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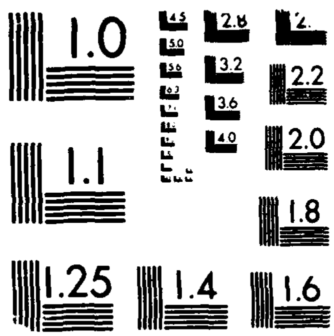

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Poisson Functionals of Markov Processes and Queueing Networks

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ABSTRACT

We present conditions under which a point process of certain jump times of a Markov process is a Poisson process. One result is that if the Markov process is stationary and the compensator of the point process in reverse time has a constant intensity a, then the point process is Poisson with rate a. A classical example is that the output flow from a M/M/1 queueing system is Poisson. We also present similar Poisson characterizations of more general marked point process functionals of a Markov process. These results yield easy-to-use criteria for a collection of such processes to be multi-variate Poisson or marked Poisson with a specified dependence or independence. We give several applications to queueing systems, and indicate how our results extend to functionals of non-Markovian processes.

Keywords and phrases: Poisson process, multivariate compound Poisson process, functionals of Markov processes, queueing networks, time reversal.

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1. Introduction

There are a variety of point processes associated with the jump times of a Markov process. For instance, in a Markovian network of queues, one might be interested in the point process of times at which units move between two sectors of the network. More generally, if the network has synchronous movements of items, one might be interested in the marked point process of the times at which batches of units move between two sectors and the numbers of units in the batches (the batch size being the "mark" of the time of the movement). One can formulate such a point process as a functional of the Markov process representing the network. The typical aim is to describe the behavior of the point process in terms of the characteristic of the Markov process. Some immediate questions in this regard are: Is such a point process Poisson (or marked Poisson)? Is a collection of these point processes multi-variate Poisson (or marked Poisson); and what are the dependencies, if any, among them?

These are the issues this study addresses. We begin in Section 2 by presenting conditions under which a point process of certain jump times of a Markov process is a Poisson process. It is well known that a simple point process on the real line is Poisson with rate a if its compensator has the constant intensity a (Theorem 2.3(i)). We present a reverse-time version of this (Theorem 2.3(ii)). It says that if the Markov process is stationary and the compensator of the point process in reverse time has the constant intensity a, then the point process is Poisson with rate a.

This is an easy-to-use criterion for establishing whether a point process of jump times is Poisson. We also give necessary conditions for this Poissonness. In Section 3, we present similar conditions under which more general functionals of a pure jump Markov process are marked Poisson processes. Some of our results overlap those of Melamed (1979), Brémaud (1981), Variaya and Walrand (1981), and Disney and Kiessler (1987). Melamed and Disney and Kiessler derive their results using a Markov renewal argument and Brémaud, Variaya and Walrand use a filtering argument. We use a simpler approach based on the notion of the time reversibility of the compensator of a point process. This approach lays bare the characteristics underlying the Poisson property, and it readily extends to the more general settings in Sections 3-5.

The applications in this area have been primarily for queueing systems. Burke (1956) and Reich (1957) showed that, in a stationary M/M/1 queueing system, the output flow is a Poisson process with the same rate as the Poisson input flow. Similarly, the exit flows from the queues in a Jackson network are independent Poisson processes; this is discussed in the references in the preceding paragraphs and in Kelly (1979), Disney and König (1985) and Whittle (1986). In Sections 2-4, we discuss applications identifying Poisson, compound Poisson and multivariate-Poisson flows in single queues and in queueing networks with dependent nodes. We end in Section 5 by indicating how our results extend to Markov processes with general state spaces and to functionals of semi-Markovian processes.

2. Jump Times of a Markov Process that Form a Poisson Process

Let $X = \{X_t : t \in R\}$ be a Markov process with a countable state space $\mathcal{S}$ and transition rates

$$q(x,y) = \lim_{t \downarrow 0} P\{X_t = y \mid X_0 = x\}/t, \qquad x \neq y,$$

and $q(x,x) = 0$. We indicate later how our results extend to a general state space. We adopt the standard assumption that

$$q(x) := \sum_y q(x,y) < \infty, \; x \in \mathcal{S},$$

and that each sample path of X is right continuous and has a finite number of jumps in any finite time period. Then the sojourn time of X in a state x is exponential with mean $q(x)^{-1}$ and, at the end of the sojourn, X jumps to some state y with probability $q(x,y)/q(x)$, $y \in \mathcal{S}$. For convenience, we assume that X is irreducible.

We shall study the point process N on R defined by

$$N(A) = \sum_{t \in A} f(X_{t-}, X_t) \tag{2.1}$$

where A is a Borel set in R and $f: \mathcal{S} \times \mathcal{S} \to \{0,1\}$ with $f(x,x) = 0$, $x \in \mathcal{S}$. The N(A) is the number of jumps of X from some x to some y for which $f(x,y) = 1$ in the time period A. Any such f is an indicator function $f(x,y) = 1((x,y) \in S)$ of a subset S of $\mathcal{S} \times \mathcal{S}$ that does not contain pairs of identical values; the N(A) would then be the number of transitions of X in the period A that take place in the transition set S. Clearly N(A) is finite when A is bounded and it may be infinite when A is unbounded.

We shall frequently use the function

$$\alpha(x) = \sum_y q(x,y) f(x,y), \qquad x \in \mathcal{S}. \tag{2.2}$$

First note that the mean measure of N is

(2.3) $$EN(A) = \int_A E\alpha(X_t)dt$$

which is Levy's formula (see for instance Benveniste and Jacod (1973)). When X is stationary, an easy check shows that N is stationary (i.e. for any $A_1,\ldots,A_n$, the distribution of $N(A_1+t),\ldots, N(A_n+t)$ is independent of t). Consequently, $EN(s,t] = a(t-s)$, $s < t$ in R, where

(2.4) $$a = EN(0,1] = \sum_{x,y} \pi(x)q(x,y)f(x,y)$$

and $\pi(x) = P\{X_t=x\}$ is the equilibrium (or stationary) distribution of X. Here $0 \le a \le \infty$. Also, with probability one, $t^{-1}N(s,s+t] \to a$ as $t \to \infty$.

