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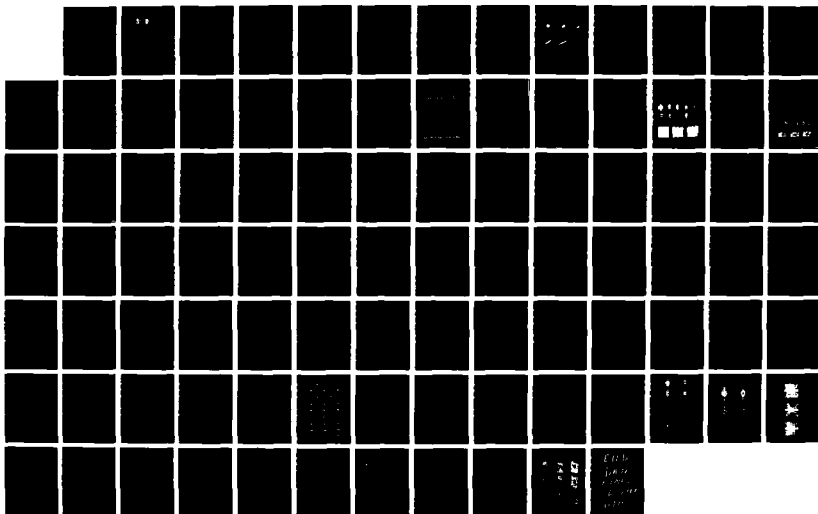
NUMERICAL SIMULATION OF FUEL DROPLET INTERACTIONS AND
BREAKUP(U) NAVAL RESEARCH LAB WASHINGTON DC J TISHKOFF
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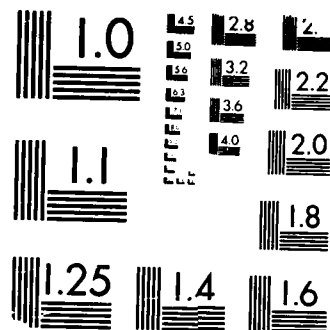
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19. ABSTRACT (Continue on reverse if necessary and identify by block number) The objective of the research in this program was to develop Lagrangian methods on triangular grids and apply these methods to the calculation of life-history and dynamics of fuel droplets. During the research period, major advances were made both in numerical technology and in the solution of problems so difficult that they have not been attempted before. With respect to numerical technology, the two-dimensional code SPLISH was converted to a VAX and then to a CRAY computer. New graphics systems were developed. Further testing of the basic SPLISH hydrodynamic algorithms as well as the surface tension algorithm were performed on internal gravity and capillary waves. A reorganization of the computer code, itself is nearly complete. This will make the code user-friendly and protable. Now it should be much easier to use, and therefore useable on many new kinds of problems. First previously calculated flows of the distortion and breakup of a droplet due to differences in flow velocities between the droplet and the external media were recomputed					
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NUMERICAL SIMULATION OF FUEL DROPLET
INTERACTIONS AND BREAKUP

Final Report to the
Air Force Office of Scientific Research
Dr. Julian Tishkoff

October 1, 1984 - September 30, 1987

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SUMMARY

The objective of the research in this program was to develop Lagrangian methods on triangular grids and apply these methods to the calculation of life-history and dynamics of fuel droplets. During the research period, major advances were made both in numerical technology and in the solution of problems so difficult that they have not been attempted before.

With respect to numerical technology, the two-dimensional code SPLISH was converted to a VAX and then to a CRAY computer. New graphics systems were developed. Further testing of the basic SPLISH hydrodynamic algorithms as well as the surface tension algorithm were performed on internal gravity and capillary waves. A reorganization of the computer code itself is nearly complete. This will make the code user-friendly and portable. Now it should be much easier to use, and therefore useable on many new kinds of problems.

First, previously calculated flows of the distortion and breakup of a droplet due to differences in flow velocities between the droplet and the external media were recomputed to verify the conversion. Then a number of calculations of droplet distortion and breakup due to shear flows were made. Qualitative comparisons to experimental results were made for the case when the droplet density and external fluid density were nearly equal. Our calculation and the experiments by Mason and coworkers showed small droplets torn off the large drop by the forces in the shear flow. A preliminary calculation of a droplet-droplet collision shows the distortion of droplets before they collide. Forced-flow and inflow-outflow boundary conditions, needed to do quantitative comparisons to experimental shear flows, were added to the model.

RESEARCH OBJECTIVES

Our major research objective was to develop models that could be used to give the basic information needed for constructing appropriate models for dense sprays. This is necessary because the limiting assumptions in current drop models imply that

the predictions given by spray models in dense or high-pressure spray regimes are not as accurate as they could be. Better drop models should be formulated using information about collision and breakup rates relating to droplet size and velocity. However such rate calculations are complicated by a rich variety of breakup, collision, and coalescence modes that can occur in different pressure and velocity regimes. For example, collisions can result in coalescence, elastic rebound, inelastic scattering with increased drop deformation and oscillation or shattering into two or more droplets. In addition, rapidly rotating or highly deformed drops may split into several drops, shattering may occur at critical pressures, and hydrodynamic breakup of drops may occur through several modes, including the bag breakup mode and droplet stripping due to velocity gradients in the exterior flow field. Drop transport is directly affected by these phenomena, and the presence of the drops affects the flow field in both the mean flow and turbulent fluctuations.

Our approach was to develop the numerical technology associated with Lagrangian methods on triangular grids to the point where it could be used to model the life-history and dynamics of fuel droplets. In particular these methods would be used to determine the conditions under which fuel droplets greatly distort and shatter, as would occur in shear flows and droplet collisions.

WORK PERFORMED

The basis of the numerical method is a Lagrangian convection algorithm which uses a triangular grid instead of a quadrilateral grid. In this method, the grid dynamically restructures itself according to preprogrammed criteria. In addition, new triangles are added or old triangles are deleted to change the resolution, also according to preprogrammed criteria. Triangle deletion allows the evolution of singly connected regions to multiconnected regions, such as would occur when droplets break up, or the evolution of multiconnected regions to singly connected regions, such as would occur during droplet collision. Since the algorithm is Lagrangian, interfaces can be tracked

explicitly. Surface tension forces and viscous forces are included in addition to the Lagrangian convection.

Conversion of the Triangular Lagrangian Code SPLISH to Other Computers

In order to begin the work on this project, we had to move the code SPLISH originally, written for the TI-ASC, first to a VAX and then to the CRAY computer. In addition to converting the code, we had to develop different graphics packages for the new computers. This was an extensive effort. Along the way, many of the original calculations done under NASA support were redone and checked. In particular, the droplet oscillation test calculations and one of the calculations of a flow past a droplet were performed. These calculations are reported in detail in Appendix B, which is a copy of the paper to be published in 1988 in the Journal of Computational Physics, and to a lesser extent in Appendix A, which is paper number AIAA-87-0539 from the AIAA Aerospace Sciences meeting of January, 1987.

Detailed Quantitative Tests of the Surface Tension Algorithm

Developing an accurate algorithm to model the effects of interfacial surface tension forces was much more difficult than we expected. We believe that this had never been satisfactorily addressed in the literature. We tested many different approaches, including the one we finally used, as documented in Appendix B. At first we used an $n = 2$ normal mode droplet oscillation as a test of the accuracy of the surface tension algorithm. The algorithm we finally used was accurate enough to allow us to compute a number of periods of oscillation of the droplet. However, the convergence properties of the algorithm were tested by comparing theoretical and computational calculations of the properties of internal capillary waves. The result was that the surface tension algorithm is first order in the mesh size, although the convection algorithm is second order.

There is one other result implied by our tests of the surface tension algorithm. The algorithm works well when the surfaces are not too highly distorted. For highly distorted surfaces the time and length scales associated with the surface tension phenomenon are much smaller than the convective time and length scales. For droplet breakup the physics of interest happens on the convective scales. For distorted surfaces many more points and time steps are needed to compute the motion due to the surface tension forces accurately, than are needed to compute the convective motion accurately.

