\mathbf{x}}_i \bar{\mathbf{x}}_i'\right\} \\ \cdot \text{etr}\left\{-1/2 \sum_{i=1}^k n_i A_i^{-1} \zeta_i \zeta_i'\right\}$$

$$\begin{aligned}
& \cdot \text{etr} \left\{ 1/2 \sum_{i=1}^k n_i (\Sigma_i^{-1} \bar{\mathbf{x}}_i + A_i^{-1} \zeta_i) (\Sigma_i^{-1} \bar{\mathbf{x}}_i + A_i^{-1} \zeta_i)' (\Sigma_i^{-1} + A_i^{-1})^{-1} \right\} \\
& \cdot \int \text{etr} \left\{ -1/2 \sum_{i=1}^k n_i (\Sigma_i^{-1} + A_i^{-1}) (\mu_i - (\Sigma_i^{-1} + A_i^{-1})^{-1} \right. \\
& \quad \left. (\Sigma_i^{-1} \bar{\mathbf{x}}_i + A_i^{-1} \zeta_i)) (\mu_i - (\Sigma_i^{-1} + A_i^{-1})^{-1} (\Sigma_i^{-1} \bar{\mathbf{x}}_i + A_i^{-1} \zeta_i))' \right\} \\
& \cdot d\mu_1 \cdots d\mu_k \\
= & \prod_{i=1}^k |\Sigma_i^{-1} A_i + I_p|^{-1/2} \text{etr} \left\{ -1/2 \sum_{i=1}^k n_i A_i^{-1} \zeta_i \zeta_i' \right\} \\
& \cdot \text{etr} \left\{ -1/2 \sum_{i=1}^k n_i \Sigma_i^{-1} \bar{\mathbf{x}}_i \bar{\mathbf{x}}_i' \right\} \\
& \cdot \text{etr} \left\{ 1/2 \sum_{i=1}^k n_i (\Sigma_i^{-1} \bar{\mathbf{x}}_i + A_i^{-1} \zeta_i) (\Sigma_i^{-1} \bar{\mathbf{x}}_i + A_i^{-1} \zeta_i)' (\Sigma_i^{-1} + A_i^{-1})^{-1} \right\}
\end{aligned}$$

The crux of the problem now is to evaluate the denominator of the expression in (1.4), namely $\int f(\mathbf{x}; \theta) \pi_0(d\theta)$ for suitable π_0 's. Towards this end, first note that

$$\begin{aligned}
(2.3) \quad H\mu_i &= \xi \iff \\
\mu_i &= H'(HH')^{-1}\xi + (I_p - H'(HH')^{-1}H)\eta_i \\
&\text{for arbitrary } \eta_i \in E^p
\end{aligned}$$

so that Θ_0 can be rewritten as

$$\begin{aligned}
(2.4) \quad \Theta_0 &= \{(\mu_1, \dots, \mu_k) : \mu_i = H'(HH')^{-1}\xi + (I_p - H'(HH')^{-1}H)\eta_i, \\
&\quad \eta_i \in E^p, i = 1, \dots, k, \xi, \eta_1, \dots, \eta_k \text{ all arbitrary},
\end{aligned}$$

$$H : s < p \text{ unknown, rank}(H) = s < p, s \text{ known}\}.$$

Our choice of π_0 on Θ_0 corresponds to choosing suitable priors for $H \in \mathcal{H}$ $\{H : s < p - \text{rank}(H) = s\}$, $\xi \in E^p$ and $\eta_i \in E^p$, $i = 1, \dots, k$. Throughout the following, as far as H is concerned, we assume that π_0 assigns all its measure to the subset $\mathcal{H} = \{H : s < p - H = (I, : 0)\}$ of $\mathcal{H}$. It remains to specify priors of ξ and $\eta_1, \dots, \eta_k$. This is done below.

