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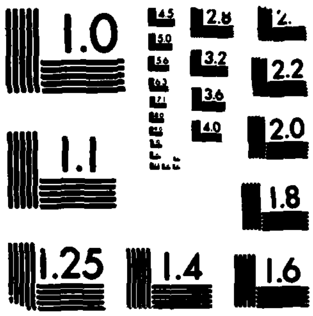

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Consider a signal $x(t) = f(t) + w(t)$, $0 \le t \le 1$. Here the noise $w(t)$ is an independent process, and $f(t)$ is a function with only finitely many jumps, satisfies a Lipschitz' condition between any two consecutive jumps. This paper gives an algorithm to determine the number, locations and magnitudes of the jumps of $f(t)$. The consistency and speeds of convergence are obtained. (Keywords; stochastic processes; distributions; estimation; convergence).

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DETECTION OF THE NUMBER, LOCATIONS AND MAGNITUDES OF JUMPS

Y. Q. YIN

§1. Introduction. Jump detection or change-point detection is a very important problem in statistics and engineering. The problem can be stated in the following way:

Let $x(t) = f(t) + w(t)$ be a stochastic process, $0 \leq t \leq 1$. Here $w(t)$ is the noise process, $Ew(t) = 0$, and $f(t) = Ex(t)$ is a function with only finitely many discontinuities $t_1, \cdots t_q$. Suppose these discontinuities are all interior points of $[0,1]$ and for each $i = 1, \cdots q$, $f(t_i + 0)$, $f(t_i - 0)$ exist and $f(t_i + 0) \neq f(t_i - 0)$. For definiteness, we suppose f is left continuous everywhere. Our problem is to estimate

1. the number q of discontinuities,
2. the positions $t_1, \cdots, t_q$ of these discontinuities;
3. the magnitudes $f(t_i + 0) - f(t_i - 0)$ of the jumps, $i = 1, \cdots, q$;

based on a sample $x(\frac{k}{n})$, $k = 0, 1, \cdots n$.

In this paper, under mild conditions, we will give an algorithm to estimate q, i. e. , define an estimator $\hat{q}$ of q. For each $i = 1, \cdots q$, we give an algorithm to estimate the position of the ith discontinuity point t_i, i. e. , define an estimator $\hat{t}_i$ of t_i; we also define an estimator $D_{n\hat{t}_i}$ for the jump at the ith discontinuity point.

We will prove that all these estimators are strongly consistent, or in other words, these estimators converge to the corresponding parameters as n, the number of sample points, tends to infinity, with probability one.

We also get the speed of convergence, for example for $\hat{t}_i$ we get

$$|\hat{t}_i - t_i| = O\left(\frac{(\ln n)^{1+\alpha}}{n}\right)$$

for any $\alpha > 0$.

The complexity of computation of our algorithms is $O(n \log n)$ approximately.

The basic hypotheses are two:

1. $w(t)$ is a gaussian white noise.
2. There exists a positive constant $K > 0$, such that

$$|f(t) - f(s)| \leq K|t-s|, \text{ if no } t_i \text{ are in } [s,t].$$

The condition 1. can be relaxed to nongaussian white noises, but it would be difficult to relax 2.

In Section 2, we define the algorithms. In Section 3 we give the proofs.

Works on this topic mostly concentrate on the single jump problem, see the references listed at the rear of this paper. Especially no work has been done on the case when the number of jumps is unknown.

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§2. The Method for Detecting the Change Points. Let $x(t) = f(t) + w(t)$ be a stochastic process, $0 \leq t \leq 1$. Here $w(t)$ is a Gaussian white noise. $f(t) = Ex(t)$ has only finitely many discontinuities $t_1, \cdots, t_q$. All these discontinuities are in $(0,1)$ and of the first kind, i. e., $f(t_i + 0)$, $f(t_i - 0)$ exist for $i = 1, \ldots, q$.

Besides, suppose that there is a constant $K > 0$ such that $|f(s) - f(t)| \leq K|s-t$ for any interval $[s,t] \subset [0,1]$ not containing any discontinuities. Let

$$d_i = f(t_i + 0) - f(t_i - 0) \neq 0, \qquad i = 1, \cdots, q,$$

and suppose

$$|d_1| \geq |d_2| \geq \cdots \geq |d_q|, \text{ and if } |d_i| = |d_j|, \text{ and } i < j, \text{ then } t_i < t_j.$$

Let $m = m(n) \uparrow \infty$, $m/n \to 0$. Define

$$D_{nk} = \frac{x(\frac{k+1}{n}) + \cdots + x(\frac{k+m}{n})}{m} - \frac{x(\frac{k-1}{n}) + \cdots + x(\frac{k-m}{n})}{m},$$

for $m \leq k \leq n - m$.