One can use N to model a variety of event occurrences of X by appropriate selections of f. For instance, suppose X takes values in $\mathcal{X} = Z_+^m$, the m-dimensional vectors with nonnegative integer-valued entries. Then N records the downward jumps of X when $f(x,y) = 1$ iff $x \neq y$ and $x_j \ge y_j$, $j=1,\ldots,m$. Similarly, N records the jumps of X at which the maximum component does not change when $f(x,y) = 1$ iff $x \neq y$ and $\max_j x_j = \max_j y_j$.

We shall investigate conditions under which N is a Poisson process. We say that <u>the future of N is independent of the past of X</u>, denoted $N_+ \perp\!\!\!\perp X_-$, if $\{N(A): A \subset [t,\infty)\}$ is independent of $\{X_u: u \le t\}$, $t \in R$. ($\{X_u: u \le t\}$ can be replaced simply by X_t since X is Markovian). Similarly, $N_- \perp\!\!\!\perp X_+$ denotes that <u>the past of N is independent of the future of X</u>. Our first result is that these conditions are sufficient for N to be a Poisson process; the N might be a non-stationary Poisson process. We write "N is $\mathcal{P}(a)$" to mean that N is a stationary (time-homogeneous) Poisson process with rate a. The degenerate case a = 0 corresponds to N = 0; the case a = ∞ is not possible.

Theorem 2.1. If $N_+ \perp\!\!\!\perp X_-$ or $N_- \perp\!\!\!\perp X_+$, then N is a Poisson process. In this case, N is $\mathscr{P}(a)$ if and only if $E\alpha(X_t) = a$, $t \in R$.

Proof. The N is a Poisson process if it is simple (i.e., $N(\{t\}) = 0$ or 1, $t \in R$), it has no fixed atoms and has independent increments (i.e., $N(A_1),\dots,N(A_n)$ are independent for disjoint $A_1,\dots,A_n$); see p. 58 of Kallenberg (1983). Since the probability is zero that X has a jump at any specified time, it follows that N is simple with no fixed atoms. Now suppose $N_+ \perp\!\!\!\perp X_-$. Then, for any $s < t$ in R,

$$P\{N(s,t] = n \mid N(A) : A \subset (-\infty,s]\}$$
$$= E[P\{N(s,t] = n \mid X_r : r \le s\}] = P\{N(s,t] = n\}.$$

Thus N has independent increments and hence N is Poisson. This conclusion also follows when $N_- \perp\!\!\!\perp X_+$ since

$$P\{N(s,t] = n \mid N(A) : A \subset [t,\infty)\}$$
$$= E[P\{N(s,t] = n \mid X_u : u \ge t\}] = P\{N(s,t] = n\}.$$

The second assertion on the stationarity of N follows from (2.3).□

Remark 2.2. From the proof, it is clear that the first assertion of Theorem 2.1 is true for any pure jump stochastic process X; the Markovian property is used only for the second assertion. Also, both assertions are true when X is a Markov process that is not time homogeneous (its transition rates are time dependent). Melamed (1979) showed that $N_+ \perp\!\!\!\perp X_-$ implies that N is Poisson; his argument relies on the property that X and $\{(N(s,t], X_t): t \ge s\}$ are Markovian.

We now discuss conditions on the parameters of the Markov process X under which N is $\mathcal{P}(a)$. When X is stationary with equilibrium distribution π, we shall frequently use the function

$$\alpha^*(x) = \pi(x)^{-1} \sum_y \pi(y)q(y,x)f(y,x), \qquad x \in \mathcal{S}. \tag{2.5}$$

Theorem 2.3. (i) If $\alpha(x) = a$, $x \in \mathcal{S}$, then $N_+ \perp\!\!\!\perp X_-$ and N is $\mathcal{P}(a)$. (ii) If X is stationary with equilibrium distribution π, and $\alpha^*(x) = a$, $x \in \mathcal{S}$, then $N_- \perp\!\!\!\perp X_+$ and N is $\mathcal{P}(a)$.

Proof. Assertion (i) is a special case of Theorem 18.9 in Liptser and Shiryayev (1978) (or Theorem T5 in Brémaud (1981)), which says that a simple point process on R is $\mathcal{P}(a)$ if its compensator has the non-random intensity a. In our setting, for each $s \in R$, the process

$$M_t = N(s,s+t] - \int_s^{s+t} \alpha(X_u)du \tag{2.6}$$

is an $\mathcal{F}_t$-martingale, where $\mathcal{F}_t = \sigma(X_u: \ s \le u \le s+t)$; the process $A_t = \int_s^{s+t} \alpha(X_u)du$ is the compensator of $N(s, s+t]$ and $\alpha(X_t)$ is the intensity of this compensator.

To prove assertion (ii), consider the process $X_t^* = \bar{X}_{-t}$, $t \in R$, where $\bar{X}_t = X_{t-}$. The X^* is the right-continuous time-reversal of X. Each sample path of X^* is the same as a sample path of X traversed in the opposite direction, and vice versa. Since X is stationary, it follows that X^* is an irreducible, stationary Markov process and its transition rates are

$$q^*(x,y) = \pi(x)^{-1}\pi(y)q(y,x), \qquad x,y \in \mathcal{S}.$$

Define the point process N^* on R by

$$N^*(A) = \sum_{t \in A} f^*(X_{t-}^*, X_t^*)$$

where $f^*(x,y) = f(y,x)$, $x,y \in \mathcal{S}$. Clearly $N^*(A) = N(-A)$, for each A, and so N^* is the time-reversal of N. Consequently, N is $\mathcal{P}(a)$ if and only if N^* is $\mathcal{P}(a)$.

