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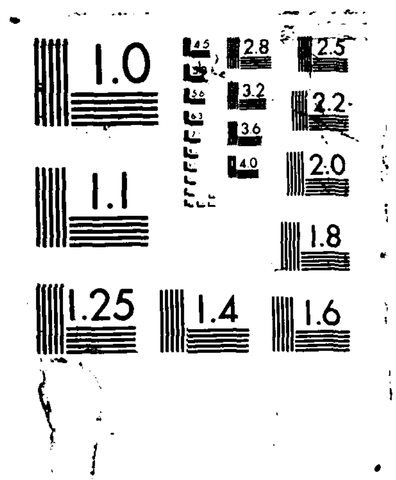
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INFERENCE FOR A NONLINEAR SEMIMARTINGALE REGRESSION MODEL

by

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INFERENCE FOR A NONLINEAR SEMIMARTINGALE REGRESSION MODEL

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Abstract

~~Consider~~ the semimartingale regression model

$$X(t) = X(0) + \int_0^t Y(s)\,\alpha(s, Z(s))\,ds + M(t),$$

where Y, Z are observable covariate processes, α is a (deterministic) function of both, time and the covariate process Z, and M is a square integrable martingale. Under the assumption that i.i.d. copies of X, Y, Z are observed continuously over a finite time interval, inference for the function $\alpha(t,z)$ is investigated. An estimator $\hat{A}$ for the time integrated $\alpha(t,z)$ and a kernel estimator of $\alpha(t,z)$ itself are introduced. For X a counting process, $\hat{A}$ reduces to the Nelson-Aalen estimator when Z is not present in the model. Various forms of consistency are proved, rates of convergence and asymptotic distributions of the estimators are derived. Asymptotic confidence bands for the time integrated $\alpha(t,z)$ and a Kolmogorov-Smirnov-type test of equality of α at different levels of the covariate are given.

1. Introduction.

A useful way of modelling the dependence of a counting process N on a covariate process Z was given by Aalen (1975). In his multiplicative intensity model N is supposed to have an intensity λ given by

$$\lambda(t) = \alpha(t)\, Z(t),$$

where α is an unknown, deterministic function of time (the hazard function). In the present paper we study inference for counting processes with intensities having general dependence on a covariate process Z, as in

$$\lambda(t) = Y(t)\, \alpha(t, Z(t)), \tag{1.1}$$

where α is an unknown, deterministic function of *both* time and the covariate process Z. The covariate process Y is taken to be an indicator process, assuming the value 1 when the counting process is under observation, zero otherwise.

An important example of our model arises in survival analysis. Suppose that the conditional hazard function $h(t\,|Z)$ for the survival time T of an individual given the covariate process Z has the form $h(t\,|Z) = \alpha(t, Z(t))$. The observable portion of the individual's lifetime is given by $\tilde{T} = \min(T, C)$, where C is a censoring time. We observe $\tilde{T}$, $\delta = I(T \le C)$ and $Z(t)$ for $t \le \tilde{T}$. Let $N(t) = I(\tilde{T} \le t, \delta = 1)$, the counting process with a single jump at an uncensored survival time. If T and C are conditionally independent given Z then N has intensity (1.1), where $Y(t) = I(\tilde{T} \ge t)$ is the indicator that the individual is "at risk" at time t. We shall introduce an estimator $\hat{A}$ for the time-integrated conditional hazard function $A(\cdot, z) = \int_0^{\cdot} \alpha(s, z)\, ds$ which, in the special case of a time-independent covariate Z, coincides with an estimator proposed in unpublished work of Beran (1981). Dabrowska (1987) recently obtained a weak convergence result for Beran's estimator by proving a "conditional" analogue of Theorem 4 of Breslow and Crowley (1974). We obtain asymptotic results for our estimator by using a martingale approach (in particular, Rebolledo's martingale central limit theorem) which enables us to give quite simple proofs and to avoid the restrictive assumption of time-independent covariate Z.

For another example of our model, consider a pure jump process describing the motion of a particle on a finite state space $\{1, 2, ..., m\}$. Let the intensity $\alpha_{ij}(t, s)$ of transition from state i to state j depend on the (calendar) time t and on the time s spent in state i since the last jump. Let $Y_i(t)$ be the indicator that the particle is in state i at time t. Then the counting process $N_{ij}(t)$ which registers the number of transitions from state i to state j up to time t has intensity

$$\lambda(t) = Y_i(t-)\, \alpha_{ij}(t, L(t-)), \tag{1.2}$$

where $L(t)$ is the length of time which at time t has elapsed since the last jump. In the terminology of Markov renewal processes (Pyke,1961), $L(t)$ is the backward recurrence time. In the case that

α_{ij} only depends on calendar time t, inference for α_{ij} has been studied by Aalen (1975, 1978). In the case that α_{ij} only depends on the backward recurrence time $L(t)$, N_{ij} is a Markov renewal process for which inference has been studied by Gill (1980).

The results of the paper will be developed for the nonlinear semimartingale regression model

$$X(t) = X(0) + \int_0^t Y(t)\,\alpha(s, Z(s))ds + M(t), \tag{1.3}$$

where M is a square integrable martingale and Y, Z, α are as before. This includes diffusion processes as well as the counting processes mentioned above. An estimator $\hat{A}(t,z)$ of the time-integrated conditional "hazard" function

$$A(t,z) = \int_0^t \alpha(s,z)\,ds$$

will be introduced. For counting processes our estimator $\hat{A}$ coincides with the Nelson-Aalen estimator if the covariate process Z is constant. A kernel estimator $\hat{\alpha}$ of α will be obtained from $\hat{A}$, as was done by Ramlau-Hansen (1983) for the Nelson-Aalen estimator.

The estimators $\hat{\alpha}$ and $\hat{A}$ are defined in Section 2. Consistency and asymptotic distribution results for $\hat{\alpha}$ and $\hat{A}$ are given in Section 3. In Section 4 we derive confidence bands for $A(\cdot, z)$ at any fixed level z of the covariate and introduce a Kolmogorov-Smirnov type statistic for testing the hypothesis that $A(\cdot, z_1)$ and $A(\cdot, z_2)$ coincide (equivalently $\alpha(\cdot, z_1)$ and $\alpha(\cdot, z_2)$ coincide) at different levels z_1, z_2. Technical lemmas used in the proofs of the main results are given in Section 5.

2. The Estimators.

$(\Omega, \mathcal{F}, P)$ will denote a complete probability space and $(\mathcal{F}_t, t \in [0,1])$ a nondecreasing right-continuous family of sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F}_0$ contains all P-null sets in $\mathcal{F}$. All processes are indexed by $t \in [0,1]$. The process $M = (M(t), \mathcal{F}_t)$ is assumed to be a square integrable martingale with mean zero and paths which are right-continuous on $[0,1)$ with left limits on $(0,1]$. Suppose that the quadratic characteristic $\langle M \rangle$ of M has the form

$$\langle M \rangle(t) = \int_0^t \gamma(s, Z(s), Y(s))\,ds, \tag{2.1}$$

where γ is a bounded, measurable function. The covariate processes Y and Z are assumed to be predictable and Y is an indicator process. For simplicity, Z is supposed to be scalar valued. We assume that the processes X, Y, Z and M are related by the equation (1.3) which can be written in differential form

$$dX(t) = Y(t)\,\alpha(t, Z(t))\,dt + dM(t), \tag{2.2}$$

where α is a bounded Borel function. In the counting process case

$$\gamma(t, Z(t), Y(t)) = Y(t)\,\alpha(t, Z(t)).$$

In the diffusion process case (without censoring) we have $Y(t) \equiv 1,\ Z(t) = X(t)$,

$$M(t) = \int_0^t \sigma(s, X(s))\,dW(s),$$

$$\gamma(t, Z(t), Y(t)) = \sigma^2(t, X(t)),$$

where W is a Wiener process, $\sigma^2(t,z)$ is the infinitesimal variance of the diffusion and $\alpha(t,z)$ is the drift function.

