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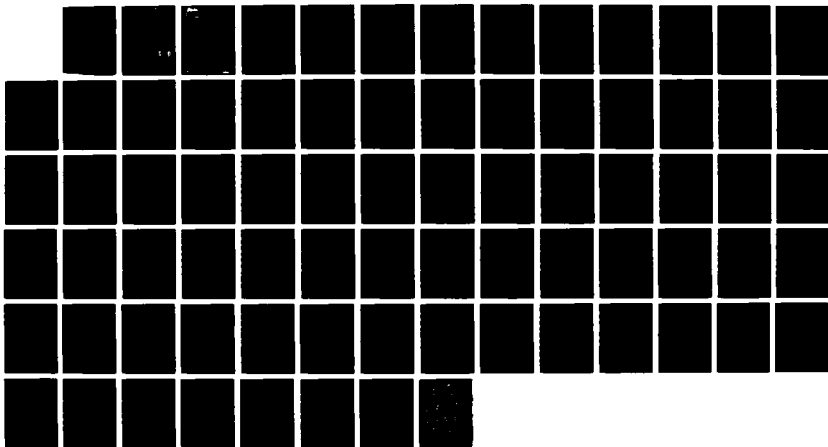
ESTIMATION AND CONTROL OF DISTRIBUTED MODELS FOR
CERTAIN ELASTIC SYSTEMS. (U) OKLAHOMA UNIV NORMAN DEPT
OF MATHEMATICS L W WHITE 30 SEP 87 AFOSR-TR-88-0245
AFOSR-84-0271

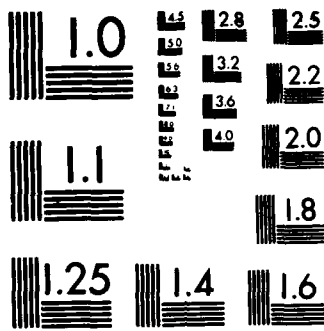
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AFOSR-TR-88-0240

Estimation and Control of Distributed Models
for Certain Elastic Systems Arising in Large Space Structures

AFOSR Grant No. AFOSR-84-0271

Final Report
September 30, 1987

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SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION Unclassified		1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY UNCLASSIFIED		3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release, distribution unlimited
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE		
4. PERFORMING ORGANIZATION REPORT NUMBER(S)		5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR- 88-0249

6a. NAME OF PERFORMING ORGANIZATION University of Oklahoma	6b. OFFICE SYMBOL <i>(If applicable)</i>	7a. NAME OF MONITORING ORGANIZATION AFOSR
6c. ADDRESS (City, State and ZIP Code) Norman, Ok 73019		7b. ADDRESS (City, State and ZIP Code) BLDG #410 Bolling AFB, DC 20332-6448

8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR	8b. OFFICE SYMBOL <i>(If applicable)</i> NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFOSR-84-0271
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8c. ADDRESS (City, State and ZIP Code) BLDG #410 Bolling AFB, DC 20332-6448	10. SOURCE OF FUNDING NOS.			
	PROGRAM ELEMENT NO. 61192F	PROJECT NO. 2304	TASK NO. A1	WORK UNIT NO.
11. TITLE (Include Security Classification) Estimation and Control of Distr.Models				

12. PERSONAL AUTHOR(S) Luther White				
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13a. TYPE OF REPORT Final	13b. TIME COVERED FROM 7/1/84 TO 9/30/87	14. DATE OF REPORT (Yr., Mo., Day) 87/09/30	15. PAGE COUNT 75
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16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB. GR.	

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

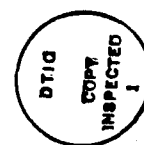
The goal of this research was to study estimation and control of elastic systems composed of beams and plates. Specifically, the research considered the problem of locating the optimal placement of controllers on a beam or plate and the problem of controlling general three-dimensional elastic models that incorporate nonlinear friction and contact laws in the boundary conditions. This final report summarizes those results.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED <input checked="" type="checkbox"/> SAME AS RPT	21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED
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22a. NAME OF RESPONSIBLE INDIVIDUAL James M Crowley, Maj, USAF	22b. TELEPHONE NUMBER <i>(Include Area Code)</i> 202-767- 5025	22c. OFFICE SYMBOL UNCLASSIFIED
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1. Introduction.

The goal of this project has been to study the estimation and control of elastic systems composed of beams and plates. This general goal has served as a guide for our research over the last several years. We have made substantial progress in our investigation of topics within this and related areas.

For control we have studied two basic problems. The first is that of locating the optimal placement of controllers on a beam or a plate in the case of either static or dynamic models. The second involves the control of general three space dimensional elastic models that incorporate nonlinear friction and contact laws in their boundary conditions.

In estimation and identification we have obtained results concerning the estimation of elastic coefficients in beams and plates for static and dynamic models. Further, we have investigated the estimation of certain damping terms arising in plate models. Finally, in related work we have considered the determination of diffusion parameters in parabolic equations, and conditions for bijectivity of the parameter to state mapping for discrete approximations of certain elliptic boundary value problems.

Our work has sought generally to explore the theoretical analysis of the distributed models, algorithms for numerical implementation, and an approximation theory relating the two.

2. Control Problems

Let Ω be an open bounded domain in \mathbb{E}^2 with a Lipschitz boundary Γ which is occupied by a plate. We consider dynamic models of the form

$$(2.1) \quad u_{tt} + Bu_t + Au = \beta(t)\Phi \text{ in } \Omega \times (0, T)$$

where

$$B\varphi = \Delta(b_0\Delta\varphi) - \nabla \cdot (b_1\nabla\varphi) + b_2\varphi$$

and

$$A\varphi = \Delta(a_0\Delta\varphi).$$

We have assumed simply supported boundary conditions although our results carry over to other boundary conditions as well. Coefficients a_0, b_0, b_1, b_2 in $L^\infty(\Omega)$ satisfy assumptions such as $a_0 \geq \nu > 0$, $b_i \geq 0$ while Φ is taken to belong to $H^{-2}(\Omega)$. Defining the control functional J to be for example,

$$(2.2) \quad J(\beta) = \int_0^T \int_{\Omega} (u(x, t; \beta) - z(x, t))^2 dx dt + \int_0^T (\epsilon_1 \beta_t^2 + \epsilon_2 \beta^2) dt$$

with $\epsilon_1, \epsilon_2 \geq 0$ and $\epsilon_1 + \epsilon_2 > 0$, we consider the problem: find β_0 in $H^1(0, T)(L^2(0, T))$ such that

$$(2.3) \quad J(\beta_0) = \text{infimum}\{J(\beta) : \beta \in H^1(0, T)(L^2(0, T))\}$$

The actuator is modeled by means of the term Φ . In fact, Φ may be chosen as a Dirac measure δ_{x_0} for $x_0 \in \Omega$ or as a convolution with a function $\phi = \phi(\cdot - x_0)$ for $x_0 \in \Omega$ with $\phi \in L^2(\Omega)$ (or smoother) and $\text{supp}\phi \subset \{x : |x - x_0| \leq \gamma\}$ where $\gamma < \text{dist}(x_0, \Gamma)$.

The existence of a unique solution to problem (2.1)-(2.3) follows from standard arguments [2]. Of interest in this investigation is the dependence of the optimal control β_0 and the functional $J(\beta_0)$ upon Φ . In particular, for $\Phi = \delta_{x_0}$ or $\Phi = \phi(\cdot - x_0)$ the uniqueness of the optimal control implies that the mappings

$$x_0 \mapsto \beta(x_0) \text{ and } x_0 \mapsto j(x_0) = J(\beta_0(x_0))$$

are well defined. One may now consider properties of these mappings. Indeed, if continuity can be established, then given any closed subset F of Ω there is a best point \hat{x} of control among those points of F in the sense $j(\hat{x}) \leq \{j(x) : x \in F\}$. If differentiability can be established, then it may be used to construct algorithms to find the optimal control location. Roughly, we show that if the difference quotients

$$\Psi_\epsilon = (\Phi(x_0 + \epsilon h) - \Phi(x_0))/\epsilon$$

converge in $H^{-2}(\Omega)$ to an element Ψ , then the mappings

$$x_0 \mapsto \beta(x_0) \text{ and } x_0 \mapsto j(x_0)$$

are differentiable, cf. [3]. Furthermore, the derivative of $\beta(x_0)$ is shown itself to be a solution of an optimal control problem. In fact the derivatives of $\beta(x_0)$ may be associated with a sequence

of optimal control problems. The length of the sequence is dependent upon the differentiability of Φ . These properties are used to compute optimal points of control in several test cases using descent algorithms for beams and plates in [9,10]. Similar considerations are given to the static case in [8].

During the grant period we also studied the control of certain models of three dimensional elastic bodies. These models are in the form of hyperbolic variational inequalities and incorporate nonlinear normal and frictional interface laws in through their boundary conditions. They were developed by J.T. Oden and his coworkers [5,6] and seem to produce results that agree well with experimental observations [5]. Since these models are rather complicated, we refer the reader to Appendix A for their formulation.

We have obtained results on optimal control by means of distributed body forces and boundary forces. The existence of optimal controls we have proved for the variational inequalities directly. Since certain functionals within the variational inequality are not differentiable, we consider regularizations of the functionals and of Sobolev type. For the resulting problems we obtain optimality and regularity conditions for solutions. Using these results, we proceed to obtain convergence results for finite element approximations to the regularized problems and iterated limit theorems relating convergence to the optimal controls of the variational inequality see Appendix A.

3. Identification and Estimation.

Let Ω be an open bounded domain in \mathbb{R}^2 with a Lipschitz boundary Γ and let

$$(3.1)(a) \quad Au = \Delta(a_0 \Delta u) - \nabla \cdot (a_1 \nabla u) + a_2 u$$

or

$$(3.1)(b) \quad Au = \sigma \Delta(a_0 \Delta u) + (1 - \sigma)((a_0 u_{xx})_{xx} + (a_0 u_{yy})_{yy} + 2(a_0 u_{xy})_{xy})$$

and

$$(3.2) \quad u_{tt} + (\Delta(b_0 \Delta) - \nabla \cdot (b_1 \nabla) + b_2)u_t + Au = f \text{ in } \Omega \times (0, T)$$

accompanied with appropriate boundary and initial conditions. Equations (3.1)-(3.2) along with appropriate initial and boundary conditions prescribe a mathematical model of a thin plate commonly called the Kirchhoff plate model [7]. Also, terms are included in (3.2) to model damping. The choices of these terms are motivated from analogous beam models [4].

The estimation problem may be formulated as follows. Given information z , which we view as a member of an observation space Z (Hilbert space), we seek to determine one or more of the parameters $q = \{a_i, b_i : i = 0, 1, 2\}$ from within a prescribed subset Q_{ad} of a Hilbert space Q that produces a solution $u(q)$ of (3.2) that "matches" z . One approach to this problem sometimes called the (regularized) output least square method is formulated as a minimization problem:

Find $q_0 \in Q_{ad}$ such that

$$(3.3)(a) \quad J(q_0) = \infimum\{J(q) : q \in Q_{ad}\}$$

where for $\beta \geq 0$

$$(3.3)(b) \quad J(q) = \|Cu(q) - z\|_Z^2 + \beta \|q\|_Q^2$$

and C is the observation operator taking a solution $u(q)$ to its image $Cu(q)$ in the observation space Z . The set Q_{ad} is specified by constraints that assure problem (3.2) is well-posed and that guarantee sufficient compactness to imply existence of a solution to (3.3).

To solve these problems numerically one must approximate the boundary value problems by a system of equations and approximate parameters in some consistent manner. The adaptation of the constraints to the discrete problem is then necessary. We have investigated this problem for static beams and certain static plates in [11,12] and [13]. These papers consider penalization and regularization techniques to incorporate constraints. Useful here is an analysis of the regularity of the optimal estimators of these problems and the use of linear splines to approximate parameters. Since linear splines preserve pointwise bounds, such pointwise L^∞ -constraints easily carry over to finite dimensional problems. Use of this technique however restricts the class of problems that may be considered for plates to those in which the domain Ω is a rectangle and the parameters are expressible as a tensor product of H^1 spaces. This second restriction is not so great since such spaces are dense in $C^0(\Omega)$. However, the theory is still essentially a one dimensional theory.

For the general case we have studied a two dimensional theory [13,14] in which Q_{ad} is assumed to be in $H^2(\Omega)$. For this case there typically is not a finite dimensional approximating basis

in $H^2(\Omega)$ that preserves pointwise bounds and remains in $H^2(\Omega)$. The regularity is determined by appropriately applying a generalized version of the Kuhn-Tucker Theorem and analyzing the regularity of the solutions of the resulting variational equations. Using these properties we may appropriately define finite dimensional sets of parameters to obtain convergence theorems.

For example in [14] we consider the problem

$$(3.4) \quad \begin{aligned} u_{tt} + Au &= f \quad \text{in } \Omega \times (0, T) \\ u &= \frac{du}{dn} = 0 \quad \text{on } \Gamma \times (0, T) \\ u(0) &= u_0 \quad \text{in } \Omega \\ u_t(0) &= u_1 \quad \text{in } \Omega \end{aligned}$$

where A satisfies (3.1) with $a_1 = a_2 = 0$ and σ is Poisson's constant in $(0, \frac{1}{2})$. Of great importance in this analysis is the determination of regularity properties of the optimal estimator. If Γ is C^4 and

$$\begin{aligned} f &\in H^1(0, T; L^2(0, T)) \cap H^2(0, T; H^{-2}(\Omega)) \\ u_0 &\in D(A) \\ u_1 &\in H_0^2(\Omega) \end{aligned}$$

and

$$z \in L^2(0, T; L^2(\Omega)),$$

consider problem (3.3) with

$$Q_{ad} = \{a \in H^2(\Omega) : a(x) \geq \nu > 0 \text{ in } \Omega\}.$$

Of course, other functionals J may be considered as well by matching at time T or at points ξ within Ω . We obtain regularity results indicating that the optimal parameter a_0 belongs to $H^{2+\delta}(\Omega)$ for $\delta \in [0, 1)$. This result is then useful in formulating finite dimensional approximating problems based on Galerkin approximation of (3.4) and in specifying finite element subspaces in $H^2(\Omega)$ to approximate the coefficients a .

Numerical examples have been studied for this problem in $\Omega = (0, 1) \times (0, 1)$ and $T = 0.5$. Tensor products of cubic B -splines on a regular mesh adjusted to boundary conditions are used to approximate u . Again, tensor products of cubic B -splines on regular mesh are used to approximate

a. The resulting system of ordinary differential equations is approximated using a symmetric time differencing technique. The following examples use 49 basis elements to approximate the state u and 25 to approximate a . The parameter β equals 10^{-8} , and the test coefficient is

$$a_{test} = 1 + (x^2 + y^2)/2$$

with observation

$$z(x, y, t) = 16x^2(1-x)^2y^2(1-y)^2 \cos(t)$$

The forcing function f is generated using a_{test} and z . Using a quasi-Newton method for minimization with initial guess $a = .25$, an approximation for a_{test} with relative L^2 error of 2.7% is obtained after 7 iterations.

For a similar case with

$$a_{test} = 1/(4((x - \frac{1}{2})^2 + (y - \frac{1}{2})^2) + 1)$$

an approximation for a_{test} with relative L^2 error of 2.5% is obtained after 5 iterations.

We have analyzed similar problems for the estimation of damping terms as in (3.2) with $b_1 = b_2 = 0$ and have obtained regularity and approximation results [15]. With a test coefficient

$$b_{test} = 1/(4((x - \frac{1}{2})^2 + (y - \frac{1}{2})^2) + 1)$$

with initial guess having relative L^2 error 16% after 4 iterations we obtain a relative L^2 error of 3.8% with $\beta = 10^{-12}$.

In other work we have studied the bijectivity or identifiability of the parameter-to-state mappings for finite dimensional version of elliptic boundary value problems [1]. In other work with K. Kunisch we have studied estimation of time and spatially dependent diffusion coefficients in parabolic equations see Appendix B. In this paper we obtain regularity properties of the estimated diffusion coefficient and also determine sufficient conditions for strict complementarity.

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"Control of dynamic models of beams and plates with small damping: Location of actuators," submitted.

"Control of certain dynamic models of beams and plates: Location of actuators," to appear *Proc. VII International Conf. on Nonlinear Analysis and Applications.* V. Lakshmikantham ed., Marcel Dekker, to appear.

"Regularity properties in parameter estimation of diffusion coefficients in elliptic boundary value problems," with K. Kunisch, *Applicable Analysis* 21 (1986), pp. 71-88.

"Regularity properties and strict complementarity of the output-least-squares approach to parameter estimation in parabolic equations," with K. Kunisch, submitted.

"Estimation of flexural rigidity in a Kirchhoff Plate Model," submitted.

"On the estimation of L^∞ diffusion coefficients in parabolic equations," with S. Gutman, submitted.

"Estimation of flexural rigidity in a dynamic plate model," submitted.

"Dynamics and control of viscoelastic solids with contact and friction effects," with J.T. Oden, in preparation.

"On the determination of optimal control locations on a static Kirchhoff plate," submitted.

