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RANDOM PERTURBATIONS OF SOME SPECIAL MODELS

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Yuri Kifer

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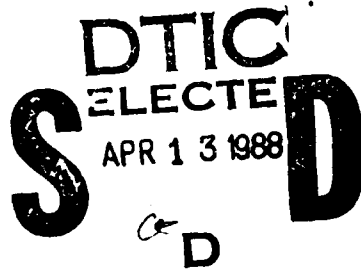
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In this paper we shall study studies the convergence of invariant measures for random perturbations of maps of an interval and a Lorentz's type model dynamical system. These models lack the shadowing property for some pseudo-orbits. Misiurewicz's map treated in Section 2 is also not uniformly expanding. However, we shall see how to modify the approach of [ki[in order to overcome these complications. (Keyworks: one dimensional transformations)

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Random Perturbations of Some Special Models

In this paper we shall study the convergence of
invariant measures for random perturbations of maps of an
interval and a Lorentz's type model dynamical system.
These models lack the shadowing property for some
pseudo-orbits. Misiurewicz's map treated in Section 2 is
also not uniformly expanding. However, we shall see how to
modify the approach of [Ki] in order to overcome these
complications.

1. Random perturbations of one-dimensional transformations.

In this section we shall discuss models of random
perturbations of maps of an interval. We shall treat
specifically the case of the Lasota-Yorke [LaY] piecewise
smooth expanding transformations.

Suppose that $F:[0,1] \to [0,1]$ is a map of the unit
interval $M=[0,1]$ into itself. Random perturbations X_n^ϵ
of F are being defined in the same way as in [Ki] and [KK]
though we shall impose certain restrictions on measures Q_x^ϵ
and transition probabilities $P^\epsilon(x,\cdot)$ appearing there. We
shall be interested in the weak convergence of invariant
measures μ^ϵ of Markov chains X_n^ϵ with
transiton probabilities $P^\epsilon(x,\cdot)$. In the same way as in

[Ki] and [KK] we shall assume that all Q_x^ϵ have densities q_x^ϵ with respect to the Lebesgue measure on $M = [0,1]$, i.e., for any Borel set $\Gamma \subset M$,

$$Q_x^\epsilon (\Gamma) = \int_\Gamma q_x^\epsilon (y) \, dy, \qquad (1.1)$$

and so $P^\epsilon(x, \cdot) = Q_{Fx}^\epsilon (\cdot)$ has the density

$$p^\epsilon(x,y) = q_{Fx}^\epsilon (y). \qquad (1.2)$$

Before we proceed with our further assumptions let us discuss first another model due to Boyarsky [Boy] and considered later also by Golosov [Gol] and Collet [Col]. They assume that $q_x^\epsilon(y) = q^\epsilon(y-x)$, i.e., this density depends only on the difference $(y-x)$. In view of (1.1) and (1.2) any invariant measure μ^ϵ of X_n^ϵ has a density ρ^ϵ with respect to the Lebesgue measure which satisfies the equation

$$\rho^\epsilon(z) = \int_M \rho^\epsilon(x) \; p^\epsilon(x,z) \, dx, \qquad (1.3)$$

and so if $q_x^\epsilon (y) = q^\epsilon (y-x)$ then

$$\rho^\epsilon(z) = \int_M \rho^\epsilon(x) \; q^\epsilon(z - Fx) \, dx \qquad (1.4)$$

$$= \int_M \left[\sum_{x \in F^{-1}y} \rho^\epsilon(x) \; |F'(x)|^{-1} \right] q^\epsilon(z - y) \, dy$$

provided the derivative $F'(x)$ exists for almost all with respect to the Lebesgue measure points $x \in M = [0,1]$. Introduce the operator Φ acting on integrable functions

g on [0,1] by the formula

$$\Phi g(y) = \sum_{x \in F^{-1}y} (g(x)|F'(x)|^{-1}) \qquad (1.5)$$

which is called the Frobenius-Perron operator of the map F. This operator describes the transformation of the density of an absolutely continuous measure under the action of F, i.e., if $g = \dfrac{d\upsilon}{dx}$, $\upsilon \in \mathscr{P}(M)$ then Φg is the density of the measure ζ such that $\zeta(\Gamma) = \upsilon(F^{-1}\Gamma)$ for any Borel set $\Gamma \subset M$. Thus if there exists an absolutely continuous F-invariant measure then the density of this measure must be a fixed point of the operator Φ. Vice versa any fixed point of Φ which is an integrable function turns out to be the density of an absolutely continuous invariant measure. By this reason the study of the Frobenius-Perron operator plays a decisive part in many works concerning absolutely continuous measures of one-dimensional transformations (see Lasota and Yorke [LaY], Misiurewicz [Mi], Collet and Eckmann [CE1] and [CE2]).

Define another operator $\mathscr{P}^\epsilon$ acting on integrable functions by the formula

$$\mathscr{P}^\epsilon g(z) = \int_M (\Phi g(y))\, q^\epsilon(z-y)\, dy \qquad (1.6)$$

which may be called the Frobenius-Perron operator of random perturbations. In view of (1.4) and (1.5) the density ρ^ϵ is a fixed point of the operator $\mathscr{P}^\epsilon$ whose explicit representation (1.6) in a convolution form enables one to obtain uniform in ϵ estimates of its fixed points essentially in the same way as one estimates variations of fixed points of the operator Φ itself. By this technique Boyarsky [Boy], Golosov [Gol], and Collet [Col] showed for certain types of maps F that limits of invariant measures μ^ϵ of X_n^ϵ as $\epsilon \to 0$ must be absolutely continuous. Moreover, by this method one can show the convergence of

the densities ρ^ϵ, as well. However, the condition $q_x^\epsilon(y) = q^\epsilon(y-x)$ is rather restrictive and, as we shall see, it excludes interesting models where X_n^ϵ is obtained by means of a composition of maps chosen independently at random from a parametric family. Besides, the above approach cannot be generalized to arbitrary manifolds.

Next, we shall specify our conditions which are a one-dimensional version of Assumption II.1.1.

Assumption 1.1 (a) Transition probabilities of Markov chains X_n^ϵ have the form $P^\epsilon(z, \cdot) = Q_{Fz}^\epsilon(\cdot)$ with Q_x^ϵ satisfying (1.1);

(b) There exist constants α, $C > 0$, $\alpha < 1$ and a family of non-negative functions $\{r_x(\xi), x \in M = [0,1], \xi \in \mathbb{R}^1 = (-\infty, \infty)\}$ such that

$$q_x^\epsilon(y) \leq C\epsilon^{-1} e^{-\frac{\alpha}{\epsilon}\text{dist}(x,y)} \quad \text{for all} \quad x, y \in M, \quad (1.7)$$

where

$$\text{dist}(x,y) = \min(|y-x|, |y-x+1|, |y-x-1|), \quad (1.8)$$

and

$$q_x^\epsilon(y) \leq (1+\epsilon^\alpha)\epsilon^{-1} r_x(\tfrac{1}{\epsilon}\sigma(x,y)) \quad (1.9)$$

provided $\text{dist}(x,y) \leq \epsilon^{1-\alpha}$, where $\sigma(x,y)$ equals one of the numbers $(y-x)$, $(y-x+1)$, or $(y-x-1)$ so that $|\sigma(x,y)| = \text{dist}(x,y)$;

(c) The functions $r_x(\xi)$, $x \in M$, $\xi \in \mathbb{R}^1$ satisfy
   (i) $\int_{\mathbb{R}^1} r_x(\xi) \, d\xi = 1$,

   (ii) $r_x(\xi) \leq C e^{-\alpha|\xi|}$ for α, $C > 0$

        independent of x and ξ,

(iii)   There exists C>0 such that if

$V_x^+ = \{\xi : r_x(\xi) > 0\}$ and $\partial V_x^+(\delta)$ denotes the δ-neighborhood in $\mathbb{R}^1$ of the boundary ∂V_x^+ of V_x^+ then

$$\int_{\partial V_x^+(\delta)} r_x(\xi) \, d\xi \leq C\delta, \qquad (1.10)$$

and

$$r_x(\xi) \leq r_y(\zeta) + C\rho + \chi_{\partial V_x^+(C\rho)}(\xi) r_x(\xi) \qquad (1.11)$$

where $\rho = \rho((x,\xi),(y,\xi)) = \text{dist}(x,y) + |\zeta - \xi|$.