Choose the conditional density of $(\eta_1, \dots, \eta_k)$, given ξ , as

$$(2.5) \quad d\pi_0(\eta_1, \dots, \eta_k | \xi) = d\eta_1 \dots d\eta_k \cdot (2\pi)^{-kr/2} \prod_{i=1}^k [(1 - n_i)P_i]^{-1/2} \\ \exp\left\{-\frac{1}{2} \sum_{i=1}^k n_i P_i^{-1} (\eta_i - (\eta_i^0 + \Delta_i \xi)) (\eta_i - (\eta_i^0 + \Delta_i \xi))'\right\}$$

and the marginal density of ξ as

$$(2.6) \quad d\pi_0(\xi) = d\xi \cdot (2\pi)^{-r/2} Q^{-1/2} \exp\left\{-\frac{1}{2} Q^{-1} (\xi - \xi^0) (\xi - \xi^0)'\right\}$$

where $\eta_i^0 : p \times 1$, $\xi^0 : s \times 1$, $\Delta_i : p \times s$, $P_i : p \times p$, p.d., $Q : s \times s$, p.d. are to be suitably chosen. Let

$$\beta_i = -\Sigma_i^{-1} \bar{\mathbf{x}}_i, \quad i = 1, \dots, k, \\ \beta = \sum_{i=1}^k n_i \Sigma_i^{-1} \bar{\mathbf{x}}_i = \sum_{i=1}^k n_i \beta_i, \\ \Psi_1 = (I, : 0) : s \times p,$$

$$\begin{aligned}
(2.7) \quad \Psi_2 &= \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{p-s} \end{pmatrix} : p \times p \\
\Psi_3 &= \Psi_1 \left(\sum_{i=1}^k n_i \Sigma_i^{-1} \right) \Psi_1' : s \times s, \\
\Psi_{4i} &= \Psi_2 \Sigma_i^{-1} \Psi_2 : p \times p, \quad i = 1, \dots, k, \\
\Psi_{5i} &= \Psi_2 \Sigma_i^{-1} \Psi_1' : p \times s, \quad i = 1, \dots, k.
\end{aligned}$$

Then the denominator of the expression in (1.4) can be simplified as

$$\begin{aligned}
(2.8) \quad \int f(\mathbf{x}; \theta) \pi_0(d\theta) = & (2\pi)^{-pk/2} \left(\prod_{i=1}^k n_i \right)^{p/2} \prod_{i=1}^k |P_i|^{-1/2} (2\pi)^{-(s+2)/2} |Q|^{-1/2} \\
& \cdot \text{etr} \left\{ -1/2 \sum_{i=1}^k n_i \Sigma_i^{-1} \bar{\mathbf{x}}_i \bar{\mathbf{x}}_i' \right\} \text{etr} \left\{ -1/2 Q^{-1} \xi^{(0)} \xi^{(0)'} \right\} \\
& \cdot \text{etr} \left\{ -1/2 \sum_{i=1}^k n_i P_i^{-1} \eta_i^0 \eta_i^{0'} \right\} \\
& \cdot \int \text{etr} \left\{ -1/2 (\Psi_3 + Q^{-1}) \xi \xi' + 1/2 (\Psi_1 \beta + Q^{-1} \xi^{(0)}) \xi' \right. \\
& \quad \left. + 1/2 \xi (\Psi_1 \beta + Q^{-1} \xi^{(0)})' \right\} \\
& \cdot \text{etr} \left\{ -1/2 \sum_{i=1}^k n_i [(\Psi_{4i} + P_i^{-1}) \eta_i \eta_i' - (\Psi_2 \beta_i - \Psi_{5i} \xi + \right. \\
& \quad \left. P_i^{-1} (\eta_i^0 + \Delta_i \xi)) \eta_i' - \eta_i (\Psi_2 \beta_i - \Psi_{5i} \xi + P_i^{-1} (\eta_i^0 + \Delta_i \xi))'] \right\} \\
& \cdot \text{etr} \left\{ -1/2 \sum_{i=1}^k n_i P_i^{-1} (\eta_i^0 \xi' \Delta_i' + \Delta_i \xi \eta_i^{0'} + \Delta_i \xi \xi' \Delta_i') \right\} \\
& \cdot d\xi d\eta_1 \cdots d\eta_k
\end{aligned}$$