Appendix A

Dynamics and control of viscoelastic solids with
contact and friction effects

**DYNAMICS AND CONTROL OF
VISCOELASTIC SOLIDS WITH
CONTACT AND FRICTION EFFECTS**

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Abstract. In this paper, we study the control of the motion of viscoelastic bodies resisted by contact and frictional forces. The contact and friction effects are modeled by new interface laws that characterize friction phenomena on dry rough surfaces. The system under study is an arbitrary body composed of a linearly viscoelastic material which is subjected to body forces and tractions and which is in contact with a rough surface of a neighboring body. Under mild conditions on the data, we are able to prove the existence of an optimal control of motion of such systems in $L^2(O,T;V)$ through a program of applied body and surface forces.

1. INTRODUCTION

The dynamics and control of distributed systems subjected to frictional resistive forces has been an open problem in partial differential equations for many years, largely owing to the absence of a meaningful existence theory for dynamical systems with friction. The mathematical difficulties associated with Coulomb friction in problems of elastodynamics were pointed out in 1972 by Duvaut and Lions (see [1]) and has been the subject of much study in the intervening years. The finite-dimensional case has proved to be somewhat more tractable, and conditions for the existence of solutions of discrete dynamical problems with friction have been recently reported by Lotstedt [2] and Jean and Pratt [3]. The complexity of the problem can be appreciated by reviewing the work on dynamics of systems with frictionless contacts by, for example, Schatzman [4], Carrero and Pascali [5], Lötstedt [6], Amerio and Prouse [7], Amerio [8], and others.

In recent papers, Oden and Martins [9,10] pointed out that one of the principal sources of mathematical difficulty was the definition of frictional stresses on the contact surfaces characterized by Coulomb's law. However, an overwhelming volume of experimental data accumulated over a half century suggests that this law is inadequate for modeling actual contacts and resistive forces on deformable bodies. By characterizing the actual normal compliance of elastic interfaces, a constitutive equation for an interface can be developed which yields results in agreement with a sizable collection of experimental results on static and dynamic friction [9]. Moreover, the use of such interface constitutive laws in mathematical models of elastostatics, elastodynamics, and viscoelastodynamics problems with friction produces a tractable theory: results on the existence and local uniqueness of solutions to static problems in elasticity with the Oden-Martins [9] nonlinear friction laws were recently established by Rabier et al. [11] and of dynamic problems on elasticity and viscoelasticity by Martins and Oden [10]. These new theories and results set the stage for the study of the optimal control of such systems, taken up in the present paper.

In the present study, we establish the existence and uniqueness of a class of optimal controls for a broad class of problems in the dynamics of linear viscoelastic bodies with contact and friction laws of the type introduced by Oden and Martins [9]. We are able to show, under very mild restrictions on the data, that for a control of the form $\beta(t)H$, H being a functional characterizing all external forces on the system, a $\beta \in H^1(0,T)$ can be found which minimizes a functional relating the L^2 -distance between solutions to the viscoelastodynamics problem and an arbitrary target motion $z(\cdot, t)$ over a time interval $[0,T]$.

Following this Introduction, we introduce in Section 2 notations and record preliminary results on the formulation of the contact problem in viscoelastodynamics. In Section 3, a weak or variational formulation of the problem is presented and an existence result from [10] is recorded. The optimal control problem is introduced in Section 4 and the major results are established in Section 5.

We believe that these results are the first of this type on optimal control of distributed-parameter systems in contact with friction.

2. PRELIMINARIES

We begin by considering a metallic body, the interior of which is a bounded open domain Ω in \mathbb{R}^N ($N = 2$ or 3). The boundary Γ of Ω is smooth (e.g., at least Lipschitz continuous) contains open subsets Γ_D , Γ_F , and Γ_C such that $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_F \cup \bar{\Gamma}_C$, $\text{meas}(\bar{\Gamma}_\alpha \setminus \bar{\Gamma}_\alpha) = 0$, $\alpha \in \{D, F, C\}$. Material particles (points) in Ω with cartesian coordinates x_i , $1 \leq i \leq N$, are denoted x , the volume measure by dx , and the surface area measure by ds , and a unit exterior vector normal to Γ by n .

We shall assume that the body is composed of a linear viscoelastic material, the mechanical response of which is characterized by the constitutive law,

$$\sigma_{ij} = E_{ijkl} u_{k,l} + C_{ijkl} \dot{u}_{k,l} \quad 1 \leq i, j, k, l \leq N \quad (2.1)$$

where σ_{ij} ($= \sigma_{ji}$) are the cartesian components of the Cauchy stress tensor, $E_{ijkl} = E_{ijkl}(x)$ and $C_{ijkl} = C_{ijkl}(x)$ are the arrays of elastic and viscoelastic parameters at point $x \in \Omega$, $u_k = u_k(x, t)$ are the components of displacement at x at time $t \in [0, T] \subset \mathbb{R}$, $u_{k,l} = \partial u_k / \partial x_l$ and $(\dot{\cdot})$ denotes differentiation with respect to time. The body is subjected to body forces of intensity b per unit volume, to surface tractions t on Γ_F , and is possibly in contact with a moving neighboring body

on the contact surface Γ_C which moves relative to Ω at a velocity \dot{U}_T tangent to Γ_C . The partial differential equations, inequalities, and conditions governing the behavior of the body are listed as follows:

Linear Momentum

$$\sigma_{ij} (u, \dot{u})_{,j} + b_i = \rho \ddot{u}_i \quad \text{in } \Omega \times (0, T) \quad (2.2a)$$

where $\sigma_{ij} (u, \dot{u})$ is the expression on the right-hand side of (2.1).

Boundary Conditions

$$u_i = 0 \quad \text{on } \Gamma_D \times (0, T) \quad (2.2b)$$

$$\sigma_{ij} (u, \dot{u}) n_j = t_i \quad \text{on } \Gamma_F \times (0, T) \quad (2.2c)$$

Contact Interface Conditions

$$\sigma_n (u, \dot{u}) = -C_n h (u_n)^m n \quad (2.2d)$$

$$u_n \leq g \Rightarrow \sigma_T (u, \dot{u}) = 0$$

$$|\sigma_T (u, \dot{u})| \leq C_T h (u_n)^m T$$

and

$$|\sigma_T (u, \dot{u})| < C_T h (u_n)^m T$$

$$\Rightarrow w_T = 0$$

$$u_n > g$$

$$|\sigma_T (u, \dot{u})| = C_T h (u_n)^m T$$

\Rightarrow there exists $\lambda > 0$
such that

$$w_T = -\lambda \sigma_T (u, \dot{u})$$

$$\text{on } \Gamma_C \times (0, T) \quad (2.2e)$$

Initial Conditions

$$u(x,0) = u_0(x)$$

$$\dot{u}(x,0) = u_1(x)$$

$$x \in \Omega$$

(2.2f)

In (2.2a), $\rho = \rho(x)$ is the mass density and in (2.2d) and (2.2e),

$$\sigma_n(u, \dot{u}) = \text{the normal stress (contact stress) on } \Gamma_C$$

$$= \sigma_{ij}(u, \dot{u}) n_i n_j$$

$$h(u_n) = \text{the approach of the two contact surfaces}$$

$$= (u_n - g)_+$$

with $u_n = u_i n_i$ the normal component of displacement, g the initial gap between surfaces, and

$(\cdot)_+$ the positive part of the indicated argument ($\phi_+ = \max(0, \phi)$).

$$\sigma_{Ti}(u, \dot{u}) = \text{the tangential (frictional) stress component on } \Gamma_C,$$

$$= \sigma_{ij}(u, \dot{u}) n_j - n_i \sigma_n(u, \dot{u})$$

$$\dot{w}_T = \text{the relative (slip) velocity of the contact surfaces.}$$

$$= \dot{u}_T - \dot{U}_T$$

where $u_{Ti} = u_i - n_i u_n$.

In (2.2d) and (2.2e), C_n, m_n, C_T, m_T are material constants characterizing the mechanical response of the interface. Equation (2. d) is the power-law constitutive equation for the normal compliance of the interface; (2.2e) defines the corresponding tangential response characteristics. Justification for the use of such interface laws is given in ODEN and MARTINS [9]. These laws correspond to a generalized Coulomb law in which the coefficient of friction is dependent on the deformation and is of the form, $\mu = C|\sigma_n(u, \bar{u})|^\alpha$, with $\alpha = (m_T/m_n) - 1$ and $C = C_T/C_n^{m_T/m_n}$.

3. A VARIATIONAL FORMULA

We now record a weak formulation of Problem (2.2), in which regularity requirements on u and on the data are relaxed. We introduce the spaces of admissible functions,

$$V = \{ v = (v_1, v_2, \dots, v_N) \in (H^1(\Omega))^N \mid v = 0 \text{ a.e. on } \Gamma_D \} \quad (3.1a)$$

$$H = (L^2(\Omega))^N \quad (3.1b)$$

$$V' = \text{topological dual of } V \quad (3.1c)$$

The space V is equipped with the usual norm,

$$\|v\| = \left\{ \int_{\Omega} (v_{ij} v_{ij} + v_i v_i) dx \right\}^{1/2}$$

and (\cdot, \cdot) and $|\cdot|$ denote the L^2 -inner product and norm on H , and $\langle \cdot, \cdot \rangle$ denotes duality pairing on $V' \times V$. Throughout, we assume for simplicity that

$$\rho(x) = 1 \text{ and } \text{meas}(\Gamma_D) > 0. \quad (3.2)$$

In (3.1a), v on the boundary is interpreted in the usual sense of traces of $H^1(\Omega)$ - functions.

The material coefficients in (2.1) are assumed to satisfy the following conditions:

$$\begin{aligned}
 & E_{ijkl}, C_{ijkl} \in L^\infty(\Omega) \\
 & E_{ijkl} = E_{jikl} = E_{ijlk} = E_{klij} \\
 & C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij} \\
 & \exists \alpha_E, \alpha_C > 0 \text{ such that} \\
 & E_{ijkl} A_{kl} A_{ij} \geq \alpha_E A_{ij} A_{ij} ; C_{ijkl} A_{kl} A_{ij} \geq \alpha_C A_{ij} A_{ij} \\
 & \text{for every symmetric matrix } A_{ij} \in \mathbb{R}^{N \times N}
 \end{aligned} \tag{3.3}$$

Whenever (3.3) hold, the bilinear forms $a: V \times V \rightarrow \mathbb{R}$ and $c: V \times V \rightarrow \mathbb{R}$ (representing the internal virtual work of the stress σ_{ij}) defined by

$$a(v, w) = \int_{\Omega} E_{ijkl} v_{k,l} w_{i,j} dx \tag{3.4a}$$

$$C(v, w) = \int_{\Omega} C_{ijkl} v_{k,l} w_{i,j} dx \tag{3.4b}$$

for $v, w \in V$, are continuous and V -elliptic, i.e., there exist positive constants $M_E, M_C, \alpha_E, \alpha_C$ such that for any w, v in V ,

$$|a(w, v)| \leq M_E \|w\| \|v\| , |c(w, v)| \leq M_C \|w\| \|v\| \tag{3.5a}$$

$$a(v, v) \geq \alpha_E \|v\|^2 , c(v, v) \geq \alpha_C \|v\|^2 \tag{3.5b}$$

We also assume that

$$1 \leq m_n, m_T \begin{cases} < +\infty & \text{if } N = 2 \\ \leq 3 & \text{if } N = 3 \end{cases} \quad (3.6)$$

We assume that

$$b(t) \in H \text{ and } t(t) \in (L^{q'}(\Gamma_C))^N \quad (3.7a)$$

$$C_n, C_T \in L^\infty(\Gamma_C), \quad C_n, C_T \geq 0 \text{ a.e. on } \Gamma_C \quad (3.7b)$$

$$g \in L^q(\Gamma_C) \quad (3.7c)$$

where

$$q = 1 + \max(m_n, m_T), \quad q' = q/(q-1)$$

$b(t)$ denotes $x \rightarrow b(x, t)$; $t(t)$ denotes $s \rightarrow t(s, t)$

where $s = (s_1, s_2, \dots, s_n)$ is a point on Γ (in particular Γ_F). Then the work done by the external forces is defined by the functional,

$$\begin{aligned} F(t) \in V' \\ \langle F(t), v \rangle = \int_{\Omega} b(t) \cdot v \, dx + \int_{\Gamma_F} t(t) \cdot v \, ds \end{aligned} \quad (3.8)$$

Also, to characterize the prescribed slip velocity \dot{U}_T on Γ_C , we introduce the function

$$\begin{aligned} \Phi(t), \dot{\Phi}(t) \in V \\ \Phi_n(t), \dot{\Phi}_T(t) = \dot{U}_T(t) \text{ a.e. on } \Gamma_C \end{aligned} \quad (3.9)$$

For discussions of such decompositions of functions in V into tangential and normal components, see KIKUCHI and ODEN [12].

Finally, to characterize the work done by normal and frictional forces on Γ_C , we introduce the nonlinear forms,

$$P : V \rightarrow V'$$

$$\langle P(\mathbf{w}), \mathbf{v} \rangle = \int_{\Gamma_C} C_n h(\mathbf{w}_n)^{m_n} v_n ds \quad (3.10a)$$

$$j : V \times V \rightarrow \mathbb{R}$$

$$j(\mathbf{w}, \mathbf{v}) = \int_{\Gamma_C} C_T h(\mathbf{w}_n)^{m_T} |\mathbf{v}_T| ds \quad (3.10b)$$

for arbitrary \mathbf{w}, \mathbf{v} in V .

We are now ready to state the following weak formulation of the problem.

Find the motion $\mathbf{u}(t) : [0, T] \rightarrow V$
such that

$$\begin{aligned} & \langle \ddot{\mathbf{u}}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle + a(\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t)) \\ & + C(\mathbf{u}(t), \mathbf{v} - \dot{\mathbf{u}}(t)) + \langle P(\mathbf{u}(t)), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle \\ & + j(\mathbf{u}(t), \mathbf{v} - \dot{\Phi}(t)) - j(\mathbf{u}(t), \dot{\mathbf{u}}(t) - \dot{\Phi}(t)) \\ & \geq \langle \mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle \quad \forall \mathbf{v} \in V \end{aligned} \quad (3.11)$$

If $\partial_j j(\mathbf{u}(t), \dot{\mathbf{u}}(t) - \dot{\Phi}(t))$ denotes the partial subdifferential of the friction functional j with respect to the second argument (i.e., the velocity), then (3.10) can be written in the equivalent form,

$$\begin{aligned} & \text{Find } \mathbf{u}(t) : [0, T] \rightarrow V \text{ such that} \\ & \mathbf{F}(t) \in \partial_j j(\mathbf{u}(t), \dot{\mathbf{u}}(t) - \dot{\Phi}(t)) \\ & \quad + P(\mathbf{u}(t)) + C \dot{\mathbf{u}}(t) + K \mathbf{u}(t) + \ddot{\mathbf{u}}(t) \end{aligned} \quad (3.12)$$

where $K, C \in \mathcal{L}(V, V')$ are the operators defined by $\langle K w, v \rangle = a(w, v)$, $\langle C w, v \rangle = c(w, v)$.

We are now in a position to state a major result on the existence and uniqueness of solutions to (3.11).

Theorem 3.1 (MARTINS and ODEN [10]). Let $\text{meas } (\Gamma_D) > 0$, (3.2), (3.5), and (3.6) hold and suppose that

$$\begin{aligned} b &\in L^2(0, T; V') \\ t &\in L^2(0, T; (L^q(\Gamma_F))^N) \\ (\text{so that } F(t) &\in L^2(0, T; V')) \\ u_0 &\in V, u_1 \in H \end{aligned} \tag{3.13}$$

Then there exists a unique solution u to problem (3.11) (or, equivalently, (3.12)) such that

$$\begin{aligned} u &\in L^\infty(0, T; V) \\ \dot{u} &\in L^\infty(0, T; H) \cap L^2(0, T; V') \\ \ddot{u} &\in L^\infty(0, T; V'). \end{aligned} \tag{3.14}$$

This result is obtained using a regularization procedure to smooth the frictional boundary condition along with a standard Galerkin technique. It relies heavily on the compactness of the trace operator from V into appropriate boundary spaces. The condition (3.5), which is used in insuring the appropriate compactness properties, is remarkably confirmed by physical experiments on dry rough surfaces; see [9].

Let us introduce the Hilbert space

$$w = \{ w \in L^2(0, T; V) : w \in L^2(0, T; V') \}$$

with norm

$$\| w \| w = \left(\int_0^T (\| w(t) \|_V^2 + (\| w(t) \|_H^2) dt)^{1/2}$$

An examination of the proof of Theorem 3.1 in [10] reveals the following continuity result.