In order to define the estimators $\hat{\alpha}$ and $\hat{A}$ we need the following notation. For $z \in \mathbf{R}$, I_z denotes an interval of length w_n containing z, where $w_n \to 0$ as $n \to \infty$. Let $(X_i, Y_i, Z_i, M_i), i = 1, \ldots, n$ denote n independent copies of the generic processes X, Y, Z, M which satisfy model (2.1), (2.2). Assume that X_i and Y_i are observable continuously over the time interval [0,1] and $Z_i(t)$ is observable at least when $Y_i(t) \neq 0$. Define

$$X^{(n)}(t,z) = \sum_{i=1}^n \int_0^t I\{Z_i(s) \in I_z\} Y_i(s)\,dX_i(s), \tag{2.3}$$

$$Y^{(n)}(t,z) = \sum_{i=1}^n I\{Z_i(t) \in I_z\}\, Y_i(t). \tag{2.4}$$

As an estimator of A we propose

$$\hat{A}(t,z) = \int_0^t \frac{1}{Y^{(n)}(s,z)}\, X^{(n)}(ds, z),$$

where $1/0 \equiv 0$. Also, for $t \in (0,1)$ set

$$\hat{\alpha}(t,z) = \frac{1}{b_n} \int_0^1 K\Big(\frac{t-s}{b_n}\Big)\, \hat{A}(ds, z),$$

where K is a bounded, nonnegative kernel function with compact support, integral 1 and $b_n > 0$ is a bandwidth parameter, $b_n \to 0$.

We note that for the above estimation of A at a fixed z the processes X, Z only need to be observed at times when Z belongs to the neighborhood I_z of z. We can show that $\hat{A}$ and $\hat{\alpha}$ yield asymptotically well behaved estimators of A and α when we shrink I_z (i.e. let $w_n \to 0$) and let $b_n \to 0$ at appropriate rates as the sample size increases. If K is left continuous and of bounded variation, then by integration by parts (see Dellacherie and Meyer (1982), Chapter VIII, (19.4)) for n sufficiently large

$$\frac{1}{b_n} \int_O^1 K\Big(\frac{t-s}{b_n}\Big)\, \hat{A}(ds, z) = \frac{1}{b_n} \int_0^1 \hat{A}(s-, z)\, dK\Big(\frac{t-s}{b_n}\Big)$$

almost surely for all $t \in [t_1, t_2]$, where $0 < t_1 < t_2 < 1$. Thus, for n sufficiently large, we can choose a Lebesgue-Stieltjes version of the process $(\hat{\alpha}(t,z),\ t \in [t_1, t_2])$:

$$\hat{\alpha}(t,z) = \frac{1}{b_n} \int_0^1 \hat{A}(s-, z)\, dK\Big(\frac{t-s}{b_n}\Big). \tag{2.5}$$

This version of $\hat{\alpha}$ is used in Theorem 2(c).

3. Main Results.

We shall consider estimation of $A(t,z)$, $\alpha(t,z)$ over $0 \le t \le 1$, $0 \le z \le 1$. Let C be a set in $\mathbf{R}$ containing $\cup_{z \in [0,1]} I_z^{(n)}$ for some $n \ge 1$. The following assumptions are supposed to hold for all (t,z) belonging to $[0,1] \times C$.

(A1) For each t, the random vector $(Z(t), Y(t))$ is absolutely continuous with respect to the product of the Lebesgue and counting measure. Denote the corresponding density by $f_{Z(t)Y(t)}(z,y)$. Also, suppose that for fixed z, y this density is integrable in t.

(A2) $f_{Z(t)Y(t)}(z,1)$ is bounded away from zero.

(A3) $f_{Z(t)Y(t)}(z,1)$ is continuous as a function of t and z for each fixed y.

(B1) α, γ are continuous functions of t and z for each fixed y.

(B2) α is Lipschitz, i.e. there exists a constant K such that

$$|\alpha(t_1, z_1) - \alpha(t_2, z_2)| \le K \|(t_1 - t_2, z_1 - z_2)\|$$

for all t_1, t_2, z_1, z_2, where $\|\cdot\|$ denotes the Euclidian norm on $\mathbf{R}^2$.

THEOREM 1. (a) Suppose that A1-A3, B1 hold and $nw_n \to \infty$ as $n \to \infty$. Then

$$\sup_z E \sup_t |\hat{A}(t,z) - A(t,z)|^2 \to 0 \tag{3.1}$$

as $n \to \infty$.

(b) Suppose, in addition, that B2 holds and $nw_n^3 \to 0$ as $n \to \infty$. Then

$$nw_n E|\hat{A}(t,z) - A(t,z)|^2 \to \int_0^t h(s,z)\, ds \tag{3.2}$$

uniformly over $(t,z) \in [0,1]^2$ as $n \to \infty$ and

$$\limsup_{n\to\infty} nw_n E \sup_t |\hat{A}(t,z) - A(t,z)|^2 \le 4 \int_0^1 h(s,z)\, ds \tag{3.3}$$

uniformly over $z \in [0,1]$ as $n \to \infty$, where

$$h(s,z) = \frac{\gamma(s,z,1)}{f_{Z(s)Y(s)}(z,1)}. \tag{3.4}$$

Proof. Define

$$M^{(n)}(t,z) = \sum_{i=1}^{n} \int_0^t I\{Z_i(s) \in I_z\}\, Y_i(s)\, dM_i(s)$$

$$\alpha^{(n)}(t,z) = \sum_{i=1}^{n} I\{Z_i(t) \in I_z\}\, Y_i(t)\, \alpha(t, Z_i(t))$$

$$\gamma^{(n)}(t,z) = \sum_{i=1}^{n} I\{Z_i(t) \in I_z\}\, Y_i(t)\, \gamma(t, Z_i(t), Y_i(t)).$$

It follows from (2.1)-(2.4) that

$$X^{(n)}(dt,z) = \alpha^{(n)}(t,z)\, dt + M^{(n)}(dt,z)$$

$$d\langle M^{(n)}(\,\cdot\,,z)\rangle(t) = \gamma^{(n)}(t,z)\, dt.$$

The Doob-Meyer decomposition of $\hat{A}$ is

$$\hat{A}(t,z) = \int_0^t \frac{\alpha^{(n)}(s,z)}{Y^{(n)}(s,z)}\, ds + \int_0^t \frac{1}{Y^{(n)}(s,z)}\, M^{(n)}(ds,z). \tag{3.5}$$