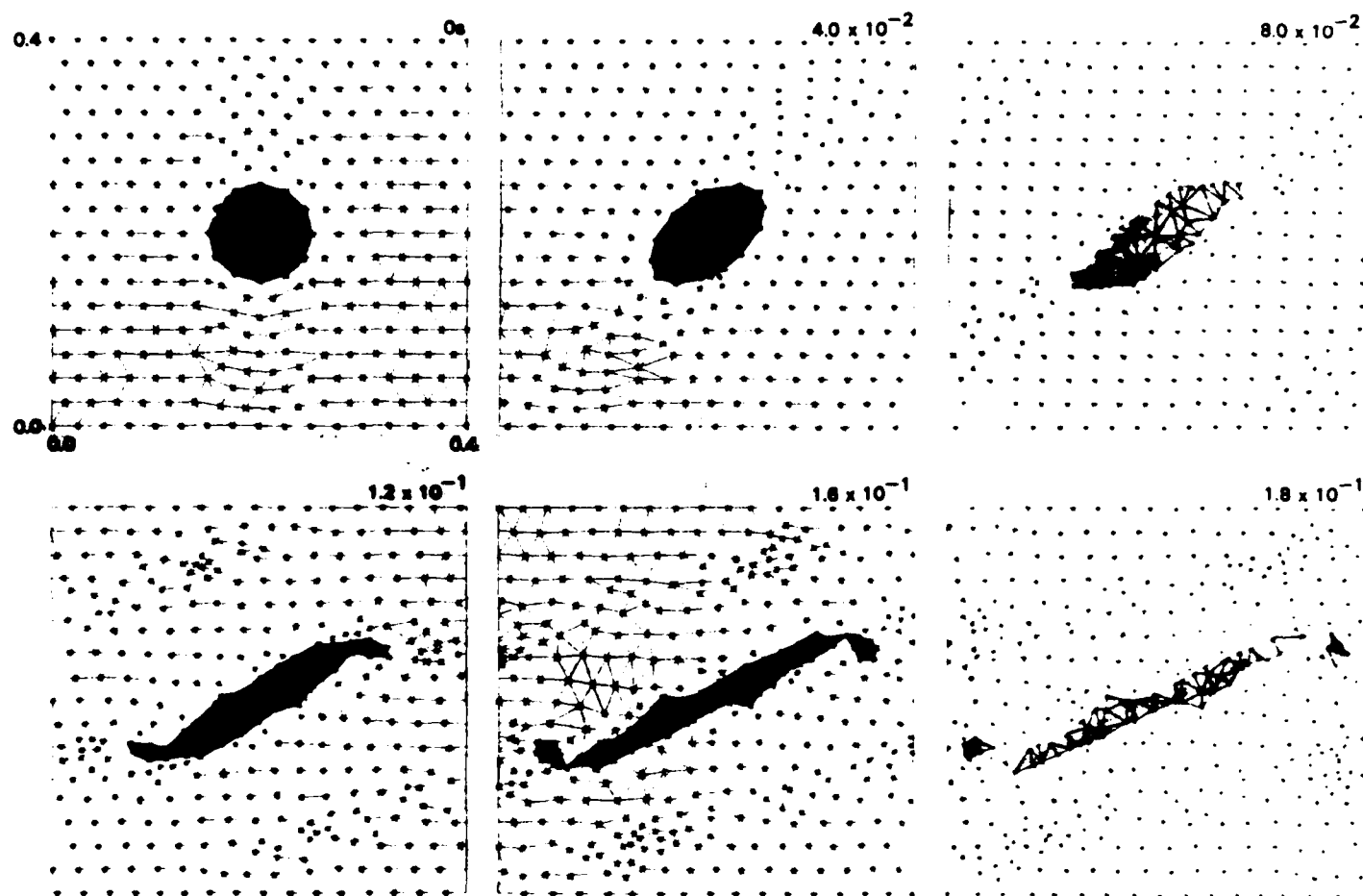
The details of this work can be found in Appendix A. These results have been presented at the American Physical Society Meeting of the Division of Fluid Dynamics, November, 1985, and at the International Symposium on Computational Fluid Dynamics, Tokyo, in August, 1985.

Calculations of Droplet Distortion and Breakup in Shear Flows

Calculations of silicon oil droplets were compared to the analytic and experimental work of Mason and co-workers. In these calculations, a drop of density 0.98 g/cc and diameter 1 mm was placed in an initial shear flow prescribed by $v_x = G(y - y_d)$, for points outside the droplet and $v_x = 0$ for points inside the droplet. The parameter y_d is the y -coordinate of the center of the droplet and G gives the magnitude of the shear. The y -coordinate of the velocity is initially zero everywhere.

Figure 1 shows the time evolution of one such calculation in which the viscosity and small surface tension coefficients were small. The experimental results for this case showed that the droplet stretches in the direction of the flow and small droplets break from the tips. This is similar to the calculated results shown in Figure 1. The small droplets shed in the calculation were larger than those shed in the experiments. The discrepancy is due to the resolution of the numerical calculation, in which we have only a few triangles available to represent the shed droplets. More resolved calculations, which we are currently doing, should provide a more solid basis for comparison with the experiments.

FIGURE 1



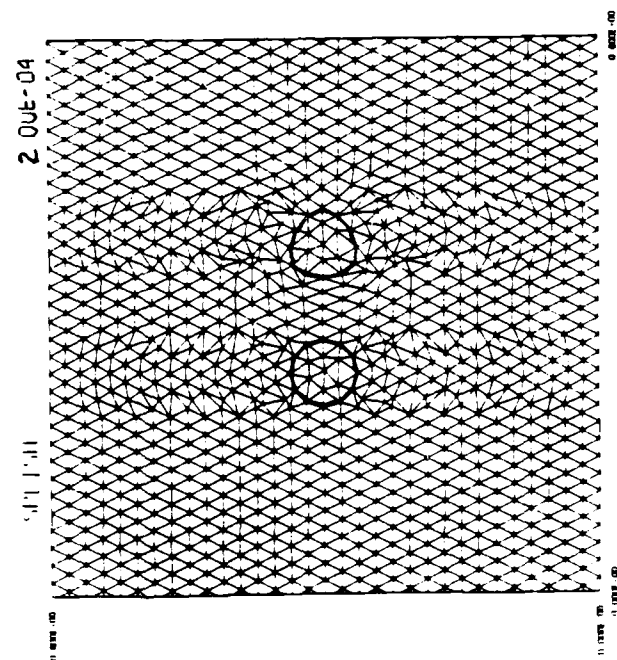
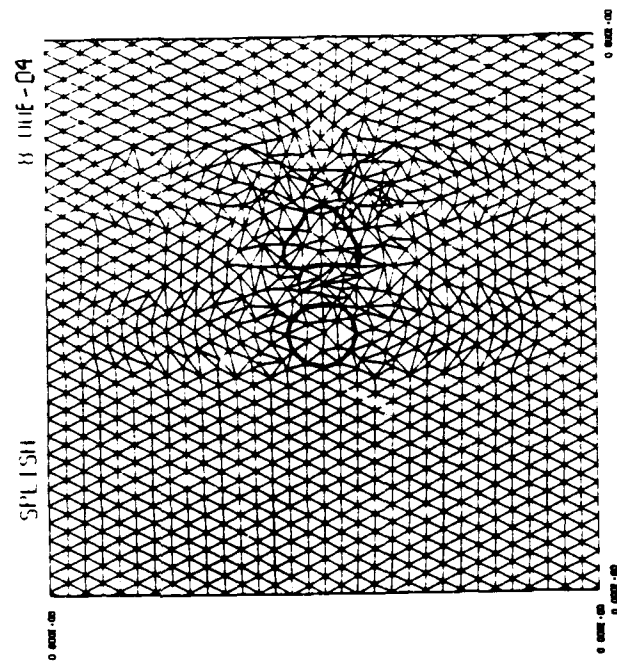
New and unique calculations of the breakup of a dense fuel droplet in a hot air shear flow were also performed. Here the stretching of the drop in the direction of the shear is much less pronounced due to the large density of the drop relative to the background air. Better resolution has allowed us to see smaller droplets being shed. Currently, very *resolved* calculations are being performed. The results of these calculations have been reported at the 1986 SIAM National Meeting, the American Physical Society Meeting of the Division of Fluid Dynamics, November, 1986, and the Aerospace Sciences meeting of the American Institute of Aeronautics and Astronautics, January, 1987. A detailed discussion of one of these calculations can be found in Appendix A.

Calculations of Droplet-Droplet Collisions

A preliminary calculation of a droplet-droplet collision was performed. This problem consists of a stationary target droplet and a moving projectile droplet. In the calculation, the two kerosene droplets were placed in a background fluid of hot air. The target droplet is initially at rest, while the background fluid and the projectile droplet on the left were initialized with a potential flow of 1 m/s about the target droplet. Figure 2 shows the initial phases of this calculation. As the projectile nears the target, both droplets deform and present nearly flat faces to each other. The lower parts of the droplets begin to merge first. This merger is accomplished numerically by the deletion of small triangles of background fluid between the two droplet interfaces. Since the new interface now has a drastic change in curvature, the surface tension forces try to expel the background fluid upwardly from between the two main sections of the kerosene mass. This process produced large numerically generated fluctuations in the pressures in that region.

Analyzing why this happened provides important information about problems in the basic algorithm and potential solutions of these problems. First, there are potential errors in the procedure used to delete triangles near interfaces. These would result in a phase lag in the pressures at newly created interfaces. A second possibility has its

FIGURE 2



origin in the values of the triangle velocities after vertex deletion. The triangle velocities determine the rate of expansion or contraction of the volumes about each vertex. The pressures react to prevent this change. When a triangle is deleted, the velocities of the neighboring triangles must be altered to maintain conservation of momentum and circulation. If these velocities are changed in such a way as to produce a huge volume expansion, we would obtain the symptoms we observe.

These possibilities are being evaluated, several proposed fixes are being tested, and the calculation will proceed once we have an answer to the problem.

Addition of Boundary Conditions

The experiments of Mason and coworkers of droplet shears were performed primarily with highly viscous silicon oils with the shear flow maintained by the motion of the containment device. Since the viscous terms are sinks for energy in the SPLISH algorithm, highly viscous flows damp very quickly to no flow at all. Therefore boundary conditions which simulate the motion of the experimental container have been added to the SPLISH algorithm in order to compare the code results to the experiments. To implement this boundary condition a layer of triangles is placed just outside the computational region on the moving boundary. These triangles have a density equal to that just inside the moving boundary and are given a prescribed velocity parallel to the boundary. Through the viscous terms in the momentum equation, this kinetic energy is then transferred to the background fluid. To model the experiments, we now set the initial flow to zero everywhere except on the exterior boundary triangles. The shear flow experiments used impulsive starts, which corresponds to giving the boundary layer of triangles a constant velocity initially, and ramp starts, which corresponds to giving the boundary layer of triangles a linear time dependent velocity up to the desired boundary velocity.

We are also implementing inflow-outflow boundary conditions. There are problems with doing this, some particular to our formulation, and others because it is simply a

very difficult problem (see, for example, Oran and Boris, 1987). One problem particular to our formulation is geometric and associated with the addition and deletion of mesh points at the boundaries. Another is associated with conservation of circulation and vertex cell area because of the incomplete vertex cells for those vertices on the boundaries. A generic problem is the physical boundary conditions themselves which determines how variables just inside the boundary, as well as the variables at newly created vertices, are time advanced. The problem is made more difficult by the irregular mesh, since there is essentially no literature on inflow-outflow for incompressible flows differenced over a general connectivity mesh. Research on boundary conditions in computational fluid dynamics is an important ongoing area of numerical research.