$$\begin{aligned}
&= (2\pi)^{-pk/2} \left(\prod_{i=1}^k n_i \right)^{p/2} \prod_{i=1}^k |P_i|^{-1/2} (2\pi)^{-s_i/2} |Q|^{-1/2} \\
&\quad \cdot \text{etr} \left\{ -1/2 \sum_{i=1}^k n_i \Sigma_i^{-1} \bar{\mathbf{x}}_i \mathbf{x}_i' \right\} \text{etr} \left\{ -1/2 Q^{-1} \xi^0 \xi^{0'} \right\} \\
&\quad \cdot \text{etr} \left\{ -1/2 \sum_{i=1}^k n_i P_i^{-1} \eta_i^0 \eta_i^{0'} \right\} \\
&\quad \cdot \text{etr} \left\{ 1/2 \sum_{i=1}^k n_i (\Psi_2 \beta_i + P_i^{-1} \eta_i^0) (\Psi_2 \beta_i + P_i^{-1} \eta_i^0)' \right. \\
&\quad \cdot (\Psi_{4i} + P_i^{-1})^{-1} \left. \right\} \text{etr} \left\{ 1/2 \tau \tau' \Xi^{-1} \right\} \\
&\quad \cdot \int \text{etr} \left\{ -1/2 \sum_{i=1}^k n_i (\Psi_{4i} + P_i^{-1}) (\eta_i + (\Psi_{4i} + P_i^{-1})^{-1} (\Psi_2 \beta_i + \Psi_{5i} \xi + P_i^{-1} \right. \\
&\quad \cdot (\eta_i^0 + \Delta_i \xi))) (\eta_i + (\Psi_{4i} + P_i^{-1})^{-1} (\Psi_2 \beta_i + \Psi_{5i} \xi + P_i^{-1} (\eta_i^0 + \Delta_i \xi)))' \right\} \\
&\quad \cdot \text{etr} \left\{ -1/2 \Xi (\xi - \Xi^{-1} \tau) (\xi - \Xi^{-1} \tau)' \right\} \\
&\quad \cdot d\xi d\eta_1 \cdots d\eta_k \\
&= \prod_{i=1}^k |P_i \Psi_{4i} + I_p|^{-1/2} |Q|^{-1/2} |\Xi|^{-1/2} \text{etr} \left\{ -1/2 Q^{-1} \xi^0 \xi^{0'} \right\} \\
&\quad \cdot \text{etr} \left\{ -1/2 \sum_{i=1}^k n_i \Sigma_i^{-1} \bar{\mathbf{x}}_i \mathbf{x}_i' \right\} \text{etr} \left\{ -1/2 \sum_{i=1}^k n_i P_i^{-1} \eta_i^0 \eta_i^{0'} \right\} \text{etr} \left\{ 1/2 \tau \tau' \Xi^{-1} \right\} \\
&\quad \cdot \text{etr} \left\{ 1/2 \sum_{i=1}^k n_i (\Psi_2 \beta_i + P_i^{-1} \eta_i^0) (\Psi_2 \beta_i + P_i^{-1} \eta_i^0)' (\Psi_{4i} + P_i^{-1})^{-1} \right\}
\end{aligned}$$

where

$$\begin{aligned}
(2.9) \quad \Xi &= \Psi_3 + Q^{-1} + \sum_{i=1}^k n_i \Delta_i' P_i^{-1} \Delta_i \\
&\quad + \sum_{i=1}^k n_i (P_i^{-1} \Delta_i + \Psi_{5i})' (\Psi_{4i} + P_i^{-1})^{-1} (P_i^{-1} \Delta_i + \Psi_{5i})
\end{aligned}$$

and

$$(2.10) \quad \tau = \Psi_1 \beta + Q^{-1} \xi^0 - \sum_{i=1}^k n_i \Delta_i' P_i^{-1} \eta_i^0 \\ + \sum_{i=1}^k n_i (P_i^{-1} \Delta_i - \Psi_{5i})' (\Psi_{4i} + P_i^{-1})^{-1} (\Psi_{2i} \beta + P_i^{-1} \eta_i^0).$$