Therefore

$$|\hat{A}(t,z) - A(t,z)|^2 = I_1(t) + I_2(t) + I_3(t),$$

where

$$I_1(t) = \Big(\int_0^t \frac{\alpha^{(n)}(s,z)}{Y^{(n)}(s,z)}\, ds - \int_0^t \alpha(s,z)\, ds\Big)^2$$

$$I_2(t) = \Big(\int_0^t \frac{1}{Y^{(n)}(s,z)}\, M^{(n)}(ds,z)\Big)^2$$

$$I_3(t) = 2I(t_1)I(t_2).$$

Now

$$E \sup_t I_1(t) \le \int_0^1 E\left|\frac{\alpha^{(n)}(s,z)}{Y^{(n)}(s,z)} - \alpha(s,z)\right|^2 ds$$

and by Lemmas 4 and 5

$$E\left[\frac{\alpha^{(n)}(s,z)}{Y^{(n)}(s,z)} - \alpha(s,z)\right]^2 = o(1)$$

uniformly in s, z as $n \to \infty$ if α is continuous. Also

$$nw_n E\left[\frac{\alpha^{(n)}(s,z)}{Y^{(n)}(s,z)} - \alpha(s,z)\right]^2 = o(1) \tag{3.6}$$

uniformly in s, z as $n \to \infty$ if α is Lipschitz and $n\,w_n^3 \to 0$. Next, by Doob's inequality

$$nw_n E \sup_t I_2(t) \le 4nw_n E\left[\int_0^1 \frac{1}{Y^{(n)}(s,z)}\, M^{(n)}(ds,z)\right]^2 = 4nw_n \int_0^1 E\,\frac{\gamma^{(n)}(s,z)}{\left(Y^{(n)}(s,z)\right)^2}\, ds,$$

where the r.h.s. of the last equality tends to $4\int_0^1 h(s,z)ds$ uniformly in z as $n \to \infty$ by Lemma 6. This proves (3.1),(3.3). To show (3.2) we observe that

$$nw_n E\, I_2(t) = nw_n E\left[\int_0^t \frac{1}{Y^{(n)}(s,z)}\, M^{(n)}(ds,z)\right]^2 = nw_n \int_0^1 E\,\frac{\gamma^{(n)}(s,z)}{\left(Y^{(n)}(s,z)\right)^2}\, ds.$$

This completes the proof.

The next result can be viewed as the analogue of Theorem 1 for the estimator $\hat{\alpha}$.

THEOREM 2. (a) Suppose that A1-A3, B1 hold and $b_n \sim w_n$, $nw_n^2 \to \infty$ as $n \to \infty$. Then

$$E\left[\hat{\alpha}(t,z) - \alpha(t,z)\right]^2 \to 0$$

for every $t \in (0,1)$ uniformly in z as $n \to \infty$ and

$$\int_0^1 E\left[\hat{\alpha}(t,z) - \alpha(t,z)\right]^2 dt \to 0$$

uniformly in z as $n \to \infty$.
(b) Suppose, in addition, that B2 is satisfied and $nw_n^4 \to 0$ as $n \to \infty$. Then

$$nw_n^2 E\left[\hat{\alpha}(t,z) - \alpha(t,z)\right]^2 \to \kappa h(s,z)$$

for every $t \in (0,1)$ uniformly in z as $n \to \infty$ and

$$nw_n^2 \int_0^1 E\left[\hat{\alpha}(t,z) - \alpha(t,z)\right]^2 dt \to \kappa \int_0^1 h(t,z)dt$$

uniformly in z as $n \to \infty$, where $\kappa = \int_{-\infty}^{\infty} K^2(u)\,du$.
(c) Suppose A1-A3, B1, B2 hold, K is left continuous, of bounded variation and $nw_n \to \infty$, $nw_n^3 \to 0$, $nw_n b_n^2 \to \infty$. Let $0 < t_1 < t_2 < 1$. Then, for the version of $\hat{\alpha}$ given by (2.5),

$$\sup_z E\Big(\sup_{t \in [t_1,t_2]} |\hat{\alpha}(t,z) - \alpha(t,z)|\Big) = O\Big(\frac{1}{\sqrt{nw_n}\, b_n}\Big) + O(b_n).$$

Proof. From the decomposition (3.5) it follows that

$$[\hat{\alpha}(t,z) - \alpha(t,z)]^2 = [I_1(t)]^2 + [I_2(t)]^2 + 2[I_1(t)I_2(t)],$$

where

$$I_1(t) = \frac{1}{b_n}\int_0^1 K\Big(\frac{t-s}{b_n}\Big)\frac{1}{Y^{(n)}(s,z)}\, M^{(n)}(ds,z)$$

$$I_2(t) = \frac{1}{b_n}\int_0^1 K\Big(\frac{t-s}{b_n}\Big)\frac{\alpha^{(n)}(s,z)}{Y^{(n)}(s,z)}\, ds - \alpha(t,z).$$

Now

$$nw_n^2 E\,[I_1(t)]^2 = nw_n^2\, E\int_0^1 \Big(\frac{1}{b_n} K\Big(\frac{t-s}{b_n}\Big)\Big)^2 \frac{\gamma^{(n)}(s,z)}{(Y^{(n)}(s,z))^2}\, ds.$$

Under the hypothesis of part (a) it follows from Lemma 1 and Lemma 3 that $nw_n^2 E\,[I_1(t)]^2$ is uniformly bounded, hence $E\,[I_1(t)]^2 \to 0$ and $\int_0^1 E\,[I_1(t)]^2\, dt \to 0$ as $n \to \infty$. Under the hypothesis of part (b) it follows from Lemma 8 that $nw_n^2 E\,[I_1(t)]^2 \to \kappa\, h(t,z)$ for all $t \in (0,1)$ uniformly in z as $n \to \infty$. The bounded convergence theorem implies that $nw_n^2 \int_0^1 E\,[I_1(t)]^2\, dt \to \kappa \int_0^1 h(t,z)\, dt$ uniformly in z as $n \to \infty$. By the Cauchy-Schwarz inequality, $E\,|I_1(t)I_2(t)| \le \{E\,[I_1(t)]^2\, E\,[I_2(t)]^2\}^{1/2}$, the proof of parts (a), (b) will be complete if we we show that

$$E\,[I_2(t)]^2 = \begin{cases} o(1) & \text{under the conditions of part (a) for all } t \in (0,1) \text{ uniformly in } z \\ O(w_n^2) & \text{under the conditions of part (b) for all } t \in (0,1) \text{ uniformly in } z \end{cases}$$

and similarly for $\int_0^1 E\,[I_2(t)]^2\, dt$. This is done in Lemma 7. To prove (c) we observe that $\hat{\alpha}(t,z) - \alpha(t,z) = I_3(t,z) + I_4(t,z)$, where

$$I_3(t,z) = \frac{1}{b_n}\int_0^1 (\hat{A}(s-,z) - A(s-,z))\, dK\Big(\frac{t-s}{b_n}\Big)$$

$$I_4(t,z) = \frac{1}{b_n}\int_0^1 A(s-,z)\, dK\Big(\frac{t-s}{b_n}\Big) - \alpha(t,z).$$

Now

$$|I_3(t,z)| \le (2/b_n)\, V(K) \sup_{0\le s\le 1} |\hat{A}(s-,z) - A(s-,z)|,$$

where $V(K)$ is the total variation of K. Thus, by Theorem 1(b)

$$\sup_z E \sup_{0\le t\le 1} |I_3(t,z)| = O\Big(\frac{1}{\sqrt{nw_n}\, b_n}\Big).$$

By integration by parts, for n sufficiently large

$$I_4(t,z) = \frac{1}{b_n}\int_0^1 K\Big(\frac{t-s}{b_n}\Big)\,A(ds,z) - \alpha(t,z) = \frac{1}{b_n}\int_0^1 K\Big(\frac{t-s}{b_n}\Big)\,\alpha(s,z)\,ds - \alpha(t,z).$$

Therefore, by B2, $|I_4(t,z)| = O(b_n)$ uniformly in $(t,z) \in [t_1,t_2] \times [0,1]$. This completes the proof of the theorem.