Remark 1.1. The definition (1.8) of the distance means that we consider the periodic boundary conditions, i.e., that we identify the endpoints 0 and 1. Another boundary condition which can be treated by our method is the reflection condition in the endpoints 0 and 1. This means that (1.9) remains the same for either $x \in [\epsilon^{1-\alpha}, 1-\epsilon^{1-\alpha}]$ or $x < \epsilon^{1-\alpha}$ and $y \geq x$ or $x > 1-\epsilon^{1-\alpha}$ and $y \leq x$. But if $x < \epsilon^{1-\alpha}$ and $y < x$ then one assumes

$$\epsilon q_x^\epsilon (y) \left(r_x\left[\frac{y-x}{\epsilon}\right] + r_x\left[-\frac{(x+y)}{\epsilon}\right] \right)^{-1} \leq 1 + \epsilon^\alpha, \qquad (1.12)$$

and if $x > 1-\epsilon^{1-\alpha}$ and $y > x$ then

$$\epsilon q_x^\epsilon (y) \left(r_x\left[\frac{y-x}{\epsilon}\right] + r_x\left[\frac{2-(x+y)}{\epsilon}\right] \right)^{-1} \leq 1 + \epsilon^\alpha . \qquad (1.13)$$

In this case (1.7) should be replaced by

$$q_x^\epsilon (y) \leq C\epsilon^{-1} \exp\left(-\frac{\alpha}{\epsilon} |y-x|\right) \qquad (1.14)$$

if $|y-x| > \epsilon^{1-\alpha}$ and $\epsilon > 0$ is small enough. We can treat also the situation when $\text{dist}(x,y) = |x-y|$ and $q_x^\epsilon (y)$ equals zero unless $x \in [0,1]$ and y belongs to an open neighborhood U of $[0,1]$. This must be complemented by the condition $FU \subset [0,1]$ which yields Markov chains X_n^ϵ defined on U. Boundary conditions do not influence decisively the study of corresponding random perturbations and related proofs differ only in details.

In this section we shall work with transformation satisfying the following conditions.

Assumption 1.2. A map F is piecewise C^2 and expanding, i.e., there exist points $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{v+1} = 1$ such that the restrictions of F to the open intervals (a_{i-1}, a_i), $i = 1, \cdots, v + 1$, $v \geq 1$ are C^2 functions which can be extended to the closed intervals $[a_{i-1}, a_i]$ as C^2 functions (taking at the endpoints right or left derivatives), and

$$\inf_x |F'(x)| = \lambda > 1 \qquad (1.15)$$

where the infinum is taken over all $x \in [0,1]$ for which the derivative $F'(x)$ exists.

Under Assumption 1.2 F is known to have invariant measures which are absolutely continuous with respect to the Lebesgue measure on $[0,1]$ (see Lasota and York [LaY] and Cornfeld, Fomin, and Sinai [CFS], § 4 of Chapter 7). Li and York [LiY] showed that in the above situation there exist at most v ergodic absolutely continuous F-invariant probability measures. In particular if $v = 1$ then one has only one absolutely continuous F-invariant probability measure.

In order to avoid certain complications we shall assume that F is continuous with respect to the metric defined by (1.8), in particular, $F(0) = F(1)$. We shall establish the following result.

Theorem 1.1. *Suppose that random perturbations* X_n^ϵ *of a map* $F:[0,1]\ [0,1]$ *meet the conditions of Assumption 1.1,* F *is continuous with respect to the dist-metric, and* F *satisfies Assumption 1.2. Then all weak limits as* $\epsilon \to 0$ *of probability invariant measures* μ^ϵ *of Markov chains* X_n^ϵ *are absolutely continuous with respect to the Lebesgue measure* $[0,1]$*. In particular, if* $\upsilon = 1$ *in Assumption 1.2 then the invariant measures* μ^ϵ *weakly converge as* $\epsilon \to 0$ *to the unique absolutely continuous F-invariant probability measure* μ*.*

Before the proof we shall discuss certain points connected with Theorem 1.1 for the one-parameter family of tent maps

$$F_s x = \begin{cases} sx & \text{if } 0 \le x \le \frac{1}{2} \\ s(1-x) & \text{if } \frac{1}{2} \le x \le 1 \end{cases} \qquad (1.16)$$

where $1 < s \le 2$. These maps meet conditions of Assumption 1.2 with $\upsilon = 1$ and $a_1 = \frac{1}{2}$. First, notice that in general these maps do not have the shadowing property. Indeed, let $F = F_{\sqrt{2}}$. Then $c = 2-\sqrt{2}$ is a repelling fixed point of F and $F^3(\frac{1}{2}) = c$. Take the δ-pseudo-orbit $x_0 = \frac{1}{2}$, $x_1 = F_x$, $x_2 = F_{x_1}$, $x_3 = c$, $x_4 = c+\delta$, $x_5 = F(x_4), \cdots$, $x_{k+1} = Fx_k, \cdots$, $k = 4,5,\cdots$. Consider the interval $I = \{x: |x - \frac{1}{2}| \le \frac{1}{4}(3-2\sqrt{2})\}$ then $F^3(I)$ is the interval whose left endpoint is c, and so $F^4(I)$ is the interval whose right endpoint is c. Hence if $y \in I$ then $x_4 > c \ge F^4 y$. Since $|x_{k+1}- F^{k+1}y| = \sqrt{2}|x_k- F^k y|$ for $k \ge 4$ provided $\frac{1}{2} \ge x_k \ge \frac{\sqrt{2}}{2}$ and $\frac{1}{2} \ge F^k y \ge \frac{\sqrt{2}}{2}$ we conclude that the orbit of y cannot shadow in any reasonable sense the δ-pseudo-orbit $x_0, x_1, x_2, \cdots$ when δ is small enough. Therefore we shall need some substitution for the shadowing property when proving Theorem 1.1.

For the family of tent maps F_s, $1 < s \leq 2$ we can consider the following model of random perturbations. Suppose that $1 < s_o < 2$ and let φ_1^ϵ, $\varphi_2^\epsilon, \cdots$ be independent random variables with the same uniform distribution on the interval $[-\epsilon, \epsilon]$ where $0 < \epsilon \leq \epsilon_o < \min\left[s_o-1, \dfrac{s_o(2-s_o)}{(s_o+2)}\right]$. Consider the Markov chains

$$X_n^\epsilon = F_{s_o + \varphi_n^\epsilon} \circ \cdots \circ F_{s_o + \varphi_1^\epsilon} x \ . \qquad (1.17)$$

Let $a_\epsilon = s_o(1 - \frac{1}{2}(s_o + \epsilon)) - \epsilon$ and $b_\epsilon = \frac{1}{2}(s_o + \epsilon)$ then $0 < a_\epsilon < \frac{1}{2} < b_\epsilon < 1$. Moreover if $x \in [a_\epsilon, b_\epsilon]$ then $X_n^\epsilon \in [a_\epsilon, b_\epsilon]$ for all $n = 1, 2, \cdots$. Thus we can study the invariant measures of X_n^ϵ on the interval $[a_\epsilon, b_\epsilon]$. Notice that transition probabilities of Markov chains X_n^ϵ have the form

$$P^\epsilon(x, \Gamma) = \int_\Gamma q_{Fx}^\epsilon(y) \ dy$$

where

$$q_z^\epsilon(y) = \begin{cases} s_o(2\epsilon z)^{-1} & \text{if } |y-z| \leq \epsilon z s_o^{-1} \\ 0 & \text{if } |y-z| > \epsilon z s_o^{-1} \end{cases} \qquad (1.18)$$

Thus this type of random perturbations considered on $[a_{\epsilon_o}, b_{\epsilon_o}]$ with $\epsilon < \epsilon_o$ satisfy conditions of Assumption 1.1 and an application of Theorem 1.1 yields that invariant measures μ^ϵ of X_n^ϵ on $[a_{\epsilon_o}, b_{\epsilon_o}]$ weakly converge as $\epsilon \to 0$ to the unique F_{s_o}-invariant absolutely continuous probability measure. Remark, that $q_z^\epsilon(y)$ given by (1.18) depends essentially on z, and so it cannot be represented as $q^\epsilon(y-z)$.