Theorem 3.2. Let $\text{meas } (\Gamma_D) > 0$, (3.2), (3.5), and (3.6) hold and suppose that $\{F_K\}_{K=1}^{\infty}$ is a sequence in $L^2(0,T;V')$ such that

$$F_K \rightarrow F \text{ weakly in } L^2(0,T;V').$$

Then corresponding sequence of solutions $u(F_K)$ of (3.12) have the property that

$$u(F_K) \rightarrow u(F) \text{ weakly in } w$$

and thus

$$u(F_K) \rightarrow u(F) \text{ in } L^2(0,T;H).$$

Theorems 3.1 and 3.2 provide the basis for a study of the optimal control of distributed systems of the type (3.11). We begin by introducing the notation

$$z \in L^2(0,T;H) \text{ a prescribed target notation for } t \in (0,T), \quad (3.15a)$$

$$F(t) \in V'$$

with

$$F(t) = (t) \Theta, \beta \in H'(0,T) \quad (3.15b)$$

where Θ is a prescribed functional in V' defined by normalized external forces

$$\langle \Theta, v \rangle = \int_{\Omega} b \cdot v \, dx + \int_{\Gamma_F} t \cdot v \, ds \quad (3.15c)$$

Here $\beta = \beta(t)$ serves as a control parameter and the functional

$$J : H'(0,T) \rightarrow R$$

given by

$$J(\beta) = \int_0^T (\|u(\cdot, t; \beta) - z(t)\|_H^2 dt + \gamma \|\beta\|_{H'(0,T)}^2) \quad (3.16a)$$

where $u(\cdot, t; \beta)$ is the solution of (3.12) for the forcing function given in (3.16).

The optimal control problem that we study here is given as follows.

$$\text{Given } z \in L^2(0,T;H) \text{ find } \beta_0 \in H'(0,T) \quad (3.16b)$$

such that

$$J(\beta_0) = \text{inimum} \{ J(\beta) : \beta \in H'(0,T) \}.$$

The following existence result is a consequence of Theorem 3.2.

Theorem 3.3. Let $(\Gamma_D) > 0$, (3.2), (3.5), (3.6) and (3.16) hold. Then there exists a solution to problem (3.16).

Proof. Let $\{\beta_i\}_{i=1}^{\infty}$ be a minimizing sequence for (3.16). Then there exists a subsequence and $\{\beta_i\}_{i=1}^{\infty}$ such that

$$\beta_i \rightarrow \beta_0 \text{ weakly in } H'(0,T).$$

From Theorem 3.2 we see that

$$u(\beta_i) \rightarrow u(\beta_0) \text{ weakly in } L^2(0,T;H).$$

That β_0 is a solution of (3.16) follows from the weak lower semicontinuity of the mapping

$$\beta \rightarrow \|\beta\|_{H'(0,T)}.$$

We have established the existence of optimal controls for a class of control problems governed by variational inequalities modeling contact problems with frictional effects. Our goal now is to determine an approximation theory for these problems. Thus, we formulate a class of approximating regularized problems in which the frictional condition is regularized similar to the approach for existence [10]. However, in addition we include a Sobolev regularization term. We determine convergence behavior of the optimal controls for the inclusion of the Sobolev term. This enables us to obtain regularity properties for the optimal controls for regularized problems. These properties allow us to determine limiting behavior for optimal control problems over finite dimensional subspaces of $H'(0,T)$.

4. Regularized Problems

In this section we consider the formulation of control problems governed by variational problems that are regularizations of (3.11), c.f. [10]. In contrast to the regularization used in the paper by Oden and Martins to establish the existence of a solution to (3.11), we regularize the frictional contribution with a smoother function and we introduce a Sobolev regularization as well. This is done to permit us to determine optimality conditions for the optimal controls of the regularized problems. From these conditions we may determine regularity properties of the controls for the regularized problem. We also obtain convergence behavior of the optimal controls as the regularization is allowed to vanish. We begin by introducing the functions

$$\Psi_\varepsilon : \mathbb{R}^N \rightarrow \mathbb{R}, \varepsilon > 0$$

that satisfy the following properties

$$\Psi_\varepsilon \in C^2(\mathbb{R}^N, \mathbb{R}) \text{ for every } \varepsilon > 0 \quad (4.1)(i)$$

$$0 \leq \Psi_\varepsilon(v) \leq \varepsilon \text{ for every } \varepsilon > 0 \text{ and } v \in \mathbb{R}^N \quad (4.1)(ii)$$

$$\Psi_\varepsilon(\theta w + (1 - \theta)v) \leq \theta \Psi_\varepsilon(w) + (1 - \theta) \Psi_\varepsilon(v) \quad (4.1)(iii)$$

for every $\varepsilon > 0$, $(w, v) \in \mathbb{R}^N \times \mathbb{R}^N$, and $\theta \in [0, 1]$

$$\text{There exists positive constant } D_1 \text{ and a positive function } \varepsilon \rightarrow D(\varepsilon) \quad (4.1)(iv)$$

such that for every $\varepsilon > 0$ and $(w, v, u) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$,

$$|\Psi'_\varepsilon(w)(v)| \leq D_1 |v|$$

$$|\Psi''_\varepsilon(w)(v, w)| \leq D_1(\varepsilon) |v| |w|$$

There exists a positive constant $D^2 > 0$ such that for every $\varepsilon > 0$ and $v \in \mathbb{R}^N$

$$\|\Psi_\varepsilon(v) - |v|\| \leq D_2 \varepsilon$$

We define the regularized functional

$$j_\varepsilon : V \times V \rightarrow \mathbb{R}$$

by

$$j_\varepsilon(w, v) = \int_{\Gamma_C} C_T [(w_n - g)_+]^{m_T} \Psi_\varepsilon(v_T) ds$$

We observe that the derivative with respect to the second argument of $j_\varepsilon(\cdot, \cdot)$ is given by

$$\langle J_\varepsilon(w, v), z \rangle = \langle \partial_v j_\varepsilon(w, v), z \rangle$$

so that

$$\langle J_\varepsilon(w, v), z \rangle = \int_{\Gamma_C} C_T [(w_n - g)_+]^{m_T} \Psi_\varepsilon(v_T)(z_T) ds \quad (4.2)$$

The variational formulation of the regularized problem of interest in this study is given as the following.

Find a function $t \rightarrow u_\varepsilon(t)$ of $[0, T]$ into V such that for any $v \in V$

$$\begin{aligned} \langle \ddot{u}_\varepsilon(t), v \rangle + \varepsilon a(\ddot{u}_\varepsilon(t), v) + c(u_\varepsilon(t), v) + a(\dot{u}_\varepsilon(t), v) + \langle P(u_\varepsilon(t), v) \rangle \\ + \langle J_\varepsilon(u_\varepsilon(t), \dot{u}_\varepsilon(t) - \dot{\Phi}(t)), v \rangle = \langle f(t), v \rangle \end{aligned} \quad (4.3)(i)$$

with initial conditions

$$u_\varepsilon(0) = u_0 \in V \quad (4.3)(ii)$$

$$\dot{u}_\varepsilon(0) = u_1 \in V$$

and

$$f \in L^2(0, T; V) \tag{4.3}(iii)$$

The proof of the existence of a unique solution to (4.3) is essentially the same as that given in [10]. We use the following convergence result the validity of which follows from a study of the proof of convergence of u_ε to u in [10]. It should be noted that for $f \in B$ a bounded subset of $L^2(0, T; V)$ that the sets

$$\{ u_\varepsilon(f) : f \in B, \varepsilon > 0 \}$$

and

$$\{ \dot{u}_\varepsilon(f) : f \in B, \varepsilon > 0 \}$$

are bounded in $L^\infty(0, T; V)$ and $L^\infty(0, T; H) \cap L^2(0, T; V)$, respectively.

Proposition 4.1. Let $\text{meas}(\Gamma_D) > 0$, (3.2), (3.5), (3.6) and (4.1) hold and suppose that $f_i \rightarrow f$ weakly in $L^2(0, T; V)$ and $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$. Then

$$u_{\varepsilon_i}(f_i) \rightarrow u(f) \text{ weak star } (?) \text{ in } L^\infty(0, T; V)$$

$$\dot{u}_{\varepsilon_i}(f_i) \rightarrow \dot{u}(f) \text{ weak star } (?) \text{ in } L^\infty(0, T; H) \text{ and weakly in } L^2(0, T; V).$$

Along similar lines but with $\varepsilon > 0$ fixed we have the continuity result similar to Theorem 3.2.

Proposition 4.2. Let $\varepsilon > 0$ be fixed and let $\text{meas}(\Gamma_D) > 0$, (3.2), (3.5), (3.6) and (4.1)

hold. If $f_u \rightarrow f$ weakly in $L^2(0, T; V')$, then $u_\varepsilon(f_u) \rightarrow u_\varepsilon(f)$ weakly in W as $n \rightarrow \infty$.

The control problem for these regularized problems we give as follows.

Find $\beta_\varepsilon \in H(0, T)$ such that

$$J^\varepsilon(\beta_\varepsilon) = \text{infinum } \{ J^\varepsilon(\beta) : \beta \in H(0, T) \} \quad (4.4)(a)$$

where

$$J^\varepsilon(\beta) = \int_0^T \| u_\varepsilon(\cdot, t; \beta) - z(t) \|^2_H dt + \gamma \| \beta \|^2_{H(0, T)} \quad (4.4)(b)$$

As a consequence of Proposition 4.2, we have the result.

Corollary 4.1. Let the assumption of Proposition 4.2 hold. There exists a solution β_ε to problem (4.4).

Remark 4.1. For β fixed we note that

$$J^\varepsilon(\beta) \geq J^\varepsilon(\beta_\varepsilon).$$

Furthermore, since from Proposition 4.2 we see that $u_\varepsilon(\beta) \rightarrow u(\beta)$ in $L^2(0, T; H)$, it follows that

$$J^\varepsilon(\beta) \rightarrow J(\beta)$$

as $\varepsilon \rightarrow 0$. The set

$$B = \{ \beta_\varepsilon : \varepsilon > 0 \}$$

of optimal controls of (4.4) is bounded in $L^2(0, T)$ and thus is weakly precompact in $L^2(0, T)$.

Theorem 4.2. Let $\text{meas } (\Gamma_D) > 0$, (3.2), (3.5), (3.6) and (4.1) hold and let $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$. Then any weak $L^2(0, T)$ limit point β_0 of the sequence $\{\beta_{\varepsilon_i}\}_{i=1}^{\infty}$ is a solution of (3.16).

Proof. By the above remark there exists a subsequence again $\{\beta_{\varepsilon_i}\}_{i=1}^{\infty}$ such that

$$\beta_{\varepsilon_i} \rightarrow \beta_0 \text{ weakly in } H(0, T)$$

as $i \rightarrow \infty$. From Proposition 4.1 it follows that

$$u_{\varepsilon_i}(\beta_{\varepsilon_i}) \rightarrow u(\beta_0) \text{ in } L^2(0, T; H).$$

Thus, we see that

$$\lim J^\varepsilon(\beta_0) \geq \lim J^{\varepsilon_i}(\beta_{\varepsilon_i}) \geq J(\beta_0)$$

as $i \rightarrow \infty$ and β_0 is a solution of (3.16).

5. Optimality Conditions for Regularized Problems

Having established existence of optimal controls for regularized problems and their convergence properties as $\varepsilon \rightarrow 0$, we now seek to determine their regularity. We begin by stating a simple result that is a consequence of the mean value theorem.

Lemma 5.1. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$\phi(x) = [(x - g)_+]^m$$

for $m \geq 1$ and $g \in \mathbb{R}$. Then for $x_1, x_2 \in \mathbb{R}$

$$|\phi(x_1) - \phi(x_2)| \leq C(m) (|x_1 - g|^{m-1} + |x_2 - g|^{m-1}) |x_1 - x_2|.$$

Useful estimates for P and j may now be obtained as Corollaries of Lemma 5.1.

Corollary 5.1. Let m_n satisfy (3.6). Then

$$|\langle P(u_1) - P(u_2), v \rangle| \leq C(m_n) [\|u_{1N}\|_V^{m_n-1} + \|u_{2N}\|_V^{m_n-1} + \|g\|_{L^{m_n+1}(\Gamma_c)}^{m_n-1}] \|u_{1N} - u_{2N}\|_V \|v\|_V$$

Proof. From Lemma 5.1 it follows that

$$\begin{aligned} |\langle P(u_1) - P(u_2), v \rangle| &= \left| \int_{\Gamma_c} C_n ([(u_{1N} - g)_+]^{m_n} - [(u_{2N} - g)_+]^{m_n}) v_N ds \right| \\ &\leq \int_{\Gamma_c} C(m_n) (|u_{1N} - g|^{m_n-1} + |u_{2N} - g|^{m_n-1}) |u_{1N} - u_{2N}| |v_N| ds \\ &\leq C(m_n) \left[\left(\int_{\Gamma_c} |u_{1N} - g|^{m_n+1} ds \right)^{m_n-1/m_n+1} + \left(\int_{\Gamma_c} |u_{2N} - g|^{m_n+1} ds \right)^{m_n-1/m_n+1} \right] \\ &\quad \left[\left(\int_{\Gamma_c} |u_{1N} - g|^{m_n+1} ds \right)^{1/m_n+1} \left(\int_{\Gamma_c} |v_N|^{m_n+1} ds \right)^{1/m_n+1} \right] \end{aligned}$$

Hence, we see that

$$\begin{aligned} |\langle P(u_1) - P(u_2), v \rangle| &\leq C(m_n) [\|u_{1N}\|_{L^{m_n+1}(\Gamma_c)}^{m_n-1} + \|u_{2N}\|_{L^{m_n+1}(\Gamma_c)}^{m_n-1} \\ &\quad + \|g\|_{L^{m_n+1}(\Gamma_c)}^{m_n-1}] \|u_{1N} - u_{2N}\|_{L^{m_n+1}(\Gamma_c)} \|v_N\|_{L^{m_n+1}(\Gamma_c)} \end{aligned}$$

Recalling that, under the assumption (3.6), $H^{1/2}(\Gamma_c)$ imbeds continuously into $L^q(\Gamma_c)$ and $V=H^1(\Omega)$ maps continuously into $H^{1/2}(\Gamma_c)$, we have the result.

In a similar manner, we have for j ,

Corollary 5.2. Let m_T satisfy (3.6). Then

$$|j(u_1, v) - j(u_2, v)| \leq C(m_T) [\|u_1\|_V^{m_T-1} + \|u_2\|_V^{m_T-1} + \|g\|_{L^{m_n+1}(\Gamma_c)}^{m_T-1}] \|u_1 - u_2\|_V \|v\|_V.$$

Let $h \in L^2(0, T)$ and define

$$w_\delta = (u(\beta + \delta h) - u(\beta)) / \delta \quad \text{for } \delta > 0.$$

We note that $w = w_\delta$ satisfies

$$\begin{aligned} & (\dot{w}(t), v) + \varepsilon a(\dot{w}(t), v) + c(\dot{w}(t), v) + a(w(t), v) + \\ & \quad + 1/\delta \langle P(u(t; \beta + \delta h) - P(u(t; \beta)), v \rangle + \quad (5.1) \\ & + 1/\delta \langle J(u(t; \beta + \delta h), \dot{u}(t; \beta + \delta h) - \dot{\Phi}(t)), v \rangle - \langle J(u(t; \beta), \dot{u}(t; \beta) - \dot{\Phi}(t)), v \rangle) \\ & = h(t) \langle \Theta, v \rangle \end{aligned}$$

We now use Corollary 5.1 to obtain for $\delta > 0$ the estimate

$$\begin{aligned} 1/\delta | \langle P(u(t; \beta + \delta h) - P(u(t; \beta)), v \rangle | & \leq \\ & \leq C(m_n) (\|u(t; \beta + \delta h)\|_V^{m_n-1} + \|u(t; \beta)\|_V^{m_n-1} + \|g\|_{L^{m_n+1}}^{m_n-1}) \|w(t)\|_V \|v\|_V \end{aligned}$$

and

$$1/\delta | \langle J(u(t; \beta + \delta h), \dot{u}(t; \beta + \delta h) - \dot{\Phi}(t)), v \rangle - \langle J(u(t; \beta), \dot{u}(t; \beta) - \dot{\Phi}(t)), v \rangle |$$

$$= | 1/\delta \int_{\Gamma_c} C_T [(u_n)(t; \beta + \delta h) - g]_+^{m_T} \Phi'(u(t; \beta + \delta h) - \dot{\Phi}(t)(v)) - [(u_n)(t; \beta) - g]_+^{m_T} \Phi'(u(t; \beta) - \dot{\Phi}(t)(v)) ds |$$

Now set

$$A = \frac{1}{\delta} \int_{\Gamma_c} C_T [(u_n)(t; \beta + \delta h) - g]_+^{m_T} - [(u_n)(t; \beta) - g]_+^{m_T} \Phi'(u(t; \beta + \delta h) - \dot{\Phi}(t)(v)) ds |$$

and

$$B = \frac{1}{\delta} \int_{\Gamma_c} C_T [(u_n)(t; \beta) - g]_+^{m_T} (\Phi'(u(t; \beta + \delta h) - \dot{\Phi}(t)(v)) - \Phi'(u(t; \beta) - \dot{\Phi}(t)(v))) ds |$$