In the above Δ_i 's, P_i 's, and Q are chosen so that Ξ is *p.d.*. Hence, combining (2.2) and (2.8), and taking the logarithm of the ratio of the expressions in (2.2) and (2.8), the Bayes test turns out to be of the form

$$(2.11) \quad \text{tr} \left\{ \sum_{i=1}^k n_i (\Sigma_i^{-1} \bar{x}_i + A_i^{-1} \zeta_i) (\Sigma_i^{-1} \bar{x}_i + A_i^{-1} \zeta_i)' (\Sigma_i^{-1} + A_i^{-1})^{-1} \right\} \\ - \text{tr} \left\{ \sum_{i=1}^k n_i (\Psi_{2i} \beta + P_i^{-1} \eta_i^0) (\Psi_{2i} \beta + P_i^{-1} \eta_i^0)' (\Psi_{4i} + P_i^{-1})^{-1} \right\} \\ - \text{tr} \{ \tau \tau' \Xi^{-1} \} \\ > c'.$$

We now choose P_i , Q , A_i , Δ_i , ξ^0 , η_i^0 , and ζ_i suitably subject to Ξ being *p.d.* to reduce (2.11) to a nice form.

Let Σ_i be partitioned as

$$(2.12) \quad \Sigma_i = \begin{pmatrix} \Sigma_{i(11)} & \Sigma_{i(12)} \\ \Sigma_{i(21)} & \Sigma_{i(22)} \end{pmatrix}$$

where $\Sigma_{i(11)} : s \times s$, $\Sigma_{i(22)} : (p-s) \times (p-s)$, and let Σ_i^{-1} be similarly

partitioned into

$$(2.13) \quad \Sigma_i^{-1} = \begin{pmatrix} \Sigma_i^{11} & \Sigma_i^{12} \\ \Sigma_i^{21} & \Sigma_i^{22} \end{pmatrix} \\ = \begin{pmatrix} \Sigma_{i(11\ 2)}^{-1} & \Sigma_{i(11)}^{-1} \Sigma_{i(12)} \Sigma_{i(22\ 1)}^{-1} \\ \Sigma_{i(22)}^{-1} \Sigma_{i(21)} \Sigma_{i(11\ 2)}^{-1} & \Sigma_{i(22\ 1)}^{-1} \end{pmatrix}$$

where $\Sigma_{i(11\ 2)} = \Sigma_{i(11)} + \Sigma_{i(12)} \Sigma_{i(22)}^{-1} \Sigma_{i(21)}$ and $\Sigma_{i(22\ 1)} = \Sigma_{i(22)}$

$\Sigma_{i(21)} \Sigma_{i(11)}^{-1} \Sigma_{i(12)}$, $i = 1, \dots, k$.

Choose

$$(i) \quad P_i = \begin{pmatrix} P_{i(11)} & \mathbf{0} \\ \mathbf{0} & \Sigma_{i(22\ 1)} \end{pmatrix}, \quad i = 1, \dots, k,$$

where $P_{i(11)}$ is any $s \times s$, p.d. matrix;

$$(ii) \quad Q = \left(\sum_{i=1}^k n_i \Sigma_{i(11)}^{-1} \right)^{-1};$$

$$(iii) \quad A_i = \begin{pmatrix} \Sigma_{i(11)} & \Sigma_{i(12)} \\ \Sigma_{i(21)} & \Sigma_{i(22)} \end{pmatrix}, \quad i = 1, \dots, k;$$

$$(iv) \quad \Delta_i = P_i \Psi_{5i}, \quad i = 1, \dots, k;$$

$$(v) \quad \xi'' = Q \left(\sum_{i=1}^k n_i \Psi_{5i}' \eta_i'' \right);$$

$$(vi) \quad \zeta_i = A_i (\Sigma_i^{-1} + A_i^{-1}) \Psi_2 (\Psi_{4i} + P_i^{-1})^{-1} P_i^{-1} \eta_i'', \quad i = 1, \dots, k,$$

and (vii) η_i'' , $i = 1, \dots, k$, arbitrary.