Hence the Boyarsky's approach based on the Frobenius-Perron

operator and described at the beginning of this section does not work in this situation. If $s_0 = 2$ then the above type of random perturbations formally does not satisfy Assumption 1.1 since in this case we cannot exclude the point 0 which is the fixed point for all F_s, $1 < s < 2$, and so $P^\epsilon(0, \cdot)$ will not have density at all. Moreover, the unit mass at 0 is the invariant measure for any X_n^ϵ, $\epsilon > 0$, and so Theorem 1.1 is not true, as stated, on the whole interval $[0,1]$. However, if $s_0 = 2$ and φ_1^ϵ, φ_2^ϵ, $\cdots$ are independent random variables uniformly distributed on $[-\epsilon, 0]$ with $\epsilon > 0$ small enough, then a careful study by our method of approaches of X_n^ϵ close to 0 yields that invariant measures μ^ϵ of X_n^ϵ having no atom at 0 weakly converge as $\epsilon \to 0$ to the Lebesque measure on $[0,1]$ which is F_2-invariant.

Next, we shall prove a version of the shadowing property which will be sufficient for our purposes.

Lemma 1.1. *Suppose that* x_0, x_1, $x_2, \cdots, x_n$ *is a* δ-*pseudo-orbit, i.e., (I.4.1) holds true with* dist *defined by (1.8) and* δ *small enough. If*

$$\min_{0 \le i \le n-1} \quad \min_{0 \le j \le \upsilon+1} \quad \mathrm{dist}(x_i, a_j) \ge \delta(\lambda-1)^{-1} \quad (1.19)$$

with $a_0, \cdots, a_{\upsilon+1}$ *introduced in Assumption 1.2 and* λ *from (1.15), then there exists a point* $y \in [0,1]$ *such that*

$$\mathrm{dist}(F^i y, x_i) \le \delta(\lambda-1)^{-1} \text{ for all } i = 0, \cdots, n. \quad (1.20)$$

Proof. Notice that the failure of shadowing which we exhibited in the case of tent maps is due, in fact, to the existence of points in $[0,1]$ having no preimages. But under our conditions a δ-pseudo-orbit may contain such points only if it contains also points close to $a_j, j = 0, \cdots, \upsilon+1$ which we prohibit. By this reason if (1.19) holds true then for any $k = 0, 1, \cdots, n-1$ the interval J_k

$= \{z: \text{dist}(z,x_k) \leq \delta(\lambda-1)^{-1}\}$ contains a connected component of the set $F^{-1}J_{k+1}$ provided δ is small enough.

Then the intersection $\bigcap_{k=0}^{n} F^{-k}J_k$ is not empty and any point belonging to this intersection can play the role of y in (1.20). □

Next, we proceed as follows.

Lemma 1.2. *There exists* $C > 0$ *such that for any* $\gamma > 0$, $\gamma < 1$, *each* $x \in [0,1]$, *and an interval* $Q \subset [0,1]$ *one has*

$$I_o^{\epsilon}(\epsilon^{1-\gamma}, n, x, Q) \leq C \text{ mes } Q \qquad (1.21)$$

provided $(\log \epsilon)^4 \geq n \geq (\log \epsilon)^2$ *and* ϵ *is small enough, where* mes *is the Lebesque measure on* $[0,1]$ *and*

$$I_o^{\epsilon}(\rho, n, x, \Gamma) = P_x^{\epsilon}\{ \min_{0 \leq k \leq n-1} \min_{0 \leq j \leq v+1} \text{dist}(X_k^{\epsilon}, a_j) \geq \rho \text{ and } X_n^{\epsilon} \in \Gamma \}.$$

Proof. We employ the same arguments as in [Ki] for the case of expanding transformations. Put

$$I_1^{\epsilon}(\rho, \delta, n, x, \Gamma) = P_x^{\epsilon}\{ \text{dist}(FX_k^{\epsilon}, X_{k+1}^{\epsilon}) < \delta \text{ and } \qquad (1.22)$$

$$\min_{0 \leq k \leq n-1} \min_{0 \leq j \leq v+1} \text{dist}(X_k^{\epsilon}, a_j) \geq \rho \text{ and } X_n^{\epsilon} \in \Gamma \}.$$

Then similarly to Lemma II.1.1 it follows that

$$|I_o^{\epsilon}(\epsilon^{1-\gamma}, n, x, Q) - I_1^{\epsilon}(\epsilon^{1-\gamma}, \epsilon^{1-\beta}, n, x, Q)| \qquad (1.23)$$

$$\leq (\text{mes } Q) \exp(-\alpha/3\epsilon^{\beta})$$

provided ϵ is small enough.

If mes $Q \geq \epsilon$ then one can choose points $v_1, \cdots, v_{\ell_{\epsilon}}$

such that

$$Q \subset \bigcup_{1 \le i \le \ell_\epsilon} U_\epsilon(v_i) \quad \text{and} \quad \text{mes } Q \ge \frac{1}{2} \sum_{1 \le i \le \ell_\epsilon} \text{mes } U_\epsilon(v_i) \qquad (1.24)$$

where $U_\rho(v) = \{w: \text{dist}(w,v) < \rho\}$.

If mes $Q < \epsilon$ then we put $\ell_\epsilon = 1$ and take Q itself in place of $U_\epsilon(v_1)$. Thus

$$I_1^\epsilon(\epsilon^{1-\gamma}, \epsilon^{1-\beta}, n, x, Q) \le \sum_{1 \le i \le \ell_\epsilon} I_1^\epsilon(\epsilon^{1-\gamma}, \epsilon^{1-\beta}, n, x, U_\epsilon(v_i)). \quad (1.25)$$

The probability $I_1^\epsilon(\epsilon^{1-\gamma}, \epsilon^{1-\beta}, n, x, U_\epsilon(v_i))$ involves only $\epsilon^{1-\beta}$-pseudo-orbits $\omega = (x_o, \cdots, x_n)$, $x_k = X_k^\epsilon$, which do not approach the points $a_j, j = 0, \cdots, \upsilon+1$ closer than $\epsilon^{1-\gamma}$, and so if $\gamma > \beta$, say $\gamma = 3\beta$, and ϵ is small enough then we can employ Lemma 1.1 to find a point $y = y^\omega$ whose orbit shadows this $\epsilon^{1-\beta}$-pseudo-orbit in the sense of (1.20) with $\delta = \epsilon^{1-\beta}$. We proceed in the same way as in the proof of Theorem II.4.1 by choosing points $z_{ijk}, j \le \epsilon^{-2\beta}+1$ such that one point z_{ijk} is taken in each connected component of the intersection

$$(\overline{U_{j\epsilon}(x) \setminus U_{(j-1)\epsilon}(x)}) \cap F^{-n} U_\epsilon(v_i) \quad . \quad \text{Since } n \ge (\log \epsilon)^2$$

then we conclude similarly to (II.4.14) that

$$I_1^\epsilon(\epsilon^{1-3\beta}, \epsilon^{1-\beta}, n, x, U_\epsilon(v_i)) \le \qquad (1.26)$$

$$\sum_{j \le \epsilon^{-2\beta}, k} I_2^\epsilon(\epsilon^{1-2\beta}, n, x, z_{ijk}, U_\epsilon(v_i))$$

where the sum is over z_{ijk} such that the orbit $F^\ell z_{ijk}, \ell = 0, \cdots, n$ does not approach the points $a_j, j = 0, \cdots, \upsilon+1$ closer than $\epsilon^{1-2\beta}$, and

$$I_2^\epsilon(\rho,n,x,z,\Gamma)=P_x^\epsilon(\text{dist}(X_\ell^\epsilon,F^\ell z) \leq \rho \text{ for all } \ell=0,\cdots,n \quad (1.27)$$