The geometrical aspects of inflow-outflow boundary conditions we are now using have been tested and work well. The algorithm deletes those triangles with no vertex within the computational region and adds triangles at the inflow boundary whenever any boundary vertex is completely within the computational region. The next step is both easier and more difficult: placing more physically correct values of the physical variables on the new triangles and vertices. At the inflow boundary, placing a prescribed value of velocity on a bordering triangle may not produce a vanishing divergence on a nearby interior vertex. This means that the inflow triangle velocities need to be rotated slightly to produce the appropriate incompressible flow. On the outflow boundary, the physical variables need to be updated in a manner that will not produce numerically reflected waves off that boundary. The form of the physical outflow boundary conditions that would be best to use in our case are the Orlansky outflow condition as formulated by Chan in his article *Finite Difference Simulation of the Planar Motion of a Ship*. The positive aspect of his approach is that is upwind so there are no reflected waves and hence we do not need to introduce artificial damping.

Extensions of the SPLISH Algorithm to Three Dimensions

Extensions of the Lagrangian algorithm on triangular grids to a Lagrangian algorithm on tetrahedral grids has always been a major goal of the work. However, as we progressed into the problem, we realized what a major effort in algorithm development is required. Our initial plans were that we would have additional funding from other sources that we use for algorithm development, which, and in the last year of this proposal, would allow us to calculate the properties of three-dimensional droplets instead of two-dimensional cylinders.

The major algorithmic problem with Lagrangian tetrahedra involve the reconnections. In two-dimensions, the test for reconnection of a side is merely the comparisons of two sets of opposing angles in the quadrilateral formed by the triangles on either side, and the reconnection is merely the swap of the diagonals of the quadrilateral. Reconnection in two dimensions is local to the side in question. In three dimensions, there is no equivalent diagonal swap. Instead, one must consider several tetrahedra and faces simultaneously, in order to get the optimum grid connectivity. Currently, Dr. Martin Fritts of SAIC has been working on this problem in collaboration with and with funding from Lawrence Livermore National Laboratories and Los Alamos National Laboratories. However, we at NRL are still somewhat skeptical about being able to apply the three-dimensional SPLISH analog in an efficient way in the near future.

Therefore we have taken two other approaches. The first still involves triangles and tetrahedra: we have successfully developed a finite-element version of the high-order monotone scheme Flux-Corrected Transport, which uses triangles instead of quadrilaterals as the basic elements. This is basically an Eulerian method, although a new arbitrary Lagrangian-Eulerian (ALE) now exists. This work is being carried out at NRL by Dr. Rainald Löhner under the sponsorship of both DARPA and NASA. These methods now work well and are incorporated in production codes. Combined with a front-tracking algorithm, this approach could now be used to calculate droplet dynamics in three dimensions.

The second numerical approach is still in the conceptual stage. This involves using the Monotonic Lagrangian Grid representation, developed by Jay Boris at NRL, as a basis for Lagrangian fluid dynamics. This is an approach we hope to evaluate in the next couple of years.

Continued Efforts to Make the Code More User-Friendly

Our objective here is to produce a computer code which is more flexible in the problems it can handle as well as more portable. The program SPLISH has the potential to solve a large class of problems. However, because of the developmental nature of the work, the code became a patchwork of additions and fixes. In order to apply the algorithm with some facility to a variety of problems, some efforts in structured programming was required. The code itself must be suitably structured so that the modifications can be made in a straightforward manner. Approximately 80% of this task is complete. The only major items remaining to be upgraded are the vertex deletion algorithms for small triangles. This will be complete in FY'88.

Discussion

During the contract period, major advances have been made in this program both in numerical technology and in the numerical solution of problems so difficult that they have not been attempted before. The physical problems are the droplet breakup in shear flows and droplet-droplet collisions. The numerical problems involve accurate formulations of surface tension, boundary conditions, and grid restructuring algorithms on the Lagrangian triangular grid. This coming year, in FY88, the results of the droplets in shear flows will be written up and submitted to the Journal of Fluid Mechanics. The calculations of droplet-droplet collisions will be continued until the numerical questions are answered and a case is run.

While the calculations of droplet-droplet collisions has only just begun, the prospects for significant research results is great. The recent experimental results C.K.

Law from the University of California, Davis and from Nassar Ashgriz from the State University of New York at Buffalo will allow us some benchmark examples for further code verification before proceeding to cases for which there are no experimental results. We have already started talking with Professor Law and we are planning a case to simulate.

The code reorganization started under the sponsorship of AFOSR will have the most lasting benefit. With the capability of easy modification of a computationally complex algorithm, the number of problems attacked by this approach could multiply. One such problem is the behavior of a reacting liquid-gas interface. The computational aspects of this problem is suited to the SPLISH algorithm. Algorithms for the condensation of the gas (or equivalently the evaporation of the liquid) would have to be added. This is a problem of current interest to the Navy in connection with liquid-metal combustion.

PERSONNEL

The funding for this contract has supported the research of Drs. David Fyfe and Elaine Oran. Dr. Martin Fritts, who currently works for Science Applications, participated in the program as a consultant during fiscal year 1985. In addition, Dr. J.P. Boris and Dr. Rainald Löhner regularly consulted on this work.

PRESENTATIONS AND PUBLICATIONS

Simulations of Two-Dimensional Fuel Droplet Flows, M.J. Fritts, D.E. Fyfe, and E.S. Oran, *Prog. Astro. Aero.* 95, 540-553, *Dynamics of Flames and Reactive Systems*, edited by J.R. Bowen, N. Manson, A.K. Oppenheim, and R.I. Soloukhin, AIAA, 1984.

Application of Two-Dimensional Techniques to Combustion, invited presentation at Heidelberg University, Heidelberg, West Germany, 1984.

- Numerical Simulations of Fuel Droplet Flows using a Lagrangian Triangular Mesh, M.J. Fritts, D.E. Oran, and E.S. Oran, *Ninth International Conference on Numerical Methods in Fluid Dynamics* pp. 219-224, Springer-Verlag, New York, 1985.
- Multidimensional Simulation of Flames and Detonations, E.S. Oran, invited presentation at the 10th International Colloquium on Dynamics of Explosions and Reactive Systems, Berkeley, CA, August, 1985.
- Droplet Flows with Surface Tension, D. Fyfe, E. Oran and M. Fritts, the American Physical Society, Meeting of the Division of Fluid Dynamics, November, 1985.
- Droplet Deformation and Breakup in Shear Flows, D.E. Fyfe, E.S. Oran and M.J. Fritts, SIAM 1986 National Meeting, July, 1986.
- Numerical Simulation of Reactive Flows, invited presentation at NASA-Lewis Research Laboratory, Cleveland, OH, 1986.
- Combustion Computation, Wright Aeronautical Laboratories, invited presentation at Wright-Patterson AFB, OH, 1986.
- Numerical Simulation of Droplets in Shear Flows, D.E. Fyfe, and E.S. Oran, American Physical Society Meeting of the Division of Fluid Dynamics, November, 1986.
- Numerical Simulation of Droplet Oscillations, Breakup, and Distortion*, D. E. Fyfe, E. S. Oran and M. J. Fritts, AIAA 25th Aerospace Meeting, January, 1987. AIAA Paper No. 87-0539.
- Droplet Deformation and Breakup in Shear Flows, D.E. Fyfe, E.S. Oran and M.J. Fritts, SIAM Meeting on Numerical Combustion, March, 1987.
- Numerical Simulation of Reactive Flows, invited presentation the 25th AIAA Aerospace Sciences Meeting, Reno, 1987.
- Some Challenges in the Numerical Simulation of Laminar and Turbulent Flames, invited presentation at Princeton University, 1987.

The book *Numerical Simulation of Reactive Flow* by E.S. Oran and J.P. Boris, Elsevier, 1987. Large portions of this were based on work done in and material developed during the course of this project.

Surface Tension and Viscosity with Lagrangian Hydrodynamics on a Triangular Mesh, D.E. Fyfe, M.J. Fritts and E.S. Oran, to appear in J. Comp. Phys., 1988.

SCIENTIFIC INTERACTIONS

During the research period, we worked with Dr. Ann Karagozian from University of California, Los Angeles, to modify her theoretical methods to describe the break up of droplets due to shear flows around the droplet. The theoretical model includes surface tension and viscosity and comes up with a droplet break up criterion.

We have been talking to Dr. Jack Hansman from MIT, who has visualizations of the breakup of water droplets in a shear flow. These show that the droplet clearly loses its spherical shape before it goes unstable and breaks up. His results are excellent for comparing with our calculations.

We have begun to interact with Dr. Nassar Ashgriz from the State University of New York at Buffalo. He has excellent experiments which we can model.

We have been talking extensively to Professor C.K. Law, from the University of California, Davis, about simulating some of the conditions of his droplet-droplet collision experiments.

We have been consulting with Dr. Josette Belan from the Jet Propulsion Laboratory about a range of parameters over which it would be of use to do a parametric study of droplet breakup.

During fiscal year 1985, we started a dialogue with Dr. Murial Ishakawa, who is a fluid theorist for one of the NASA shuttle flights. The experiments carried out were by Dr. Taylor Wang, of JPL, who was interested in droplet oscillations and oscillations of droplets within droplets. Our model was ideal for doing calculations of his experiments, but due to lack of personnel and funding, no calculations of

their experiments were ever completed. The net results was that we now have some algorithms for droplet rotation and droplets within droplets, but they have not been applied to any problem.

During the research effort, Gopal Patnaik started working with our group at NRL. He did his dissertation with Dr. William Sirignano, now at University of California, Irvine. Dr. Patnaik developed and did the calculations for the droplet model and has consulted on our droplet program.

APPENDIX A: AIAA paper number 87-0539

AIAA'87

AIAA-87-0539

**Numerical Simulation of Droplet
Oscillations, Breakup, and Distortion**

D. E. Fyfe, E. S. Oran and M. J. Fritts,
Naval Research Lab., Washington, DC

AIAA 25th Aerospace Sciences Meeting

January 12-15, 1987/Reno, Nevada

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NUMERICAL SIMULATION OF DROPLET OSCILLATIONS, BREAKUP, AND DISTORTION

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Abstract

Numerical algorithms for surface tension and viscosity are presented in the context of a Lagrangian treatment of incompressible hydrodynamics with a dynamically restructuring grid. A test problem consisting of an oscillating droplet is described. Calculations of droplet distortion and breakup in an initially constant flow and in an initial external shear flow are presented.