Now observe that $\alpha^*(x) = \sum_y q^*(x,y)f^*(x,y)$. That is, α^* is the function α for the processes X^*, N^*. Then an application of (i) to X^*, N^* says that $N^*_+ \perp\!\!\!\perp X^*_-$ and N^* is $\mathcal{P}(a)$. But these statements are eqivalent to the respective statements that $N_- \perp\!\!\!\perp X_+$ and N is $\mathcal{P}(a)$, since X^*, N^* are the time reversals of X,N.□

Remarks. (1) In Theorem 2.3, statement (i) is the well-known property of Markov processes that N is $\mathcal{P}(a)$ if the intensity $\alpha(X_t)$ of N's compensator equals the constant a. Statement (ii) is simply a reversed time version of (i): it is (i) in terms of X and N viewed in reverse time. The α^* is the reversed time version of α in the sense that $\alpha^*(X^*_t)$ is the intensity of the compensator of N in reverse time. Also, $N_- \perp\!\!\!\perp X_+$ in (ii) is the time reversal of $N_+ \perp\!\!\!\perp X_-$ in (i). (2) Note that the process X in (i) may be transient, recurrent or non-stationary; but in (ii), X must be stationary. (3) In consulting the references for Theorem 2.3(i), one can see that this result is true for a non-Markovian process X for which α: $\mathcal{X} \to R_+$ is a function such that the process M_t in (2.6) is an $\mathcal{F}_t$-martingale. Similarly, it follows that Theorem 2.3(ii) is true for a non-Markovian process X if α^*: $\mathcal{X} \to R_+$ is a function such that, for each $s \epsilon R$,

$$M^*_t = N[s-t,s) - \int_{s-t}^{s} \alpha^*(X_u)du$$

is an $\mathcal{F}^*_t$-martingale where $\mathcal{F}^*_t = \sigma(X_u: s-t \leq u < s)$.

In applications, the Poisson property of N due to $\alpha(x) = a$, $x \in \mathcal{X}$, is usually foreseen, while the Poisson property of N due to $\alpha^*(x) = a$, $x \in \mathcal{X}$, might not be anticipated. Here is an example.

Example 2.4. M/M/1 and Batch Service Queues. Suppose the Markov process X has the state space Z_+ and transition rates

$$q(n,n+1) = \lambda \qquad n \in Z_+$$

$$q(n,n-K) = \mu \qquad n \geq K$$

$$q(n,0) = \mu \qquad n < K$$

where λ, μ and K are positive. This process represents the number of customers in a queueing system in which customers arrive singly at the rate λ and are served in batches such that when $n \geq K$ customers are in the system, then batches of K customers depart at the rate μ; and when $n < K$ customers are present, then all of the customers depart at the rate μ. When K = 1, this is the M/M/1 queueing system.

Implicit in the description of this queueing system, the point process N of customer arrivals is $\mathcal{P}(\lambda)$, regardless of whether X is transient or recurrent. Indeed, this follows from Theorem 2.3 (i) since N is defined by (2.1) with $f(n,n') = 1$ iff $n' = n + 1$ and

$$\alpha(n) = \sum_{n'} q(n,n')f(n,n') = q(n,n+1) = \lambda.$$

Now, suppose that N denotes the point process of times at which batches of size K depart from the system. This is defined by (2.1) with $f(n,n') = 1$ iff $n' = n - K$ and $n \geq K$. In this case,

$$\alpha(n) = q(n,n-K)1(n \geq K) = \mu 1(n \geq K).$$

This depends on n, and so Theorem 2.3 (i) does not ensure that N is Poisson. However, assume that X is stationary. Necessarily, $\lambda < \mu K$ and the equilibrium distribution of X is

$$\pi(n) = r^n(1 - r), \quad n \geq 0,$$

where $r \in (0,1)$ is the unique root of

$$\mu r^{K+1} - (\lambda + \mu)r + \lambda = 0$$

(see Section 3.2 of Gross and Harris (1985)). Clearly

$$\begin{aligned} \alpha^*(n) &= \pi(n)^{-1} \sum_{n'} \pi(n')q(n',n)f(n',n) \\ &= \pi(n)^{-1} \pi(n + K)q(n + K,n) \\ &= \mu r^K = \mu + \lambda(1 - 1/r). \end{aligned}$$

Thus, we conclude by Theorem 2.3 (ii) that the process N is $\mathscr{P}(\mu + \lambda(1 - 1/r))$. One would probably not anticipate this result from the description of the process, or even from earlier work in this area. For the special case in which X is the M/M/1 queue, we have $K = 1$, $r = \lambda/\mu$, and so N, which is the departure process, is $\mathscr{P}(\lambda)$. Burke (1956) and Reich (1957) were the first ones to prove this.

Example 2.5. Queues With Compound Poisson Arrivals and Poisson Departures. Suppose the Markov process has the state space Z_+ and transition rates

$$\begin{aligned} q(n, n + m) &= \lambda_n p^{m-1}(1 - p) \qquad && m \geq 1, \quad n \in Z_+ \\ q(n, n - 1) &= \mu_n && n \geq 1 \end{aligned}$$

where λ_n, μ_n are positive and $0 < p < 1$. This process represents the number of customers in a queueing system in which batches of customers arrive at the rate λ_n when n customers are present and the number of customers in a batch has a geometric distribution with parameter p. The customers depart at the rate μ_n when n are in the system. The equilibrium distribution of X is

$$\pi(n) = \pi(0)\lambda_0\mu_1^{-1}\cdots\mu_n^{-1} \prod_{k=1}^{n-1} (\lambda_k + p\mu_k), \qquad n \geq 1,$$

provided the sum of these terms over n is finite, which we assume is true (see Kook (1988)).