$$\text{and } X_n^\epsilon \in \Gamma) = \int_{U_\rho(Fz)} \cdots \int_{U_\rho(F^{n-1}z)} \int_{U_\rho(F^nz)\cap\Gamma}$$

$$q_{Fx}^\epsilon(y_1)\cdots q_{Fy_{n-1}}^\epsilon(y_n)dy_1\cdots dy_n.$$

Let $2\beta < \alpha$ then by (1.9) taking into account that the orbit of $z = z_{ijk}$ stays $\epsilon^{1-2\beta}$-apart from 0 and 1 we obtain that

$$I_2^\epsilon(\epsilon^{1-2\beta},n,x,z,U_\epsilon(v_i)) \leq (1+\epsilon^\alpha)^n \int_{U_{\epsilon^{1-2\beta}}(Fz)} \quad (1.28)$$

$$\cdots \int_{U_{\epsilon^{1-2\beta}}(F^{n-1}z)} \int_{U_{\epsilon^{1-2\beta}}(F^nz)\cap U_\epsilon(v_i)} \epsilon^{-1}r_{Fx}\left[\frac{y_1-Fx}{\epsilon}\right]$$

$$\epsilon^{-1}r_{Fy_1}\left[\frac{y_2-Fy_1}{\epsilon}\right]\cdots\epsilon^{-1}r_{Fy_{n-1}}\left[\frac{y_n-Fy_{n-1}}{\epsilon}\right]dy_1\cdots dy_n.$$

Since F has a bounded second derivative apart from the points $a_j, j=0,\cdots,v+1$ and intervals $(y_\ell,F^\ell z)$ do not contain these points then

$$|y_{\ell+1} - Fy_\ell - (y_{\ell+1}-F^{\ell+1}z)+F'(F^\ell z)(y_\ell - F^\ell z)| \leq \epsilon^{2-5\beta} \quad (1.29)$$

provided ϵ is small enough and $\ell = 0,\cdots,n-1$. Next, we set $\eta_\ell = y_\ell - F^\ell z$ and replace in the right hand side of (1.28) each $r_{Fy_\ell}\left[\frac{y_{\ell+1}-Fy_\ell}{\epsilon}\right]$ by $r_{F^{\ell+1}z}\left[\frac{(\eta_{\ell+1}-F'(F^\ell z)\eta_\ell}{\epsilon}\right]$ which acccording to (1.11) and (1.29) may decrease the right hand side of (1.28) by no more than a positive power of ϵ provided $0 < \beta < \frac{1}{5}$. In view of (1.15) we can

employ the one-dimensional counterparts of Propositions II.2.1 and II.3.11 to derive (1.21) from (1.28) and the above arguments precisely in the same way as in the proof of Theorem II.4.1 using obvious simplifications due to the one-dimensional situation. □

To complete the proof of Theorem 1.1 we shall need the following result which enables us to estimate the probabilities of arriving to small neighborhoods of the points $a_j, j=0, \cdots, v+1$ which we had to exclude in Lemma 1.2.

Lemma 1.3. *There exists $C > 0$ such that if $\gamma > 0$ is small enough then for any $x, y \in [0,1]$ and $k \geq \log(\frac{1}{\epsilon})$*

$$P^\epsilon(k, x, U_{\epsilon^{1-\gamma}}(y)) \leq C\epsilon^\gamma \quad . \tag{1.30}$$

Proof. By the Chapman-Kolmogorov formula for any $\ell < k$ one has

$$P^\epsilon(k, x, U_{\epsilon^{1-\gamma}}(y)) = \int_{[0,1]} P^\epsilon(k-\ell, x, dz) P^\epsilon(\ell, z, U_{\epsilon^{1-\gamma}}(y)) \tag{1.31}$$

$$\leq \sup_{z \in [0,1]} P^\epsilon(\ell, z, U_{\epsilon^{1-\gamma}}(y)) \quad ,$$

and so if (1.30) will be proved for $k = \ell$ then it will remain true for any $k \geq \ell$.

Without loss of generality we can assume that $\lambda > 2$ in (1.15). Indeed, we can always choose an integer $r > 0$ such that $\lambda^r > 2$ and then pass to Markov chains $Y_n^\epsilon = X_{nr}^\epsilon$ which are random perturbations of the map F^r satisfying Assumption 1.1 since F is Lipschitz continuous. The assertion of Lemma 1.3 proved for Y_n^ϵ will imply the desired assertion for X_k^ϵ itself. Thus we assume that $\lambda > 2$.

Under Assumption 1.2 we have

$$\sup_{x} |F'(x)| = D < \infty \quad . \tag{1.32}$$

Put

$$\ell = [\tfrac{1}{2}\beta \, (\log(D + 1))^{-1} \, \log(\tfrac{1}{\epsilon})] \quad , \tag{1.33}$$

where $1 > \beta > 0$ and $[\cdot]$ means the integral part. Then

$$(D + 1)^{\ell+1} \leq \epsilon^{-\beta/2} \quad . \tag{1.34}$$

In the same way as in the previous lemma we can pass to $\epsilon^{1-\beta}$-pseudo-orbits making only a negligible mistake in our estimates. In view of (1.32), (1.34), and the continuity of F every $\epsilon^{1-\beta}$-pseudo-orbit $y_0 = x, y_1, \cdots, y_\ell$ satisfies

$$\text{dist}(F^i x, y_i) < \epsilon^{1-2\beta} \text{ for all } i=0,1,\cdots,\ell \quad . \tag{1.35}$$

We shall introduce points $z_i, i=0,\cdots,\ell$ by setting $z_i = F^i x$ if $\min\limits_{0 \leq j \leq \upsilon+1} \text{dist}(a_j, F^i x) \geq \epsilon^{1-2\beta}$ and $z_i = a_j$ if $\text{dist}(a_j, F^i x) < \epsilon^{1-2\beta}$. For ϵ small enough these define points z_i uniquely. By (1.35),

$$\max_{0 \leq i \leq \ell} \text{dist}(z_i, y_i) < 2\epsilon^{1-2\beta} \tag{1.36}$$

but for $z_i = F^i x$ we have a better inequality (1.35). Thus taking into account (1.7), (1.9), and the Chapman-Kolmogorov formula we can write

$$P^\epsilon(\ell, x, \Gamma) \leq I_3^\epsilon(\ell, x, \Gamma) + \exp(-\alpha/3\epsilon^\beta) \tag{1.37}$$

where $2\beta < \alpha$,

$$I_3^\epsilon(\ell,x,\Gamma) = P_x^\epsilon\{X_i^\epsilon \in U^{(i)} \text{ for all } i=0,1,\cdots,\ell \text{ and } X_\ell^\epsilon \in \Gamma\} \quad (1.38)$$

$$= \int_{U^{(0)}} \cdots \int_{U^{(\ell-1)}} \int_{U^{(\ell)} \cap \Gamma} q_{Fx}^\epsilon(y_1) q_{Fy_1}^\epsilon(y_2) \cdots q_{Fy_{\ell-1}}^\epsilon(y_\ell)$$

$$dy_1 \cdots dy_\ell \le (1+\epsilon^\alpha)^\ell \int_{U^{(0)}} \cdots \int_{U^{(\ell-1)}} \int_{U^{(\ell)} \cap \Gamma} \epsilon^{-1} r_{Fx}\left[\frac{\sigma(Fx,y_1)}{\epsilon}\right]$$

$$\times \epsilon^{-1} r_{Fy_1}\left[\frac{\sigma(Fy_1,y_2)}{\epsilon}\right] \cdots \epsilon^{-1} r_{Fy_{\ell-1}}\left[\frac{\sigma(Fy_{\ell-1},y_\ell)}{\epsilon}\right] dy_1 \cdots dy_\ell \quad,$$

where $U^{(i)} = \{v: \text{ dist}(v,z_i) < \epsilon^{1-2\beta}\}$ if $z_i = F^i x$ and $U^{(i)} = \{v: \text{ dist}(v,z_i) < 2\epsilon^{1-2\beta}\}$ if $z_i \neq F^i x$.