REMARK. The conditions on w_n, b_n of part (c) of Theorem 2 are satisfied for the sequences $b_n = n^{-\beta}$ and $w_n = n^{-\delta}$, where $1/3 < \delta < 1$ and $0 < \beta < (1-\delta)/2$.

Asymptotic distribution results for $\hat{A}$ will be established under additional assumptions on X that will make functional central limit theorems for martingales easily applicable to the martingale part of (3.5). In the sequel $D[0,1]$ denotes the space of real valued functions on $[0,1]$ which are right continuous on [0,1) and whose left limits exist on $(0,1]$, equipped with the Skorohod topology. Also, $D[0,1]^l$ denotes the product space of ι copies of $D[0,1]$. For an account of weak convergence in $D[0,1]$ we refer to Billingsley (1968).

THEOREM 3. Suppose that X has continuous sample paths or is a counting process, A1-A3, B1-B2 hold and $nw_n \to \infty$, $nw_n^3 \to 0$ as $n \to \infty$. Then for $z_1, ..., z_l \in [0,1]$, all distinct, the process

$$\big(\sqrt{nw_n}(\hat{A}(t,z_r) - A(t,z_r)), t \in [0,1]\big)_{r=1}^{l}$$

converges weakly in $D[0,1]^l$ as $n \to \infty$ to the Gaussian process

$$\big(U(t,z_r), t \in [0,1]\big)_{r=1}^{l}$$

with zero mean and covariance function

$$\mathrm{Cov}(U(t_1,z_{r_1}), U(t_2,z_{r_2})) = \delta_{r_1,r_2}\int_0^{t_1\wedge t_2} h(s,z_{r_1})\,ds$$

(where δ_{r_1,r_2} denotes the Kronecker symbol).

Proof. By decomposition (3.5)

$$\sqrt{nw_n}(\hat{A}(t,z_r) - A(t,z_r)) = \sqrt{nw_n}(I_3(t) + I_4(t)),$$

where

$$I_3(t) = \int_0^t \frac{\alpha^{(n)}(s,z_r)}{Y^{(n)}(s,z_r)}\,ds - \int_0^t \alpha(s,z_r)\,ds,$$

$$I_4(t) = \int_0^t \frac{1}{Y^{(n)}(s,z_r)}\,M^{(n)}(ds,z_r).$$

Observe that

$$nw_n E \sup_t (I_3(t))^2 \leq nw_n \int_0^1 E\left[\frac{\alpha^{(n)}(s,z_r)}{Y^{(n)}(s,z_r)} - \alpha(s,z_r)\right]^2 ds \to 0$$

as $n \to \infty$ by (3.6). Therefore it will be sufficient to show that

$$(\sqrt{nw_n} I_4(t))_{r=1}^l \to (U(t,z_r))_{r=1}^l \tag{3.7}$$

weakly in $D[0,1]^l$ as $n \to \infty$. If X has continuous sample paths, so does the square integrable martingale $I_4(t)$ and by Liptser and Shiryayev (1980) it will be sufficient to show that

$$nw_n \langle I_4 \rangle(t) \xrightarrow{P} \int_0^t h(s,z)\, ds \tag{3.8}$$

for all t, z_r as $n \to \infty$. Since

$$\left\langle \int_0^{\cdot} \frac{1}{Y^{(n)}(s,z_r)} M^{(n)}(ds,z_r) \right\rangle(t) = \int_0^t \frac{\gamma^{(n)}(s,z_r)}{(Y^{(n)}(s,z_r))^2}\, ds,$$

(3.8) follows from Lemma 6. If X is a counting process, by Rebolledo (1978) we will have to verify, in addition, the Lindeberg condition

$$nw_n \int_0^1 \frac{\gamma^{(n)}(s,z_r)}{(Y^{(n)}(s,z_r))^2} I\{\sqrt{nw_n} \frac{1}{Y^{(n)}(s,z_r)} > \epsilon\}\, ds \xrightarrow{P} 0,$$

for all $\epsilon > 0$. But by application of the Cauchy-Schwarz inequality

$$E\, nw_n \frac{\gamma^{(n)}(s,z_r)}{(Y^{(n)}(s,z_r))^2} I\{\sqrt{nw_n} \frac{1}{Y^{(n)}(s,z_r)} > \epsilon\}$$

$$\leq \left\{E\left[nw_n \frac{1}{Y^{(n)}(s,z_r)}\right]^4\right\}^{1/2} \left\{E\left[\frac{\gamma^{(n)}(s,z_r)}{nw_n}\right]^4 P[\sqrt{nw_n} \frac{1}{Y^{(n)}(s,z_r)} > \epsilon]\right\}^{1/4}.$$

By Corollary 1 and Lemma 3 the r.h.s. above tends to zero for all z_r if for all s, z_r

$$P[\sqrt{nw_n} \frac{1}{Y^{(n)}(s,z_r)} > \epsilon] \to 0.$$

This follows from Lemma 3 and the Markov inequality. So far we have only shown weak convergence in $D[0,1]$ for each z_r. Note that $(I_4(t))_{r=1}^l$ is a vector of square integrable martingales that are orthogonal whenever $I_{z_{r_1}} \cap I_{z_{r_2}} = \emptyset$, which is true for sufficiently large n. Therefore (3.7) follows from the previous arguments by application of the Cramér-Wold device.

The next theorem gives the asymptotic finite dimensional distributions of the estimator $\hat{\alpha}$.

THEOREM 4. Suppose that X has continuous sample paths or is a counting process, A1-A3, B1, B2 hold and $b_n \sim w_n$, $nw_n^2 \to \infty$, $nw_n^4 \to 0$ as $n \to \infty$. Then for all $z_1, \ldots, z_l \in [0,1], t_1, \ldots, t_k \in (0,1)$, all distinct,

$$(\sqrt{n}w_n(\hat{\alpha}(t_j, z_r) - \alpha(t_j, z_r)))_{j=1,r=1}^{k,l}$$

converges in distribution to the Gaussian random array $(V(t_j, z_r))_{j=1,r=1}^{k,l}$ with mean zero and covariance

$$\mathrm{Cov}(V(t_{j_1}, z_{r_1}), V(t_{j_2}, z_{r_2})) = \delta_{j_1,j_2}\delta_{r_1,r_2}\,\kappa h(t_{j_1}, z_{r_1}).$$

Proof. By decomposition (3.5)

$$\sqrt{n}w_n(\hat{\alpha}(t_j, z_r) - \alpha(t_j, z_r)) = \sqrt{n}w_n(I_5 + I_6),$$

where

$$I_5 = \frac{1}{b_n}\int_0^1 K\Big(\frac{t_j - s}{b_n}\Big)\frac{\alpha^{(n)}(s, z_r)}{Y^{(n)}(s, z_r)}\,ds - \alpha(t_j, z_r),$$

$$I_6 = \frac{1}{b_n}\int_0^1 K\Big(\frac{t_j - s}{b_n}\Big)\frac{1}{Y^{(n)}s, z_r)}\,M^{(n)}(ds, z_r).$$