Let h_n be a sequence of positive numbers, $h_n \to 0$, $\dfrac{m_n}{nh_n} \to 0$, for definiteness, let $h_n = (\ln_3 n / \ln_2 n)^{1/3}$, $m_n = \ln n (\ln_2 n)^{2/3} (\ln_3 n)^{1/3}$ where $\ln_1 n = 1 \wedge \ln n$, $\ln_k n = \ln_1(\ln_{k-1} n)$.

Define

$$I_1 = \arg\max_k \left\{ |D_{nk}| - \frac{k}{n} h_n \right\},$$

$$I_2 = \arg\max_k \left\{ |D_{nk}| - \frac{k}{n} h_n : |k - I_1| > 4m \right\},$$

$$I_3 = \arg\max_k \left\{ |D_{nk}| - \frac{k}{n} h_n : |k - I_1| > 4m \text{ and } |k - I_2| > 4m \right\}, \cdots.$$

If the definition is not unique, we choose the smallest one. At first, we state the main theorem, which will be proved in Section 3.

Theorem 3.1. (1) $\left|\frac{I_i}{n} - t_i\right| \leq \frac{2m}{n}$ for all large n, a. s, $i = 1, \cdots, q$.

(2) $D_{nI_i} \to d_i$, a. s., $i = 1, \cdots, q$.

(3) $D_{nI_i} = O(h_n)$, a. s. for $i > q$.

We see from this theorem that the quantities defined above are strongly consistent estimators of the change points $t_1, \cdots, t_q$ and the magnitudes d_i of changements. Furthermore we get the convergence rate

$$\left|\frac{I_i}{n} - t_i\right| \leq \frac{2m}{n}, \quad \text{for large } n, \text{ a. s.}$$

from (1).

Theorem 3.1 does not supply a method to estimate the integer q, explicitly. But based on Theorem 3.1, we can construct a strongly consistent estimator of q in the following manner.

Let

$$(4) \qquad G_{nk} = \frac{1}{2^{k+1}}|D_{n,I_{k+1}}| + \frac{1}{2^{k+2}}|D_{n,I_{k+2}}| + \cdots + kc_n.$$

Here $c_n > 0$ with the properties $c_n \to 0$ and $h_n/c_n \to 0$. Let

$$\hat{q}_n = \arg\min_k G_{nk}.$$

Suppose we have proved Theorem 3.1. We are going to prove

Theorem 2.1. $\hat{q}_n \to q$, a. s.

Proof. If $k < q$, then, almost surely, as $n \to \infty$

$$\begin{aligned} G_{nk} - G_{nq} \geq \frac{1}{2^{k+1}}|D_{nI_{k+1}}| + \cdots + \frac{1}{2^q}|D_{nI_q}| - \frac{1}{2^k}h_n \\ + (k-q)c_n \to \frac{1}{2^{k+1}}|d_{k+1}| + \cdots + \frac{1}{2^q}|d_q| > 0, \end{aligned}$$

by (2) of Theorem 3.1. That means $\hat{q}_n \neq k$ for large n.

If $k > q$, then, noticing $|D_{nI_i}| \downarrow$, as $i \uparrow$, by (3) of Theorem 3.1, we have for a constant $C > 0$,

$$\begin{aligned} G_{nk} - G_{nq} &= -\frac{1}{2^{q+1}}|D_{nI_{q+1}}| - \cdots - \frac{1}{2^k}|D_{nI_k}| + (k-q)c_n \\ &\geq -\frac{1}{2^q}|D_{nI_{q+1}}| + (k-q)c_n \\ &\geq -\frac{C}{2^q}h_n + (k-q)c_n = c_n\left((k-q) - \frac{Ch_n}{2^{q+1}c_n}\right) \\ &\geq c_n\frac{1}{2}(k-q) > 0 \end{aligned}$$

for n sufficiently large, almost surely. In this case $\hat{q}_n \neq k$ either. So, almost surely, for n sufficiently large, $\hat{q}_n = q$.

§3. Proof of the Main Theorem. At first we prove an elementary lemma.