We cannot proceed precisely in the same way as in the proof of Lemma 1.2 using (1.29) since the derivative of F may have discontinuities at the points $a_j, j=0,\cdots,\upsilon+1$.

Let $F_+'(z_i)$ and $F_-'(z_i)$ be the right and the left derivatives of F at z_i, respectively. Put $\eta_i = y_i - z_i$, $b_i = z_{i+1} - Fz_i$, $A(\eta_i) = F_+'(z_i)$ if $\eta_i \ge 0$ and $A(\eta_i) = F_-'(z_i)$ if $\eta_i < 0$, $i = 0,\cdots,\ell$. Then for ϵ small enough and $y_i \in U^{(i)}$, $i = 1,\cdots,\ell$ one has

$$|\sigma(Fy_i,y_{i+1}) - \eta_{i+1} + A(\eta_i)\eta_i - b_i| \le \epsilon^{2-5\beta}, \quad (1.39)$$

$i = 0,\cdots,\ell-1$ since by our choice of the points z_i the map F can be extended as a C^2 function into each closed interval $[y_i,z_i]$.

Replace in the right hand side of (1.38) each $r_{Fy_i}\left[\frac{\sigma(Fy_i,y_{i+1})}{\epsilon}\right]$ by $r_{z_{i+1}}\left[\frac{\eta_{i+1}-A(\eta_i)\eta_i+b_i}{\epsilon}\right]$.

According to (1.11) and (1.39) this substitution may decrease the right hand side of (1.38) by no more than a positive power of ϵ provided $\beta < \frac{1}{5}$. These will lead to

an expression which can be bounded by the integral

$$I_4^\epsilon(\ell,x,\Gamma) = \int_{\mathbb{R}^1} \cdots \int_{\mathbb{R}^1} \int_\Gamma \epsilon^{-1} r_{z_1}\left[\frac{\eta_1+b_0}{\epsilon}\right]$$

$$\times \epsilon^{-1} r_{z_2}\left[\frac{\eta_2-A(\eta_1)\eta_1+b_1}{\epsilon}\right] \tag{1.40}$$

$$\times \cdots \times \epsilon^{-1} r_{z_\ell}\left[\frac{\eta_\ell-A(\eta_{\ell-1})\eta_{\ell-1}+b_{\ell-1}}{\epsilon}\right] d\eta_1\cdots d\eta_\ell .$$

Define inductively

$$\Xi_{k+1}^\epsilon = A(\Xi_k^\epsilon)\ \Xi_k^\epsilon + \epsilon\theta_{k+1}- b_k \tag{1.41}$$

where $\Xi_0^\epsilon = 0$ and $\theta_1,\theta_2,\cdots$ are independent random variables with the distributions

$$P\{\theta_k \in \Psi\} = \int_\Psi r_{z_k}(\eta)d\eta . \tag{1.42}$$

It is easy to see that Ξ_k^ϵ $k = 0,1,\cdots$ is a nonhomogeneous Markov chain whose transition density from $\Xi_k^\epsilon = \eta$ to $\Xi_{k+1}^\epsilon = \eta$ has the form $\epsilon^{-1} r_{z_{k+1}}\left[\frac{\zeta-A(\eta)\eta+b_k}{\epsilon}\right]$. Thus

$$I_4^\epsilon(\ell,x,\Gamma) = P\{\Xi_\ell^\epsilon \in \Gamma\} . \tag{1.43}$$

By (1.41),

$$\Xi_\ell^\epsilon = \epsilon \sum_{k=1}^{\ell-1} A(\Xi_{\ell-1}^\epsilon)\cdots A(\Xi_k^\epsilon)(\theta_k- \epsilon^{-1}b_{k-1}) + \epsilon\theta_\ell - b_{\ell-1}. \tag{1.44}$$

For any sequence $\kappa = (\kappa_1,\cdots,\kappa_{\ell-1})$ $\kappa_i = \pm 1$ of +1 and -1 we put $B_i(\kappa) = F_+'(z_i)$ if $\kappa_i = +1$ and $B_i(\kappa) = F_-'(z_i)$ if $\kappa_i = -1$. Denote also

$$\Xi_{\ell}^{\epsilon}(\kappa) = \epsilon \sum_{k=1}^{\ell-1} B_{\ell-1}(\kappa) \cdots B_k(\kappa)(\theta_k - \epsilon^{-1} b_{k-1}) + \epsilon \theta_{\ell} - b_{\ell-1}. \quad (1.45)$$

Since by (1.15), $|B_i(\kappa)| \geq \lambda$ for all i and κ we derive in the same way as in the proof of (II.2.7) that

$$P\{\Xi_{\ell}^{\epsilon}(\kappa) \in \Gamma\} \leq \tilde{C}\epsilon^{-1} \lambda^{-\ell} \quad \text{mes } \Gamma \quad (1.46)$$

for all $\kappa = (\kappa_1, \cdots, \kappa_{\ell-1})$. Then we obtain

$$P\{\Xi_{\ell}^{\epsilon} \in \Gamma\} \leq \sum_{\kappa} P\{\Xi_{\ell}^{\epsilon}(\kappa) \in \Gamma\} \leq \tilde{C}\epsilon^{-1} \left[\frac{2}{\lambda}\right]^{\ell} \text{mes } \Gamma . \quad (1.47)$$

Since we assume that $\lambda > 2$ then taking $\Gamma = U_{\epsilon^{1-\gamma}}(y)$ and ℓ from (1.33) we derive (1.30) from (1.31), (1.37), and the arguments following after (1.37) together with (1.43) and (1.47), provided $\gamma > 0$ is small enough. $\square$

Now we can complete the proof of Theorem 1.1. From Lemma 1.3 and the Markov property it follows that

$$P_x^{\epsilon} \left\{ \min_{n \geq k \geq \log\left[\frac{1}{\epsilon}\right]} \min_{v+1 \geq j \geq 0} \text{dist}(X_k^{\epsilon}, a_j) < \epsilon^{1-\gamma} \right\} \leq Cn\epsilon^{\gamma} . \quad (1.48)$$

Thus taking $n = n(\epsilon) = [2(\log \epsilon)^2]$ and $\ell = \ell(\epsilon) = [\log \frac{1}{\epsilon}] + 1$ we obtain from (1.21), (1.48), and the Markov property that for any interval $Q \subset [0,1]$,

$$P^{\epsilon}(n,x,Q) \leq P_x^{\epsilon}\{X_n^{\epsilon} \in Q \text{ and } \min_{n \geq k \geq \log(\frac{1}{\epsilon})} \min_{v+1 \geq j \geq 0} \quad (1.49)$$

$$\text{dist}(X_k^{\epsilon}, a_j) \geq \epsilon^{1-\gamma}\} + \epsilon^{\gamma/2} = \int_{[0,1]} P^{\epsilon}(\ell,x,dy)$$

$$I_o^{\epsilon}(\epsilon^{1-\gamma}, n-\ell, y, Q) + \epsilon^{\gamma/2} \leq C \text{ mes } Q + \epsilon^{\gamma/2}$$

provided ϵ is small enough. Hence for any invariant probability measure μ^ϵ of X_n^ϵ one has

$$\mu^\epsilon(Q) = \int_{[0,1]} d\mu^\epsilon(x) \, P^\epsilon(n,x,Q) \leq C \text{ mes } Q + \epsilon^{\gamma/2} . \qquad (1.50)$$

which being true for any interval $Q \subset [0,1]$ implies the assertion of Theorem 1.1. $\square$

Adapting the arguments of Section 2.6 to the one-dimensional piecewise C^2 and expanding maps F in the same way as above one obtains the entropy convergence result provided $\upsilon = 1$.