It follows from Lemma 7 that $\sqrt{n}w_n I_5 \xrightarrow{P} 0$ as $n \to \infty$. Therefore it will be sufficient to show that

$$(\sqrt{n}w_n I_6)_{j=1,r=1}^{k\,l} \xrightarrow{\mathcal{D}} (V(t_j, z_r))_{j=1,r=1}^{k,l}.$$

Now $I_6 = I_6(1)$, where

$$I_6(\tau) = \int_0^\tau \frac{1}{b_n}K\Big(\frac{t_j - s}{b_n}\Big)\frac{1}{Y^{(n)}(s, z_r)}\,M^{(n)}(ds, z_r),$$

and $(I_6(\tau))_{j=1,r=1}^{k,l}$ is an array of $k\,l$ square integrable martingales that are orthogonal for n sufficiently large. If X has continuous sample paths it follows from Remark 2 in Liptser and Shiryayev (1980) and the Cramér-Wold device that we only need to show $nw_n{}^2\langle I_6\rangle(1) \xrightarrow{P} \kappa h(t_j, z_r)$ as $n \to \infty$ for all t_j, z_r. This is done in Lemma 8. If X is a counting process we have to verify, in addition, that

$$nw_n^2\int_0^1\Big(\frac{1}{b_n}K\Big(\frac{t-s}{b_n}\Big)\Big)^2\frac{\gamma^{(n)}(s,z)}{(Y^{(n)}(s,z))^2}\,I\Big\{\sqrt{n}w_n\frac{1}{b_n}K\Big(\frac{t-s}{b_n}\Big)\frac{1}{Y^{(n)}(s,z)} > \epsilon\Big\}\,ds \xrightarrow{P} 0$$

as $n \to \infty$ for all $\epsilon > 0$. As in the proof of Theorem 3 this follows from

$$P[\sqrt{n}w_n\frac{1}{b_n}K\Big(\frac{t-s}{b_n}\Big)\frac{1}{Y^{(n)}(s,z)} > \epsilon] \le P[\sqrt{n}\frac{1}{Y^{(n)}(s,z)} > \tilde{\epsilon}]$$

(for some $\tilde{\epsilon} > 0$, since K is bounded)

$$= P[nw_n\frac{1}{Y^{(n)}(s,z)} > \tilde{\epsilon}\sqrt{n}w_n] \to 0$$

(by Lemma 3 and the Markov inequality).

This completes the proof of the theorem.

4. Confidence Bands and Hypothesis Tests.

In order to use the previous theorems for inference, an estimate of $h(t,z)$ (as defined in (3.4)) and of $\int_0^t h(s,z)\,ds = H(t,z)$ is needed. The following theorem provides consistent estimators for both of these quantities in the case that X is a counting process.

THEOREM 5. Suppose that A1-A3, B1, B2 hold, X is a counting process. Define

$$\hat{H}(t,z) = nw_n \int_0^t \frac{1}{\left(Y^{(n)}(s,z)\right)^2} X^{(n)}(ds,z),$$

for $t \in [0,1]$,

$$\hat{h}(t,z) = \frac{1}{b_n} \int_0^1 K\left(\frac{t-s}{b_n}\right) \hat{H}(ds,z),$$

for $t \in (0,1)$. If $b_n \sim w_n$ $nw_n^2 \to \infty$ as $n \to \infty$ then

$$E\,|\,\hat{h}(t,z) - h(t,z)| \to 0$$

for all $t \in (0,1)$ uniformly in $z \in [0,1]$ as $n \to \infty$. If $nw_n \to \infty$ then

$$E \sup_t |\hat{H}(t,z) - H(t,z)| \to 0$$

uniformly in $z \in [0,1]$ as $n \to \infty$.

Proof. By decomposition (3.5)

$$|\hat{h}(t,z) - h(t,z)| \le I_1(t) + I_2(t),$$

where

$$I_1(t) = \left| \frac{nw_n}{b_n} \int_0^1 K\left(\frac{t-s}{b_n}\right) \frac{\alpha^{(n)}(s,z)}{(Y^{(n)}(s,z))^2}\, ds - h(t,z) \right|,$$

$$I_2(t) = \left| \frac{nw_n}{b_n} \int_0^1 K\left(\frac{t-s}{b_n}\right) \frac{1}{(Y^{(n)}(s,z))^2} M^{(n)}(ds,z) \right|.$$

By Lemma 8 (with K^2 replaced by κK) and $b_n \sim w_n$ we have

$$\sup_z E\, I_1(t) \to 0.$$

Next

$$E\,[I_2(t)]^2 \le \frac{1}{{b_n}^2}\int_0^1 K^2\Big(\frac{t-s}{b_n}\Big)\, n^2{w_n}^2\, E\,\frac{\gamma^{(n)}(s,z)}{\big(Y^{(n)}(s,z)\big)^4}\, ds \to 0$$

for all $t \in (0,1)$ uniformly in z as $n \to \infty$ by Lemmas 1, 3. Similarly

$$|\hat{H}(t,z) - H(t,z)| = \Big|\, nw_n \int_0^t \frac{1}{\big(Y^{(n)}(s,z)\big)^2}\, X^{(n)}(ds,z) - \int_0^t h(s,z)\, ds\,\Big| \le I_3(t) + I_4(t),$$

where

$$I_3(t) = \Big|\, nw_n \int_0^t \frac{\alpha^{(n)}(s,z)}{\big(Y^{(n)}(s,z)\big)^2}\, ds - \int_0^t h(s,z)\, ds\,\Big|,$$

$$I_4(t) = \Big|\, nw_n \int_0^t \frac{1}{\big(Y^{(n)}(s,z)\big)^2}\, M^{(n)}(ds,z)\,\Big|.$$

But

$$\sup_z E \sup_t I_3(t) \le \int_0^1 \sup_{z,s} E\,\Big|\, nw_n \frac{\alpha^{(n)}(s,z)}{\big(Y^{(n)}(s,z)\big)^2} - h(s,z)\,\Big|\, ds \to 0$$

by Lemma 6 (with $\gamma^{(n)}$ replaced by $\alpha^{(n)}$) and

$$\sup_z E \sup_t\, [I_4(t)]^2 \le \int_0^1 \sup_{z,s} E\,\Big[(nw_n)^2 \frac{\gamma^{(n)}(s,z)}{\big(Y^{(n)}(s,z)\big)^4}\Big]\, ds \to 0$$

as $n \to \infty$ by Doob's inequality and Lemmas 1, 3.

In the diffusion process case, in which $\sigma^2(t,z)$ is assumed to be known, the following theorem provides consistent estimators for $h(t,z)$ and $H(t,z)$.