Lemma. Suppose $f(t)$ is defined in the interval $(t_0 - a, t_0 + a)$ for a positive number a. Suppose $f(t_0 \pm 0)$ exist and are finite. Let I_n and m_n be two sequences of positive integers such that $I_n/n \to t_0$, $m_n/n \to 0$, $m_n \to \infty$. Let

$$A_n = -\frac{1}{m}\left(f(\frac{I-m}{n}) + \cdots + f(\frac{I-1}{n}) - f(\frac{I+1}{n}) - \cdots - f(\frac{I+m}{n})\right),$$

where $I = I_n$, $m = m_n$. Then, from $\liminf_{n\to\infty} |A_n| \geq |f(t_0 + 0) - f(t_0 - 0)|$, we can deduce that $A_n \to f(t_0 + 0) - f(t_0 - 0)$.

Proof. Without loss of generality, suppose $\frac{I}{n} < t_0$ for all n. Let

$$k_n = max\left\{k : \frac{I+k}{n} < t_0, 0 \leq k \leq m\right\}.$$

Fix $\epsilon > 0$, $\exists \delta > 0$ such that $t_0 - \delta < t < t_0 \implies |f(t) - f(t_0 - 0)| < \epsilon$, and $t_0 < t < t_0 + \delta \implies |f(t) - f(t_0 + 0)| < \epsilon$.

Let N be such that as $n > N$, $\left|\frac{I}{n} - t_0\right| < \frac{\delta}{2}$, $\frac{m}{n} < \frac{\delta}{2}$. So, as $n > N$

$$\begin{aligned} A_n &< -f(t_0 - 0) + \epsilon + (f(t_0 - 0) + \epsilon)\frac{k_n}{m} + \left(1 - \frac{k_n}{m}\right)(f(t_0 + 0) + \epsilon) \\ &= \left(1 - \frac{k_n}{m}\right)(f(t_0 + 0) - f(t_0 - 0)) + 2\epsilon. \end{aligned}$$

In the same way

$$A_n > \left(1 - \frac{k_n}{m}\right)(f(t_0 + 0) - f(t_0 - 0)) - 2\epsilon.$$

Thus,

$$|A_n| \leq \left|1 - \frac{k_m}{m}\right| |f(t_0 + 0) - f(t_0 - 0)| + 2\epsilon.$$

Therefore we must have $\frac{k_n}{m} \to 0$, thus $A_n \to f(t_0+0) - f(t_0-0)$.

Theorem 3.1. (1) $|I_k/n - t_k| \le \frac{2m}{n}$, for all large n, a. s., for $k = 1, \cdots, q$;

(2) $D_{nI_k} \to d_k$, a. s., for $k = 1, \cdots, q$;

(3) $D_{nI_k} = O(h_n)$, a. s., for $k > q$.

Proof. 1. In this part, we prove $\left|\frac{I_1}{n} - t_1\right| \le \frac{2m}{n}$ for all large n, a. s. Introduce the following notations

$\Delta_k = \frac{1}{m}\left\{f\left(\frac{k+1}{n}\right) + \cdots + f\left(\frac{k+m}{n}\right)\right\} - \frac{1}{m}\left\{f\left(\frac{k-1}{n}\right) + \cdots + f\left(\frac{k-m}{n}\right)\right\}$,

$W_k^+ = \frac{1}{m}\left\{w\left(\frac{k+1}{n}\right) + \cdots + w\left(\frac{k+m}{n}\right)\right\}$,

$W_k^- = \frac{1}{m}\left\{w\left(\frac{k-1}{n}\right) + \cdots + w\left(\frac{k-m}{n}\right)\right\}$.

Let $\hat{k}$ be such that $|\hat{k}/n - t_1| = \min_k |k/n - t_1|$, $\hat{k}$ depends only on n.

At first, we note that for $|k/n - t_1| > 2m/n$,

$$\begin{aligned}(4)\quad & P\left(|D_{n\hat{k}}| - \tfrac{\hat{k}}{n}h_n \le |D_{nk}| - \tfrac{k}{n}h_n\right) \\ & \le P\left(|\Delta_{\hat{k}}| - |\Delta_k| + \left(\tfrac{k-\hat{k}}{n}\right)h_n \le |W_k^+| + |W_k^-| + |W_{\hat{k}}^+| + |W_{\hat{k}}^-|\right).\end{aligned}$$