Theorem 1.2. *Let* $\Pi = (Q_1, \cdots, Q_k)$ *be a partition of* $[0,1]$ *into intervals with sufficiently small lengths and suppose that* $\upsilon = 1$ *which means that invariant measures* μ^ϵ *of* X_n^ϵ *weakly converge to the unique absolutely continuous F-invariant measure* μ. *Then*

$$\lim_{\epsilon \to 0} h^\epsilon(\theta, \zeta^\epsilon) = h_\mu(F) \qquad (1.51)$$

where ζ^ϵ *is the partition of the sample space* Ω *into the sets* $\Gamma_j^\epsilon = \{\omega: X_0^\epsilon \in Q_j\}$, $h_\mu(F)$ *is the entropy of* F *relative to the measure* μ, *and* $h^\epsilon(\theta, \zeta^\epsilon)$ *is the entropy of the shift transformation* θ.

2. Misiurewicz's maps of an interval.

In this section we shall discuss certain points concerning random perturbations of one-dimensional maps satisfying Misiurewicz's conditions from [Mi]. For the detailed exposition we refer the reader to Katok and Kifer [KK].

We shall consider random perturbations X_n^ϵ satisfying Assumption 1.1 of maps F of the interval having a

non-positive Schwarzian derivative, no sinks and future
orbits of critical points staying away from critical
points. We shall restrict ourselfs to the most widely
considered one- parameter family of maps

$$F_\lambda: \quad x \to 4\lambda x(1-x) \qquad\qquad (2.1)$$

for which the above conditions are satisfied for a set of
parameters λ having cardinality of the continuum.

   Assumption 2.1. The map F_λ has the form (2.1), it
has no stable periodic orbit, and

$$\frac{1}{2} \in \mathcal{T}_\lambda = \overline{\underset{n \geq 1}{\cup} F_\lambda^n (\tfrac{1}{2})} \quad . \qquad\qquad (2.2)$$

   According to Misiurewicz [Mi] any map F_λ satisfying
Assumption 2.1 possess exactly one absolutely continuous
invariant measure μ_{F_λ} which is ergodic.

   The following result was proved in Katok and Kifer
[KK].

   Theorem 2.1. *Suppose that random perturbations* X_n^ϵ
of the map F_λ *meet the conditions of Assumption 1.1 and*

F_λ *satisfies Assumption 2.1. Then invariant measures* μ^ϵ

of X_n^ϵ *weakly converge to* μ_{F_λ} *as* $\epsilon \to 0$.

   Example 2.1. Similarly to the previous section we can
consider the following model of random perturbations
satisfying our conditions. Let $\varphi_1, \varphi_2, \cdots$ be independent
random variables with the same distribution having a smooth
density $\rho(x)$ concentrated on $[-1,1]$. Suppose that λ_0

is a fixed parameter such that $\frac{1}{2} < \lambda_0 < 1$ and the map
F_{λ_0} satisfies the above conditions. Then for $\epsilon < 1-\lambda_0$
the composition of independent random transformations
$F_{\lambda_0 + \epsilon \varphi_i}$, $i=1,2,\cdots$ generates a Markov chain

$X_n^\epsilon = F_{\lambda_0 + \epsilon \varphi_n} \circ \cdots \circ F_{\lambda_0 + \epsilon \varphi_1} x$ which, considered on the
invariant interval $[4\lambda_0 (\lambda_0 + \epsilon)(1-\lambda_0 - \epsilon) - \epsilon, \ \lambda_0 + \epsilon]$, belongs to

the class of random perturbations satisfying our
conditions. The case $\lambda_0 = 1$ must be studied separately
since then we cannot exclude from the consideration the
point 0 which is fixed for all F_λ. The transition
probability of X_n^ϵ can be written in the form

$$P^\epsilon(x,\Gamma) = P\{F_{\lambda_0+\epsilon\varphi_i}x\epsilon\Gamma\} = P\{\epsilon\varphi_i \in (x(1-x))^{-1}\Gamma - \lambda_0\} \quad (2.3)$$

$$= (4\epsilon x(1-x))^{-1} \int_\Gamma \rho\left[\frac{1}{\epsilon}\left[\frac{y}{4x(1-x)} - \lambda_0\right]\right] dy \quad .$$

Thus the transition density $p^\epsilon(x,y) = (4\epsilon x(1-x))^{-1}$
$\rho\left[\frac{1}{\epsilon}\left[\frac{y-F_{\lambda_0}x}{4x(1-x)}\right]\right]$ does not have the form $q^\epsilon(y-F_{\lambda_0}x)$ needed
for an application of the Frobenius-Perron operator method
described at the beginning of the previous section.

   Remark 2.1. The stability of measures μ_{F_λ} with
respect to random perturbations is especially interesting
in view of the fact that in general there is no stability
with respect to deterministic perturbations in this case.
Indeed, consider F_λ with λ close to 1. Clearly, F_1
satisfies Assumption 2.1 and it has absolutely continuous
invariant measure with the density $\pi^{-1}(x(1-x))^{-1/2}$. Put
$n_\lambda = \min(n>1: F_\lambda^n(\frac{1}{2}) \geq \frac{1}{2})$. Since $F_1^n(\frac{1}{2}) = 0$ for all $n>1$
then if $F_\lambda^{n_\lambda}(\frac{1}{2}) > \frac{1}{2}$ by the continuity one can find $\beta(\lambda)$
such that $1 > \beta(\lambda) > \lambda$ and $F_{\beta(\lambda)}^{n_\lambda}(\frac{1}{2}) = \frac{1}{2}$. Hence $\frac{1}{2}$ is a
periodic point of $F_{\beta(\lambda)}$ and its orbit is attracting since
$F_\lambda^{\cdot}(\frac{1}{2}) = 0$ for any λ. Thus we obtained a sequence $\lambda_k \uparrow 1$
such that any F_{λ_k} has an attracting periodic
orbitcontaining $\frac{1}{2}$ and only one point of this orbit can be
to the right of $\frac{1}{2}$. The invariant measure υ_{λ_k} supported
by this periodic orbit is stable with respect to random

perturbations since the complement of its basin of attraction has zero Lebesque measure (see Collet and Eckmann [CE1], Proposition II.5.7). On the other hand, measures ν_k do not converge as $\lambda_k \to 1$ to the smooth invariant measure of F_1 since the above periodic orbits have only one point to the right of $\frac{1}{2}$ and so all weak limits of ν_k have support in the interval $[0, \frac{1}{2}]$. Similar examples can be constructed for $\lambda_k \to \lambda_o \neq 1$ with λ_o satisfying Assumption 2.1.

The maps F_λ do not necessarily have the shadowing property for all pseudo-orbits. However one can obtain the following result (see Katok and Kifer [KK], Lemma 2.3).

Lemma 2.1. *Suppose that* F_λ *satisfies Assumption 2.1 and let* $x_o, \cdots, x_n$ *be a* ϵ^β*-pseudo-orbit of* F_λ, *i.e., (I.1.4) holds true with* $F = F_\lambda$ *and dist defined by (1.8). There exists a constant* $C > 0$ *depending only on* F_λ *such that if* $0 \leq \gamma \leq \beta/2$ *and*

$$|x_k - \tfrac{1}{2}| \geq 2C\epsilon^\gamma, \quad k = 0, \cdots, n-1 \qquad (2.4)$$

then one can find a point y *so that*

$$\text{dist}(F_\lambda^k y, x_k) \leq C\epsilon^{\beta - \gamma}, \quad k = 0, \cdots, n. \qquad (2.5)$$

Since $F_\lambda'(\frac{1}{2}) = 0$ then, of course, the maps F_λ are not expanding. However, Assumption 2.1 yields some substitution for expanding which turns out to be sufficient both for Lemma 2.1 and other aspects of our approach.

Lemma 2.2. *Suppose that* F_λ *satisfies Assumption 2.1. There exists* $\eta > 1$ *such that for any* $\rho > 0$ *one can find an integer* $M_\rho > 0$ *so that*

$$|(F_\lambda^{M_\rho})'(x)| \geq \eta \quad provided \quad \min_{0 \leq i < M_\rho} |F_\lambda^i x - \tfrac{1}{2}| \geq \rho \qquad (2.6)$$

and

$$|(F_\lambda^n)`(x)| \quad \text{provided} \quad \text{dist}(F_\lambda^n x, \mathcal{T}_\lambda) \geq \rho \qquad (2.7)$$

for any $x \in [0,1]$ *and* $n \geq 1$.