From (4.1) (w) we see that for A we have

$$A \leq D_1 \int_{\Gamma_c} C_T m_T (|u_n(t; \beta + \delta h) - g|^{m_T-1} + |u_n(t; \beta) - g|^{m_T-1}) |w(t)(v)| ds | \quad (5.3)$$

and for B

$$B \leq D_1(\epsilon) \int_{\Gamma_c} C_T |u_n(t; \beta) - g|^{m_T} |w(t)(v)| ds$$

We now make the restriction of assumption (3.6)

$$1 \leq m_T \begin{cases} < +\infty & \text{if } N = 2 \\ \leq 2 & \text{if } N = 3 \end{cases} \quad (5.4)$$

Under the assumption (5.2) we have that

$$B \leq \hat{D}_1(\epsilon) C_T \left(\int_{\Gamma_c} |u_n(t; \beta) - g|^{m_T+2} ds \right)^{m_T/m_T+2} \frac{\|w\|}{L^{m_n+2}(\Gamma_c)} \frac{\|v\|}{L^{m_n+2}(\Gamma_c)}$$

$$\leq C(\epsilon, m_T) (\|u(\beta)\|_V^{m_T} + \|g\|_{L^{m_n+1}(\Gamma_c)}) \|\dot{w}\|_V \|v\|_V \quad (5.5)$$

Thus, we see that from (5.1) - (5.3) and (5.5),

$$\begin{aligned} & (\ddot{w}(t), v) + \epsilon a(\ddot{w}(t), v) + c(\dot{w}(t), v) + a(w(t), v) \\ & \leq |h(t)| \langle \Theta, v \rangle + C(\delta) (\|w(t)\|_V + \|\dot{w}(t)\|_V) \|v\|_V \end{aligned} \quad (5.6)$$

where $C(\delta)$ depends on the parameter of the problem as well as $u(\beta)$, g , and ϵ and is bounded for $\delta > 0$ in a bounded set. Now setting $v = \dot{w}(t)$ we have

$$\begin{aligned} & d/dt (\|\dot{w}(t)\|_H^2 + \epsilon a(\dot{w}(t), w(t)) + a(w(t), w(t)) + 2C(\dot{w}(t), \dot{w}(t))) \\ & \leq 2|h(t)| \langle \Theta, w(t) \rangle + 2\hat{C}(\delta) (\|w(t)\|_V + \|\dot{w}(t)\|_V + \|\dot{w}(t)\|_V^2) \\ & \leq C_1 |h(t)|^2 + C_2(\delta) (\|w(t)\|_V^2 + \|\dot{w}(t)\|_V^2) \end{aligned} \quad (5.7)$$

We may now obtain from Gromwall's inequality that

$$\|\dot{w}_\delta(t)\|_H^2 + \|\dot{w}_\delta(t)\|_V^2 + \|w_\delta(t)\|_V^2 + \int_0^t \|\dot{w}_\delta(\tau)\|_V^2 dt \leq C(\delta) \int_0^t |h(\tau)|^2 dt \quad (5.7)$$

where $C(\delta)$ is a function of δ sending bounded sets in \mathbb{R}^+ into bounded sets in \mathbb{R}^+ . In a similar manner, using $v = w(t)$, we may show boundedness of $w(t)$ in $L^2(0, T; V)$. Hence, we see that there is a sequence $\delta_i \rightarrow 0$ such that

$$\begin{aligned} w_{\delta_i} & \rightarrow w \text{ weak star in } L^\infty(0, T; V) \\ \dot{w}_{\delta_i} & \rightarrow w \text{ weak star in } L^\infty(0, T; V) \\ \ddot{w}_{\delta_i} & \rightarrow w \text{ weak star in } L^2(0, T; V) \end{aligned} \quad (5.9)$$

With this preparation, we have the following result: We denote by $\partial u_\varepsilon(\beta)(h)$ the variation of $u_\varepsilon(\beta)$ with increment h .

Proposition 5.1. Let $\text{meas}(\Gamma_D) > 0$, (3.2), (3.5), (3.6), (4.1) and (5.4) hold. Then $w = \partial u_\varepsilon(\beta)(h)$ exists and satisfies the variational equation

$$\begin{aligned}
 & (\ddot{w}(t), v) + \varepsilon a(\dot{w}(t), v) + C(\dot{w}(t), v) + a(w(t), v) + \int_{\Gamma_C} m_n [(u_n - g)_+]^{m_n-1} w_n(t), v_n \, ds + \\
 & + \int_{\Gamma_C} \{ C_T [(u_n - g)_+]^{m_T} \Psi''(\dot{u}_T - \dot{\Phi})(w_T, v_T) + \\
 & + m_T C_T [(u_n - g)_+]^{m_T-1} w_n \Psi'(\dot{u}_T - \dot{\Phi})(v_T) \} \, ds = h(t) \langle \Theta, v \rangle
 \end{aligned} \tag{5.10}(a)$$

for any $v \in V$ and with initial condition

$$w(0) = \dot{w}(0) = 0. \tag{5.10}(b)$$

Based on estimates similar to (5.7), it is straightforward to obtain the following result.

Lemma 5.2. Let (4.1) hold. Then there exists a unique solution w to problem (5.9).

Proof (of Proposition 5.1). From (5.8) it is clear that the first four terms of (5.1) converge to the first four terms of (5.9)(a). For the boundary terms we use the compactness of [10] to see that

$$w_{\partial_i} \rightarrow w \quad \text{in } L^2(0, T; L^q(\Gamma_C))$$

and

$$\dot{w}_{\partial_i} \rightarrow \dot{w} \quad \text{in } L^2(0, T; L^q(\Gamma_C))$$

Furthermore, from (5.7) it follows that

$$u(\beta + \delta h) \rightarrow u(\beta) \quad \text{in } H^1(0, T; V)$$

and therefore

$$u(\beta + \delta h) \rightarrow u(\beta) \text{ in } L^\infty(0, T; L^q(\Gamma_c))$$

where $1 \leq q \leq 4$ for $N=3$ and $1 \leq q$ for $N=2$.

Having observed the above convergence properties, we note that the boundary integral terms convey by the dominated convergence theorem.

Remark 5.1. We point out that the Sobolev regularization is used to obtain inequality (5.8). It is necessary since the coefficient of the $\|w(t)\|_{V^2}$ term in inequality (5.7) cannot be made small.

Using the results from Proposition 5.1, we may now calculate the variation of J .

Corollary 5.3. Let $\text{meas}(\Gamma_D) > 0$, (3.2), (3.5), (3.6), (4.1) and (5.4) hold and let J^ϵ be given by (4.4)(b). Then $\partial J^\epsilon(\beta)(h)$ is given by

$$\partial J^\epsilon(\beta)(h) = 2 \int_0^T (u_\epsilon(t; \beta) - z(t), w(t, h))_H dt + 2 \gamma(\beta, h)_{H^1(0, T)} \quad (5.11)$$

where w is the solution of (5.10).

Having determined the differentiability of the functional J^ϵ , we have the result.

Proposition 5.1. Let $\text{meas}(\Gamma_D) > 0$, (3.2), (3.5), (3.6), (4.1) and (5.4) hold. If β_ϵ is a solution of (4.4), then

$$\int_0^T (u_\epsilon(t; \beta) - z(t), w(t, h))_H dt + \gamma(\beta_\epsilon, h)_{H^1(0, T)} = 0 \quad (5.12)$$

for all $h \in L^2(0, T)$.

We may now use Proposition 5.1 to obtain regularity of β_ϵ , $\epsilon > 0$.

Theorem 5.1. Let $\text{meas}(\Gamma_D) > 0$, (3.2), (3.5), (3.6), (4.1) and (5.4) hold. If β_ϵ is a solution of (4.4), then $\beta_\epsilon \in H^2(0, T)$.

Proof. Consider the linear mapping

$$h \rightarrow \alpha(h) = \int_0^T (u_\epsilon(t; \beta_\epsilon) - z(t), w(t, h))_H$$

of $H'(0, T)$ into \mathbb{R} . We note from Proposition 4.2 and from (5.8) that

$$|\alpha(h)| \leq \|u_\epsilon(\cdot, \beta_\epsilon) - z(\cdot)\|_{L^2(0, T; H)} C \|h\|_{L^2(0, T)}$$

and thus in a continuous linear function on $L^2(0, T)$. It follows by the Riesz(?) Representation theorem [16] that there exists $\tilde{\alpha} \in L^2(0, T)$ such that

$$\alpha(h) = (\tilde{\alpha}, h)_{L^2(0, T)}.$$

From (5.12) we see that for every $h \in H'(0, T)$

$$(\beta_\epsilon, h)_{H'(0, T)} = -1/\gamma (\tilde{\alpha}, h). \quad (5.13)$$

We conclude from equation (5.13) that in fact

$$-\beta_{\epsilon u} + \beta_\epsilon = -1/\gamma \tilde{\alpha}$$

with

$$\beta_\epsilon(0) = \beta_\epsilon(T) = 0.$$

Hence, $\beta_\epsilon \in H^2(0, T)$.

6. Approximation

We begin by giving the Galerkin formulation for problem (4.3). Thus, let $S^N \subset V$ with basis $\{B_i\}_{i=1}^N$ and let

$$u^N(x, t) = \sum_{i=1}^N \mu_K(t) B_i(x)$$

be such that for $\epsilon \geq 0$ fixed and $v \in S^N$

$$\begin{aligned}
& \langle \bar{u}_\varepsilon^N(t), v \rangle + \varepsilon a(\bar{u}_\varepsilon^N(t), v) + c(u_\varepsilon^N(t), v) + a(\dot{u}_\varepsilon^N(t), v) + \langle P(u_\varepsilon^N(t), v) + \\
& \quad + (J_\varepsilon(u_\varepsilon(t), \dot{u}_\varepsilon(t) - \frac{f(t)}{\varepsilon}), v) = \beta(t) \langle \Theta, v \rangle
\end{aligned} \tag{6.1(a)}$$

with initial conditions

$$\begin{aligned}
u_\varepsilon^N(0) &= u_0^N \\
\dot{u}_\varepsilon^N(0) &= u_1^N
\end{aligned} \tag{6.1(b)}$$

where u_0^N and u_1^N converge to u_0 and u_1 in V , respectively, as $N \rightarrow \infty$. The following is not difficult to show.

Proposition 6.1. Let $\text{meas}(\Gamma_D) > 0$, (3.2), (3.5), (3.6), (4.1) and (5.4) hold and let $\varepsilon > 0$ be fixed. Let $\{\beta_i\}_{i=1}^\infty$ be a sequence in $H'(0,T)$ such that $\beta_i \rightarrow \beta$ weakly in $H'(0,T)$ and let $N_i \rightarrow \infty$.

Then

$$u_\varepsilon^{N_i}(\beta_i) \rightarrow u_\varepsilon(\beta) \text{ in } L^2(0,T;H)$$

as $i \rightarrow \infty$.

To approximate the controls, let L^M be a subspace in $H'(0,T)$. Suppose there is a continuous linear mapping $I^M: H'(0,T) \rightarrow L^M$ such that

$$\|u - I^M u\|_{H'(0,T)} \leq C \|u\|_{H'(0,T)}$$

and for $u \in H^2(0,T)$

$$\|u - I^M u\|_{H'(0,T)} \leq \delta(M) \|u\|_{H'(0,T)}$$

where $C \geq 0$ and $\delta(M) \geq 0$ with $\delta(M) \rightarrow 0$ as $M \rightarrow \infty$. Both C and $\delta(M)$ are independent of u .

The problems we consider are given by the following:

Find $\beta_2^{M,N} \in L^M$ such that

$$J_M^e(\beta_2^{M,N}) = \text{infinum } (J_M^e(\beta) : \beta \in L^N) \quad (6.3)(a)$$

where

$$J_M^e(\beta) = \int_0^T \|u^M(t; \beta) - z(t)\|_H^2 dt + \gamma \|\beta\|_{H'(0,T)}^2 \quad (6.3)(b)$$

Remark 6.1. The existence of solutions $\beta_e^{M,N}$ of (6.3). Moreover, these solutions are bounded in $H'(0,T)$.

Theorem 6.1. Let $\tilde{\beta}_e$ be a weak $H'(0,T)$ limit point of $\beta_e^{M_i, N_i}$ for $M_i \rightarrow \infty$ and $N_i \rightarrow \infty$. Then $\tilde{\beta}_e$ is a solution of (4.4).

Proof. Since $\beta_e^{M_i, N_i} \rightarrow \beta_e$ weakly in $H'(0,T)$ as $i \rightarrow \infty$, it follows from Proposition 6.1 that

$$u_e^{M_i, N_i}(\beta_e^{M_i, N_i}) \rightarrow u_e(\tilde{\beta}_e)$$

in $L^2(0,T)$ as $i \rightarrow \infty$. Thus, we see that

$$\lim_{i \rightarrow \infty} J_e^{N_i, M_i}(\beta_e^{M_i, N_i}) \geq J_e(\tilde{\beta}_e)$$

Now from Theorem 5.1, $\beta_e \in H'(0,T)$ and, thus from (6,2), we see that

$$J_e^{N_i, M_i}(\beta_e) \geq J_e^{N_i, M_i, N_i}(\beta_e^{M_i, N_i})$$

and

$$J_e(\beta_e) = \lim_{i \rightarrow \infty} J_e^{N_i, M_i, N_i}(\beta_e^{M_i, N_i}) \geq J_e(\tilde{\beta}_e).$$

We conclude that $\tilde{\beta}_\varepsilon$ is a solution of (4.4).

Remark 6.1. It is now clear that an iterated limit theorem follows from Theorem 4.2 and Theorem 6.1.

Theorem 6.2. Let $\varepsilon_i \rightarrow 0$. Then there exist sequences $M_i N_i \rightarrow (\infty, \infty)$ as $i \rightarrow \infty$ such that

$$\text{w-lim}_{\varepsilon_i \rightarrow 0} (\text{w-lim}_{i \rightarrow \infty} \beta_{\varepsilon_j}^{M_i N_i}) = \beta_0$$

where β_0 is a solution of (3.16) and where the w-limits are weak limits in $H'(0,T)$. Also,

$$\lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} J_{\varepsilon_j} (\beta_{\varepsilon_j}^{M_i N_i}) = J(\beta_0).$$

Acknowledgement. The support of this work by the U. S. Air Force Office of Scientific Research under Contract F49620-86-C-0051 under grant No. AFOSR-84-0271 is gratefully appreciated.

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Appendix B

Regularity properties and strict complementarity
of the output-least-squares approach to parameter
estimation in parabolic equations.

**REGULARITY PROPERTIES AND STRICT COMPLEMENTARITY OF
THE OUTPUT-LEAST-SQUARES APPROACH TO PARAMETER
ESTIMATION IN PARABOLIC EQUATIONS**

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* Work in part supported by the Fonds zur Förderung der wissenschaftlichen Forschung, AUSTRIA, S3206.

** Work supported in part by AFOSR Grant number AFOSR-84-0271.

1. Introduction.

In this paper we study properties of the output-least-squares formulation for the estimation of the diffusion coefficient a in

$$(1.1) \quad u_t = (au_x)_x + f,$$

together with appropriate initial- and boundary conditions, from data z . More precisely, assume that (1.1) is a model of a system S where the coefficient a is unknown. To determine a we assume the availability of data z of S . The task arises of determining a^* such that $u(a^*)$ best fits the data z . The most common approach to this parameter estimation problem is the output least squares formulation, given by

$$(P) \quad \text{minimize } |u(a) - z|^2$$

over some set of admissible parameters A , where $u(a)$ satisfies (1.1) and the distance $u(a) - z$ is taken in an appropriate norm. This problem has received a considerable amount of attention (see [3,4,5,11,12,18] et al). These investigations are primarily concerned with the approximation of the infinite dimensional problem (P), with numerical aspects of the finite dimensional approximating problems as well as with establishing techniques to cope with the illposedness of the estimation problem, by which we mean the lack of continuous dependence of the solutions of (P) on perturbations in z . Predominantly among the latter is the regularization technique, in which (P) is replaced by

$$(P)^{\beta} \quad \text{minimize } |u(a) - z|^2 + \beta |a|^2, \text{ over } A,$$

where β is a small positive parameter and $|\cdot|$ denotes an appropriate norm. It appears that optimization theoretic aspects of (P) and $(P)^{\beta}$ have not yet been studied in detail and we shall make a step in this direction.

Two constraints will be involved in defining A : a pointwise lower bound on the elements in A guarantees that the differential operator appearing in (1.1) is elliptic and a norm bound is used to establish existence of solutions of (P). Due to the lack of radial unboundedness of the mapping $a \rightarrow u(a)$, (P) may have no solution unless the elements in A are bounded in an appropriate sense or unless a regularization term, as in $(P)^{\beta}$, is used.

In section 2 we collect results on regularity properties of the solution of (1.1). Special attention is given to the fact that these results are obtained under weak smoothness requirements on the coefficient a . In sections 3 and 4 we show that due to the norm constraint or the regularization term, the solutions \hat{a} of (P) experience certain local as well as global regularity properties, by which we mean that \hat{a} is more regular than the functions in the set of admissible parameters A . While section 3 is devoted to the case where a is a function of the spatial variable only, we treat space and time dependent variables in section 4. Besides being of interest in their own right, regularity properties of this kind proved to be useful for elliptic estimation problems: in [12] rate of convergence results for Galerkin approximations of the output-least-squares formulation of estimation problems was proved and this rate depends on the regularity of the solutions of the infinite dimensional problem, and in [14] a priori knowledge of the regularity of the solution improved results for the penalty method formulation of the norm constraint.