Introduction

In principle, a Lagrangian formulation of the hydrodynamics equations is particularly attractive for numerical calculations. Each discretized fluid element is tracked as it evolves through the interaction with its changing environment and with external forces. The local interactions can be represented without nonphysical numerical diffusion. Conservation laws are simple to express since there are no fluxes out of the fluid element boundaries. The paths of the fluid elements are themselves a flow visualization. It thus appears to be the natural approach to transient hydrodynamics with free surfaces, interfaces, or sharp boundaries.

In practice, the use of Lagrangian methods in numerical simulations has generally been restricted to "well-behaved" flows. Shear, fluid separation, or even large amplitude motion produce severe grid distortion. These distortions arise because grid points can move far enough that their near-neighbors change in the course of a calculation. When differential operators are approximated over a mesh which is distorting, the approximations may become inaccurate. Attempting to regain accuracy through regridding and interpolating physical quantities onto the new grid introduces numerical diffusion into the calculation.

We first describe the numerical technique for Lagrangian calculations using a restructuring triangular mesh [1] for incompressible, two-dimensional Cartesian flows. The major advance of this approach is that the grid automatically adapts and refines itself to maintain accuracy for discretized operators in a manner that is nondiffusive. The algorithms have been implemented in the code SPLISH, which has been applied to calculations of nonlinear waves [2, 3], flows over obstacles [4], Kelvin-Helmholtz instabilities [5], Rayleigh-Taylor instabilities [6], Couette flows and Taylor vortex flows [7]. We then describe the new surface tension algorithms, and describe applications of the code to calculations of droplet oscillations and distortions due to the presence of background flows.

Basic Elements of Lagrangian Triangular Grids

Consider a two-dimensional space which is divided into triangular cells. A section of this mesh shown in Figure 1, which shows an interface between fluid type I and fluid type II. In Figure 1a, a particular triangle j is highlighted by heavy lines and the various components of the triangle are labeled. Three vertices, V_1 , V_2 , and V_3 , are connected consecutively by sides S_1 , S_2 , and S_3 . The direction of labeling around each triangle is counterclockwise and the z axis is directed out of the page. Since the mesh can be irregularly connected, an arbitrary number of triangles can meet at each vertex. We can define a cell surrounding a vertex, as shown in Figure 1b, by the shaded region surrounding V_3 . The borders of such vertex-centered cells are determined by constructing line segments joining the centroid of each triangle with the midpoints of the two triangle sides connected to the vertex, for all triangles surrounding that vertex. This definition of a vertex cell equally apportions the area of a triangle to each of its three vertices and provides a simple, efficient way to evaluate the finite difference operators.

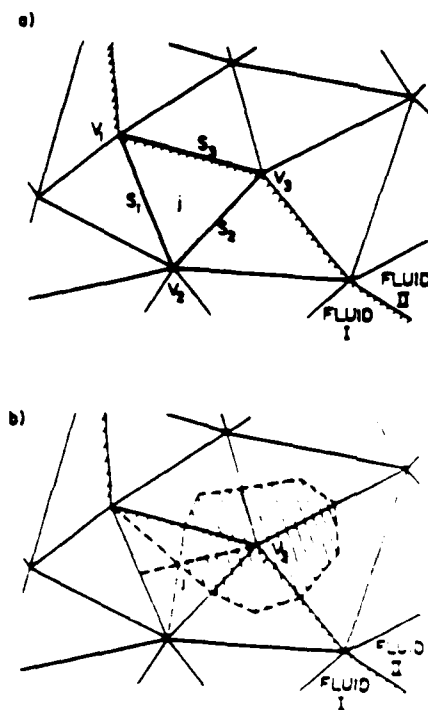


Figure 1. A section of a triangular grid showing a) a material interface, b) a vertex-cell.

The approach we describe is a conservative integral, control volume approach on a triangular grid that uses an integral formulation to derive the difference algorithms. We use the expressions for the integral of the gradient of a scalar function, f , and the divergence and curl of a vector field, $\mathbf{v}$, in two Cartesian dimensions,

$$\int_A \nabla f \, dA = \oint_C f \, d\mathbf{l} \times \hat{\mathbf{z}} \quad (1)$$

$$\int_A \nabla \cdot \mathbf{v} \, dA = \oint_C \mathbf{v} \cdot (d\mathbf{l} \times \hat{\mathbf{z}}) \quad (2)$$

$$\int_A \nabla \times \mathbf{v} \, dA = \oint_C \mathbf{v} \cdot d\mathbf{l} \, \hat{\mathbf{z}}, \quad (3)$$

where A is the region enclosed by the curve C and $d\mathbf{l}$ is the vector arc length around C in the counterclockwise direction.

A triangle-centered quantity is assumed to be piecewise constant over the triangles with discontinuities occurring at the triangle sides, and a vertex-centered quantity is assumed to be piecewise linear over the triangles. If we want to form a triangle-centered derivative, we use the triangle as the area A and the sides of the triangle for the curve C in Eqs. (1) - (3). If we want to form a vertex-centered derivative, we use the vertex-centered cell as the area A . We approximate the area integral on the left side of Eq. (1) - (3) by the area of the vertex-centered cell times the value of the derivative at the vertex. We approximate the line integral using the value on each triangle and the appropriate vector length through the triangle. This approach is described in more detail in Fyfe et al. [8].

The basic equations for inviscid incompressible hydrodynamics are

$$\frac{d\rho}{dt} = 0, \quad (7)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (8)$$

$$\rho \frac{d\mathbf{v}}{dt} + \nabla p = \mathbf{f}_e. \quad (9)$$

In two dimensions the fluid density ρ , pressure p , and velocity $\mathbf{v}$ are assumed to vary with x , y , and t . The term $\mathbf{f}_e$ represents external forces applied to the fluid, for example, forces due to gravity.

In this formulation, it is important to consider which of the physical variables, ρ , $\mathbf{v}$, and p , should be defined as vertex-centered quantities and which should be defined as triangle-centered quantities. Choosing these correctly ensure the correct conservation properties. We have found that prescribing velocities as triangle-centered quantities makes the formulation of conservation of circulation straightforward. Prescribing the densities on triangles and pressures at vertices allows conservation of vertex cell areas.

The numerical integration procedure for velocities uses a split-step algorithm. The velocities are advanced a half timestep, the grid is advanced a full timestep, and then the velocities are advanced forward the other half timestep:

$$\mathbf{v}_j^{1/2} = \mathbf{v}_j^o - \frac{\delta t}{2\rho_j} (\nabla p)_j^o + \frac{\delta t}{2\rho_j} \mathbf{f}_e, \quad (10)$$

$$\mathbf{v}_i^{1/2} = \frac{1}{2}(\mathbf{v}_i^o + \mathbf{v}_i^n), \quad (11)$$

$$\mathbf{x}_i^n = \mathbf{x}_i^o + \delta t \mathbf{v}_i^{1/2}, \quad (12)$$

$$\tilde{\mathbf{v}}_j^{1/2} = \mathbf{R}(\{\mathbf{x}_j^o\}, \{\mathbf{x}_j^n\}) \cdot \mathbf{v}_j^{1/2}, \quad (13)$$

$$\mathbf{v}_j^n = \tilde{\mathbf{v}}_j^{1/2} - \frac{\delta t}{2\rho_j} (\nabla p)_j^n + \frac{\delta t}{2\rho_j} \mathbf{f}_e. \quad (14)$$

The vertex velocity $\mathbf{v}_i$ in Eq. (11) is obtained from the area-weighted $\mathbf{v}_j$ determined in the previous iteration.

The transformation $\mathbf{R}$ results from the requirement of conservation of circulation. Equation (13), which produces conservation of circulation over vertex cell volumes, is a consequence of this approach. It reflects numerically the fact that the triangle velocities must be altered as the grid rotates and stretches. The transformation $\mathbf{R}$ is derived by considering the circulation about each vertex. Conservation of vorticity then takes the form of the operator $\mathbf{R}$ which preserves the value of the circulation about each vertex as the grid changes.

The pressures $\{p_i^n\}$ in Eq. (14) are derived from the condition that the new velocities $\{\mathbf{v}_j^n\}$ should be divergence-free at the new timestep, satisfying Eq. (5). The pressure Poisson equation is derived from Eq. (14) by setting $(\nabla \cdot \mathbf{v}_j^n) = 0$ to obtain a pressure p_i^n , such that

$$(\nabla \cdot \frac{\delta t}{2\rho_j} (\nabla p)_j^n)_i = (\nabla \cdot \mathbf{v}_j^{1/2})_i + (\nabla \cdot \frac{\delta t}{2\rho_j} \mathbf{f}_e)_i, \quad (15)$$

Both terms in Eq. (15) are straightforward to evaluate, since the divergence is taken over triangle-centered quantities. Two features of the Poisson equation, Eq. (15), are noteworthy. First, it is derived from $\nabla^2 \phi = \nabla \cdot \nabla \phi$, as in the continuum case. Second, the left-hand-side results in the more familiar second-order accurate templates for the Laplacians (such as the five-point formula) derived for homogeneous fluids and regular mesh geometries.