For the proof we refer the reader to Misiurewicz [Mi], Theorem 1.3 and to Katok and Kifer [KK], Lemma 2.2.

Under Assumption 2.1 the map F_λ becomes expanding in the sense that $|(F_\lambda^n)`(x)|$ grows exponentially fast in n for points x whose orbit stay away from $\mathcal{T}_\lambda$. Indeed, suppose that $F_\lambda^k x \neq \frac{1}{2}$ for all $k = 0, 1, \cdots, n$. While $F_n^k x$ is not too close to $\frac{1}{2}$ then the derivative grows exponentially fast by (2.6). If for some k, $|F_\lambda^k x - \frac{1}{2}| = \rho$ then

$$|(F_\lambda^{k+1})`x| = 8\lambda\rho |(F_\lambda^k)`x| \qquad (2.8)$$

and

$$(2.9) \qquad \text{dist}(F_\lambda^{k+1} x, \mathcal{T}_\lambda) \leq \text{dist}(F_\lambda^{k+1} x, F_\lambda(\tfrac{1}{2})) = 4\lambda\rho^2 .$$

Thus in view of (2.2) and (2.9) in order to have another chance to get close to $\frac{1}{2}$ the orbit must accumulate the derivative of order ρ^{-2} which according to (2.6) will take of order $\log(\frac{1}{\rho})$ steps. If $\ell_\rho = C_1 \log(\frac{1}{\rho})$ is this number of steps then $|(F^{k+\ell_\rho+1})`x| = |(F_\lambda^k)`x| C_2 \rho^{-1}$ $= |(F_\lambda^k)`x| C_2 (e^{1/C_1})^{\ell_\rho}$ which again leads to the exponential growth.

Still, proceeding with our method one has to face certain complications due to small derivatives of F_λ near $\frac{1}{2}$. Lemma 2.1 enables us to employ the linearization procedure if we restrict ourselfs to paths of X_n^ϵ which

are ϵ^β-pseudo-orbits staying outside of the $2C\epsilon^\gamma$-neighborhood of the point $\frac{1}{2}$. However, this will lead to orbits of F_λ which may approach $\frac{1}{2}$ as close as $C(2\epsilon^\gamma + \epsilon^{\beta-\gamma})$, and so the derivatives of F_λ^k may be sometimes that small. By this reason a direct counterpart of Proposition II.2.1 will not work here. The following result proved in Appendix to Katok and Kifer [KK] saves the situation.

Lemma 2.3. *Suppose in addition to Assumption 1.1 that for each $x \in [0,1]$ the number of points of discontinuity of $r_x(\xi)$ in ξ is bounded by a number N independent of x and on each interval of continuity $r_x(\xi)$ is Lipschitz continuous in ξ. For arbitrary points $x_1, \cdots, x_n \in [0,1]$ let $\theta_1, \cdots, \theta_n$ be independent random variables with distribution functions $P(\theta_i \leq a) = \int_{-\infty}^{a} r_{x_i}(\xi)\,d\xi$. Then there exist $C, \kappa > 0$ independent of $x_1, \cdots, x_n$ and n such that for any nonzero numbers $a_1, \cdots, a_n$ the distribution function of the random variable*

$$\left[\sum_{1 \leq i \leq n} a_i^2 \right]^{-1/2} \sum_{1 \leq i \leq n} a_i(\theta_i - E\theta_i)$$

has the derivative, i.e., the probability density function, satisfying

$$r_{x_1, \cdots, x_n}^{a_1, \cdots, a_n}(\zeta) \leq Ce^{-\kappa|\zeta|}$$

where $E\theta_i$ is the expectation of θ_i.

We discussed here only few arguments involved in the proof of Theorem 2.1 which is pretty long and can be found in Katok and Kifer [KK].

Remark 2.2. One can adapt the arguments of Section 2.6 and prove Theorem 1.2 of the previous section also for maps F_λ satisfying Assumption 2.1.

Remark 2.3. Another class of maps with a critical point (and so not uniformly expanding) possessing absolutely continuous invariant measures was studied by Collet and Eckmann [CE2]. For instance, for a set of

parameters having a positive Lebesque measure the one-parameter family of maps F_λ: $[0,1] \to [0,1]$ given by the formula

$$F_\lambda x = \begin{cases} 1-2|x - \frac{1}{2}| & \text{if} \quad |x - \frac{1}{2}| \geq \lambda \\ 1-\lambda-(x - \frac{1}{2})^2 \lambda^{-1} & \text{if} \quad |x - \frac{1}{2}| \leq \lambda, \end{cases}$$

$0 < \lambda < \frac{1}{2}$ satisfies the conditions of [CE2]. Collet [Col] studied random perturbations of Boyarsky's type for this class of maps employing the Frobenius-Perron operator method described at the beginning of Section 1.1. It is not difficult to adapt the machinery of Katok and Kifer [KK] in order to prove Theorem 2.1 for this class of maps employing results of Appendices A and E from Collet [Col] which actually provide necessary dynamical prerequisites for our approach similar to Section 2 of Katok and Kifer [KK].

3. **Lorenz's type models.**

In this section we shall discuss random perturbations of model dynamical systems which are believed to describe main features of the Lorenz attractor (see Guckenheimer and Holmes [GH] or Sparrow [Sp]).

In 1963 E. Lorenz [Lo] published a paper describing a qualitative study by numerical integration of the following three-dimensional system of ordinary differential equations with three parameters $\sigma, r, b > 0$,

$$\frac{dx}{dt} = \sigma(y-x)$$

$$\frac{dy}{dt} = rx - y - xz \qquad (3.1)$$

$$\frac{dz}{dt} = xy - bz$$

derived from a model of fluid convection. Computer experiments indicated that for certain choice of parameters $\sigma, r,$ and b the flow F^t generated by (4.1) has an attractor (called now Lorenz's) where orbits of F^t

exhibit a chaotic behavior.

The divergence of the vector field $(\sigma(y-x),\ rx-y-xz,\ xy-bz)$ equals $-(\sigma+1+b)$, and so F^t contracts the volume by $e^{-(\sigma+1+b)t}$ for $t>0$. Furthermore, consider the Lyapunov function $V(x,y,z) = rx^2+\sigma y^2+\sigma(z-2r)^2$ then

$$\frac{dV(F^t(x,y,z))}{dt}\bigg|_{t=0} = -2\sigma(rx^2+y^2+bz^2-2brz) . \qquad (3.2)$$

Let c be the maximum of V in the bounded domain where $\frac{dV}{dt} \geq 0$. If $\delta > 0$ is small enough then it is easy to see that all orbits of F^t eventually enter the bounded ellipsoid $\mathcal{E} = \{(x,y,z):\ V(x,y,z) \leq c + \delta\}$. Thus we conclude that all orbits tend towards a bounded set of zero volume (see Sparrow [Sp], Appendix C).

Let X_t^ϵ be diffusion random perturbations of the flow F^t described in Example II.1.3. For any $\rho>0$ we can consider Markov chains $Y_n^{\epsilon,\rho} = X_{n\rho}^\epsilon$ which are random perturbations of $F = F^\rho$. In view of (3.2) it is easy to see that conditions of Theorem I.1.7 are satisfied for Markov chains $Y_n^{\epsilon,\rho}$. Thus all their invariant measures have support in $\mathcal{E}$ and when $\epsilon \to 0$ then all weak limits of these measures are supported by a bounded set of zero volume. In particular, this is true for invariant measures μ^ϵ of diffusion processes X_t^ϵ.

The most popular choice of parameters leading to what is usually called the Lorenz attractor is $\sigma = 10$, $r = 28$ and $b = 8/3$. The origin O is the stationary point of the hyperbolic type for the system (3.1). It has the two-dimensional stable manifold $W^s(O)$ and the one-dimensional unstable manifold consisting of two branches Γ_1 and Γ_2. The plane $\Pi = \{(x,y,z):\ z = 27\}$ contains two more hyperbolic fixed points O_1 and O_2 which have one-dimensional stable manifolds which are lines

contained in Π and two-dimensional unstable manifolds transverse to Π.