Because $\left|\frac{k}{n} - t_1\right| > \frac{2m}{n}$, the points $\frac{k-m}{n}, \cdots, \frac{k-1}{n}, \frac{k+1}{n}, \cdots, \frac{k+m}{n}$ are all on the same side of t_1. If n is larger than some nonrandom number $N_1 > 0$, for any k with $\left|t_1 - \frac{k}{n}\right| > \frac{2m}{n}$, the interval $\left[\frac{k-m}{n}, \frac{k+m}{n}\right]$ can contain at most one discontinuous point t_i. There are three possibilities:

(a) No t_i in $\left[\frac{k-m}{n}, \frac{k+m}{n}\right]$,

(b) $t_i \in \left[\frac{k-m}{n}, \frac{k+m}{n}\right]$, $|d_i| = |d_1|$, then $t_i > t_1$, and $k > \hat{k}$,

(c) $t_i \in \left[\frac{k-m}{n}, \frac{k+m}{n}\right]$, $|d_i| < |d_1|$.

For case (a) above,

$$|\Delta_{\hat{k}}| \ge |d_1| - 2K\frac{m}{n}, \quad |\Delta_k| \le K\frac{m+1}{n}$$

so,

$$|\Delta_{\hat{k}}| - |\Delta_k| + \frac{k-\hat{k}}{n}h_n \ge |d_1| - 3K\frac{m+1}{n} + \frac{k-\hat{k}}{n}h_n > \frac{1}{2}|d_1|,$$

for $n \ge N_2 > 0$, N_2 is a nonrandom constant.

For case (b),

$$|\Delta_{\hat{k}}| \geq |d_1| - 2K\frac{m}{n}, \quad |\Delta_k| \leq |d_1| + 2K\frac{m}{n},$$

so, for a constant $c > 0$,

$$|\Delta_{\hat{k}}| - |\Delta_k| + \frac{(k-\hat{k})}{n}h_n \geq -4K\frac{m}{n}$$
$$+ (t_i - t_1)h_n + o(h_n) \geq ch_n$$

when $n \geq N_3 > 0$, N_3 is nonrandom, c is a positive constant.

For case (c),

$$|\Delta_{\hat{k}}| - |\Delta_k| + \frac{(k-\hat{k})}{n}h_n \geq |d_1| - |d_i| + o(1)$$
$$\geq \frac{1}{2}(|d_1| - |d_i|) > 0$$

when $n \geq N_4 > 0$, N_4 is nonrandom.

Thus, for $n \geq max(N_1, N_2, N_3, N_4)$, and for some constants $c_1 > 0$, $c_2 > 0$,

$$P\left(|D_{n\hat{k}}| - \frac{\hat{k}}{n}h_n \leq |D_{nk}| - \frac{k}{n}h_n\right)$$
$$\leq 4P\left(b_1 h_n \leq \left|\frac{v_1 + \cdots + v_m}{m}\right|\right)$$
$$\leq 4P\left(\frac{b_1}{\sigma}h_n\sqrt{m} \leq |z|\right)$$
$$\leq 8\frac{1}{\sqrt{2\pi}\frac{b_1}{\sigma}h_n\sqrt{m}}e^{-\frac{1}{2}\frac{b_1^2}{\sigma^2}h_n^2 m}$$
$$\leq b_2 e^{-b_3 \ln n \ln_3 n} = b_2 n^{-b_3 \ln_3 n}$$

if we choose $h_n = \left(\frac{\ln_3 n}{\ln_2 n}\right)^{1/3}$, $m = \ln n(\ln_2 n)^{2/3}(\ln_3 n)^{1/3}$. Here b_1, b_2, b_3 are positive constants, $\ln_2 x = \ln\ln x$, $\ln_3 x = \ln\ln x$. Thus, the series

$$\sum_n P\left(|D_{n\hat{k}}| - \frac{\hat{k}}{n}h_n \leq |D_n k| - \frac{k}{n}h_n, \text{for some } k \text{ with } \left|\frac{k}{n} - t_1\right| > \frac{2m}{n}\right)$$
$$\leq \sum_n nP\left(|D_{n\hat{k}}| - \frac{\hat{k}}{n}h_n \leq |D_{nk}| - \frac{k}{n}h_n\right) < \infty.$$