Viscosity modifies Eq. (6), so that now

$$\rho \frac{d\mathbf{v}}{dt} + \nabla p = \mathbf{f}_e + \mu \nabla^2 \mathbf{v}. \quad (16)$$

Discretization of the additional term in the momentum equation follows the same approach as the discretization of the other terms. Since the velocity is a triangle-centered quantity, we need a discrete vertex-centered gradient operator and a discrete triangle-centered divergence operator. The finite difference equations, Eqs. (10) and (14), can be modified to account for the additional term in the momentum equation by

$$\mathbf{v}_j^{1/2} = \mathbf{v}_j^o - \frac{\delta t}{2\rho_j} (\nabla p)_j^o + \frac{\delta t}{2\rho_j} \mathbf{f}_e + \frac{\mu_j \delta t}{2\rho_j} (\nabla^2 \mathbf{v}_j^o), \quad (17)$$

$$\mathbf{v}_j^n = \tilde{\mathbf{v}}_j^{1/2} - \frac{\delta t}{2\rho_j} (\nabla p)_j^n + \frac{\delta t}{2\rho_j} \mathbf{f}_e + \frac{\mu_j \delta t}{2\rho_j} (\nabla^2 \mathbf{v}_j^n). \quad (18)$$

These equations are implicit in the velocities, just as the original Eqs. (10) - (14) are. As in the inviscid case, we solve by iteration.

In Lagrangian calculations, the grid may distort to the point where grid restructuring is necessary. The derivations of the reconnection and vertex addition and deletion algorithms are done through the control volume approach and the use of triangle velocities. For all the algorithms used, the divergence and curl taken about each vertex are both identically conserved for grid reconnections and vertex addition.

The accuracy of a general triangular mesh is diminished by large obtuse angles within triangles. With reconnections, accuracy can be recovered by ensuring that large obtuse angles are preferentially eliminated. Of the many ways of formulating a reconnection algorithm, we have chosen one based on requirements for solving the pressure Poisson equation. Since the equation is solved by iteration, we want the iteration to converge as rapidly as possible. Mathematically, convergence is assured if the finite difference equation is diagonally dominant. This requirement translates to a relation between two of the angles of each triangle. The reconnection algorithm preferentially eliminates large angles in triangles, since the diagonal is chosen which divides the largest opposing angles. Interface sides are never allowed to reconnect. In such cases vertex addition algorithms are needed.

Vertex addition algorithms are also needed where the flow naturally depletes vertices. For vertex addition, satisfaction of conservation integrals is particularly simple. The vertex added at the centroid of a triangle subdivides that triangle into three smaller triangles. A vertex added to the midpoint of a side subdivides the two adjacent triangles into four smaller triangles. If the new triangle velocities are all the same as the velocity of the subdivided triangles, all conservation laws are satisfied. Since the reconnection algorithm is also conservative, subsequent reconnections to other vertices ensure that the only effect of the addition is an increase in resolution.

The case is not as obvious for vertex deletion. Reconnections can be used to surround any interior vertex within a triangle. The vertex is then removed and the new larger triangle given a velocity which is the area-weighted sum of the old velocities, redistributes circulation in accordance with area coordinates.

Surface Tension

The surface tension at an interface between two materials depends on the curvature of the interface. In the conventional numerical representation of surface tension, it is cast into a finite-difference form by fitting vertices on the material interface to some parametric function. This function is then used to find an estimate of local curvature. Once the curvature is known, a surface tension force is evaluated and used to accelerate interface vertices.

This scheme fails in SPLISH for two reasons. First, the interface vertices are accelerated directly by surface tension forces evaluated on the vertices. Since velocities are centered on triangles in SPLISH, the velocity field sees the effect of the acceleration a half-timestep later, unless a secondary calculation is made. As a result, the pressure calculated within the droplet is inconsistent with that

found from the surface tension formula. Second, since the pressure gradient forces and surface tension forces are not calculated in the same manner, numerical errors result which grow with each timestep.

Both of these problems are eliminated by a different formulation of surface tension, in which a surface tension potential is used to generate the forces. The surface tension force is formulated as a gradient of a potential present only at the surfaces. With this method, the pressure gradient forces are calculated in the same manner and on the same grid as the forces derived from the surface tension potential. Therefore both the surface tension potential and the pressure are dynamically similar, and the physical pressure drop across the interface must exactly cancel the surface tension forces. Preliminary aspects of this work were described by Fritts et al. [8,9].

The finite-difference algorithms for surface tension are straightforward. The surface tension forces are included through Laplace's formula for the pressure jump across an interface [11],

$$p_i - p_o = \sigma/R, \quad (16)$$

where p_i is the pressure just inside the droplet at the interface, p_o is the pressure just outside the droplet at the interface, σ is the surface tension coefficient associated with the two media which define the interface, and R is the radius of curvature in the two-dimensional plane. The radius of curvature is positive at points on the interface where the droplet surface is convex (a circle is convex everywhere) and negative when the surface is concave. These pressure jumps are included in the Poisson equation for the pressure. The average pressure, $(p_i + p_o)/2$, is computed at the interface vertices. From the average pressure and the pressure jump, we can compute a pressure gradient centered on triangles, both inside and outside the surface. This pressure gradient is used in the momentum equation.

The radius of curvature is computed from a parametric cubic spline interpolant to the interface vertices. Past calculations of droplets oscillating due to surface tension forces [12,13] also use cubic spline interpolation. However, they divided the surface into at least four segments (the top, bottom, right and left sides of the droplet) to produce an interpolant on each segment. Each interpolant was matched at the joints to produce an overall curve. The parametric interpolant used here does not require this special matching. We generate the twice differentiable periodic spline interpolants, $\mathbf{r}(s) = (x(s), y(s))$ as prescribed by deBoor [14].

The spline fit is also used for regridding. When the regridding algorithm calls for the bisection of a triangle side which borders the two media, a new vertex is added on the spline interpolant between the vertices. This is done rather than bisecting the straight-line segment, since a straight-line bisection introduces spurious interface oscillations. Bisecting the spline maintains a better overall shape for the interface.

Oscillating Droplets

In order to test the algorithm for surface tension in SPLISH, we calculated the oscillation of a droplet due to surface tension. We have extended Rayleigh's linear theory [15] of small amplitude droplet oscillations to include the presence of an external fluid,

$$\omega^2 = (n^3 - n) \frac{\sigma}{(\rho_d + \rho_e)a^3}, \quad (17)$$

where ω is the frequency, ρ_d is the droplet density, ρ_e is the density of the external fluid, and the surface of the droplet is given in polar coordinates by

$$r = a + \epsilon \cos(n\theta), \quad (18)$$

where a is the unperturbed radius of the jet, and n prescribes the mode of oscillation in the plane with amplitude ϵ .

Figure 2 is a composite of frames from a calculation in which $\epsilon = 0.2a = 0.0025$. In this calculation there are 17 vertices in each direction along the exterior boundaries, 12 vertices on the droplet interface and a total of 313 vertices initially in the calculation. The computational domain is 0.1 cm on a side. The left and right boundaries are periodic while the top and bottom boundaries are solid walls. The timestep is $\delta t = 2.5 \times 10^{-3}$ s. The figures show four and a half oscillations of the droplet. We can see that as the calculation proceeds, no new vertices have been added, but in fact some have been subtracted. This was the case because the initial gridding was adequate to represent the droplet shape. From these calculations, the period of oscillation is $\tau_{12} = 1.35 \times 10^{-3}$ s, compared to the theoretical value of 1.13×10^{-3} s. Figures 3 shows the initial oscillation for a more resolved case in which there are 28 vertices surrounding the droplet.

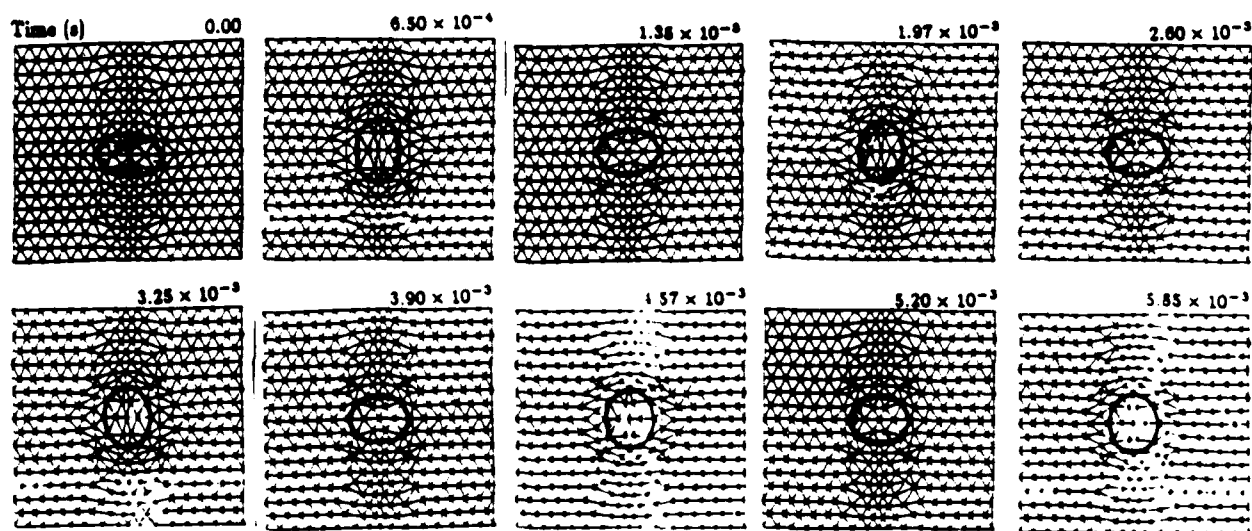


Figure 2. A composite of frames from a calculation of an $n = 2$ normal mode droplet oscillation with 12 vertices around the droplet: $\rho_e = 1$ g/cc, $\rho_d = 2$ g/cc, $\sigma = 30$ dynes/cm, $a = 0.0125$ cm. Each frame is 0.1×0.1 cm².