In section 5 we give conditions under which the norm constraint is in fact active and then we produce a class of examples for which the formulation of (P) without norm constraint on A has no solution. The proof of the activeness of the norm constraint is achieved by showing that the associated Lagrange multiplier is nontrivial. From comparison with estimation problems for elliptic equations we expect that nontriviality of the Lagrange multiplier will have important consequences for parabolic equations as well. Among them are the stability of the solutions of (P) with respect to perturbations in z . This will be studied elsewhere.

Throughout we use standard notation. The norms of inner products of elements in function spaces are indicated with a subscript, as for example $\|\cdot\|_{H^2}$. However, if the function space is L^2 then we may drop its notation and we use $\langle \cdot, \cdot \rangle$ for the inner product in L^2 and $\|\cdot\|$ for the norm in L^2 and in \mathbb{R}^n .

2. Preliminaries.

In this section we summarize some results concerning the output least squares and the regularized output least squares formulation of

$$(2.1) \quad \begin{cases} u_t = (au_x)_x + f & \text{in } Q \\ u(x,0) = \phi(x) & \text{in } \Omega \\ u(0,t) = u(1,t) = 0 & \text{for } t \in (0,T) \end{cases}$$

where $\Omega = (0,1)$ and $Q = \Omega \times (0,T)$, $T > 0$. Depending on the cases when a is a function of x only, or a function of x and t , we define the admissible parameter sets

$$A = \{a = a(x) : a \in H^1(0,1), a(x) \geq \nu > 0, \|a\|_{H^1} \leq \mu, \nu < \mu < \infty\}$$

or

$$A = \{a = a(x,t) : a \in H^2(0,1), a(x,t) \geq \nu > 0, \|a\|_{H^2} \leq \mu, \nu T < \mu < \infty\}$$

The requirements $0 < \nu < \mu$ and $\nu T < \mu$ guarantee that A is not empty. The set A is given the topology of H^1 or H^2 respectively. We shall employ the following fit-to-data criteria:

$$J_1(a) = \int_0^T \int_0^1 |u(x,t;a) - z_1(x,t)|^2 dx dt + B \|a\|_A^2$$

and

$$J_2(a) = \int_0^1 |u(x,T;a) - z_2(x)|^2 dx + B \|a\|_A^2,$$

where $B \geq 0$, $z_1 \in L^2(Q)$, $z_2 \in L^2(\Omega)$ and $\|\cdot\|_A$ stands for $\|\cdot\|_{H^1}$ if we consider only x -dependent coefficients and for $\|\cdot\|_{H^2}$ if the coefficients depend on x and t . Finally $u = u(x,t;a)$ denotes the solution of (2.1) depending as a function of x and t on the coefficient a . The least squares problems that we shall investigate are given by

$$(P_1) \quad \min J_1(a) \text{ over } A.$$

By putting $\mu = \infty$ or $B = 0$ this formulation includes the case without norm bound on the parameter set or without regularization in the cost functional. If the dependence of J_1 or (P_1) on B or μ is relevant

we denote this by additional indices, as for instance $(P_1)^B$. First some existence and regularity properties for the equation (2.1) are summarized. Unless specified otherwise it is assumed that $\phi \in L^2(\Omega)$, $f \in L^2(0,T;H^{-1})$ and $a \in A$. Henceforth we shall write $L^2(X)$ for $L^2(0,T;X)$ when X is a Banach space. Let

$$W(0,T) = \{u : u \in L^2(H^1), u_t \in L^2(H^{-1})\}$$

endowed with $(\|u\|_{L^2(H^1)}^2 + \|u_t\|_{L^2(H^{-1})}^2)^{1/2}$ as a norm. It is well known that $W(0,T)$ is a Hilbert space which embeds continuously into $C(L^2)$ [19, p. 19] and compactly into $L^2(C)$. Here we used the following

Lemma 2.1 [22, p. 271]. Let X_0, X, X_1 be Banach spaces such that $X_0 \subset X \subset X_1$, where the injections are continuous, X_i are reflexive for $i = 0, 1$ and X_0 embeds compactly in X . Set

$$Y = \{v \in L^2(X_0) : v' \in L^2(X_1)\},$$

with $\|v\|_Y^2 = \|v\|_{L^2(X_0)}^2 + \|v'\|_{L^2(X_1)}^2$. Then the injection of Y into $L^2(X)$ is compact.

Proposition 2.1. For $a = a(x,t) \in A$ there exists a unique solution u of (2.1) in $W(0,T)$.

Proof. Existence and uniqueness of $u \in L^2(H^1)$ is shown in [16, p. 121]. Using this in (2.1) we find $u \in W(0,T)$. Moreover we have the estimate [16, p. 118].

$$(2.2) \quad \|u\|_{L^2(H^1)} \leq \hat{k}(\|\phi\|_{L^2} + \|f\|_{L^2(H^{-1})}),$$

where \hat{k} depends on ν and μ but is otherwise independent of a .

Proposition 2.2. If $\phi \in L^2(\Omega)$, $f \in L^1(L^2)$ and a is time independent so that $a = a(x) \in H^1(\Omega)$, $a(x) \geq \nu > 0$, then the solution of (2.1) satisfies $u \in L^1(C^1)$.

Proof. Using a semigroup theoretic argument let A in L^2 be defined by $\text{dom}(A) = H_0^1 \cap H^2$ and $Av = (av_x)_x$. It is wellknown that A is the infinitesimal generator of an analytic semigroup $S(t)$ and that the solution u of (2.1) can be expressed as

$$(2.3) \quad u(t) = S(t)\phi + \int_0^t S(t-s)f(s)ds.$$

In the following estimates we use the fact that the fractional power space $\text{dom}(A^\alpha)$ endowed with $|A^\alpha v|$ as a norm embeds continuously into C^1 if $\alpha \in (3/4, 1)$, [9, p. 39]. Recall that $|A^\alpha S(t)| \leq t^{-\alpha}$ for $f \geq 0$. Since

$$|S(t)\phi|_{L^1(C^1)} \leq k \int_0^T |A^\alpha S(t)\phi| dt < \infty, \text{ where } k \text{ is an embedding constant,}$$

we find that $S(t)\phi \in L^1(C^1)$. Moreover using the convolution theorem we have

$$\begin{aligned} \int_0^T \int_0^t |S(t-s)f(s)|_{C^1} ds dt &\leq k \int_0^T \int_0^t |A^\alpha S(t-s)f(s)| ds dt \\ &\leq k \|f\|_{L^1(L^2)} \int_0^T \int_0^t s^{-\alpha} ds dt \\ &\leq \frac{k}{(1-\alpha)(2-\alpha)} \|f\|_{L^1(L^2)} T^{2-\alpha}. \end{aligned}$$

Using these estimates in (2.3) implies the claim.

Proposition 2.3. If $f \in L^2(L^2)$, $\phi \in H_0^1$ and $a = a(x,t) \in \mathcal{A} \subset H^2$, then $u \in L^2(H^2 \cap H_0^1)$, $u_t \in L^2(L^2)$, and $u \in C(H_0^1)$.

Proof. This result is stated without proof in [17, p. 180]. Since the other wellknown references require stronger $(W^{1,\infty})$ regularity assumptions on a and since regularity of a is the focus of our considerations, we outline the proof.

Employing the mean value theorem for integrals one can prove that

$$(2.4) \quad \int_0^T |a_t(t, \cdot)|_{L^\infty(0,1)} dt \leq \tilde{K} \|a\|_{H^2(Q)}$$

and

$$(2.5) \quad \left(\int_0^1 |a_x(x, \cdot)|_{L^\infty(0, T)}^2 dx \right)^{1/2} \leq \bar{K} |a|_{H^2(Q)},$$

for a constant \bar{K} independent of $a \in H^2$.

We recalled before that $u \in L^2(H^1)$. Using (2.4) we can now refer to [17, p. 178] to conclude that $u \in L^2(H^1)$ and $u_t \in L^2(L^2)$. Next observe that observe that $(au_x)_x - a_x u_x = u_t - f - a_x u_x$ and the right hand side of this expression is in $L^2(L^1)$. Consequently $au_{xx} = (au_x)_x - a_x u_x$ is in $L^2(L^1)$ and $u \in L^2(W^{2,1})$. Recall that $W^{2,1}$ embeds continuously into C^1 . This implies that $u \in L^2(C^1)$. Next an estimate for u_{xx} in $L^2(L^2)$ can be derived:

$$\begin{aligned} \|u_{xx}\|_{L^2(L^2)} &\leq \|au_{xx}\|_{L^2(L^2)} \leq \|u_t\|_{L^2(L^2)} + \left(\int_0^T \int_0^1 |a_x u_x|^2 dx dt \right)^{1/2} \\ &\leq \|u_t\|_{L^2(L^2)} + \left(\int_0^T \|u_x(\cdot, t)\|_{L^\infty(0,1)}^2 \int_0^1 |a_x(x, \cdot)|_{L^\infty(0, T)}^2 dx dt \right)^{1/2} \\ &\leq \|u_t\|_{L^2(L^2)} + \bar{K} \|u\|_{L^2(C^1)} \|a\|_{H^2}, \end{aligned}$$

where we used (2.5) in the last step. Thus $u \in L^2(H^2)$. Employing this fact in (2.1) we obtain $u_t \in L^2(L^2)$. The final claim follows from [19, p. 19].

To develop the final regularity result that will be required we need to express (2.1) in an evolution operator theoretic setting. For $t \in [0, T]$ let the operators $A(t)$ be given by

$$\text{dom } A(t) = H^2 \cap H_0^1$$

with

$$A(t)\phi = (a(t, \cdot)\phi_x)_x \quad \text{for } \phi \in H^2 \cap H_0^1,$$

and for $\lambda \in \mathbb{C}$ let

$$R(\lambda; A(t)) = (\lambda - A(t))^{-1},$$

whenever $R(\lambda; A(t)) \in \mathcal{L}(L^2(\Omega))$. Here $\mathcal{L}(L^2(\Omega))$ denotes the set of bounded linear operators on $L^2(\Omega)$. The following lemma will be required.

Lemma 2.2.

(a) If $a \in H^2(\mathbb{Q})$, then $a \in C^{1/2}(0, T; H^1(\Omega))$, which is the space of Hölder continuous functions with exponent 1/2 and values in H^1 . Let $a = a(x, t) \in A$.

(b) There exist constants \underline{c} and \hat{c} such that

$$\underline{c} \|\phi\|_{H^2(\Omega)} \leq \|A(t)\phi\|_{L^2(\Omega)} \leq \hat{c} \|\phi\|_{H^2(\Omega)},$$

for all $\phi \in H^2 \cap H_0^1$ and $t \in [0, T]$,

(c) $\|R(\lambda; A)\|_{\mathcal{L}(L^2(\Omega))} \leq \frac{1}{|\lambda|}$ for all $\lambda \in \mathbb{C}$ with $\text{Re } \lambda > 0$ and $t \in [0, T]$,

(d) there exists a constant L such that

$$\|(A(t) - A(s))A(\tau)^{-1}\|_{\mathcal{L}(L^2(\Omega))} \leq L|t-s|^{1/2}$$

for all $t, s, \tau \in [0, T]$.

From the proof it can be seen that \underline{c} , \hat{c} and L in (b) and (d) can be taken independently of $a \in A$, if μ in the definition of A is finite.

Proof of Lemma 2.2.

(a) Let $t, s \in [0, T]$. Then

$$\begin{aligned} \left[\int_0^t |a_x(t, x) - a_x(s, x)|^2 dx \right]^{1/2} &\leq \left[\int_0^t \int_0^t |a_{xt}(\sigma, x)|^2 d\sigma dx \right]^{1/2} \\ &\leq |t-s|^{1/2} \left[\int_0^1 \int_0^1 |a_{xt}(s, x)|^2 ds dx \right]^{1/2} \leq |t-s|^{1/2} \|a\|_{H^2(a)}, \end{aligned}$$

and thus (a) follows.

(b) The second inequality in (b) is obvious. The first estimate is fairly standard as well and we only sketch the proof to convince the reader that \underline{c} can be taken uniformly with respect to $t \in [0, T]$. Suppressing the dependence of a on x we find

$$\int_0^1 a^2(t) \phi_{xx}^2 dx \leq \int_0^1 (A(t)\phi)^2 dx - 2 \int_0^1 a(t) a_x(t) \phi_x \phi_{xx} dx - \int_0^1 a_x^2 \phi_x^2 dx$$

$$\leq \int_0^1 (A(t)\phi)^2 dx + \frac{1}{2} \int_0^1 a^2(t) \phi_{xx}^2 dx + 2 \int_0^1 a_x^2(t) \phi_x^2 dx - \int_0^1 a_x^2(t) \phi_x^2 dx.$$

Hence it follows that

$$\frac{1}{2} \int_0^1 a^2(t) \phi_{xx}^2 dx \leq \int_0^1 (A(t)\phi)^2 dx + \int_0^1 a_x^2(t) \phi_x^2 dx$$

and since $a \in A$

$$\nu \int_0^1 \phi_{xx}^2 dx \leq 2 \int_0^1 (A(t)\phi)^2 dx + 2 \|\phi_x\|_{L^\infty}^2 \int_0^1 a_x^2(t) dx.$$

From (a) and the Ehrlingsche Lemma [23, p. 118] there exists for every $\varepsilon > 0$ a constant $c(\varepsilon)$ such that

$$\nu \|\phi_{xx}\|^2 \leq 2\|A(t)\phi\|^2 + 2(\varepsilon \|\phi_{xx}\|^2 + c(\varepsilon) \|\phi_x\|^2) \|a\|_{H^2(Q)}.$$

For an appropriate choice of \tilde{c} this implies

$$(2.6) \quad \|\phi_{xx}\| \leq \tilde{c} \|A(t)\phi\|$$

for all $\phi \in H^2 \cap H_0^1$ and $t \in [0, T]$. The estimate $\|\phi\|_{H^1} \leq \hat{c} \|A(t)\phi\|$ for some constant \hat{c} uniformly in $\phi \in H^2 \cap H_0^1$ and $t \in [0, T]$ is easily verified and together with (2.6) this implies (b).

(c) For $\phi \in H^2 \cap H_0^1$ we have

$$(\phi, A(t)\phi) \leq -\nu \|\phi_x\|^2,$$

and thus semigroup theory implies that for every $t \in [0, T]$, $A(t)$ generates a semigroup $\exp A(t)$ on $L^2(Q)$ satisfying $\|\exp A(t)\| \leq e^{-\nu t}$ [20; pp. 8, 14]. This implies (c) (compare [20, p. 11]).

(d) The operator $A(\tau)$, $\tau \in [0, T]$ are homeomorphisms between $H^2 \cap H_0^1$ and L^2 and in view of (a) this homeomorphisms are uniform with respect to $\tau \in [0, T]$. Thus it suffices to show

$$\|(A(t) - A(s))\phi\|_{L^2} \leq L|t-s|^{1/2} \|\phi\|_{H^2}$$

for all $\phi \in H^2 \cap H_0^1$, and $t, s \in [0, T]$. We have

$$\begin{aligned}
 \|(\Lambda(t) - \Lambda(s))\phi\|_{L^2} &\leq \|a_x(t) - a_x(s)\phi_x\|_{L^2} + \|(a(t) - a(s))\phi_{xx}\|_{L^2} \\
 &= \left[\int_0^1 \|a_x(t) - a_x(s)\phi_x\|^2 dx \right]^{1/2} + \left[\int_0^1 \|a(t) - a(s)\phi_{xx}\|^2 dx \right]^{1/2} \\
 &\leq \sqrt{2} \|\phi\|_{H^2} \left[\int_0^1 \int_s^t |a_{xt}(\sigma, x)|^2 d\sigma dx \right]^{1/2} + \sup_x |a(t, x) - a(s, x)| \|\phi\|_{H^2} \\
 &\leq \|\phi\|_{H^2} \left[\sqrt{2} \left(\int_0^1 \int_s^t |a_{xt}(\sigma, x)|^2 d\sigma dx \right)^{1/2} |t-s|^{1/2} + \sup_x \int_s^t |a_t(\sigma, x)| d\sigma \right] \\
 &\leq \|\phi\|_{H^2} |t-s|^{1/2} [\sqrt{2} \|a\|_{H^2} + \sqrt{2} \|a\|_{H^2}].
 \end{aligned}$$

This estimate implies the claim.