Suppose that N is the point process of customer departures. This is defined by (2.1) with $f(n,n') = 1$ iff $n' = n - 1$. Clearly,

$$\alpha^*(0) = \pi(0)^{-1}\pi(1)q(1,0) = \lambda_0$$

$$\alpha^*(n) = \pi(n)^{-1}\pi(n+1)q(n+1,n) = \lambda_n + p\mu_n, \qquad n \geq 1.$$

Thus, if X is stationary and $\lambda_0 = a$ and $\lambda_n + p\mu_n = a$, $n \geq 1$, then the departure process N is $\mathcal{P}(a)$.

We end this section with elaborations on Theorem 2.3 that establish necessary conditions for N to be Poisson.

Theorem 2.6. When X is recurrent, the following statements are equivalent.

(i) $N_+ \perp\!\!\!\perp X_-$ and N is $\mathcal{P}(a)$.

(ii) $\alpha(x) = a, \quad x \in \mathcal{X}$.

(iii) $N_+ \perp\!\!\!\perp X_-$ and $EN(s,t] = a(t-s), \quad s < t$ in R.

(iv) $E[\alpha(X_t) \mid X_s = x] = a, \quad x \in \mathcal{X}, \quad s < t$ in R.

Proof. Theorem 2.3 ensures that (ii) implies (i). Clearly (i) implies (iii). Now, if (iii) holds, then, for $s < t$ in R,

$$a(t-s) = EN(s,t] = E[N(s,t] \mid X_s = x]$$
$$= \int_s^t E[\alpha(X_u) \mid X_s = x]du$$

the last equality being Lévy's formula. Taking the derivative of this with respect to t yields (iv). Finally, if equation (iv) holds, then taking the derivative of it with respect to t and letting $t \downarrow s$, we obtain

$$\sum_y \alpha(y)q(x,y)/q(x) = \alpha(x) \qquad \text{for each } x.$$

This says that α is a harmonic function of the Markov matrix $\{q(x,y)/q(x)\}$. Now, this matrix is recurrent under the assumption that X is recurrent. But we know (see for instance Section 7.2 of Çinlar (1975)) that harmonic functions of irreducible, recurrent Markov chains are constant. This observation and equation (iv) imply (ii).□

Theorem 2.7. When X is stationary with equilibrium distribution π, the following statements are equivalent.

(i) $N_- \perp\!\!\!\perp X_+$ and N is $\mathcal{P}(a)$.

(ii) $\alpha^*(x) = a, \quad x \in \mathfrak{X}$.

(iii) $N_- \perp\!\!\!\perp X_+$ and $EN(0,1] = a$.

(iv) $E[\alpha^*(X_s) \mid X_t = x] = a, \quad x \in \mathfrak{X},\ s < t$ in $\mathbb{R}$.

(v) $\alpha^*(x) = \sum_y \alpha(y)\pi(y) = a, \quad x \in \mathfrak{X}$.

Proof. The equivalence of (i) - (iv) follows from Theorem 2.6 applied to the reversed-time processes N^*, X^* defined in the proof of Theorem 2.3. Furthermore, (v) obviously implies (ii). And (ii) implies (v) since N is $\mathcal{P}(a)$ by (i) and by (2.4), we have

$$a = EN(0,1] = \int_0^1 E\alpha(X_t)dt = \sum_y \alpha(y)\pi(y).$$

Thus (i) - (v) are equivalent statements.□

Remark 2.8. Statement (v), relating α^* to α, is the "independence condition" that is the focus of Melamed (1979), Brémaud (1981) and Varaiya and Walrand (1981). Their main result, which is implicit in Theorem 2.7, is as follows: If X is stationary with equilibrium distribution π, then $N_- \perp\!\!\!\perp X_+$ if and only if $\alpha^*(x) = \sum_y \alpha(y)\pi(y)$, $x \in \mathfrak{X}$. In this case, N is $\mathcal{P}(a)$ where a is the preceding sum.

Although we are considering processes N and X defined on the entire real line, our results herein also apply to such processes defined on any time interval I in R. Indeed, consider the process extended to all of R and then the results apply to these extended processes and hence to their restrictions to I.

3. General Poisson Functionals of Markov Processes

We have been studying the point process N that records the times at which the Markov process X jumps from some x to some y with $f(x,y) = 1$. We shall now study a more general point process that records additional information at these jumps. Suppose that h is a function from $\mathcal{X} \times \mathcal{X}$ to some space $\mathcal{X}'$. Consider the marked point process M on $R \times \mathcal{X}'$ defined by

$$M(A\times B) = \sum_{t\in A} f(X_{t-},X_t)1(h(X_{t-},X_t) \in B) \tag{3.1}$$

where f is as above, $A \times B$ is in the σ-field of $R \times \mathcal{X}'$ and $1(\cdot \in B)$ is the indicator function of B. This M is a functional of X that records the "mark" $h(X_{t-},X_t)$ at each jump time t of X at which $f(X_{t-},X_t) = 1$. As a point process, M is simple and $M(\{t\} \times \mathcal{X}') = 0$ or 1, $t \in R$. Also, $M(A \times \mathcal{X}') = N(A)$ records the times at which M records the marks.

In this section, we identify conditions on X under which M is Poisson with $EM((s,t] \times B) = a(t-s)F(B)$, where $a \geq 0$ and F is a probability measure on $\mathcal{X}'$. In this case, the process N of occurrence times of the marks of M, is $\mathcal{P}(a)$, and each mark has the distribution F. We call such an M a <u>marked Poisson process</u> with rate a and mark distribution F, and we simply say M is $\mathcal{MP}(a,F)$.