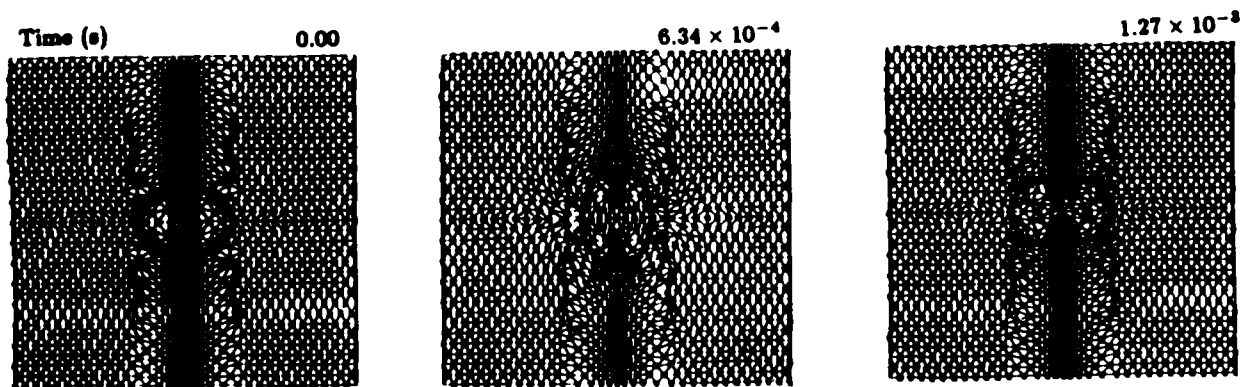


Figure 3. A composite of frames from a calculation of an $n = 2$ normal mode droplet oscillation with 28 vertices around the droplet.

Incompressible Flow about a Droplet

In this section we present some calculations of forced, asymmetric drop oscillations induced by flow around a droplet. These calculations include both the effects of viscosity and surface tension. The capability of studying such flows for viscous droplets in shear flows is the motivation for developing the viscosity and surface tension algorithms.

The initial conditions we used specified an initially steady-state potential flow about a periodic series of cylinders. Again, the boundary conditions on the left and right sides are periodic, and the upper and lower boundary conditions are reflecting walls. Initially, a perfectly circular droplet is at rest in a background flow. A physical situation modelled by such an initialization might occur if the flow velocity were ramped up to its final value before any significant structure could develop in the flow, and before the droplet could pick up any substantial velocity. Basically, it is a smooth start for the calculation. Previously we had performed calculations which began with an impulsive start, but found that as a result there was a large amount of momentum transferred across the droplet interface early in the calculation.

Table 1. Conditions for Flow around Droplet Calculation

density of kerosene	0.82 g/cc
density of air	0.0013 g/cc
surface tension (STP)	30 dynes/cm
viscosity of kerosene	1.8 centipoise
viscosity of air	0.018 centipoise
air velocity	120 m/s
initial droplet velocity	0.0 m/s
droplet radius	125 microns

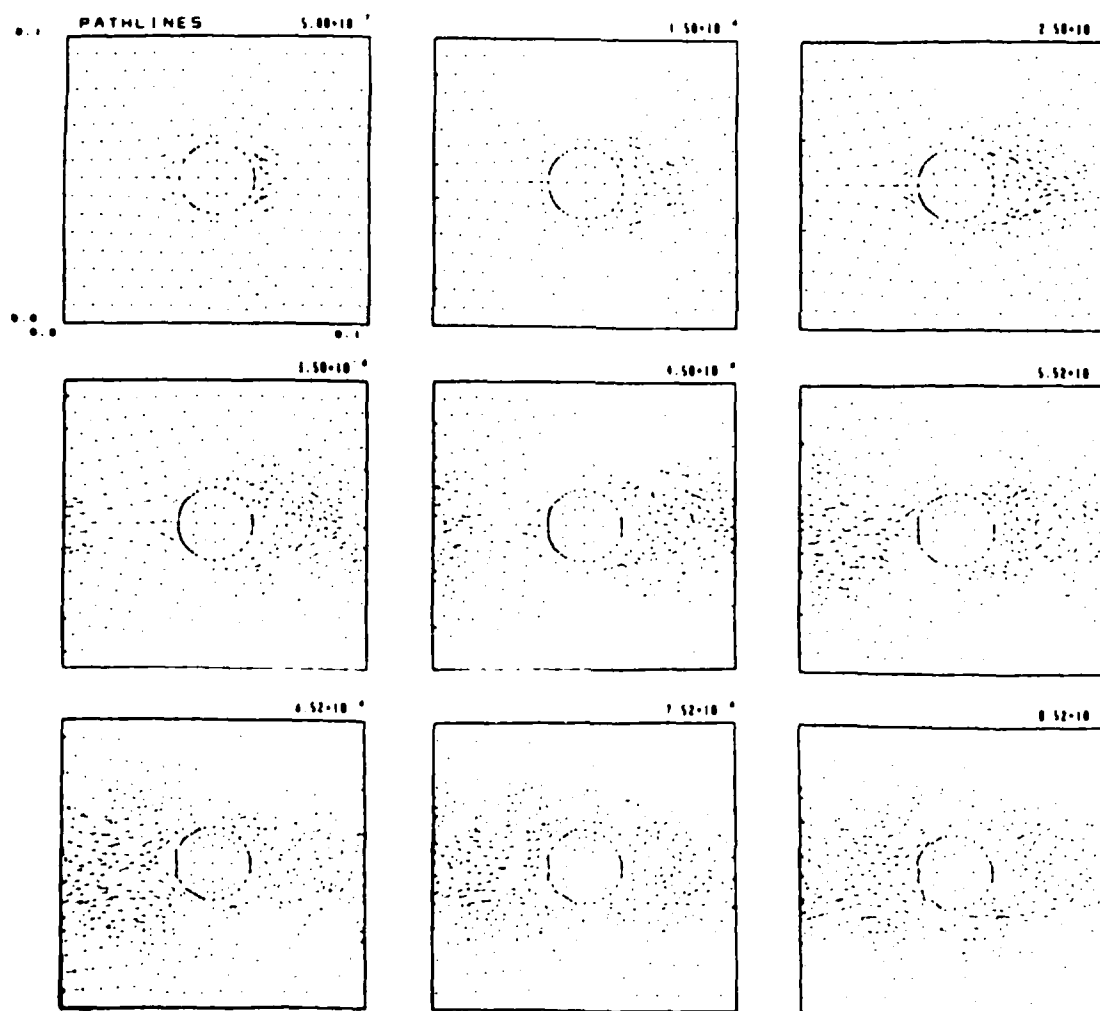


Figure 4. Pathlines from a calculation of a flow around a kerosene droplet at flow velocity of 120 m/s and $Re \approx 2000$.

The calculations presented here model the forced fluid flow due to a fast air stream about an initially stationary kerosene droplet. The physical parameters, given in Table 1, are appropriate for a combustor environment. A total of 309 vertices were used to initialize the problem, with 12 vertices at the droplet interface. Figure 4 follows the evolution of pathlines in the internal and external flow fields through a series of timesteps. For an air velocity of 120 m/s and a droplet radius of 125 microns, the corresponding Reynolds number is roughly 2000. The pathlines are defined by the paths of vertices over five timesteps. By the last frame of Figure 4, the fluid originally to the left of the droplet has progressed through the mesh and interacted with the face of the (next) droplet.

The first clear indication of the development of the recirculation region is seen in the fourth frame of Figure 4, which shows a pair of counter-rotating vortices. The recirculation zone continues to develop throughout the calculation, although at times the vortex pair is not as evident due to the deletion and addition of vertices, which interrupts the continuity of the pathlines. By the last frame, another pair of vortices is forming near the droplet, and the original pair has been shed.

Distortions in the face of the droplet are evident in at least the seventh frame. These distortions occur because the curvature has increased and the streamlines in the external flow are condensed by the approaching wake. The internal velocities are small compared to the external flow rates and therefore cannot be distinguished as pathlines. However, indication of the (small) internal recirculation may be obtained by comparing internal vertex positions at various timesteps.

Figure 5 shows the grid at times in the calculation corresponding to those in Figure 4. During the course of the calculation, a great deal of vertex addition and deletion has occurred. Vertex addition, however, is most noticeable in the wake of the droplet and around the droplet interface. Whereas there were 300 vertices at the beginning of the calculation, there are 450 at the end.

As seen in Figure 5, the computational grid needs further refinement at this time because the perturbations cannot be resolved by the limits set on minimum triangle size originally chosen for the calculation. A sign that the calculation is under-resolved is that one of the crests of the