As an immediate consequence of Lemma 2.2 and the results in [20; V.6 and V.7] we have:

Proposition 2.4. The operators $\Lambda(t)$ with $a = a(x, t) \in A$, generate an evolution system $U(t, s)$, $0 \leq s \leq t \leq T$ on $L^2(\Omega)$. If f is Hölder continuous in t with values in $L^2(\Omega)$, then the unique solution $u(t)$ of (2.1), given by

$$u(t) = U(t, s)\phi + \int_0^t U(t, \sigma)f(\sigma)d\sigma$$

satisfies $u' \in C(\delta, T; L^2(\Omega))$ and $u \in C(\delta, T; H^2 \cap H_0^1)$ for every $\delta \in (0, T)$.

We now turn to the existence question related to (P_1) . Observe that A is a bounded, closed and convex subset of $H^1(Q)$. Therefore it is also weakly closed and weakly sequentially compact in $H^1(Q)$.

Proposition 2.5. If $\mu < \infty$ or $B > 0$ then there exists a solution \tilde{a} of $(P_1)^{B, \mu}$ for $i = 1, 2$.

Proof. We consider the case $i = 2$ when a is a function of x and t . The cases when $i = 1$ or when the coefficient is a function of x only are treated quite similarly. Let a_j be a minimizing sequence, so that

$$J_2(a_j) \rightarrow \inf_A J_2(a) = d \text{ as } j \rightarrow \infty.$$

If $\mu < \infty$, then (a_j) is a bounded subset of H^2 . If $B > 0$ then for every $\varepsilon > 0$

$$\|a_j\|_{H^2(Q)} \leq \sqrt{\frac{d}{B}} + \varepsilon$$

for all sufficiently large j . In either case (a_j) is a bounded subset of $H^2(Q)$. Thus there exists a subsequence a_{j_k} of a_j converging weakly in $H^2(Q)$ to some $\bar{a} \in H^2(Q)$. Since A is weakly closed $\bar{a} \in A$. From (2.3) and since (a_{j_k}) is also a bounded subset of $C(Q)$, there follows the existence of a constant \bar{k} independent of a_{j_k} such that

$$(2.6) \quad \|u(a_{j_k})\|_{W(0,T)} \leq \bar{k}(\|\phi\|_{L^2(\Omega)} + \|f\|_{L^2(H^{-1})}).$$

By (2.6) there exist $u \in W(0,T)$ and a subsequence of a_{j_k} , again denoted by a_{j_k} , such that $u(a_{j_k}) \rightarrow u$ weakly in $W(0,T)$. Henceforth we suppress the index k . Observe that due to the weak convergence of a_j to \bar{a} in $H^2(Q)$ it follows that $a_j \rightarrow \bar{a}$ strongly in $C(Q)$. Let $w \in \mathcal{D}(0,T;H_0^1)$, which is the space of all infinitely differentiable functions vanishing at the end-points with values in H_0^1 . Let $\langle \cdot, \cdot \rangle_{L^2(Q)}$ denote the inner product $L^2(Q)$ and take the limit in

$$\langle u_t(a_j), w \rangle_{L^2(Q)} = -\langle a_j u(a_j)_x, w_x \rangle_{L^2(Q)} + \langle f, w \rangle_{L^2(Q)}$$

to obtain

$$(2.7) \quad \langle u_t, w \rangle_{L^2(Q)} = -\langle \bar{a} u_x, w_x \rangle_{L^2(Q)} + \langle f, w \rangle_{L^2(Q)}.$$

Since $\mathcal{D}(0,T;H_0^1)$ is dense in $W(0,T)$ [19, p. 11], (2.7) holds for all $w \in W(0,T)$. Moreover $W(0,T)$ embeds continuously into $C(0,T;L^2)$ and therefore $u(0) = \phi$. Together with (2.6) and uniqueness of the solution to (2.1) in $W(0,T)$ we have $u = u(\bar{a})$. Finally by the weak lower

semicontinuity of the norm

$$d = \underline{\lim} J_2(a_j) = \underline{\lim} (\|u(\cdot, T; a_j) - z_2\|_{L^2(\Omega)}^2 + \beta \|a_j\|_{H^2(Q)}^2)$$

$$\geq \underline{\lim} \|u(\cdot, T; a_j) - z_2\|_{L^2(\Omega)}^2 + \beta \underline{\lim} \|a_j\|_{H^2(Q)}^2$$

$$\geq \|u(\cdot, T; \bar{a}) - z_2\|_{L^2(\Omega)}^2 + \beta \|\bar{a}\|_{H^2(Q)}^2.$$

Therefore \bar{a} is a solution of (P_2) and the claim is verified.

3. Regularity for spatially dependent diffusion coefficients.

In this section we examine the case in which a is a function of x only. Thus $A = \{a \in H^1(\Omega) : a(x) \geq \nu, \|a\|_{H^1} \in \mu\}$ and the regularization term in J_1 has the form $\beta \|a\|_{H^1(\Omega)}^2$. For \bar{a}_1 , a (local or global) solution of $(P_1)^\beta$, we define the open set

$$U = U(\bar{a}_1) = \{x \in \Omega : \bar{a}_1(x) > \nu\},$$

and assume throughout that U is not empty. Recall that $U(\bar{a}_1)$ can be written as a countable union of open intervals. Throughout the first part of this section we consider the case of the regularized fit-to-data criterion, i.e. $\beta > 0$. We have the following results:

Theorem 3.1. Let \bar{a}_1 be a solution of $(P_1)^\beta$ with $\beta > 0$ (and $\nu < \mu \in \cdot$ arbitrary). Then $\bar{a}_1 \in H^2(U)$. If moreover $\phi \in H_0^1$ and $f \in L^2(L^2)$, then $\bar{a}_2 \in H^3(U)$.

Theorem 3.2. Let \bar{a}_2 be a solution of $(P_2)^\beta$ with $\beta > 0$ (μ arbitrary). Then $\bar{a}_2 \in W^{2,1}(U)$. If $\phi \in H_0^1$ and $f \in L^2(L^2)$ then $\bar{a}_2 \in H^2(U)$. If moreover $z_2 \in H_0^1$ then $\bar{a}_2 \in H^3(U)$.

In addition to these local regularity properties of \bar{a}_i we obtain the following global regularity property for \bar{a}_i :

Theorem 3.3. Let $\beta > 0$ and let \bar{a}_i be a solution of $(P_i)^\beta$, $i = 1, 2$. Then \bar{a}_x is of bounded variation and in particular $\bar{a} \in H^{1+s}(0,1)$ for every $s \in [0, 1/2)$.

The case $\beta = 0$ is considered at the end of this section.

To prove these theorems we prepare several preliminaries. First the Fréchet derivative of the parameter-to-output mappings $\phi_1: A \subset H^1 \rightarrow W(0,T)$, given by $\phi_1(a) = u(a)$ and $\phi_2: A \subset H^1 \rightarrow L^2(Q)$ given by $\phi_2(a) = u(\cdot, T; a)$ is characterized. For $a \in A$ and $h \in H^1$ consider:

$$v_t = (av_x)_x + (hu_x(a))_x \quad \text{in } Q,$$

$$(3.1) \quad v(x,0) = 0 \quad \text{in } \Omega$$

$$v(0,t) = v(1,t) = 0.$$

Since $(hu_x(a))_x \in L^2(0,T;H^{-1})$ Proposition 2.1 implies the existence of a unique solution $v \in W(0,T)$ of (3.1). If moreover $\phi \in H_0^1$ and $f \in L^2(L^2)$ then by Proposition 2.3 $u(a) \in L^2(H^2)$ and one can conclude that $(hu_x(a))_x \in L^2(L^2)$. Applying Proposition 2.3 once again we find that $v \in L^2(H^2)$ with $v_t \in L^2(L^2)$. The Fréchet derivative of ϕ_1 in a in direction h is denoted by $\phi_1'(a;h)$.

Proposition 3.1. a) The Fréchet derivative of ϕ_1 in $a \in A$ in direction $h \in H^1$ is given by the unique solution v of (3.1), i.e. $\phi_1'(a;h) = v$.
 b) Similarly $\phi_2'(a;h) = v(\cdot, T)$.

Proof. We shall use the implicit function theorem [6, p. 115] and define $F: H^1(\Omega) \times W(0,T) \rightarrow L^2(0,T;H^{-1}) \times L^2(\Omega)$ by

$$F(a,u) = (u_t - (au_x)_x - f, u(0) - \phi).$$

One can easily show that F is continuous and that $F_u(a,u)$, $a \in A$, is surjective from $W(0,T)$ to $L^2(0,T;H^{-1}) \times L^2$. Moreover the partial Fréchet derivative of F with respect to a in direction h is given by $F_a(a,u)(h) = ((hu_x)_x, 0)$. Clearly $(a,u) \rightarrow F_a(a,u)$ from $H^1(\Omega) \times W(0,T)$ to $\mathcal{L}(H^1, L^2(H^{-1}) \times L^2(\Omega))$ is continuous. Here $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from X to Y . Thus by the implicit function theorem $a \rightarrow \phi_1(a) = u(a)$ is Fréchet differentiable from H^1 to $W(0,T)$. Its value can now be calculated directly from (2.1) and is found to be characterized by (3.1). As for the Fréchet derivative of ϕ_2 , observe that $u \rightarrow u(\cdot, T)$ from $W(0,T)$ to $L^2(\Omega)$ is a bounded linear mapping. Therefore $\phi_2'(a;h) = v(\cdot, T)$. From Proposition 3.1 it follows that $J_i(a)$, $i = 1, 2$ are Fréchet differentiable with respect to a . The Fréchet derivative of J_i at $a \in A$ in direction $h \in H^1$ is given by

$$J_i'(a;h) = 2\langle u(a) - z_i, v \rangle_{L^2(\Omega)} + 2B\langle a, h \rangle_{H^1(\Omega)}$$

and

$$J_2'(a;h) = 2\langle u(a)(\cdot, T) - z_2, v_2 \rangle_{L^2(Q)} + 2B\langle a, h \rangle_{H^1(Q)}.$$

Integration by parts and use of (3.1) yields an alternative representation for $J_1'(a;h)$.

Proposition 3.2. Let $a \in A$ and $h \in H^1(Q)$. Then

$$J_1'(a;h) = -2 \int_{\Omega} \int_0^T u_x(a) p_x(a) dt h dx + 2B\langle a, h \rangle_{H^1(Q)},$$

and

$$J_2'(a;h) = -2 \int_{\Omega} \int_0^T u_x(a) q_x(a) dt h dx + 2B\langle a, h \rangle_{H^1(Q)},$$

where

$$(3.2) \quad \begin{cases} -p_t - (ap_x)_x = u(a) - z_1 & \text{in } Q, \\ p(x, T) = 0 & \text{in } \Omega, \\ p(0, t) = p(1, t) = 0 & \text{for } t \in (0, T), \end{cases}$$

and

$$(3.3) \quad \begin{cases} -q_t - (aq_x)_x = 0 & \text{in } Q, \\ q(x, T) = u(x, T; a) - z_2(x) & \text{in } \Omega, \\ q(0, t) = q(1, t) = 0 & \text{for } t \in (0, T). \end{cases}$$

Remark 3.1. a) From Proposition 2.3 we find $p \in L^2(H^2)$ with $p_t \in L^2(L^2)$.

b) From Proposition 2.2 it follows that $q \in L^1(C^1) \cap W(0, T)$.

c) If $z_2 \in H_0^1$, $\phi \in H_0^1$ and $f \in L^2(L^2)$ then $u(\cdot, T) - z_2 \in H_0^1$ and therefore $q \in L^2(H^2)$ with $q_t \in L^2(L^2)$.

Lemma 3.1. Let $w(x) = \int_0^T u_x(x,t)r_x(x,t)dt$.

- a) If $u \in W(0,T)$ and $r \in W(0,T)$ then $w \in L^1$.
- b) If $u \in W(0,T)$ and $r \in L^2(H^2)$ then $w \in L^2$.
- c) If $u_x \in C(L^2)$ and $r \in L^1(C^1)$ then $w \in L^2$.
- d) If $u \in L^2(H^2)$ and $r \in L^2(H^2)$ then $w \in H^1$.

Remark 3.2. Throughout this paper and specifically in the proof of Lemma 3.1 we use the following fact which is a special case of [8, Theorem 17; p. 198]. If $F \in L^1(0,T;L^p(\Omega))$, $1 \leq p \leq \infty$, then there exists an integrable (with respect to the product Lebesgue measure on $[0,T] \times \Omega$) function f , uniquely determined up to measure zero, such that $f(t,\cdot) = F(t)$ for almost every $t \in [0,T]$. Moreover $f(\cdot,x)$ is integrable for a.e. $x \in \Omega$ and $\int_0^1 f(t,x)dx$ as a function of x equals the element $\int_0^1 F(t)dt$ of $L^p(\Omega)$.

Proof of Lemma 3.1. Part a) is obvious. To verify b) observe that

$$\begin{aligned} \int_0^1 \int_{\Omega} |u_x r_x| dt dx &\leq \int_0^1 \left(\int_{\Omega} |u_x|^2 dt \cdot \int_{\Omega} |r_x|^2 dt \right) dx \\ &\leq \sup_{x \in \Omega} \int_0^1 |r_x|^2 dt \|u_x\|_{L^2(\Omega)}^2 \leq \text{const } \|r\|_{L^2(H^2)} \|u\|_{W(0,T)}. \end{aligned}$$

To verify c) we use Minkowski's integral inequality [21, p. 271] in the first step:

$$\begin{aligned} \left(\int_0^1 \int_{\Omega} |u_x r_x| dt dx \right)^{1/2} &\leq \int_0^1 \left(\int_{\Omega} |u_x r_x| dt dx \right)^{1/2} dt \\ &\leq \int_0^1 \|r_x(\cdot,t)\|_C \cdot \|u_x(\cdot,t)\|_{L^2} dt \leq \|r\|_{L^1(C^1)} \|u_x\|_{C(L^2)}. \end{aligned}$$

Part d) is left to the reader.

Lemma 3.2. Let \bar{I} be an open interval in Ω , $\delta > 0$ and $a \in H^1(\Omega)$. If further $\psi \in L^1$ (or L^2 or H^1) and

$$\int_{\bar{I}} [\delta(a_x h_x + ah) + \psi h] dx = 0 \text{ for all } h \in H^1_0(\bar{I}),$$

then $-a_{xx} + a = -\frac{1}{\delta} \psi$,

and

$$a \in W^{2,1}(\bar{I}) \text{ (or } H^2(\bar{I}), \text{ or } H^3(\bar{I})).$$

Proof. By assumption

$$\int_{\bar{I}} (\delta a_x - \delta \int_0^x a \, ds - \int_0^x \psi) h_x \, dx = 0$$

for all $h \in H^1(\bar{I})$. Therefore $a_x = \int_0^x (a + \frac{\psi}{\delta}) \, ds + \text{const.}$ and therefore $a_{xx} = a + \frac{\psi}{\delta}$. The remaining assertion now follows easily.

The proof of Theorems 3.1 - 3.3 employs a Lagrange multiplier formulation of (P_1) . As another preparatory result we therefore discuss the regularity condition for the constraint set A . Let $G_1: H^1(\Omega) \rightarrow \mathbb{R}$ and $G_2: H^1(\Omega) \rightarrow C(\Omega)$ be defined by

$$G_1(a) = |a|_{H^1}^2 - \mu^2 \text{ and } G_2(a) = \nu - a,$$

and put $G(a) = (G_1(a), G_2(a))$ where $G: H^1 \rightarrow \mathbb{R} \times C$. We have

$$G'_1(a; h) = 2\langle a, h \rangle_{H^1} \text{ and } G'_2(a; h) = -h.$$

By C_+ we denote the cone of nonnegative functions in C , by \mathbb{R}_+ the nonnegative reals and C_+^\sharp stands for the dual cone of C_+ , i.e.:

$$C_+^\sharp = (\lambda^* \in C^*: \lambda^*(\psi) \geq 0 \text{ for all } \psi \in C_+).$$

Observe that $A = (a \in H^1: -G(a) \in \mathbb{R}_+ \times C_+) = (a \in H^1: G(a) \leq 0)$.

Lemma 3.3. Every admissible point $a \in A$ satisfies the regular point condition. That is

$$(3.4) \quad 0 \in \inf (G(a) + \text{range } G'(a; H^1) + \mathbb{R}_+ \times C_+).$$

Proof. This result is verified in [15]. For the sake of completeness and since the proof is short we repeat it here. We will show that every (r, ψ) is contained in the set appearing in (3.4) provided that $|r|_{\mathbb{R}} + |\psi|_C < \varepsilon = \min\left(\frac{1}{4|a|_{L^2}}, \frac{1}{2}\right)(\mu^2 - \nu^2)$. Note that (3.4) is equivalent to

$$(3.5) \quad 0 \in \inf(|a|_{H^1}^2 - \mu^2 + 2\langle a, h \rangle_{H^1} + \mathbb{R}_+, \nu - a - h + C_+; h \in H^1).$$

Let $\Psi = \psi - \min \psi$ and note that $\Psi \in C_+$. We put $h = \nu - a - \min \psi \in H^1$. Then $\psi = \nu - a - h + \Psi$, which is of form of the elements in the second component of (3.5). Next we need to find $\rho \in \mathbb{R}_+$ such that

$$(3.6) \quad r = |a|_{H^1}^2 - \mu^2 + 2\langle a, h \rangle_{H^1} + \rho.$$

This is equivalent to $r = -\mu^2 - |a|_{H^1}^2 + 2(a, \nu)_{L^2} - 2(a, \min \psi)_{L^2} + \rho$. But

$$\begin{aligned} 2(a, \nu)_{L^2} - 2(a, \min \psi)_{L^2} - |a|_{H^1}^2 - \mu^2 &\leq |a|_{L^2}^2 + \nu^2 - |a|_{H^1}^2 - \mu^2 - 2(a, \min \psi)_{L^2} \\ &\leq \nu^2 - \mu^2 + 2|a|_{L^2}\varepsilon \leq \frac{1}{2}(\nu^2 - \mu^2). \end{aligned}$$

Therefore

$$\rho \geq r + \frac{1}{2}(\mu^2 - \nu^2) \geq \frac{1}{2}(\mu^2 - \nu^2) - |r| \geq \frac{1}{2}(\mu^2 - \nu^2) - \varepsilon \geq 0.$$

Thus ρ in (3.4) is nonnegative and the lemma is verified.