The approach we used in Section 2 for deriving Poisson characterizations of N readily extends to yield marked Poisson characterizations of **M**. The following results are analogues of Theorems 2.1, 2.3 and 2.7; the analogue of Theorem 2.6 can be seen from Theorem 3.3 and hence is not displayed. Their proofs are omitted since they follow the same line of reasoning as their counterparts in Section 2. Here we let

$$\alpha(x,B) = \sum_y q(x,y)f(x,y)1(h(x,y) \in B)$$

$$\alpha^*(x,B) = \pi(x)^{-1} \sum_y \pi(y)q(y,x)f(y,x)1(h(y,x) \in B).$$

Theorem 3.1. If $M_+ \perp\!\!\!\perp X_-$ or $M_- \perp\!\!\!\perp X_+$, then M is a Poisson process. In this case, M is $\mathcal{MP}(a,F)$ if and only if $E\alpha(X_t,B) = aF(B)$, for each t and B.

Theorem 3.2. (i) If $\alpha(x,B) = aF(B)$, for each x and B, then $M_+ \perp\!\!\!\perp X_-$ and M is $\mathcal{MP}(a,F)$. (ii) If $\alpha^*(x,B) = aF(B)$, for each x and B, then $M_- \perp\!\!\!\perp X_+$ and M is $\mathcal{MP}(a,F)$.

Theorem 3.3. When X is stationary with equilibrium distribution π, then the following statements are equivalent.

(i) $M_- \perp\!\!\!\perp X_+$ and M is $\mathcal{MP}(a,F)$.

(ii) $\alpha^*(x,B) = aF(B)$, for each x and B.

(iii) $M_- \perp\!\!\!\perp X_+$ and $EM((0,1] \times B) = aF(B)$ for each x and B.

(iv) $E[\alpha^*(X_s,B) \mid X_t=x] = aF(B)$, for each x, B and $s < t$ in R.

(v) $\alpha^*(x,B) = \sum_y \alpha(y,B)\pi(y) = aF(B)$, for each x and B.

Point processes of the form (3.1) are useful for representing multivariate and compound point processes. Indeed, suppose one is interested in the n-dimensional compound point process $(M_1,\ldots,M_n)$ defined by

(3.2) $$M_j(A) = \sum_{t\in A} f(X_{t-},X_t)h_j(X_{t-},X_t)$$

where $h_j: \mathfrak{X} \times \mathfrak{X} \to R$ (or any other group). These processes contain the same information as M defined in (3.1) with $\mathfrak{Y}' = R^n$ and $h(x,y) = (h_1(x,y),\ldots,h_n(x,y))$. More precisely, there is a one-to-one correspondence between point processes M on $R \times R^n$ as in (3.1) and n-dimensional compound point processes $(M_1,\ldots,M_n)$ as in (3.2). Note that M on $R \times R^n$ is $\mathcal{MP}(a,F)$ if and only if its corresponding $(M_1,\ldots,M_n)$ is an n-dimensional compound Poisson process with rate a and atom distribution F on R^n. In this case,

$$P\{M_1(s,t] \in B_1,\ldots,M_n(s,t] \in B_n\}$$
$$= \sum_{k=0}^{\infty} F^{k*}(B_1\times\ldots\times B_n)a^k(t-s)^k\ e^{-a(t-s)}/k!.$$

And each M_j is a compound Poisson process with rate $a_j = a(1-F_j(0))$ and atom distribution F_j, where $F_1,\ldots,F_n$ are the marginal distributions of F. Here are some special cases:

C1: $M_1,\ldots,M_n$ are independent compound Poisson processes with rates $a_1,\ldots,a_n$ and atom distributions $F_1,\ldots,F_n$ if and only if $F = F_1\times\ldots\times F_n$.

C2: $(M_1,\ldots,M_n)$ is an n-dimensional Poisson process with rate a and point allocation distribution F on $\{0,1\}^n$ if and only if F has support on $\{0,1\}^n$. In this case, each M_j is $\mathcal{P}(a_j)$.

C3: $M_1,\ldots,M_n$ are independent Poisson processes with rates $a_1,\ldots,a_n$ if and only if F has support on $\{e_1,\ldots e_n\}$, where e_j is the jth unit vector with 1 in entry j and 0's elsewhere.

Remark 3.4. Each of the preceding theorems hold for these cases C1, C2, C3 when the F in the theorem is as specified in the case of interest. For instance, Theorem 3.2(ii) for case C3 reads: If $\alpha^*(x,\{e_j\}) = a_j$,

$j=1,\ldots,m$, and $\alpha^{*}(x,\{z\}) = 0$ otherwise, $x \in \mathcal{X}$, then $(M_1,\ldots,M_n)_{-} \perp\!\!\!\perp X_{+}$ and $M_1,\ldots,M_n$ are independent Poisson processes with respective rates $a_1,\ldots,a_n$.

An obvious application of Theorem 3.2(i) shows that the process of customer arrivals in Example 2.5, when $\lambda_n = \lambda$, is a compound Poisson process with rate λ and geometric atom distribution $p^{m-1}(1-p)$. The following is an example of a not so obvious compound Poisson flow in a queueing process.

Example 3.5. A Batch-Service Queueing System With Poisson Arrivals and Compound Poisson Departures. Suppose the Markov process has the state space Z_+ and transition rates

$$
\begin{aligned}
q(0,1) &= \lambda(1-p), \quad q(n,n+1) = \lambda, && n \geq 1, \\
q(n,n-m) &= \mu p^{m-1}(1-p) \qquad 1 \leq m < n, && n \geq 1, \\
q(n,0) &= \mu p^{n-1} && n \geq 1,
\end{aligned}
$$

where λ,μ are positive and $0 < p < 1$. This process represents the number of customers in a queueing system in which customers arrive at the rate λ and are served in batches as follows. When there are customers in the system, "buses" arrive at a rate μ to take them immediately from the queue. Busing is a common practice in computer systems and material handling systems. The number of customers each bus can take is a random variable with the geometric distribution $p^{m-1}(1-p)$, $m \geq 1$. Also, when there are no customers in the queue and a customer arrives, then with probability p there is a bus available to take the customer without delay. The equilibrium distribution of X is

$$
\pi(n) = \pi(0)(1-p)\lambda^{n}/(\mu + p\lambda)^{n}, \qquad n \geq 1,
$$

provided $\lambda < \mu + p\lambda$, which we assume is true (see Kook (1988)).