Next, one considers the Poincare return map G of the plane Π to itself. Namely, if v is a point on this plane and the integral curve containing v goes downwards when intersecting the plane Π at v then Gv is the point of the next intersection of the integral curve with Π. The map G is not defined on the intersection $W^s(0) \cap \Pi$ and G maps points approaching this intersection from one side close to $Q_1 = \Gamma_1 \cap \Pi$ while points approaching $W^s(0) \cap \Pi$ from another side are being mapped close to $Q_2 = \Gamma_2 \cap \Pi$.

By a change of coordinates we can reduce the study to the transformation G mapping the square $S = \{(x,y): |x| \leq 1, |y| \leq 1\} \subset \Pi$ into itself. The right hand side curvilinear triangle is the image of the top rectangle $S_1 = \{(x,y): |x| \leq 1, 0 < y \leq 1\}$ and the left hand side triangle is the image of the bottom rectangle $S_2 = \{(x,y): |x| \leq 1, -1 \leq y < 0\}$.

Taking into account that the system (3.1) is invariant with respect to the transformation $x \to -x$, $y \to -y$, $z \to z$ we conclude that the map G can be represented by means of 2 function $f_1(x,y)$, $g_1(x,y)$, $f_2(x,y) = -f_1(-x,-y)$, and $g_2(x,y) = -g_1(-x,-y)$ in the following way

$$G(x,y) = (f_i(x,y), g_i(x,y)) \quad \text{if} \quad (x,y) \in S_i, i = 1,2 . \quad (3.3)$$

We assume that the functions f_1 and g_1 (and so also f_2 and g_2) can be extended continuously to $\tilde{W} = \{|x| \leq 1, y=0\}$ so that

$$\lim_{y \to 0} (f_i(x,y), g_i(x,y)) = Q_i, \quad i = 1,2 . \quad (3.4)$$

The map G has also two hyperbolic fixed points O_1 and O_2. One can proceed with the ergodic theory of the map G,

as well as, of the flow F^t provided G satisfies some hyperbolicity conditions. These were not yet established rigorously. However the following conditions which according to Afraimovich, Bykov, and Shilnikov [ABS] yield the hyperbolicity of G were checked with the help of a computer by Sinai and Vul [SV]. These conditions are

$$\left\|\frac{\partial f_1}{\partial x}\right\| < 1, \quad \left\|\left[\frac{\partial g_1}{\partial y}\right]^{-1}\right\| < 1,$$

$$\left\|\left[\frac{\partial g_1}{\partial y}\right]^{-1}\frac{\partial f_1}{\partial y}\right\| \cdot \left\|\frac{\partial g_1}{\partial x}\right\| < \left[1 - \left\|\frac{\partial f_1}{\partial x}\right\|\right]\left[1 - \left\|\left[\frac{\partial g_1}{\partial y}\right]^{-1}\right\|\right], \quad (3.5)$$

$$1 - \left\|\left[\frac{\partial g_1}{\partial y}\right]^{-1}\right\| \cdot \left\|\frac{\partial f_1}{\partial x}\right\| > 2\left[\left\|\left[\frac{\partial g_1}{\partial y}\right]^{-1}\left[\frac{\partial f_1}{\partial y}\right]\right\| \cdot \left\|\left[\frac{\partial g_1}{\partial y}\right]^{-1}\right\| \cdot \left\|\frac{\partial g_1}{\partial x}\right\|\right]^{\frac{1}{2}}$$

where $\|h(x,y)\| = \sup\limits_{x,y\in S} |h(x,y)|$.

The Lorenz attractor for G is $K = \overline{\bigcap\limits_{0 \leq n < \infty} G^n S}$ and the corresponding attractor for the flow F^t can be written in the form $\Lambda = \bigcup\limits_{-\infty < t < \infty} F^t K$. The attractor K consists of smooth curves stretched along the y-axis. Bunimovich and Sinai [BS] showed that there exists a G-invariant probability measure υ on K which is absolutely continuous with respect to the Lebesgue measure generated by the length on smooth curves forming K. This property determines the measure υ uniquely and this measure possesses essentially the same properties as the well known Sinai-Bowen-Ruelle measure (see Bunimovich and Sinai [BS] and Bunimovich [Bu]). The measure υ generates the unique F^t-invariant measure μ on Λ such that

$$\left.\frac{\mu\left(\bigcup\limits_{0 \leq t \leq s} F^t \Gamma\right)}{ds}\right|_{s=0} = \upsilon(\Gamma) \qquad (3.6)$$

for any Borel subset $\Gamma \subset K$ whose distance from $W^s(O_1) \cup W^s(O_2)$ is positive.

Theorem 3.1. *Suppose that the conditions (3.5) hold true. Then invariant measures μ^ϵ of diffusion random perturbations X_t^ϵ weakly converge to μ as $\epsilon \to 0$.*

The proof of this result proceeds by the method of [Ki]. In place of processes X_t^ϵ we shall consider Markov chains $Y_n^\epsilon = X_{nr}^\epsilon$, $n=0,1,2,\cdots$ for some small but fixed $r > 0$ which are random perturbations of the diffeomorphism $F = F^r$. The number r is chosen so that an application of F^t with $|t| \leq r$ does not destroy expanding and contracting properties of the map G in the transverse and parallel to $W^s(0)$ directions, respectively. Next, if we consider the time $n = n(\epsilon)$ of order $(\log \epsilon)^2$ then we may restrict our attention to a neighborhood U of the attractor Λ.

Lemma 3.1. *Let U be a sufficiently small neighborhood of the attractor Λ. There exists a constant $C > 0$ such that if $x_0, x_1, \cdots, x_n$ is a δ-pseudo-orbit of $F = F^1$ staying in U and satisfying*

$$\min_{0 \leq i \leq n} \text{dist}(x_i, W^s(0)) > C\delta \qquad (3.7)$$

then one can find a point $y \in U$ such that

$$\max_{0 \leq i \leq n} \text{dist}(x_i, F^i y) \leq Cn\delta \qquad (3.8)$$

where $W^s(0)$ in (3.7) denotes a connected component of the stable manifold of 0 in U containing 0 and dist here is the Euclidean distance.

In order to obtain this kind of the shadowing one combines the arguments leading to the shadowing for the hyperbolic transformation G together with the corresponding arguments valid in a neighborhood of the hyperbolic fixed point 0. Namely, in the ρ-neighborhood of

$W^S(0)$ with $\rho > 0$ small but fixed the expanding and contracting in transverse and parallel to $W^S(0)$ directions, respectively, is due to the presence of the hyperbolic fixed point 0. Thus if the orbit of the flow starts in the ρ/N-neighborhood of $W^S(0)$ with N large enough and exits from the ρ-neighborhood of $W^S(0)$ then expanding and contracting will be already accumulated enough not to be destroyed until the orbit pierces S. For orbits staying outside the ρ/N-neighborhood of $W^S(0)$ we derive expanding and contracting properties along them from the corresponding hyperbolicity properties of G which follow from (3.5). The condition (3.7) enables us to avoid difficulties connected with the discontinuity of G.

Next, employing the above arguments we derive similarly to Lemma 1.2 the absolute continuity in the unstable direction of probabilities that $Y_n^\epsilon = X_{nr}^\epsilon$ arrives to a set for $n(\epsilon) \sim (\log \epsilon)^2$ steps along paths which do not approach $W^S(0)$ closer than $\epsilon^{1-\gamma}$. Since the flow F^t stretches in the transverse to $W^S(0)$ direction then in the same way as in Lemma 1.3 we conclude that for $n \geq \log(\frac{1}{\epsilon})$ and $\gamma > 0$ small enough Y_n^ϵ may belong to the $\epsilon^{1-\gamma}$-neighborhood of $W^S(0)$ with probability not exceeding ϵ^γ. After that we complete the proof of Theorem 3.1 in the same way as the proof of Theorem 1.1. We note that the technical prerequisites for our method can be found in Bunimovich and Sinai [BS] or easily derived from their arguments.

Remark 3.1. One can generalize this approach in order to apply the method to situations where some kind of hyperbolicity conditions holds true only for an appropriate return map of a flow and not for the flow itself.

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