Proof of Theorem 3.1. We only consider the case $\mu > 0$. The case $\mu = 0$ requires no essential changes from the present proof.

By the Lagrange multiplier theorem (see e.g. [24]) there exists a pair $(\lambda_1, \lambda_2^{\pm}) \in \mathbb{R}_+ \times C_+^{\pm}$ such that

$$(3.7) \quad J_1'(\bar{a}; h) + \lambda_1 G_1'(\bar{a}; h) + \langle \lambda_2^{\pm}, G_2'(\bar{a}; h) \rangle_C + B\langle \bar{a}, h \rangle_{H^1} = 0,$$

for all $h \in H^1$,

$$(3.8) \quad \lambda_1(|\bar{a}|_{H^1}^2 - \mu_2) = 0 \quad \text{and} \quad \langle \lambda_2^\#, \nu - \bar{a} \rangle_C = 0.$$

Here $\langle \cdot, \cdot \rangle_C$ denotes the duality pairing on C and we dropped the index in the notation for \bar{a}_1 . Recall that $\lambda_2^\#$ can be expressed as $\lambda_2^\# = \int_0^1 d\lambda_2$, where λ_2 is a nondecreasing function of bounded variation. From the mean value theorem for Stieltjes integrals and (3.8) it follows that λ_2 is constant on open subintervals of $U = U(\bar{a})$. Using Proposition 3.2 it can be seen that (3.7) is equivalent to

$$(3.9) \quad -2(w, h)_{L^2} + (2\lambda_1 + \beta) \langle \bar{a}, h \rangle_{H^1} + \langle \lambda_2^\#, h \rangle_C = 0$$

for all $h \in H^1$, where $w(x) = \left(\int_0^1 u_x(\bar{a}) p_x(\bar{a}) dt \right)(x)$. Let I be an open interval in U and $h \in \{h \in H^1: h = 0 \text{ on } \Omega \setminus I\}$. Then

$$(3.10) \quad -2(w, h)_{L^2(I)} + (2\lambda_1 + \beta) \langle \bar{a}, h \rangle_{H^1(I)} = 0.$$

By Remark 3.1 a) and Lemma 3.1 b) we have $w \in L^2(\Omega)$ and under the assumption $\phi \in H_0^1$ and $f \in L^2(L^2)$ it follows from Remark 3.1 a) and Lemma 3.1 d) that $w \in H^1$. Since $\beta > 0$ by assumption, we can use Lemma 3.2 with $\delta = \beta + 2\lambda_1 > 0$ to obtain

$$(3.11) \quad -\bar{a}_{xx} + \bar{a} = \frac{2}{2\lambda_1 + \beta} w \quad \text{on } I$$

and $\bar{a} \in H^2(I)$ respectively $\bar{a} \in H^3(I)$.

Since U is an open subset of Ω in dimension one, it can be expressed as a countable union of disjoint open intervals. On each of these intervals (3.11) holds and therefore $\bar{a} \in H^2(U)$, respectively $\bar{a} \in H^3(U)$ if the stronger regularity assumptions hold for f and ϕ .

Proof of Theorem 3.2. The proof is quite analogous to the one of Theorem

3.1. Here we take $w(x) = \left(\int_0^1 u_x(\bar{a}_2) q_x(\bar{a}_2) dt \right)(x)$ and by Remark 3.1 b) and Lemma 3.1 a) we find $w \in L^1$. Again we have (3.11) and therefore $\bar{a} \in$

$W^{2,1}(I)$ and consequently $\bar{a} \in W^{2,1}(U)$. If in addition $\phi \in H_0^1$ and $f \in L^2(L^2)$, then $u_x \in C(L^2)$ and $q \in L^1(C^1)$ and thus $w \in L^2$ by Lemma 3.1 c). Under the additional assumption $z_2 \in H_0^1$ we find $w \in H^1$ by Remark 3.1 c and Lemma 3.1 d). Using these facts in (3.11) we arrive at the desired result.

Proof of Theorem 3.3. Consider (3.9) with w defined as in the proof of Theorem 3.1 if $i = 1$ or as in the proof of Theorem 3.2 if $i = 2$. In either case $w \in L^1(\Omega)$. Thus from (3.9) the mapping $\bar{a}_i \rightarrow ((\bar{a}_i)_x, h_x)$ can be

extended to a bounded linear functional on C , i.e. $(\bar{a}_x, h_x) = \int_0^1 h \, d\psi$ for all $h \in H^1$ and some function ψ of bounded variation. For all $h \in H_0^1$ we thus have

$$(\bar{a}_x, h_x) = - \int_0^1 \psi \, h_x \, dx$$

which implies $\bar{a}_x = -\psi + C$ for a constant C . Therefore \bar{a} is of bounded variation and the proof is finished.

Remark 3.2. It is interesting to observe that in the case $\lambda_1 = \beta = 0$ the regularity of λ_2 (which is used for the ~~function~~ integral representation of λ_2^{\ddagger}) is determined by w . In fact, in this case $w = \dot{\lambda}_2$ and $\lambda_2 \in W^{1,1}$ by (3.9) for $i = 1$ or 2 . 29

Now let us consider the case $\beta = 0$. From (3.9) - (3.11) it is obvious that the same regularity properties can be obtained for the solutions \bar{a}_i of the unregularized problems $(P_i)^{\circ}$, $i = 1, 2$, if only $\lambda_1 \neq 0$. By (3.8) this in turn implies that the norm constraint is active, i.e. $\|a\|_{H^1} = \mu$. We recall that $U(\bar{a}_i)$ is assumed to be nonempty.

Proposition 3.3. Let $\beta = 0$, $\mu < \infty$ and let \bar{a}_i be a solution of $(P_i)^{\circ}$. If for some open interval $\bar{I} \subset \Omega$

$$(3.12) \quad \int_{\bar{I}} u_x(\bar{a}_i) p_x(\bar{a}_i) \, dt = 0 \quad \text{in } L^2(\bar{I}) \quad \text{respectively}$$

$$\int_0^I u_x(\bar{a}_2) q_x(\bar{a}_2) dt = 0 \text{ in } L^1(\bar{I}),$$

then the Lagrange multiplier associated with the norm constraint satisfies $\lambda_1 > 0$ and the regularity results of Theorem 3.1 - 3.3 hold. Moreover $\|\bar{a}_1\|_{H^1} = \mu$ in this case.

Proof. Assume that $\lambda_1 = 0$ and choose \bar{I} as in the statement of the proposition. Then by (3.10) we have $w = 0$ on \bar{I} which contradicts (3.12) and ends the proof.

It appears to be infeasible to give conditions in terms of (2.1) and the equations for p or q which guarantee (3.12). The following proposition gives a necessary condition for (3.12) to hold and thus for $\lambda_1 > 0$.

Proposition 3.4. Let $\beta = 0$ and $\mu < \infty$. If $\lambda_1 > 0$ then $u(\bar{a}_1)$ cannot coincide with z_1 as a function in $L^2(L^2)$ respectively $L^2(0)$.

Proof. Let $I \subset U$ be an open interval and assume that $u(\bar{a}_1) = z_1$, for $i = 1$ or 2 . Then $p = 0$ respectively $q = 0$. Therefore (3.12) cannot hold. By (3.10) $\lambda_1 \langle \bar{a}_1, h \rangle_{H^1(I)} = 0$ for all $h \in H^1(I)$ and thus $\lambda_1 = 0$.

4. Regularity for spatially and temporally dependent diffusion coefficients.

In this section we consider the case in which the coefficient a is a function of both x and t . Hence we take the admissible set of parameters to be

$$A = \{a \in H^2(Q) : a(x,t) \geq \nu > 0, \|a\|_{H^2} \leq \mu\}$$

and the regularization term in J_1 is of the form $\beta \|a\|_{H^2(Q)}^2$. For \bar{a}_1 a (local or global) solution of $(P_1)^\beta$ we define the open set

$$U = U(\bar{a}_1) = \{(x,t) \in \Omega : a_1(x,t) > \nu\}.$$

Throughout this section it is assumed that U is not empty. As in the case of section 3, the optimal parameter may satisfy special regularity properties either due to the regularization term or due to the norm bound. Here we state both cases simultaneously. We shall write $\Omega_1 \subset\subset \Omega_2$ if Ω_1 and Ω_2 are open subsets of a Euclidean space and if Ω_1 is compact with $\Omega_1 \subset \Omega_2$. The Lagrange multiplier associated with the norm bound is again denoted by λ_1 .

Theorem 4.1. Assume that $\mu < \infty$ and $\lambda_1 \neq 0$ or that $\beta > 0$ and let \bar{a}_1 be a solution of $(P_1)^\beta$. If $\bar{U} \subset\subset U$, $f \in L^2(Q)$ and $\phi \in H_0^1$, then $\bar{a} \in H^s(\bar{U})$.

Theorem 4.2. Assume that $\mu < \infty$ and $\lambda_1 \neq 0$ or that $\beta > 0$ and let \bar{a}_2 be a solution of $(P_2)^\beta$. If $\bar{U} \subset\subset U$, $f \in L^2(Q)$, $\phi \in H_0^1$ and $z_2 \in H_0^1$ then $\bar{a} \in H^s(\bar{U})$.

Theorem 4.3. Assume that $\mu < \infty$ and $\lambda_1 \neq 0$ or that $\beta > 0$ and let \bar{a}_i be a solution of $(P_i)^\beta$, $i = 1$ or 2 . If $\bar{Q} \subset\subset Q$ and \bar{Q} satisfies the uniform cone property, then $\bar{a}_i \in H^{2+s}(\bar{Q})$ for any $s \in (0,1)$.

In section 5 we shall specify conditions which guarantee that $\lambda_1 \neq 0$.

The verification of the above theorems, proceeds along the same lines as that presented in section 3. We indicate the salient features. First the

Fréchet derivatives of J_i , $i = 1$ or 2 , are specified.

Proposition 4.1. Let $a \in A$ and $h \in H^2(Q)$. Then

$$J_1'(a; h) = -2 \int_{\Omega} \int_0^T u_x(a) p_x(a) h \, dt dx + 2B\langle a, h \rangle_{H^2(Q)},$$

and

$$J_2'(a; h) = -2 \int_{\Omega} \int_0^T u_x(a) q_x(a) h \, dt dx + 2B\langle a, h \rangle_{H^2(Q)},$$

where $p \in W(0, T)$ and $q \in W(0, T)$ are given in (3.2) and (3.3).

Remark 4.1. Since $f \in L^2(H^{-1})$ and $\phi \in L^2$ it follows that $u \in W(0, T)$ and that $u_x \in L^2(L^2)$. By Proposition 2.3 this further implies that $p \in L^2(H^2)$ and $p_t \in L^2(L^2)$.

Lemma 4.1. a) Under the standard assumption that $f \in L^2(H^{-1})$, $\phi \in L^2$, $z_1 \in L^2(Q)$ and $z_2 \in L^2(Q)$ it follows that $u_x p_x \in L^1(Q)$ and $u_x q_x \in L^1(Q)$ and thus both these functions can be identified with elements in the dual space $(H^{1+\gamma})^*$ of $H^{1+\gamma}(Q)$ for every $\gamma > 0$.

b) If $f \in L^2(L^2)$ and $\phi \in H_0^1$ then $u_x p_x \in L^2(L^2)$.

c) If $f \in L^2(L^2)$, $\phi \in H_0^1$ and $z_2 \in H_0^1$ then $u_x q_x \in L^2(L^2)$.

Proof. Obviously $u_x p_x$ and $u_x q_x$ belong to $L^1(Q)$. Moreover, if $\psi \in L^1(Q)$ then, since $H^{1+\gamma}(Q)$ embeds continuously in $L^\infty(Q)$ for $\gamma > 0$, we see that

$$\|\psi\|_{(H^{1+\gamma})^*} = \sup_{\phi \in H^{1+\gamma}} \frac{\langle \psi, \phi \rangle_{1+\gamma}}{\|\phi\|_{H^{1+\gamma}}} = c \|\psi\|_{L^1(Q)},$$

for some constant c independent of $\psi \in L^1(Q)$. Here $\langle \cdot, \cdot \rangle_{1+\gamma}$ denotes the duality pairing between $H^{1+\gamma}$ and $(H^{1+\gamma})^*$ with L^2 as a pivot

space. To verify b) we observe that by Proposition 2.3 we find that $u \in C(H^1)$. Moreover $p \in L^2(H^2)$ by Remark 4.1 and thus it easily follows that $u_x p_x \in L^2(L^2)$. Finally c) is implied by the fact that $u \in C(H_0^1)$ and $q \in L^2(H^2)$.

To determine the regularity properties of the solutions \bar{a}_i of $(P_i)^\beta$ we again use a Lagrange formulation. First the constraints that are involved in defining the set of admissible parameters are characterized by a mapping $G = (G_1, G_2): H^2 \rightarrow \mathbb{R} \times H^{1+\gamma}(Q)$ given by

$$G_1(a) = |a|_{H^2(Q)}^2 - \mu^2, \quad G_2(a) = \nu - a.$$

Let $H_+^{1+\gamma} = \{a \in H^{1+\gamma}: a(x,t) \geq 0 \text{ on } Q\}$. For $a \in H^{1+\gamma}$ we write $a \leq 0$ if $-a \in H_+^{1+\gamma}$. Observe that $a \in A$ if and only if $G(a) \leq 0$. Since $H^{1+\gamma}$, $\gamma > 0$, embeds continuously into C , one can show with an argument similar to the one of Lemma 3.3 that every point $a \in A$ satisfies the regular point condition

$$0 \in \text{int}(G(a) + \text{range } G'(a; H^2) + \mathbb{R}_+ \times H_+^{1+\gamma}).$$

Thus for the problems $(P_i)^\beta$, $i = 1$ or 2 , there exists a Lagrange multiplier pair $(\lambda_1, \lambda_2^*) \in \mathbb{R}_+ \times (H_0^{1+\gamma})^*$ satisfying

$$(4.1) \quad -2 \int_0^T \int_\Omega u_x(\bar{a}_1) p_x(\bar{a}_1) h \, dx dt + 2 \langle \lambda_1 + \beta, \bar{a}_1, h \rangle_{H^2(Q)} + \langle \lambda_2^*, h \rangle_{1+\gamma} = 0,$$

respectively

$$(4.2) \quad -2 \int_0^T \int_\Omega u_x(\bar{a}_2) q_x(\bar{a}_2) h \, dx dt + 2 \langle \lambda_1 + \beta, \bar{a}_2, h \rangle_{H^2(Q)} + \langle \lambda_2^*, h \rangle_{1+\gamma} = 0$$

for all $h \in H^2(Q)$. Moreover the complementary condition

$$(4.3) \quad \lambda_1 (|\bar{a}_1|_{H^2}^2 - \mu^2) = 0, \quad \langle \lambda_2, \nu - \bar{a}_1 \rangle_{1+\gamma} = 0,$$

respectively

$$(4.4) \quad \lambda_1 (\|\bar{a}_2\|_{H^2}^2 - \mu^2) = 0, \quad \langle \lambda_2, \gamma - \bar{a}_2 \rangle_{1+\gamma} = 0,$$

holds. Here $(H_+^{1+\gamma})^* = \{g \in (H^{1+\gamma})^* : \langle g, a \rangle_{1+\gamma} \geq 0 \text{ for all } a \in H_+^{1+\gamma}\}$. Next we analyse further the variational equations (4.1) and (4.2). Henceforth the assumption $\lambda_1 + \beta > 0$ of Theorems 4.1 - 4.3 will be used. The bilinear form

$$B(a, h) = 2(\lambda_1 + \beta) \langle a, h \rangle_{H^2}$$

is H^2 -coercive and H^2 -bounded, i.e.

$$B(a, a) \geq 2(\lambda_1 + \beta) \|a\|_{H^2}^2$$

and

$$B(a, h) \leq 2(\lambda_1 + \beta) \|a\|_{H^2} \|h\|_{H^2}.$$

Thus by the ~~Max~~-Milgram theorem there exists a bounded linear operator $L: H^2(Q)^* \rightarrow H^2(Q)$ satisfying

$$(4.5) \quad B(L(\rho), h) = \langle \rho, h \rangle_{(H^2, *, H^2)} \quad \text{for all } h \in H^2(Q)$$

and

$$(4.6) \quad \|L(\rho)\|_{H^2(Q)} \leq \frac{1}{2(\lambda_1 + \beta)} \|\rho\|_{H^2(Q)^*}.$$