Consider the compound point process

$$D(A) = \sum_{t \in A} \max\{0, X_t - X_{t-}\}$$

that describes the total number of departures in the time period A. It records the times at which batches of customers depart and the batch sizes as well. This process corresponds to M in (3.1) with $f(n,n') = 1$ iff $n' \neq n$ and

$$h(n,n') = \begin{cases} n - n' & n' \leq n \\ 0 & n' > n \,. \end{cases}$$

Clearly, for each $n \geq 0$ and $m \geq 1$,

$$\begin{aligned} \alpha^*(n,\{m\}) &= \pi(n)^{-1} \sum_{n'} \pi(n')q(n',n)f(n',n)1(h(n',n) = m) \\ &= \pi(n)^{-1}\pi(n + m)q(n + m,n) \\ &= \lambda(1 - p)F(\{m\}), \end{aligned}$$

where $F(\{m\}) = r^{m-1}(1-r)$, $m \geq 1$, and $r = p\lambda/(\mu+p\lambda)$. Thus, if X is stationary, then Theorem 3.2(ii) implies that M is $\mathcal{MP}(\lambda(1 - p),F)$. Hence D is a compound process with rate $\lambda(1 - p)$ and geometric atom distribution F.

4. Poisson Flows in a Network of Queues

We shall consider a queueing network process defined as follows. Suppose that $X(t) = (X_1(t),\ldots,X_m(t))$, $t \in R$, is a queueing network process on m nodes, where $X_j(t)$ denotes the number of units (i.e. customers) at node j at time t. The process X takes on values $n = (n_1,\ldots,n_m)$ in Z_+^m. Let e_j denote the jth unit vector in Z_+^m with 1 in entry j and 0's elsewhere. As in Whittle (1986), we assume that units move among the m nodes such that X is a Markov process with transition rates

$$q(n, n + e_k) = \lambda_{0k} \tag{4.1}$$

$$q(n, n - e_j + e_k) = \lambda_{jk}\Phi(n - e_j)/\Phi(n), \qquad n_j \geq 1,$$

$$q(n, n - e_j) = \lambda_{j0}\Phi(n - e_j)/\Phi(n), \qquad n_j \geq 1,$$

and $q(n,n') = 0$ for all other states n'. Here $\Phi: Z_+^m \to (0,\infty)$, the λ's are nonnegative, and the subscript 0 denotes the "outside" node. Under this assumption, units enter the nodes $1,\ldots,m$ by independent Poisson processes with respective rates $\lambda_{01},\ldots,\lambda_{0m}$. When **X** is in state n, then $\sum_k \lambda_{jk}\, \Phi(n-e_j)/\Phi(n)$ is the departure rate of units from node j. The λ_{jk} is the "arc-dependent" routing intensity from node j to node k, and $\Phi(n-e_j)/\Phi(n)$ is the "system-dependent" departure intensity from node j (the ratio representing the potential difference between the system in state n and in state $n-e_j$ with one less customer at node j).

We shall assume that **X** is irreducible. This is equivalent to the irreducibility of the Markov routing matrix

$$p(j,k) = \lambda_{jk}/\sum_{\ell=0}^{m} \lambda_{j\ell}, \qquad j,k = 0,\ldots,m, \tag{4.2}$$

where $\lambda_{00} = 0$. The irreducibility of this matrix is equivalent to the existence of unique positive numbers $w_1,\ldots,w_m$ that satisfy

$$\sum_{k=0}^{m} (w_j\lambda_{jk} - w_k\lambda_{kj}) = 0, \qquad j = 0,\ldots,m, \tag{4.3}$$

where $w_0 = 1$. We also assume that $\sum_n \Phi(n) \prod_{j=1}^{m} w_j^{n_j}$ is finite. Then **X** is positive recurrent and has the equilibrium distribution (p.198 of Whittle (1986))

$$\pi(n) = c\Phi(n) \prod_{j=1}^{m} w_j^{n_j}, \qquad n \in Z_+^m, \tag{4.4}$$

where c is the normalizing constant. Hereafter, we assume that **X** is stationary.

Now, consider the point process

$$N_{j0}(A) = \sum_{t \in A} 1(X(t) = X(t-) - e_j)$$

that represents the times at which units exit the network from node j. Of course, $N_{j0} = 0$ when $\lambda_{j0} = 0$.

Theorem 4.1. The exit processes $N_{10}, \ldots, N_{m0}$ are independent Poisson processes with rates $w_1\lambda_{10}, \ldots, w_m\lambda_{m0}$, and $(N_{10}, \ldots, N_{m0})_- \perp\!\!\!\perp X_+$.

Whittle (1986) on p. 207 proved this by establishing that the reverse time process X^* of X is again a queueing network process and the exit processes of X are just the time reversals of the Poisson input processes of X^*. Theorem 4.1 also follows from Theorem 3.2(ii) and Remark 3.4 since, for each n and j,

$$\alpha^*(n, \{e_j\}) = \pi(n)^{-1}\pi(n + e_j)q(n + e_j, n) = w_j\lambda_{j0}$$

and $\alpha^*(n, \{z\}) = 0$ elsewhere.

Along with these exit processes, consider the point processes

$$N_{jk}(A) = \sum_{t \in A} 1(X(t) = X(t-) - e_j + e_k)$$

of times at which units move from node j to node k. We shall now identify sets $J \subset \{1, \ldots, m\}$ and $K \subset \{0, 1, \ldots, m\}$ such that N_{jk}, $j \in J$, $k \in K$ are independent Poisson processes. Suppose that J and K satisfy the following assumptions:

A1: Each unit that exits J can never return to J. (To verify this one need only check the possible routing under λ_{jk}.)