It may be noted that Ξ simplifies to

$$\begin{aligned}
(2.14) \quad \Xi &= \Psi_3 + Q^{-1} + \sum_{i=1}^k n_i \Psi'_{5i} P_i \Psi_{5i} \\
&= \Psi_1 \left(\sum_{i=1}^k n_i \Sigma_i^{-1} \right) \Psi'_1 + \sum_{i=1}^k n_i (\Sigma_{i(11)}^{-1} + \Psi_1 \Sigma_i^{-1} \Psi_2 P_i \Psi_2 \Sigma_i^{-1} \Psi'_1) \\
&= \Psi_1 \left(\sum_{i=1}^k n_i \Sigma_i^{-1} \right) \Psi'_1 + \sum_{i=1}^k n_i (\Sigma_{i(11)}^{-1} + \Sigma_i^{12} \Sigma_{i(22-1)} \Sigma_i^{21}) \\
&= \sum_{i=1}^k n_i (\Sigma_i^{11} + \Sigma_{i(11)}^{-1} + \Sigma_i^{11} - \Sigma_{i(11)}^{-1}) \\
&= 2 \sum_{i=1}^k n_i \Sigma_i^{11}
\end{aligned}$$

which is clearly *p.d.* and $P_{i(11)}$ does not affect the calculation.

The three terms appearing in the left hand side of (2.11) can now be evaluated. Combining the first term and the second term of (2.11), and using (vi), we have

$$\begin{aligned}
(2.15) \quad & tr \left\{ \sum_{i=1}^k n_i \bar{\mathbf{x}}_i \bar{\mathbf{x}}'_i \Sigma_i^{-1} ((\Sigma_i^{-1} + A_i^{-1})^{-1} - \Psi_2 (\Psi_{4i} + P_i^{-1})^{-1} \Psi_2) \Sigma_i^{-1} \right. \\
&+ \sum_{i=1}^k n_i \bar{\mathbf{x}}_i (\zeta'_i A_i^{-1} (\Sigma_i^{-1} + A_i^{-1})^{-1} - \eta_i^{0'} P_i^{-1} (\Psi_{4i} + P_i^{-1})^{-1} \Psi_2) \Sigma_i^{-1} \\
&+ \sum_{i=1}^k n_i \Sigma_i^{-1} ((\Sigma_i^{-1} + A_i^{-1})^{-1} A_i^{-1} \zeta_i - \Psi_2 (\Psi_{4i} + P_i^{-1})^{-1} P_i^{-1} \eta_i^0) \bar{\mathbf{x}}'_i \\
&+ \left. \sum_{i=1}^k n_i (\zeta_i \zeta'_i A_i^{-1} (\Sigma_i^{-1} + A_i^{-1})^{-1} A_i^{-1} - \eta_i^0 \eta_i^{0'} P_i^{-1} (\Psi_{4i} + P_i^{-1})^{-1} P_i^{-1}) \right\} \\
&= tr \left\{ \sum_{i=1}^k n_i \bar{\mathbf{x}}_i \bar{\mathbf{x}}'_i \Sigma_i^{-1} ((\Sigma_i^{-1} + A_i^{-1})^{-1} - \Psi_2 (\Psi_{4i} + P_i^{-1})^{-1} \Psi_2) \Sigma_i^{-1} \right\}
\end{aligned}$$

$$+ \sum_{i=1}^k n_i (\zeta_i \zeta_i' A_i^{-1} (\Sigma_i^{-1} + A_i^{-1})^{-1} A_i^{-1} - \eta_i^0 \eta_i^{0'} P_i^{-1} (\Psi_{4i} + P_i^{-1})^{-1} P_i^{-1}) \}.$$