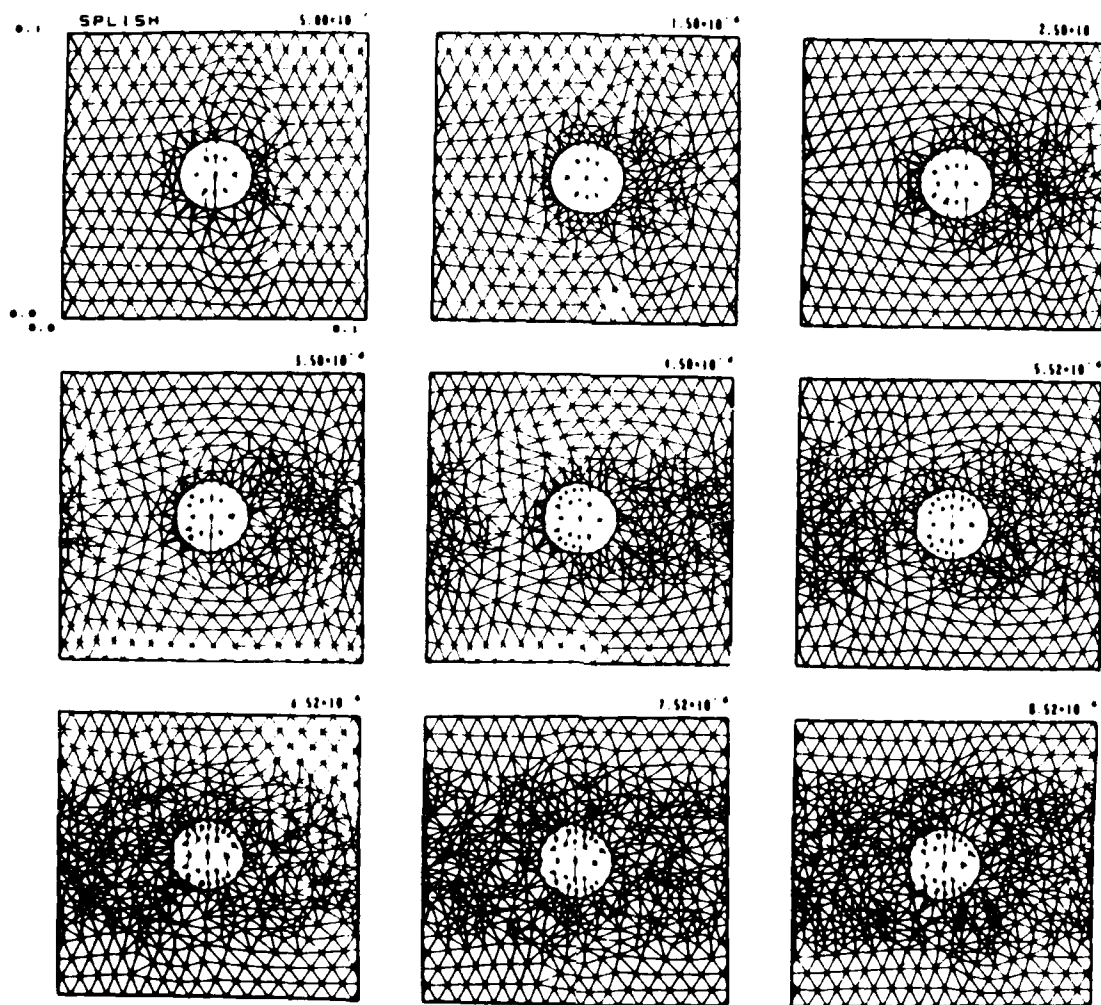


Figure 5. Frames showing the triangular grid at the same times as shown for the pathlines in Figure 4.

surface wave is spanned by a single triangle, a situation which allows no communication of that surface fluid with the interior of the droplet. In order to continue the simulation, better resolution must be obtained about the droplet surface. Another algorithm is currently being included to allow higher resolution near points of large curvature at material interfaces.

A Droplet in a Shear Flow

An important problem in atomization is how a droplet breaks up due to shear forces. To investigate this computationally, we have simulated droplets in a shear flow centered around the droplet. The initial flow is prescribed by $v_x = G(y - y_d)$ for points outside the droplet and $v_x = 0$ for points inside the droplet. The parameter y_d is the y -coordinate of the center of the drop and G gives the magnitude of the shear. The y -component of the velocity is initially zero everywhere.

Results of a such a shear on a kerosene droplet in hot air are shown in the three panels in Figures 6, for a case in which $G = 5 \times 10^3/\text{sec}$. We used 0.013g/cc as the density for hot air. The remaining physical parameters are the same as those in Table 1. Initially the droplet was round, but in Figure 6a it has already become elongated in the directions of the shear. At a later time, in Figure 6b, it has become even more elongated. Times intermediate to these two figures show that some very small droplets have already been pulled off of the large drop, but their size was so small that they were deleted from the calculations. Figure 6c shows a still later time, when small droplets have been moving off of both sides. The small droplets sometimes seem to move counter to the flow of the main shear layer. This is because a recirculation zone forms on the upper left and the lower right of the large droplet.

Summary

This paper presented the current algorithms included in the code SPLISH, a two-dimensional Cartesian Lagrangian treatment of incompressible flows with a dynamically restructuring grid. Algorithms for modelling viscosity and surface tension have been tested on a number of problems.

Detailed benchmarks of the surface tension algorithm were presented using a Rayleigh oscillating droplet for a test problem. This algorithm, based on spline fits to determine curvature, was good enough to allow the droplet to oscillate many times and still maintain a constant period. However, the amplitude calculated for the original excited mode decayed into higher modes. We believe this is due to a resonance coupling between the modes $n = 2$ and $n = 3$. A similar resonance coupling exists in three dimensions[16].

We are currently calculating droplet distortion and breakup due to differential external flows, shear flows, and droplet-droplet collisions. Some of the results of these calculations were shown in Figures 4, 5, and 6, and will be discussed more thoroughly in the presentation.

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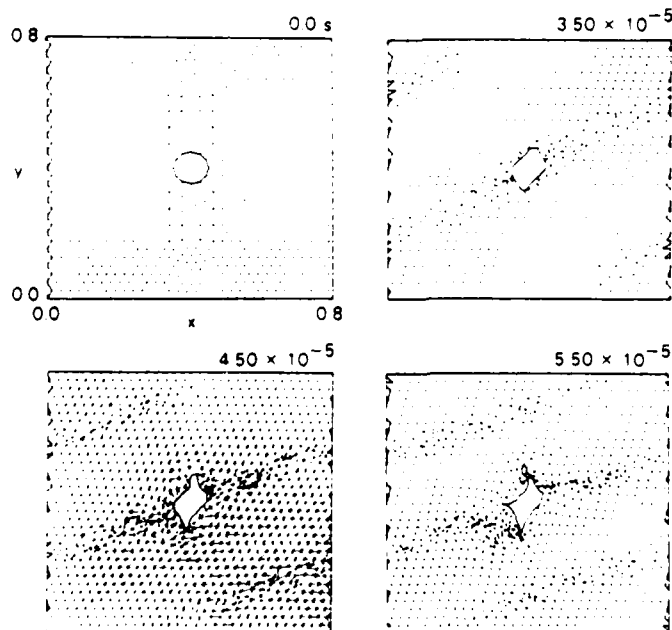


Figure 6. Four frames from the calculation of a droplet in a shear flow.

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APPENDIX B: Article to appear in J. Comp. Phys.

**Surface Tension and Viscosity with Lagrangian
Hydrodynamics on a Triangular Mesh**

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Surface Tension in Lagrangian Flows

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Abstract

Numerical algorithms for surface tension and viscosity are presented in the context of a Lagrangian treatment of incompressible hydrodynamics with a dynamically restructuring grid. New algorithms are given which update previous Lagrangian approaches in the code SPLISH. Test problems involving internal gravity and capillary waves, an oscillating droplet and a viscous shear layer are described. An example is given of a flow calculated in and around a viscous droplet with surface tension in a shear flow.

I. Introduction

In principle, a Lagrangian formulation of the hydrodynamics equations is particularly attractive for numerical calculations. Each discretized fluid element is tracked as it evolves through the interaction with its changing environment and with external forces. The local interactions can be represented without nonphysical numerical diffusion. Conservation laws are simple to express since there are no fluxes out of the fluid element boundaries. The paths of the fluid elements are themselves a flow visualization. It thus appears to be the natural approach to transient hydrodynamics with free surfaces, interfaces, or sharp boundaries.

In practice, the use of Lagrangian methods in numerical simulations has generally been restricted to "well-behaved" flows. Shear, fluid separation, or even large amplitude motion produce severe grid distortion. These distortions arise because grid points can move far enough that their near-neighbors change in the course of a calculation. When differential operators are approximated over a mesh which is distorting, the approximations may become inaccurate. Attempting to regain accuracy through regridding and interpolating physical quantities onto the new grid introduces numerical diffusion into the calculation.

This paper is a summary and update of the latest additions and modifications to a numerical technique for indefinitely extending Lagrangian calculations by using a restructuring triangular mesh, first introduced by Fritts and Boris [1]. The major advance of this approach is that the grid automatically adapts and refines itself to maintain accuracy for discretized operators in a manner that is nondiffusive. The algorithms have been implemented in the code SPLISH, which has been applied to calculations of nonlinear waves [2, 3], flows over obstacles [4], Kelvin-Helmholtz instabilities [5], Rayleigh-Taylor instabilities [6], Couette flows and Taylor vortex flows [7].

Work on Lagrangian techniques for grids which do not have fixed connectivity has recently had a renaissance. Early attempts included the PANACEA code [8] and the PAF (Particle-And-Force) algorithm [9, 10]. In the 1970's, these concepts were

improved and extended for triangular grids: triangle reconnection by Crowley [11]; MHD algorithms over a triangular mesh [12]; and adaptive triangular meshes in the work mentioned in the previous paragraph on SPLISH. During the same period work began which used Voronoi meshes for hydrodynamics calculations [13].

Recently this use of general connectivity grids has rapidly expanded, as summarized in the First International Conference on Free-Lagrange Methods [14]. Applications now include finite-difference and finite-element calculations of classic hydrodynamic instabilities, tokamak modelling, high temperature plasma physics, heat conduction, wave-structure interactions, impact deformations, and hydrodynamics problems for both compressible and incompressible fluids. Free-Lagrange methods now use quadrilateral, triangular and mixed meshes in two dimensions, tetrahedral meshes in three dimensions, Voronoi meshes in both two and three dimensions, and methods which are mesh-free.

In this paper we present the latest modifications to SPLISH (section II). These include the most recent version of the rotation operator, which conserves circulation, and the residual algorithm, which ensures conservation of the area of cells. We also introduce new algorithms for viscosity and surface tension. Including viscosity proved to be straightforward (section II). However, the search for a good enough algorithm for surface tension (section III) was more challenging and difficult. The basic problem is defining a proper curvature from a finite number of points. Because of this, the numerical approximation of surface tension forces between two fluids is conceptually quite different from approximations of convection and viscous forces. The final formulation chosen, a series of test problems, and a list of approaches that failed are detailed (section III). Finally, we combine the convective transport, surface tension, and viscosity algorithms to perform some preliminary calculations of flows in and around a viscous kerosene droplet. These calculations show vortex shedding behind the droplet, distortion of the droplet due to the shear flow, and internal droplet flows.

II. Basic Elements of Lagrangian Triangular Grids

This section is a review of the derivation of low order finite-difference approximations to the equations describing incompressible fluid motion for general triangular grids. Some of the material was originally presented by Fritts and Boris [1], and the interested reader is referred there for more detail. However, new material brings the previous paper up to date. This includes the latest version of the rotation operator, which conserves circulation, the residual algorithm, which ensures conservation of the area of cells, and the new algorithm for viscosity.