By Borel-Cantelli Lemma,

$$P\Big(\exists N, s.t. n \geq N, |\frac{k}{n} - t_1| > \frac{2m}{n}$$
$$\Longrightarrow |D_{n\hat{k}}| - \frac{\hat{k}}{n}h_n > |D_{nk}| - \frac{k}{n}h_n\Big) = 1$$

or,

$$P\Big(\exists, N, s.t. n \geq N, |D_{n\hat{k}}| - \frac{\hat{k}}{n}h_n \leq |D_{nk}| - \frac{k}{n}h_n$$
$$\Longrightarrow |\frac{k}{n} - t_1| \leq \frac{2m}{n}\Big) = 1.$$

Since $|D_{nI_1}| - \frac{I_1}{n}h_n \geq |D_{n\hat{k}}| - \frac{\hat{k}}{n}h_n$,

$$P\left(\exists N, s.t. n \geq N \Longrightarrow \left|\frac{I_1}{n} - t_1\right| \leq \frac{2m}{n}\right) = 1.$$

This proves case $k = 1$ for (1).

By

$$|D_{nI_1}| - \frac{I_1}{n}h_n \geq |D_{n\hat{k}}| - \frac{\hat{k}}{n}h_n$$

we have

$$\liminf |D_{nI_1}| \geq |d_1|.$$

So, by the elementary lemma, we get $D_{nI_1} \to d_1$ a. s.

2. In this part we prove $|\frac{I_i}{n} - t_i| \leq \frac{2m}{n}$ for all large n, a. s.. and $D_{nI_i} \to d_i$ a. s., if $2 \leq i \leq q$. But we carry out the proof only for the case $i = 2$.

Let $\hat{k}$ be such that $|\frac{\hat{k}}{n} - t_2| = \min_k |\frac{k}{n} - t_2|$, and suppose k is such that $|\frac{k}{n} - t_1| > \frac{2m}{n}$ and $|\frac{k}{n} - t_2| > \frac{2m}{n}$. Suppose for these $\hat{k}$ and k, $|D_{n\hat{k}}| - \frac{\hat{k}}{n}h_n \leq |D_{nk}| - \frac{k}{n}h_n$, so that

$$|\Delta_{\hat{k}}| - |\Delta_k| + \frac{k - \hat{k}}{n}h_n \leq |W_k^+| + |W_k^-| + |W_{\hat{k}}^+| + |W_{\hat{k}}^-|.$$

Since $|\frac{k}{n} - t_1| > \frac{2m}{n}$, $|\frac{k}{n} - t_2| > \frac{2m}{n}$, t_1 and t_2 are not in the interval $[\frac{k-m}{n}, \frac{k+m}{n}]$. $\exists N_1 > 0$ nonrandom, such that as $n \geq N_1$, $[\frac{k-m}{n}, \frac{k+m}{n}]$ contains at most one discontinuity point. There are three possibilities:

(a) No t_i in $[\frac{k-m}{n}, \frac{k+m}{n}]$,

(b) $t_i \in [\frac{k-m}{n}, \frac{k+m}{n}]$, $|d_i| = |d_2|$, of course $i > 2$, $t_i > t_2$.

(c) $t_i \in [\frac{k-m}{n}, \frac{k+m}{n}]$, $|d_i| < |d_2|$.

For (a), we have

$$|\Delta_{\hat{k}}| \geq |d_2| - K\frac{2m}{n}, \quad |\Delta_k| \leq K\frac{2m}{n},$$

and then as $n \geq N_2$, N_2 nonrandom,

$$|\Delta_{\hat{k}}| - |\Delta_k| + \frac{k-\hat{k}}{n}h_n \geq |d_2| - \frac{4Km}{n} - h_n \geq \frac{1}{2}|d_2|.$$

For (b),

$$|\Delta_{\hat{k}}| \geq |d_2| - K\frac{2m}{n}, \quad |\Delta_k| \leq |d_2| + K\frac{2m}{n},$$

and as $n \geq N_3$, N_3 nonrandom, we have

$$\begin{aligned} |\Delta_{\hat{k}}| - |\Delta_k| + \frac{k-\hat{k}}{n}h_n &\geq -K\frac{4m}{n} + \frac{k-\hat{k}}{n}h_n \\ &\geq \left(-K\frac{4m}{nh_n} + t_i - t_2 + o(1)\right)h_n \geq bh_n, \end{aligned}$$

for constant $b > 0$.