Since the second term of the last equation does not contain data, we may drop it. So (2.15) is now

$$\begin{aligned} (2.16) \quad & tr \left\{ \sum_{i=1}^k n_i (\Sigma_i^{-1} \bar{\mathbf{x}}_i) (\Sigma_i^{-1} \bar{\mathbf{x}}_i)' ((\Sigma_i^{-1} + A_i^{-1})^{-1} - \Psi_2 (\Psi_{4i} + P_i^{-1})^{-1} \Psi_2) \right\} \\ &= tr \left\{ 1/2 \sum_{i=1}^k n_i (\Sigma_i^{-1} \bar{\mathbf{x}}_i) (\Sigma_i^{-1} \bar{\mathbf{x}}_i)' \right. \\ &\quad \cdot \left(\begin{pmatrix} \Sigma_{i(11 \ 2)} & \mathbf{0} \\ \mathbf{0} & \Sigma_{i(22 \ 1)} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Sigma_{i(22 \ 1)} \end{pmatrix} \right) \left. \right\} \\ &= tr \left\{ 1/2 \sum_{i=1}^k n_i (\Sigma_i^{-1} \bar{\mathbf{x}}_i) (\Sigma_i^{-1} \bar{\mathbf{x}}_i)' \begin{pmatrix} \Sigma_{i(11 \ 2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right\}. \end{aligned}$$

Also, using (ii), (iv) and (v), the third term of (2.11) reduces to

$$\begin{aligned} (2.17) \quad & tr \{ \tau \tau' \Xi^{-1} \} \\ &= tr \left\{ 1/2 (\Psi_1 \beta) (\Psi_1 \beta)' \left(\sum_{i=1}^k n_i \Sigma_i^{11} \right)^{-1} \right\} \\ &= tr \left\{ 1/2 \beta \beta' \begin{pmatrix} (\sum_{i=1}^k n_i \Sigma_i^{11})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right\} \\ &= tr \left\{ 1/2 \left(\sum_{i=1}^k n_i \Sigma_i^{-1} \bar{\mathbf{x}}_i \right) \left(\sum_{i=1}^k n_i \Sigma_i^{-1} \bar{\mathbf{x}}_i \right)' \begin{pmatrix} (\sum_{i=1}^k n_i \Sigma_i^{11})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right\}. \end{aligned}$$

Clearly $P_{i(11)}$ does not affect the above calculations. Combining (2.16) and (2.17), (2.11) is now of the form

$$(2.18) \quad \begin{aligned} & \operatorname{tr} \left\{ \sum_{i=1}^k n_i (\Sigma_i^{-1} \bar{\mathbf{x}}_i) (\Sigma_i^{-1} \bar{\mathbf{x}}_i)' \begin{pmatrix} \Sigma_{i(11 \cdot 2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right\} \\ &= \operatorname{tr} \left\{ \left(\sum_{i=1}^k n_i \Sigma_i^{-1} \bar{\mathbf{x}}_i \right) \left(\sum_{i=1}^k n_i \Sigma_i^{-1} \bar{\mathbf{x}}_i \right)' \begin{pmatrix} (\sum_{i=1}^k n_i \Sigma_{i(11 \cdot 2)}^{-1})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right\} \\ &> c''. \end{aligned}$$

This immediately prove the following result.

THEOREM 2.1. *An admissible Bayes test of (1.1) rejects H_0 for large values of the test statistic given by the left hand side of (2.18).*

When $\Sigma_1, \dots, \Sigma_k$ are equal (and $= \Sigma$, say), (2.18) reduces to

$$(2.19) \quad \operatorname{tr} \left\{ \left(\sum_{i=1}^k n_i \bar{\mathbf{x}}_i \bar{\mathbf{x}}_i' - 1/n \left(\sum_{i=1}^k n_i \mathbf{x}_i \right) \left(\sum_{i=1}^k n_i \bar{\mathbf{x}}_i' \right) \right) \left(\Sigma^{-1} - \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Sigma_{22}^{-1} \end{pmatrix} \right) \right\} > c,$$

which is the same as

$$(2.20) \quad \operatorname{tr} \left\{ B \left(\Sigma^{-1} - \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Sigma_{22}^{-1} \end{pmatrix} \right) \right\} > c,$$

where B is defined earlier.

This leads to the following important result.