THEOREM 6. Suppose A1-A3, B1, B2 hold, X is a diffusion process. Define

$$\hat{H}(t,z) = nw_n \int_0^t \frac{\sigma^2(s,z)}{Y^{(n)}(s,z)}\, ds$$

for $t \in [0,1]$,

$$\hat{h}(t,z) = \frac{1}{b_n}\int_0^1 K\Big(\frac{t-s}{b_n}\Big)\, \hat{H}(ds\ \ z)$$

for $t \in (0,1)$. If $b_n \sim w_n$, $nw_n^2 \to \infty$ then

$$E\,|\hat{h}(t,z) - h(t,z)| \to 0$$

for all $t \in (0,1)$ uniformly in $z \in [0,1]$ as $n \to \infty$. If $nw_n \to \infty$ then

$$E \sup_t |\hat{H}(t,z) - H(t,z)| \to 0$$

uniformly in $z \in [0,1]$ as $n \to \infty$.

Proof. It follows from $b_n \sim w_n$ and Lemma 8 (where we replace $\gamma^{(n)}(s,z)$ by $\sigma^2(s,z)\, Y^{(n)}(s,z)$ and K^2 by $\kappa\, K$) that

$$E\,|\hat{h}(t,z) - h(t,z)| = E\left|\frac{nw_n}{b_n}\int_0^1 K\Big(\frac{t-s}{b_n}\Big)\,\frac{\sigma^2(s,z)}{Y^{(n)}(s,z)}\,ds - h(t,z)\right| \to 0$$

for all $t \in (0,1)$ uniformly in z. Next

$$E \sup_t |\hat{H}(t,z) - H(t,z)| \le \sup_{s,z} E\left|nw_n \frac{\sigma^2(s,z)}{Y^{(n)}(s,z)} - h(s,z)\right|,$$

which tends to zero by Lemma 6 with $\gamma^{(n)}(s,z)$ replaced by $\sigma^2(s,z)\, Y^{(n)}(s,z)$. This completes the proof of the theorem.

Confidence bands for $A(\cdot,z)$.

Under the conditions of Theorem 3

$$\sqrt{nw_n}\,\frac{\sqrt{H(1,z)}}{H(1,z)+H(t,z)}\Big(\hat{A}(t,z) - A(t,z)\Big) \to W^0\Big(\frac{H(t,z)}{H(1,z)+H(t,z)}\Big)$$

weakly in $D[0,1]$ as $n \to \infty$, where W^0 is the Brownian bridge process. By application of Theorem 5 in the counting process case or Theorem 6 in the diffusion process case we obtain the following asymptotic $100(1-\alpha)\%$ confidence band for $A(\cdot,z)$:

$$\hat{A}(t,z) \pm c_\alpha \sqrt{\frac{\hat{H}(1,z)}{nw_n}}\Big(1 + \frac{\hat{H}(t,z)}{\hat{H}(1,z)}\Big), \quad t \in [0,1],$$

where

$$P[\sup_{t\in[0,1/2]} |W^0(t)| > c_\alpha] = \alpha.$$

A table for the distribution of $\sup_{t\in[0,1/2]} |W^0(t)|$ can be found in Hall and Wellner (1980).

Testing equality of A at two different levels of the covariate.

We now introduce a test statistic for testing the null hypothesis $H_0 : A(t,z_1) = A(t,z_2)$ for all $t \in [0,1]$, where z_1, z_2 are two prechosen values of z. Define

$$A_{1\,2}(t) = A(t,z_1) - A(t,z_2)$$

$$\hat{A}_{12}(t) = \hat{A}(t, z_1) - \hat{A}(t, z_2)$$

$$H_{12}(t) = H(t, z_1) - H(t, z_2)$$

$$\hat{H}_{12}(t) = \hat{H}(t, z_1) - \hat{H}(t, z_2).$$

Then under the conditions of Theorem 3 we have that

$$\sqrt{nw_n}\frac{\sqrt{H_{12}(1)}}{H_{12}(1) + H_{12}(t)}(\hat{A}_{12}(t) - A_{12}(t)) \to W^0\Big(\frac{H_{12}(t)}{H_{12}(1) + H_{12}(t)}\Big)$$

weakly in $D[0,1]$ as $n \to \infty$. Set

$$\hat{\xi} = \sqrt{nw_n \hat{H}_{12}(1)} \sup_{t\in[0,1]}\left|\frac{\hat{A}_{12}(t) - A_{12}(t)}{\hat{H}_{12}(1) + \hat{H}_{12}(t)}\right|.$$

Then in the counting process and diffusion process cases considered above $\hat{\xi} \xrightarrow{\mathcal{D}} \xi$ as $n \to \infty$, where $\xi \stackrel{\mathcal{D}}{=} \sup_{t\in[0,1/2]} |W^0(t)|$. Therefore an asymptotic test of size α can be carried out by rejecting H_0 if and only if $\hat{\xi} > c_\alpha$, where $P(\xi > c_\alpha) = \alpha$. Finally we mention that Theorem 4 can be used to construct asymptotic χ^2-tests as in Rao (1973) for testing equality of α at any finite number of values of t and z.

5. Technical Lemmas.

LEMMA 1. Suppose that A1, A3, B1 hold and $nw_n \to \infty$ as $n \to \infty$. Then

$$E\left[\frac{1}{nw_n}\gamma^{(n)}(s,z)\right]^k = (g(s,z))^k + o(1) \tag{5.1}$$

for all nonnegative integers k, uniformly in s, z as $n \to \infty$, where

$$g(s,z) = f_{Z(s)\,Y(s)}(z,1)\,\gamma(s,z,1).$$

Proof. By the multinomial theorem

$$E\,[\gamma^{(n)}(s,z)]^k = \sum_{j_1+\cdots+j_n=k} \frac{k!}{j_1!\cdots j_n!}\prod_{i=1}^{n} E\,[I\{Z_i(s) \in I_z\}\,Y_i(s)\,\gamma(s, Z_i(s), Y_i(s))]^{j_i}.$$

Since $|Z_i(s) - z| < w_n$ implies $|\gamma(s, Z_i(s), 1) - \gamma(s, z, 1)| < \epsilon_n$ uniformly in s, z for some $\epsilon_n \to 0$ as $n \to \infty$, we have for $j_i \neq 0$

$$E\,[I\{Z_i(s) \in I_z\}\,Y_i(s)\,\gamma(s, Z_i(s), Y_i(s))]^{j_i} = E\,[I\{Z_i(s) \in I_z\}\,Y_i(s)\,(\gamma(s,z,1) + O(\epsilon_n)]^{j_i}$$

$$= (((\gamma(s,z,1))^{j_i} + O(\epsilon_n))\,E\,[I\{Z_i(s) \in I_z\}\,Y_i(s)] = I_1.$$

Also, by uniform continuity of f,

$$|E[I\{Z_i(s) \in I_z\}Y_i(s)] - w_n f_{Z(s)Y(s)}(z,1)| \le \int_{I_z} |f_{Z(s)Y(s)}(u,1) - f_{Z(s)Y(s)}(z,1)|\, du = w_n\, o(1)$$

uniformly in s and z. Therefore

$$I_1 = (\gamma(s,z,1))^{j_i} w_n f_{Z(s)Y(s)}(z,1) + w_n o(1) \tag{5.2}$$

uniformly in s, z and

$$E\left[\frac{\gamma^{(n)}(s,z)}{nw_n}\right]^k = \left(\frac{1}{nw_n}\right)^k \Big\{k! \binom{n}{k} \Big\{w_n (f_{Z(s)Y(s)}(z,1)\,\gamma(s,z,1) + o(1))\Big\}^k$$

$$+ \sum_{l=1}^{k-1} k! \binom{n}{l} \binom{k-1}{l-1} O(w_n^l)$$

$$= (g(s,z))^k + o(1).$$

COROLLARY 1. Suppose that A1, A3 hold and $nw_n \to \infty$ as $n \to \infty$. Then

$$E\left[\frac{\gamma^{(n)}(s,z)}{nw_n} - g(s,z)\right]^k \to 0 \tag{5.3}$$

as $n \to \infty$ for all $s, z, k \ge 1$,

$$\mathrm{Var}\left[\frac{\gamma^{(n)}(s,z)}{nw_n}\right] \to 0 \tag{5.4}$$

uniformly in s, z as $n \to \infty$ and

$$\mathrm{Var}\left[\frac{\gamma^{(n)}(s,z)}{nw_n}\right]^2 \to 0$$

uniformly in s, z as $n \to \infty$.