A2: The system-dependent departure intensity for each node $j \in J$ is of the form

$$\Phi(n - e_j)/\Phi(n) = \Phi_J(n_J - e_j)/\Phi_J(n_J)$$

where $n_J = (n_j : j \in J)$, and Φ_J is a positive function on such vectors.

A3: K is the largest subset of $\{0,1,\ldots,m\}$ such that each unit in K cannot enter J on a subsequent move. (Note that $0 \in K$ and $J \cap K = \{j \in J: \lambda_{j\ell} = 0,\ \ell \in J\}$.

For some networks, $J = \{1,\ldots,m\}$ may be the only set of nodes that satisfies A1. At the other extreme are networks in which each node is visited at most once by a unit, and so each subset of nodes satisfies A1. Assumption A2 is equivalent to being able to factor Φ as $\Phi(n) = \Phi_J(n_J)\Psi(n_k : k \notin J)$. Upon selecting J conforming to A1, A2, it is advantageous to select K as large as possible as we did in A3.

Let $X_J(t) = \{X_j(t): j \in J\}$, $t \in R$, denote the process X on the nodes J and let $\mathcal{X}_J$ denote its state space.

Theorem 4.2. The processes N_{jk}, $j \in J$, $k \in K$, are independent Poisson processes with respective rates $w_j\lambda_{jk}$, $j \in J$, $k \in K$. Furthermore, $\{N_{jk}: j \in J,\ k \in K\}_- \perp\!\!\!\perp (X_J)_+$.

Proof. Under the assumptions, the X_J is a queueing network process on the nodes J with transition rates $q_J(n,n')$ defined as in (4.1) with the last line replaced by

$$q_J(n, n - e_j) = \Lambda_j \Phi_J(n - e_j)/\Phi_J(n), \qquad n_j \geq 1,\ n \in \mathcal{X}_J,$$

where $\Lambda_j = \sum_{k \in K} \lambda_{jk}$. Furthermore, X_J has the equilibrium distribution

$$\pi_J(n) = c_J \Phi_J(n) \prod_{j \in J} \alpha_j^{n_j}, \qquad n \in \mathcal{X}_J.$$

This is just the sum of $\pi(n)$ in (4.4) over all n_ℓ, $\ell \notin J$. Thus, from Theorem 4.1, we know that X_J's exit processes $N_j = \sum_{k \in K} N_{jk}$, $j \in J$, are independent Poisson processes with respective rates $w_j \Lambda_j$, $j \in J$, and that $(N_j,\ j \in J)_- \perp\!\!\!\perp (X_J)_+$.

Next, observe that for each $j \in J$, the processes N_{jk}, $k \in K$, form a partition of N_j in which each point of N_j is assigned to the subprocess N_{jk} with probability λ_{jk}/Λ_j, independent of everything else. Consequently, N_{jk}, $k \in K$, are independent Poisson processes with respective rates $(\lambda_{jk}/\Lambda_j)(w_j\Lambda_j) = w_j\lambda_{jk}$, $k \in K$ (see for instance p.89 of Çinlar (1975)). This property and the preceding argument yield the assertions.□

Example 4.3. Suppose the network has a node j such that each unit passing through the network visits j at most once. Also, assume that $\Phi(n) = \Phi_j(n_j)\Psi(n_k : k \neq j)$, and let $J = \{j\}$. Then the departure processes $N_{j0}, N_{j1}, \ldots, N_{jm}$ are independent Poisson processes as in Theorem 4.2. Now, suppose that each node of the network can be visited at most once by each unit and $\Phi(n) = \Phi_1(n_1)\ldots\Phi_m(n_m)$. Then each process N_{jk} is $\mathcal{P}(w_j\lambda_{jk})$; some of these processes may be dependent. Consider the arrival processes to a fixed node k. Let J denote the set of all nodes j that can never be reached from k. Then, under the preceding supposition, the arrival processes N_{jk}, $j \in J$, are independent Poisson processes as in Theorem 4.2.

Example 4.4. Multivariate Poisson Flows. Suppose $J_1, \ldots, J_\nu$ are subsets of $\{1, \ldots, m\}$ and $K_1, \ldots, K_\nu$ are subsets of $\{0, 1, \ldots, m\}$ (the subsets need not be disjoint). Consider the process $N_i = \sum_{\substack{j \in J_i \\ k \in K_i}} N_{jk}$ of times at which units move from some node in J_i to some node in K_i. Suppose the sets $J = J_1 \cup \ldots \cup J_\nu$ and $K = K_1 \cup \ldots \cup K_\nu$ satisfy assumptions A1, A2, A3, or that they are contained in a pair of such sets. Then N_{jk}, $j \in J$, $k \in K$, are independent Poisson processes as in Theorem 4.2. Consequently, $(N_1, \ldots, N_\nu)$ is a ν-dimensional Poisson process with rate $a = \sum_{\substack{j \in J \\ k \in K}} w_j\lambda_{jk}$ and point allocation distribution

$$F(\{\sum_{i\in I} e_i\}) = a^{-1} \sum_{j,k}^{I} w_j \lambda_{jk}$$

where $I \subset \{1,\ldots,\nu\}$, and the sum is over all j in $\bigcap_{i\in I} J_i$, and k in $\bigcap_{i\in I} K_i$. The $N_1,\ldots,N_\nu$ are independent if $\bigcap_i J_i = \phi$ and $\bigcap_i K_i = \phi$.