Here $\langle \cdot, \cdot \rangle_{H^2, *, H^2}$ denotes the duality pairing between $H^2(Q)$ and $H^2(Q)^*$. Hence, by restriction, for any $\tilde{Q} \subset\subset Q$ we find

$$\|L(\rho)\|_{H^2(\tilde{Q})} \leq \frac{1}{2(\lambda_1 + \beta)} \|\rho\|_{H^2(Q)^*},$$

and L can be considered as a bounded linear operator from $H^2(Q)^*$ to $H^2(\tilde{Q})$. We shall write

$$L \in \mathcal{L}(H^2(Q)^*, H^2(\tilde{Q})),$$

where $\mathcal{L}(X, Y)$ denotes the space of all continuous linear operators between

the Banach spaces X and Y . Applying results on the interior regularity of solutions of variational elliptic boundary value problems we further obtain that for $\rho \in L^2(Q)$ and $\bar{Q} \subset\subset Q$ the solution $L(\rho)$ of (4.5) satisfies $\|L(\rho)\|_{H^2(\bar{Q})} \leq C\|\rho\|_{L^2(Q)}$, where the constant C is independent of $\rho \in L^2(Q)$ [1, p. 51-51; 23, p. 299-302]. We thus obtain

$$(4.8) \quad L \in \mathcal{L}(L^2(Q), H^2(\bar{Q})).$$

We next utilize results from interpolation theory. Since $H^2(Q)$ is continuously and densely imbedded in $L^2(Q)$ it follows that (taking $L^2(Q)$ as a pivot space) $L^2(Q)$ can be identified with a subset of $(H^2)^*$. Subsequently from (4.7), (4.8) and the interpolation theorem we find, using the notation from [19],

$$(4.9) \quad L \in \mathcal{L}([L^2(Q), H^2(Q)^*]_{\theta}, [H^2(\bar{Q}), H^2(\bar{Q})]_{\theta})$$

for every $\theta \in (0,1)$. For interpolation between dual spaces it is known [19, p. 29] that

$$[L^2(Q), H^2(Q)^*]_{\theta} = ([H^2(Q), L^2(Q)]_{1-\theta})^*,$$

with equivalent norms, and further [19, p. 40]

$$([H^2(Q), L^2(Q)]_{1-\theta})^* = H^{2\theta}(Q)^*.$$

This implies $[L^2(Q), H^2(Q)^*]_{\theta} = H^{2\theta}(Q)^*$. If \bar{Q} satisfies the uniform cone property, then the Calderon Zygmund extension theorem [23, p. 100] is applicable to the domain \bar{Q} and the results in [19, p. 40-43] can be used to obtain

$$[H^2(\bar{Q}), H^2(\bar{Q})]_{\theta} = H^{2(1-\theta)}(\bar{Q}).$$

Using these facts in (4.9) we arrive at

$$(4.10) \quad L \in \mathcal{L}(H^{2\theta}(Q)^*, H^{2(1-\theta)}(\bar{Q})),$$

for any $\theta \in (0,1)$.

Proof of Theorem 4.1. Let \hat{a}_1 be a solution of $(P_1)^{\beta}$ and assume that $\bar{U} \subset\subset U$. Then there exists a bounded domain V such that $\bar{U} \subset\subset V \subset\subset U$.

Moreover there exists $\delta > 0$ such that $\bar{a}_1(x,t) \geq \nu + \delta$ on V . Let $h \in H^2$ with compact support in V_+ satisfy $\|h\|_{L^\infty} \leq \delta$. Then $\nu + h - \bar{a}_1 \leq 0$ and by the complementarity condition (4.3)

$$\langle \lambda^*, \nu + h \rangle_{1+\gamma} = \langle \lambda^*, \nu + h - \bar{a}_1 \rangle_{1+\gamma} \leq 0.$$

Therefore it follows that

$$(4.11) \quad \langle \lambda^*, h \rangle_{1+\gamma} = 0 \quad \text{for all } h \in H^2 \text{ with compact support in } V.$$

We use this fact in (4.1) and find

$$-\langle u_x p_x, h \rangle_{L^2(V)} + (\lambda_1 + \beta) \langle \bar{a}_1, h \rangle_{H^2(V)} = 0$$

for every $h \in H^2$ with compact support in V . From Lemma 4.1 we have $u_x p_x \in L^2(L^2)$. The assumptions in Theorem 4.1 guarantee that $\lambda_1 + \beta > 0$. Thus from (4.5) and (4.8) with Q, \bar{Q} replaced by V, \bar{U} we obtain $\bar{a}_1 \in H^4(\bar{U})$. See also [23, p. 73, 299]. This ends the proof.

Proof of Theorem 4.2. We only replace (4.1), (4.3) by (4.2), (4.4) in the proof of Theorem 4.1 to obtain the desired result.

Proof of Theorem 4.3. From Lemma 4.1 we observe that $u_x p_x$ and $u_x q_x$ can be identified with elements in $H^{1+\gamma}(Q)^*$. Furthermore $\lambda_2^* \in H^{1+\gamma}(Q)^*$ as well. Therefore we have

$$\rho_1 := -\lambda_2^* + u_x(\bar{a}_1) p_x(\bar{a}_1) \in H^{1+\gamma}(Q)^*$$

and

$$\rho_2 := -\lambda_2^* + u_x(\bar{a}_2) q_x(\bar{a}_2) \in H^{1+\gamma}(Q)^*.$$

By assumption $\lambda_1 + \beta > 0$. Thus from (4.1), respectively (4.3), (4.5), and (4.10) it follows that $L(\rho_1) \in H^{2-\gamma}$ for every $\gamma \in (0,1)$. Setting $s = 1 - \gamma$ this implies the result.

5. Nontriviality of the Lagrange multiplier associated with the norm constraint.

In this section we give conditions, which guarantee that $\lambda_1 > 0$. In view of (4.3), (4.4) these conditions imply that the norm constraint is active and they constitute a specific case, where the strict complementarity conditions holds:

$\lambda_i (|\bar{a}_i|_{H^2}^2 - \gamma^2) = 0$, $i = 1$ or 2 , without both factors being zero simultaneously. We shall see that the problem of nontriviality of λ_1 is related to the question of attainability of z_1 . Here z_1 is called attainable, if $z_1 \in V_1 = \{z: z = u(a) \text{ in } W(0,T), a \in A\}$, respectively $z_2 \in V_2 = \{z: z = u(\cdot, T; a) \text{ in } L^2(\Omega), a \in A\}$, where $u(a)$ is the solution of (2.1).

We point out some of the consequences of the nontriviality of λ_1 . First, we have already seen in sections 3 and 4 that $\lambda_1 > 0$ implies regularity properties of the solutions \bar{a}_1 of (P_1) . Moreover, for elliptic estimation problems $\lambda_1 > 0$ guarantees stability of the solution of the output-least-squares formulation of the estimation problem with respect to perturbation of z or A [7] and it guarantees augmentability as used, for example, in the augmented Lagrangian approach to estimation problems that we described in [10].

Recall from (4.1) that for $B = 0$

$$(5.1) \quad -2\langle u_x p_x, h \rangle_{L^2(Q)} + 2\lambda_1 \langle \bar{a}_1, h \rangle_{H^2(Q)} + \langle \lambda_1^{\frac{1}{2}} h \rangle_{L^2(\Gamma)} = 0$$

for all $h \in H^2$ and that

$$U = U(\bar{a}_1) = \{(x, t) \in Q: \bar{a}_1(x, t) > \nu\}.$$

For the unregularized fit-to-data criterion J_1 we have the following result which guarantees $\lambda_1 > 0$. By a rectangle V in Q we mean a set of the form $[x_1, x_2] \times [t_1, t_2] \subset Q$ with $0 \leq x_1 < x_2 \leq 1$ and $0 \leq t_1 < t_2 \leq T$.

Theorem 5.1. Let $B = 0$, $f \in L^2(L^2)$, $\phi \in H_0^1$, z_1 be Hölder continuous with values in $L^2(Q)$ and assume that \bar{a}_1 is any (local or global) solution of (P_1) satisfying $\bar{a}_1 \neq \nu$ and

(a) for every rectangle $V \subset U$, f is not (a.e.) only a function of t on V ,

(b) there exists at least one rectangle $V \subset U$ such that $u(\bar{a}_1) - z_1$ is not (a.e.) only a function of t on V . Then $\lambda_1 > 0$.

Before we give the proof let us further interpret the result. First observe that the conclusion $\lambda_1 > 0$ depends on μ only in as far as \bar{a}_1 may depend on μ . The solution \bar{a}_1 of (P_1) enters into the assumptions only in (b) and through the requirement that $\bar{a}_1 \neq \mu$ (which is equivalent to $U(\bar{a}_1) \neq \emptyset$). As for (b) we point out that due to the assumptions on f and ϕ every solution $u(a)$, $a \in A$, of (2.1) is continuous on Q with $u(0,t;a) = u(1,t;a) = 0$. In the specific case that z_1 is constructed from piecewise constant interpolation of point data, (b) holds for every $a \in A$ unless $z_1 = 0$. We observe also that under the smoothness assumptions and assumption (a) of Theorem 5.1, (P_1) with the norm constraint eliminated, cannot have a solution \bar{a}_1 , which is not identically ν and which satisfies (b). In fact, if this were the case \bar{a}_1 were also a solution of the norm-constraint problem with $\mu = 2\|\bar{a}_1\|_{H^2}$ and Theorem 5.1 were applicable and gave $\lambda_1 > 0$. By the complementarity condition (4.3) this would imply $\mu = \|\bar{a}_1\|_{H^2}$ which is, of course, impossible.

Proof of Theorem 5.1. Let us assume that $\lambda_1 > 0$. Then by (5.1) we have

$$(5.2) \quad 2\langle u_x p_x, h \rangle_{L^2(Q)} = \langle \lambda_1^{\frac{1}{2}}, h \rangle_{1+\gamma} \quad \text{for all } h \in H^2.$$

Thus $h \rightarrow \langle \lambda_1^{\frac{1}{2}}, h \rangle_{1+\gamma}$ can be extended to a continuous linear functional F on $L^2(Q)$ with $F(h) = \langle 2u_x p_x, h \rangle_{L^2(Q)}$. Here we used the fact that $u \in C(H_0^1)$ and $p \in L^2(H^2 \cap H_0^1)$ by Proposition 2.3, so that $u_x p_x \in L^2(L^2)$. Since $\lambda_1^{\frac{1}{2}}$ is in the positive dual cone $(H_+^{1+\gamma})^*$ we find that

$$(u_x p_x, h)_{L^2(Q)} \geq 0$$

for all $h \in H^{1+\gamma} \cap C^+$. But $H^{1+\gamma} \cap C^+$ is dense in L_+^2 (this can be seen with a mollifier argument [ADAMS, p. 30]) and thus $(u_x p_x, h)_{L^2} \geq 0$ for all $h \in L_+^2$. This implies

$$(5.3) \quad u_x p_x \geq 0 \quad \text{a.e. on } Q.$$

Moreover, since $\langle \lambda_1^{\frac{1}{2}}, \nu - \bar{a} \rangle_{1+\gamma} = 2(u_x p_x, \nu - \bar{a})_{L^2(Q)} = 0$ we find that

$$(5.4) \quad u_x p_x = 0 \quad \text{a.e. on } U.$$

Next observe that $t \rightarrow u(t, \cdot)$ from $[0, T]$ to $L^2(\Omega)$ is Hölder continuous with exponent $1/2$. In fact,

$$|u(t) - u(s)| \leq \int_s^t |u_t(\sigma)| d\sigma \leq |t-s|^{1/2} \left(\int_0^T |u_t|^2 ds \right)^{1/2},$$

where the existence of the last integral follows from Proposition 2.3. Since z_1 is Hölder continuous with values in $L^2(\Omega)$, it follows that $u - z_1$ is Hölder continuous with values in $L^2(\Omega)$ and thus $p \in C(0, T-\delta; H^2 \cap H_0^1)$ for every $\delta \in (0, T)$ by Proposition 2.4. Specifically p_x is continuous on $[0, T-\delta] \times [0, 1]$ for every $\delta \in (0, T)$.

Now let us return to (5.4) and assume that

$$(5.5) \quad p_x(\bar{x}, \bar{t}) \neq 0 \text{ at some } (\bar{x}, \bar{t}) \in U \cap ([0, 1] \times [0, T-\delta]).$$

(Of course, it is assumed that δ is chosen sufficiently small so that $U \cap ([0, 1] \times [0, T-\delta]) \neq \emptyset$.) Then there exists a rectangle $V = V(\bar{x}, \bar{t}) = [x_1, x_2] \times [t_1, t_2]$ such that $p_x(x, t) \neq 0$ for all $(x, t) \in V$. By (5.4) it follows that $u_x(x, t) = 0$ on V and thus u_x is a function of t only on V ; i.e. $u(x, t) = c_1(t)$, where $W^{1,2}(t_1, t_2; \mathbb{R})$. Using this in (2.1) we arrive at

$$c_1'(t) = f(t, x) \text{ on } V.$$

This contradicts assumption (a) of the theorem and therefore (5.5) cannot hold at any $(\bar{x}, \bar{t}) \in U \cap ([0, 1] \times [0, T-\delta])$, where $\delta \in (0, T)$. We conclude that $p_x(x, t) = 0$ a.e. on U . Let now $V = [x_1, x_2] \times [t_1, t_2] \subset U$ be chosen as given by assumption (b). Then $p(x, t) = c_2(t)$ on V for some $c_2 \in W^{1,2}(t_1, t_2; \mathbb{R})$. Using this in the equation for p we arrive at

$$c_2'(t) = u(t, x) - z_1(x, t) \text{ on } V,$$

which contradicts (b).

Thus we have to revoke the assumption $\lambda_1 = 0$ and the theorem is proved.

Remark 5.1. We observe that in the proof of the previous theorem the role of u and p can be reserved, provided u is sufficiently regular (i.e. u_x is continuous on $[0, 1] \times [\delta, T]$, for any $\delta \in (0, T)$). Specifically one

can show: if the assumption of Hölder-continuity of z_1 is replaced by Hölder-continuity with values in $L^2(Q)$ of f and if (a), (b) in Theorem 5.1 are replaced by

(a') there exists some rectangle $V \subset U$ such that f is not (a.e.) only a function of t on V ,

(b') for every rectangle $V \subset U$, $u(\bar{a}) - z_1$ is not (a.e.) only a function of t on V ,

then $\lambda_1 > 0$.

We now consider the case of the fit-to-data criterion J_2 . We have

$$-2\langle u_x q_x, h \rangle_{L^2(Q)} + 2\lambda_1 \langle \bar{a}, h \rangle_{H^2} + \langle \lambda_2^{\frac{1}{2}}, h \rangle_{L^2(\Gamma)} = 0, \text{ for all } h \in H^2.$$

Theorem 5.2. Assume that $a^*(x,t) > \nu$ on $[0,1] \times [0,T]$ and that

(a) for every rectangle $V \subset [0,1] \times [0,T]$, f is not (a.e.) a function of t on V only,

(b) $u(x,T;\bar{a})$ is not a.e. equal to $z_2(x)$ on Ω .

Then $\lambda_1 > 0$.

Proof. Since $a^*(x,t) > \nu$, the complementarity condition implies that $\lambda_2^{\frac{1}{2}} = 0$. Therefore

$$-\langle u_x q_x, h \rangle_{L^2(Q)} + \lambda_1 \langle \bar{a}, h \rangle_{H^2} = 0 \text{ for all } h \in H^2.$$

Now assume that $\lambda_1 = 0$. Then

$$(5.6) \quad u_x q_x = 0 \text{ a.e. on } [0,1] \times [0,T].$$

From Proposition 2.4 we find that $q \in C(0, T-\delta; H^2 \cap H^1)$ for every $\delta \in (0, T)$. Specifically q_x is continuous on $[0,1] \times [0, T)$. If $q_x(\hat{x}, \hat{t}) \neq 0$ for some $(\hat{x}, \hat{t}) \in [0,1] \times [0, T)$ then $u_x = 0$ a.e. on some rectangle $V \subset [0,1] \times [0, T)$. This contradicts (a) and thus $q_x(x,t) = 0$ on $[0,1] \times [0, T)$. Now we use the equation for q and find that

$$-\int_0^1 \int_0^T q_t q \, dt dx = \int_0^1 \int_0^T (a q_x)_x q \, dx dt = 0$$

and thus $-\int_0^1 q^2(T,x)dx + \int_0^1 q^2(0,x)dx = -\int_0^1 q^2(T,x)dx = 0$. Therefore
 $u(x,T;\bar{a}) = z_2(x)$ a.e. on $(0,1)$. This contradicts (b) and thus $\lambda_1 > 0$.

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