LEMMA 2. Suppose $X \sim$ binomial (n,p), $0 < p \le 1$. Let

$$X^* = \begin{cases} 1/X, & \text{if } X > 0; \\ 0, & \text{if } X = 0. \end{cases}$$

Then for each integer $k \ge 1$

$$E[X^*]^k \le \left(\frac{k+1}{np}\right)^k.$$

Proof.

$$E\,[X^{*}]^{k} = \sum_{i=1}^{n} \frac{1}{i^{k}} \, \frac{n!}{i!\,(n-i)!}\, p^{i}\, q^{n-i} = \sum_{i=1}^{n} \frac{1}{i^{k}} \, \frac{(i+1)\cdots(i+k)}{(i+k)!} \, \frac{n!}{(n-i)!}\, p^{i}\, q^{n-i}$$

$$\leq \sum_{i=1}^{n} (k+1)^{k} \frac{n!}{(i+k)!\,(n-i)!}\, p^{i}\, q^{n-i} = \frac{(k+1)^{k}\, n!}{p^{k}\,(n+k)!} \sum_{i=1}^{n} \frac{(n+k)!}{(i+k)!\,(n-i)!} p^{i+k} q^{n-i}$$

$$\leq \left(\frac{k+1}{n\,p}\right)^{k}.$$

LEMMA 3. Suppose A1, A2 hold and $n w_n \to \infty$ as $n \to \infty$. Then

$$E\left[\frac{n w_n}{Y^{(n)}(s,z)}\right]^{k} = O(1)$$

uniformly in s, z as $n \to \infty$ for every integer $k \geq 1$, where $1/0 \equiv 0$.

Proof. Set $m = \inf_{s,z} f_{Z(s)Y(s)}(z,1)$. Then $Y^{(n)}(s,z)$ has a binomial distribution with parameters $(n, p^{(n)}(s,z))$, where $p^{(n)}(s,z) \geq m w_n$, so the previous lemma applies.Therefore

$$E\left[\frac{n w_n}{Y^{(n)}(s,z)}\right]^{k} \leq \left(\frac{(k+1)\, n w_n}{n\, m w_n}\right)^{k}.$$

In the following lemma we will use the notation $J^{(n)}(s,z) = I\{Y^{(n)}(s,z) \neq 0\}$.

LEMMA 4. Suppose that A1, A2 hold and $n w_n \to \infty$ as $n \to \infty$. Then

$$E\,|1 - J^{(n)}(s,z)|^{k} \leq \exp\{-n w_n \inf_{s,z} f_{Z(s)Y(s)}(z,1)\}$$

for each integer $k \geq 1$.

Proof. Set $m = \inf_{s,z} f_{Z(s)Y(s)}(z,1)$. Then $m > 0$ and

$$E\,|1 - J^{(n)}(s,z)|^{k} = P\,(J^{(n)}(s,z) = 0) = (1 - P\,[Z(s) \in I_z,\, Y(s) = 1])^{n}$$

$$\leq (1 - m w_n)^{n} \leq \exp\{-m\, n w_n\}.$$

LEMMA 5.

$$\left|\frac{\alpha^{(n)}(s,z)}{Y^{(n)}(s,z)} - J^{(n)}(s,z)\,\alpha(s,z)\right| = \begin{cases} o(1) & \text{uniformly in } s,z \text{ if } \alpha \text{ is continuous} \\ O(w_n) & \text{uniformly in } s,z \text{ if } \alpha \text{ is Lipschitz.} \end{cases}$$

Proof. By definition of $\alpha^{(n)}(s,z)$

$$|\alpha^{(n)}(s,z) - \alpha(s,z)\,Y^{(n)}(s,z)| \le \epsilon^{(n)}\,Y^{(n)}(s,z),$$

where $\epsilon^{(n)} = o(1)\,(= O(w_n))$ uniformly is s,z if α is continuous (if α is Lipschitz).

LEMMA 6. Suppose that A1 - A3, B1 hold and $nw_n \to \infty$ as $n \to \infty$. Then

$$\sup_{s,z} E\left|nw_n \frac{\gamma^{(n)}(s,z)}{\left(Y^{(n)}(s,z)\right)^2} - h(s,z)\right| \to 0,$$

where

$$h(s,z) = \frac{\gamma(s,z,1)}{f_{Z(s)Y(s)}(z,1)}.$$

Proof.

$$E\left|nw_n \frac{\gamma^{(n)}(s,z)}{\left(Y^{(n)}(s,z)\right)^2} - h(s,z)\right| \le I_1 + I_2,$$

where

$$I_1 = E\left|nw_n \frac{\gamma^{(n)}(s,z)}{\left(Y^{(n)}(s,z)\right)^2} - J^{(n)}(s,z)\,h(s,z)\right|$$

$$I_2 = h(s,z)\,E\,|1 - J^{(n)}(s,z)|.$$

By Lemma 4, $I_2 \to 0$ uniformly in s,z as $n \to \infty$. Now by application of the Cauchy-Schwarz inequality

$$I_1 \le \left\{E\left[\frac{nw_n}{Y^{(n)}(s,z)}\right]^4\right\}^{1/2} \{I_3\}^{1/2},$$

where

$$I_3 = E\left[\frac{\gamma^{(n)}(s,z)}{nw_n} - \left(\frac{Y^{(n)}(s,z)}{nw_n}\right)^2 h(s,z)\right]^2.$$

Also $\frac{1}{2} I_3 \le I_4 + I_5$, where

$$I_4 = E\left[\frac{\gamma^{(n)}(s,z)}{nw_n} - g(s,z)\right]^2$$

$$I_5 = C^2 E\left[\left(\frac{Y^{(n)}(s,z)}{nw_n}\right)^2 - \left(f_{Z(s)Y(s)}(z,1)\right)^2\right]^2,$$

where $C = \sup_{s,z} h(s,z)$. But

$$I_4 = \mathrm{Var}\Big[\frac{\gamma^{(n)}(s,z)}{nw_n}\Big] + o(1)$$

uniformly in s, z as $n \to \infty$ by Lemma 1 and (5.4) and

$$I_5 = \mathrm{Var}\Big[\frac{Y^{(n)}(s,z)}{nw_n}\Big]^2 + o(1)$$

uniformly in s, z as $n \to \infty$ by Lemma 1 with $\gamma(s,z,y) \equiv 1$. Therefore $I_4 \to 0$ uniformly is s, z as $n \to \infty$ by Corollary 1, and $I_5 \to 0$ uniformly in s, z as $n \to \infty$ by Corollary 1 with $\gamma(s,z,y) \equiv 1$.