Example 4.5. Networks With Several Types of Units. Consider a network as above in which each unit carries a label from a finite set $\mathcal{A}$ of types and the label may change when the unit moves. We represent this network by the process $X(t) = \{X_{aj}(t): a\in\mathcal{A},\ j=1,\ldots,m\}$, $t\in R$, where $X_{aj}(t)$ is the number of type a units at node j at time t. A typical state of X is $n = (n_{aj}: a\in\mathcal{A}, j=1,\ldots,m)$. We assume that X is a Markov process with transition rates (analogous to (4.1))

$$q(n,n + e_{bk}) = \lambda_{0,bk}$$

$$q(n,n - e_{aj} + e_{bk}) = \lambda_{aj,bk}\,\Phi(n - e_{aj})/\Phi(n), \qquad n_{aj} \geq 1,$$

$$q(n,n - e_{aj}) = \lambda_{aj,0}\,\Phi(n - e_{aj})/\Phi(n), \qquad n_{aj} \geq 1.$$

Under assumptions as above, the process X has the equilibrium distribution

$$\pi(n) = c\Phi(n) \prod_{aj} w_{aj}^{n_{aj}}.$$

The results above hold for this network - one need only use double indices aj, bk in place of j,k and consider the point processes $N_{aj,bk}$, $aj \in J$, $bk \in K$, where J, K satisfy assumptions A1, A2, A3 with double indices. For instance, suppose $A \subset \mathcal{A}$ is such that a unit with a label from $\mathcal{A}\backslash A$ can never carry a label from A. Consider the point processes of times at which units change labels from A to $\mathcal{A}\backslash A$. These processes are $N_{aj,bk}$, $aj\in J$, $bk\in K$, where $J = \{aj: a\in A,\ j=1,\ldots,m\}$ and $K = \{bk: b\in\mathcal{A}\backslash A,\ k=0,\ldots,m\}$. Clearly J satisfies A1 and K satisfies A3.

Suppose J also satisfies A2. Then the preceding point processes are independent Poisson processes with respective rates $\{w_{aj}\lambda_{aj,bk}\}$, and the past of these processes is independent of $(X_J)_+$.

The preceding results are for networks in which only one unit can move at a time. In networks with simultaneous movement of units, the flows among the nodes may be compound Poisson processes. Kook (1988) has characterized such flows using Theorem 3.2.

5. Further Generalizations

The results in Sections 2 and 3 readily extend to more general processes. We discussed some of these situations in the remarks above. Here are some more generalizations.

Markov Processes With General State Spaces. Suppose that $X = \{X_t : t \in R\}$ is a pure jump Markov process with a general state space E and associated σ-field $\mathcal{E}$, and its transition kernel is $K(x,B)$, $x \in E$, $B \in \mathcal{E}$. That is, the exponential sojourn time in state x has parameter $K(x,E)$ and the probability of X jumping from x into B is $K(x,B)/K(x,E)$. Then the results in Sections 2 and 3 hold with the sums replaced by integrals. For example, α^* is the Radon-Nikodym derivative

$$\alpha^*(x) = \int_E \pi(dy)K(y,dx)f(y,x)/\pi(dx).$$

Functionals Involving Sojourn Times. Suppose X is a Markov process as in Section 2 and

$$N(A) = \sum_{t \in A} f(X_t, X_{t-}, W_t)$$

where W_t is the waiting time in state X_{t-} and $f : \mathcal{S} \times \mathcal{S} \times R_+ \to \{0,1\}$. Then the results of Section 2 hold for this N with α, α^* defined by

$$\alpha(x) = \sum_y \int_0^\infty q(x,y) f(x,y,w) F_x(dw)$$

$$\alpha^*(x) = \pi(x)^{-1} \int_0^\infty \sum_y \pi(y) q(y,x) f(y,x,w) F_y(dw)$$

where $F_x(w) = 1 - \exp(-wq(x))$. Similarly, the results of Section 3 hold for M as a functional of W_t as well as of X_t, X_{t-}. One can generalize further by assuming that X is a semi-Markov process and replacing F_x by a general sojourn time distribution $F_{x,y}$.

References

Benveniste, A. and J. Jacod (1973). Systemes de Lévy des processus de Markov. Invent. Math. 21, 183-98.

Brémaud, P. (1981). Point Processes and Queues: Martingale Dynamics. Springer-Verlag, New York.

Burke, P. J. (1956). The output of a queueing system. Operations Res. 4, 699-704.

Çinlar, E. (1975). Introduction to Stochastic Processes. Prentice-Hall, Englewood Cliffs.

Disney, R. L. and P. C. Kiessler (1987). Traffic Processes in Queueing Networks: A Markov Renewal Approach. Johns Hopkins University Press, Baltimore.

Disney, R. L. and D. König (1985). Queueing networks: a survey of their random processes. SIAM Review, 27, 335-403.

Gross, D. and C. M. Harris (1985). Fundamentals of Queueing Theory. Wiley, New York.

Kallenberg, O. (1983). Random Measures. 3rd Ed. Akademie-Verlag, Berlin and Academic Press, New York.

Karr, A. F. (1986). Point Processes and Their Statistical Inference. Dekker, New York.

Kelly, F. P. (1979). Reversibility and Stochastic Networks. Wiley, New York.

Kook, K. (1988). Ph.D. Dissertation. Industrial and Systems Engineering, Georgia Institute of Technology.

Liptser, R. Sh. and A. N. Shiryayev (1978). Statistics of Random Processes. Vol II, Springer-Verlag, New York.

Melamed, B. (1979). On Poisson traffic processes in discrete-state Markovian systems with applications to queueing theory. Adv. Appl. Prob. 11, 218-239.

Reich, E. (1957). Waiting times when queues are in tandem. Ann. Math. Stat. 28, 768-773.

Varaiya, P. and J. Walrand (1981). Flows in queueing networks: a martingale approach. Math. Operations Res., 6, 387-404.

Whittle, P. (1986). Systems in Stochastic Equilibrium. Wiley, New York.

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