For case (c), just in the same way, we can show that as $n \geq N_4$, N_4 nonrandom,

$$|\Delta_{\hat{k}}| - |\Delta_k| + \frac{k-\hat{k}}{n}h_n \geq \frac{1}{2}(|d_2| - |d_i|) > 0.$$

therefore as $n \geq max(N_1, N_2, N_3, N_4)$,

$$\begin{aligned} P\left(|D_{n\hat{k}}| - \frac{\hat{k}}{n}h_n \leq |D_{nk}| - \frac{k}{n}h_n\right) & \\ &\leq 4P\left(b_1 h_n \leq \left|\frac{v_1 + \cdots + v_m}{m}\right|\right) \\ &\leq 4P(b_2 h_n\sqrt{m} \leq |z|) \leq b_2 n^{-b_3 \ln_2 n} \end{aligned}$$

for positive constants b_1 b_2, b_3. Here $v_1, \cdots, v_m$, z have the same meanings as before.

Therefore the series

$$\sum_n P\Big(|D_{n\hat{k}}| - \frac{\hat{k}}{n}h_n \le |D_{nk}| - \frac{k}{n}h_n, \quad \text{for some } k \text{ with } |\frac{k}{n} - t_1| > \frac{2m}{n} \text{ and } |\frac{k}{n} - t_2| > \frac{2m}{n}\Big) < \infty.$$

Thus

$$P\Big(\exists N : n \ge N \implies |D_{n\hat{k}}| - \frac{\hat{k}}{n}h_n > |D_{nk}| - \frac{k}{n}h_n \quad \text{for all } k \text{ with } |\frac{k}{n} - t_1| > \frac{2m}{n} \text{ and } |\frac{k}{n} - t_2| > \frac{2m}{n}\Big) = 1.$$

Suppose the event in the last $P(\quad)$ is true, and $|\frac{I_1}{n} - t_1| \le \frac{2m}{n}$ is true for large n. Then, $|\hat{k} - I_1| = n|\frac{\hat{k}}{n} - \frac{I_1}{n}| \ge n(|t_2 - t_1| - \frac{1}{2n} - \frac{2m}{n}) > 4m$, if n is large. So, $|D_{nI_2}| - \frac{I_2}{n}h_n \ge |D_{n\hat{k}}| - \frac{\hat{k}}{n}h_n$, and $|\frac{I_2}{n} - t_1| \le \frac{2m}{n}$ or $|\frac{I_2}{n} - t_2| \le \frac{2m}{n}$, for large n a. s. But $|\frac{I_2}{n} - t_1| \ge |\frac{I_2 - I_1}{n}| - |\frac{I_1}{n} - t_1| > \frac{4m}{n} - \frac{2m}{n} = \frac{2m}{n}$, so $|\frac{I_2}{n} - t_2| \le \frac{2m}{n}$ for large n a. s.

From $|D_{nI_2}| - \frac{I_2}{n}h_n \ge |D_{n\hat{k}}| - \frac{\hat{k}}{n}h_n$, we have $\liminf_{n\to\infty} |D_{nI_2}| \ge |d_2|$. By the lemma, we must have $D_{nI_2} \to d_2$, a. s.

3. In this part we prove that $D_{nI_{q+1}} = O(h_n)$, a. s. By definition, $|I_{q+1} - I_i| > 4m$, $i = 1, \cdots, q$ for sufficiently large n, a. s. Thus, in this case,

$$\left|\frac{I_{q+1}}{n} - t_i\right| > \frac{4m}{n} - \frac{2m}{n} = \frac{2m}{n}.$$

Therefore in $\left[\frac{I_{q+1}-m}{n}, \frac{I_{q+1}+m}{n}\right]$ there are no discontinuity points, so

$$\begin{aligned}
|D_{nI_{q+1}}| &\leq |\Delta_{I_{q+1}}| + |W^{+}_{I_{q+1}}| + |W^{-}_{I_{q-1}}| \\
&\leq K\frac{2m}{n} + |W^{+}_{I_{q+1}}| + |W^{-}_{I_{q+1}}|,
\end{aligned}$$

and

$$\begin{aligned}
&\sum P\left(|D_{nI_{q+1}}| \geq h_n\right) \\
&\qquad \leq \sum P\left(h_n \leq K\frac{2m}{n} + |W^{+}_{I_{q+1}}| + |W^{-}_{I_{q+1}}|\right) \\
&\qquad \leq b\sum_n nP\left(b_1 h_n \leq \left|\frac{v_1 + \cdots + v_m}{m}\right|\right) \\
&\qquad \leq b\sum_n nP(b_2\sqrt{m}h_n \leq |z|) < \infty.
\end{aligned}$$

Therefore

$$|D_{nI_{q+1}}| = O(h_n), \quad a.\ s.$$

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