LEMMA 7. Suppose that A1, A2 hold and for some $\theta > 0$, $nw_n^{1+\theta} \to \infty$ as $n \to \infty$. Set

$$I(t) = E\left[\frac{1}{b_n}\int_0^1 K\Big(\frac{t-s}{b_n}\Big)\frac{\alpha^{(n)}(s,z)}{Y^{(n)}(s,z)}\,ds - \alpha(t,z)\right]^2.$$

Then

$$I(t) = \begin{cases} o(1) & \text{for all } t \in (0,1) \text{ uniformly in } z \text{ if } \alpha \text{ is continuous,} \\ O(w_n^2) & \text{for all } t \in (0,1) \text{ uniformly is } z \text{ if } \alpha \text{ is Lipschitz,} \end{cases}$$

and

$$\int_0^1 I(t)\,dt = \begin{cases} o(1) & \text{uniformly in } z \text{ if } \alpha \text{ is continuous,} \\ O(w_n^2) & \text{uniformly is } z \text{ if } \alpha \text{ is Lipschitz.} \end{cases}$$

Proof. $I(t) \le 3(I_1(t) + I_2(t) + I_3(t))$, where

$$I_1(t) = E\left[\frac{1}{b_n}\int_0^1 K\Big(\frac{t-s}{b_n}\Big)\left|\frac{\alpha^{(n)}(s,z)}{Y^{(n)}(s,z)} - J^{(n)}(s,z)\,\alpha(s,z)\right|ds\right]^2$$

$$I_2(t) = E\left[\frac{1}{b_n}\int_0^1 K\Big(\frac{t-s}{b_n}\Big)\,(1 - J^{(n)}(s,z))\,\alpha(s,z)\,ds\right]^2$$

$$I_3(t) = \left(\frac{1}{b_n}\int_0^1 K\Big(\frac{t-s}{b_n}\Big)\,\alpha(s,z)\,ds - \alpha(t,z)\right)^2.$$

By Lemma 5,

$$I_1(t) = \begin{cases} o(1) & \text{uniformly in } t, z \text{ if } \alpha \text{ is continuous,} \\ O(w_n^2) & \text{uniformly in } t, z \text{ if } \alpha \text{ is Lipschitz,} \end{cases}$$

which implies

$$\int_0^1 I_1(t)\,dt = \begin{cases} o(1) & \text{uniformly in } z \text{ if } \alpha \text{ is continuous,} \\ O(w_n^2) & \text{uniformly in } z \text{ if } \alpha \text{ is Lipschitz.} \end{cases}$$

Next

$$I_2(t) \le \frac{1}{b_n}\int_0^1 K^2\Big(\frac{t-s}{b_n}\Big)\,ds\,\Big(\sup_{s,z}\alpha(s,z)\Big)^2 \sup_{s,z}\frac{1}{b_n}E\Big(1-J^{(n)}(s,z)\Big)^2.$$

But $\sup_{s,z} E\left(1-J^{(n)}(s,z)\right)^2 = O(1/(nw_n))^k$ for all integers $k \ge 1$ by Lemma 4. This and $b_n \sim w_n$ imply $w_n{}^{-2} I_2(t) = O(1/(nw_n^{1+3/k}))^k$ for $3/k < \theta$ uniformly in t, z as $n \to \infty$. Thus $I_2(t) = O(w_n^2)$ and $\int_0^1 I_2(t)\,dt = O(w_n^2)$. Finally

$$I_3(t) = \begin{cases} o(1) & \text{for all } t \in (0,1) \text{ uniformly in } z \text{ if } \alpha \text{ is coninuous,} \\ O(w_n^2) & \text{for all } t \in (0,1) \text{ uniformly in } z \text{ if } \alpha \text{ is Lipschitz.} \end{cases}$$

Uniform boundedness of $I_3(t)$ and the dominated convergence theorem imply

$$\int_0^1 I_3(t)\,dt = \begin{cases} o(1) & \text{uniformly in } z \text{ if } \alpha \text{ is continuous,} \\ O(w_n^2) & \text{uniformly in } z \text{ if } \alpha \text{ is Lipschitz.} \end{cases}$$

This proves the lemma.

LEMMA 8. Suppose that A1-A3, B1 hold and $nw_n^2 \to \infty$ as $n \to \infty$. Then for each $\tau \in [0,1]$

$$\sup_z E\left|nw_n^2 \int_0^\tau \Big(\frac{1}{b_n}K\Big(\frac{t-s}{b_n}\Big)\Big)^2 \frac{\gamma^{(n)}(s,z)}{\left(Y^{(n)}(s,z)\right)^2}\,ds - \kappa(\tau,t)\,h(t,z)\right| \to 0$$

as $n \to \infty$ for all $t \in (0,1)$, where

$$\kappa(\tau,t) = \begin{cases} 0, & \text{if } \tau < t; \\ \int_{-\infty}^0 K^2(u)\,du, & \text{if } \tau = t; \\ \kappa, & \text{if } \tau > t. \end{cases}$$

Proof. For $\tau < t$ the theorem is obvious. Suppose $\tau \ge t$. Then

$$E\left|nw_n^2 \int_0^\tau \Big(\frac{1}{b_n}K\Big(\frac{t-s}{b_n}\Big)\Big)^2 \frac{\gamma^{(n)}(s,z)}{(Y^{(n)}(s,z))^2}\,ds - \kappa(\tau,t)\,h(t,z)\right| \le I_1(t) + I_2(t),$$

where

$$I_1(t) = E\,\frac{w_n}{b_n^2}\int_0^\tau K^2\Big(\frac{t-s}{b_n}\Big)\left|nw_n\frac{\gamma^{(n)}(s,z)}{\left(Y^{(n)}(s,z)\right)^2} - h(s,z)\right|ds$$

$$I_2(t) = \left|\frac{w_n}{b_n^2}\int_0^\tau K^2\Big(\frac{t-s}{b_n}\Big)\,h(s,z)\,ds - \kappa(\tau,t)\,h(t,s)\right|.$$

Note that $w_n/b_n \to 1$. Therefore $I_1(t) \to 0$ for all $t \in (0,1)$ uniformly in z as $n \to \infty$ by Lemma 6 and $I_2(t) \to 0$ for all $t \in (0,1)$ uniformly in z as $n \to \infty$ by continuity of h.

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ABSTRACT: Consider the simimartingale regression model

$$X(t) = X(0) + \int_0^t Y(s)\alpha(s, Z(s))ds + M(t),$$

Where Y, Z are observable convariate processes, α is a (deterministic) function of both, time and the covariate process Z, and M is a square integrable martingale. Under the assumption

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) that i.i.d. copies of $X < Y < Z$ are observed continuously over a finite time interval, inference for the function $\alpha(t,z)$ is investigated. An estimator $\hat{A}$ for the time integrated $\alpha(t,z)$ and a kernel estimator of $\alpha(t,z)$ itself are introduced. For X a counting process, $\hat{A}$ reduces to the Nelson- Aalen estimator when Z is not present in the model. Various form of consistency are proved, rates of convergence and asymptotic distributions of the estimators are derived. Asymptotic confidence bands for the time integrated $\alpha(t,z)$ and a Kolmogorv-Smirnov-type test of equality of α at different levels of the covariate are given.

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