THEOREM 2.2. *An admissible Bayes test of (1.1) when $\Sigma_1 = \dots = \Sigma_k = \Sigma$ rejects H_0 for large values of the test statistic given by the left hand side of (2.19) or (2.20).*

REMARK 2.1. It is interesting to observe that the test statistic given by the left hand side of (2.19) can be interpreted as the sum of the first s roots of the matrix $B\Sigma^{-1}$. This follows because one can always take $\Sigma = I_p$ due to the nature of the hypotheses (1.1) and change B to $\Sigma^{-1/2}B\Sigma^{-1/2}$.

REMARK 2.2. It is clear from the preceding calculations that if, under H_0, π_0 is chosen as assigning all its measure to the subspace $\mathcal{H}_{i_1, \dots, i_s}$ of the form $\mathcal{H}_{i_1, \dots, i_s} = \{H : s \times p \text{ matrix } H = (0 \dots \mathbf{1}_{i_1} 0 \dots \mathbf{1}_{i_2} \dots \mathbf{1}_{i_s} \dots 0), \mathbf{1}_{i_j} = (0 \dots 1 \dots 0) : s \times 1 \text{ with } 1 \text{ at the } j\text{th position}^*, 1 \leq i_1 \leq \dots \leq i_s \leq p\}$, and appropriate changes are made accordingly in the definitions of Ψ_1, Ψ_2 , and also in the choice of P_i, Q, A_i etc., in the above priors of ξ and $\eta_1, \dots, \eta_k$, then the Bayes test rejects H_0 for large values of the test statistic

$$(2.21) \quad tr \left\{ \sum_{i=1}^k n_i (\Sigma_i^{-1} \bar{\mathbf{x}}_i) (\Sigma_i^{-1} \bar{\mathbf{x}}_i)' \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Sigma_{i(11 \ 2)} & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

* $1 \leq j \leq s$, and $\mathbf{1}_{i_j}$ as the i_j th column of H ,

$$= \text{tr} \left\{ \left(\sum_{i=1}^k n_i \Sigma_i^{-1} \bar{\mathbf{x}}_i \right) \left(\sum_{i=1}^k n_i \Sigma_i^{-1} \bar{\mathbf{x}}_i \right)' \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \left(\sum_{i=1}^k n_i \Sigma_{i(112)}^{-1} \right)^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \right\}$$

where $\Sigma_{i(112)}$ is the appropriate $s \times s$ proper submatrix of Σ_i corresponding to the rows $(i_1, \dots, i_s)$ and the columns $(i_1, \dots, i_s)$. In the special case of the equality of $\Sigma_1, \dots, \Sigma_k$, it therefore follows from (2.21) that a test which rejects H_0 for large values of the sum of any s roots of $B\Sigma^{-1}$ is admissible Bayes. In particular, the likelihood ratio test of Rao(1973) for equal covariance matrices which rejects H_0 for large values of the sum of s smallest roots of $B\Sigma^{-1}$ is admissible Bayes.

REFERENCES

- FISHER, R. A. (1939). The sampling distribution of some statistics obtained from nonlinear equations. *Annals. of Eugenics*, **9**, 238-249.
- FUJIKOSHI, Y. (1974). The likelihood ratio tests for the dimensionality of regression coefficients. *J. Mult. Anal.*, **4**, 327-340.
- KIEFER, J. and SCHWARTZ, R. (1965). Admissible Bayes character of T^2 , R^2 , and other fully invariant tests for classical multivariate normal problems. *Ann. Math. Statist.*, **36**, 747-770.
- RAO, C. R. (1973). *Linear Statistical Inference and Its Applications*. 2nd. edition. Wiley, New York.
- RAO, C. R. (1985). Tests for dimensionality and interactions of mean vectors under general and reducible covariance structures. *J. Mult. Anal.*, **16**, 173-184.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND BALTIMORE COUNTY
CATONSVILLE, MARYLAND 21228

CENTER FOR MULTIVARIATE ANALYSIS
UNIVERSITY OF PITTSBURGH
PITTSBURGH, PENNSYLVANIA 15260

~~FILED~~ MED
4 8