A. The Triangular Grid

Consider a two-dimensional space which is divided into triangular cells. A section of this mesh shown in Fig. 1, which shows an interface between fluid type I and fluid type II. In Fig. 1a, a particular triangle j is highlighted by heavy lines and the various components of the triangle are labeled. Three vertices, V_1 , V_2 , and V_3 , are connected consecutively by sides S_1 , S_2 , and S_3 . The direction of labeling around each triangle is counterclockwise and the z axis is directed out of the page. Since the mesh can be irregularly connected, an arbitrary number of triangles can meet at each vertex.

We can define a cell surrounding a vertex, as shown in Fig. 1b, by the shaded region surrounding V_3 . The borders of such vertex-centered cells are determined by constructing line segments joining the centroid of each triangle with the midpoints of the two triangle sides connected to the vertex, for all triangles surrounding that vertex. This definition of a vertex cell equally apportions the area of a triangle to each of its three vertices and provides a simple, efficient way to evaluate the finite difference operators. However, the definition of a vertex cell is arbitrary. Other definitions could be equally well employed, although they generally require additional calculations to determine cell intersection points. The integration of cell quantities may therefore involve more arithmetic operations for other definitions.

B. Finite Differences on a Triangular Grid

Finite-difference approximations for derivatives of functions defined on the triangular grid are derived from the expressions for the integral of the gradient of a scalar function, f , and the divergence and curl of a vector field, $\mathbf{v}$, in two Cartesian dimensions.

$$\int_A \nabla f \, dA = \oint_C f \, d\mathbf{l} \times \hat{\mathbf{z}} \quad (2.1)$$

$$\int_A \nabla \cdot \mathbf{v} \, dA = \oint_C \mathbf{v} \cdot (d\mathbf{l} \times \hat{\mathbf{z}}) \quad (2.2)$$

$$\int_A \nabla \times \mathbf{v} \, dA = \oint_C \mathbf{v} \cdot d\mathbf{l} \, \hat{\mathbf{z}} \quad (2.3)$$

In each of these expressions, A is the region enclosed by the curve C and $d\mathbf{l}$ is the vector arc length around C in the counterclockwise direction. The variable $\hat{\mathbf{z}}$ is a unit vector in the direction of the ignorable coordinate. By using these definitions in a conservative integral approach, the definitions for spatial derivatives described below can be naturally extended to two-dimensional axisymmetric geometry [7].

Throughout the following discussion a triangle-centered quantity is assumed to be piecewise constant over the triangles with discontinuities occurring at the triangle sides and a vertex-centered quantity is assumed to be piecewise linear over the triangles. If we want to form a triangle-centered derivative, we use the triangle as the area A and the sides of the triangle for the curve C in Eqs. (2.1)–(2.3). We then approximate the area integral by the area of the triangle times the value of the derivative on the triangle, and approximate the line integral using the trapezoidal rule on each side of the triangle. For example, the gradient of a scalar function f defined at the vertices is a triangle-centered quantity, $(\nabla f)_j$, given by

$$A_j (\nabla f)_j = \frac{1}{2} \sum_{i(j)} f_i (\mathbf{r}_{i-1} - \mathbf{r}_{i+1}) \times \hat{\mathbf{z}}, \quad (2.4)$$

where $\mathbf{r}_i = (x_i, y_i)$ is a vector coordinate for vertex i and A_j is the area of triangle j . We have also used the notation of Fritts and Boris [1] that $\sum_{i(j)}$ is interpreted as the

sum over vertices i of triangle j . In the material presented below, the index i designates vertex-centered quantities and the index j designates triangle-centered quantities.

If we want to form a vertex-centered derivative, we use the vertex-centered cell as the area A . We approximate the area integral on the left side of Eq. (2.1)–(2.3) by the area of the vertex-centered cell times the value of the derivative at the vertex. We approximate the line integral using the value on each triangle and the appropriate vector length through the triangle. For example, the curl of the vector field $\mathbf{v}$ at a vertex c is approximated by

$$A_c(\nabla \times \mathbf{v})_c = \frac{1}{2} \sum_{i(c)} \mathbf{v}_{i+1/2} \cdot (\mathbf{r}_{i+1} - \mathbf{r}_i) \hat{z}, \quad (2.5)$$

where $A_c = \frac{1}{3} \sum_{j(c)} A_j$ is the vertex-centered cell area, $\sum_{j(c)}$ is a sum over the triangles around the central vertex c , $\sum_{i(c)}$ is a sum over the vertices around vertex c , and $\mathbf{v}_{i+1/2}$ is the value of the vector field $\mathbf{v}$ on the triangle having vertices c , i , $i+1$. Similarly, the divergence of the vector field $\mathbf{v}$ at a vertex is approximated by

$$A_c(\nabla \cdot \mathbf{v})_c = \frac{1}{2} \sum_{i(c)} [\mathbf{v}_{i+1/2} \times (\mathbf{r}_{i+1} - \mathbf{r}_i)] \cdot \hat{z}. \quad (2.6)$$

C. The Equations for Incompressible, Inviscid Flow

The basic equations for inviscid incompressible hydrodynamics are

$$\frac{dp}{dt} = 0, \quad (2.7)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (2.8)$$

$$\rho \frac{d\mathbf{v}}{dt} + \nabla p = \mathbf{f}_e. \quad (2.9)$$

In two dimensions the fluid density ρ , pressure p , and velocity $\mathbf{v}$ are assumed to vary with x , y , and t . The term $\mathbf{f}_e$ represents external forces applied to the fluid, for example,

forces due to gravity. Equation (2.8), the condition for incompressibility, removes the sound waves.

Since we want our finite difference approximation to preserve the conservation properties for incompressible inviscid fluids, it is important to consider which of the physical variables, ρ , $\mathbf{v}$, and p , should be defined as vertex-centered quantities and which should be defined as triangle-centered quantities. We have found that prescribing velocities as triangle-centered quantities makes the formulation of conservation of circulation straightforward. Prescribing the densities on triangles and pressures at vertices allows conservation of vertex cell areas.

The time integration of velocities uses a second-order implicit split-step algorithm which is solved by iteration. The vertex positions are advanced using a second-order midpoint rule. Specifically, the velocities are advanced a half timestep, the grid is advanced a full timestep, and then the velocities are advanced forward the other half timestep. The complete algorithm is as follows. First compute the half-timestep triangle velocities using

$$\mathbf{v}_j^{1/2} = \mathbf{v}_j^o - \frac{\delta t}{2\rho_j}(\nabla p)_j^o + \frac{\delta t}{2\rho_j}\mathbf{f}_e, \quad (2.10)$$

where the superscript o designates the values at the old time step. We then make an initial guess for the new triangle velocities

$$\mathbf{v}_j^{n,0} = \mathbf{v}_j^{1/2}$$

and iterate

$$\mathbf{v}_i^{1/2,k} = \frac{1}{2}(\mathbf{v}_i^o + \mathbf{v}_i^{n,k-1}), \quad (2.11)$$

$$\mathbf{x}_i^{n,k} = \mathbf{x}_i^o + \delta t \mathbf{v}_i^{1/2,k}, \quad (2.12)$$

$$\tilde{\mathbf{v}}_j^{1/2,k} = \mathbf{R}(\{\mathbf{x}_i^o\}, \{\mathbf{x}_i^{n,k}\}) \cdot \mathbf{v}_j^{1/2}, \quad (2.13)$$

$$\mathbf{v}_j^{n,k} = \tilde{\mathbf{v}}_j^{1/2,k} - \frac{\delta t}{2\rho_j}(\nabla p)_j^{n,k} + \frac{\delta t}{2\rho_j}\mathbf{f}_e. \quad (2.14)$$

where the second superscript indicates the iteration number. The vertex velocity $\mathbf{v}_i^{n,k}$ in Eq. (2.11) is obtained from a weighted average of the triangle velocities $\mathbf{v}_j^{n,k}$ for those triangles having i as a vertex,

$$\mathbf{v}_i^n = \frac{\sum_{j(i)} w_j \mathbf{v}_j^n}{\sum_{j(i)} w_j}. \quad (2.15)$$

We use $w_j = \theta_j \rho_j A_j$, where θ_j is the angle (in radians) of triangle j at vertex i divided by π . The transformation $\mathbf{R}$ in Eq. (2.13) results from the requirement of conservation of circulation, and is discussed in Section D below.

The pressures $\{p_i^{n,k}\}$ in Eq. (2.14) are derived from the condition that the new velocities $\{\mathbf{v}_j^{n,k}\}$ should be divergence-free at the new timestep, satisfying Eq. (2.8). The pressure Poisson equation is derived from Eq. (2.14) by setting $(\nabla \cdot \mathbf{v}_j)^{n,k} = 0$ to obtain a pressure $p_i^{n,k}$, such that

$$\left(\nabla \cdot \frac{\delta t}{2\rho_j} (\nabla p)_j\right)_i^{n,k} = (\nabla^{n,k} \cdot \mathbf{v}_j^{1/2,k})_i + (\nabla^{n,k} \cdot \frac{\delta t}{2\rho_j} \mathbf{f}_e)_i \quad (2.16)$$

Both terms in Eq. (2.16) are straightforward to evaluate, since the divergence is taken over triangle-centered quantities. Note also that the discrete gradient operator ∇ must also carry time advancement superscripts since it depends on the current grid location. (See Eq. (2.4).) Two features of the Poisson equation, Eq. (2.16), are noteworthy. First, it is derived from $\nabla^2 \phi = \nabla \cdot \nabla \phi$, as in the continuum case. Second, the left-hand-side results in the more familiar second-order accurate templates for the Laplacians (such as the five-point formula) derived for homogeneous fluids and regular mesh geometries.

D. Conservation of Circulation

The approach we have outlined is basically a control volume approach which uses an integral formulation to derive the difference algorithms. Equation (2.13), which produces conservation of circulation over vertex cell volumes, is a consequence of this approach. It reflects numerically the fact that the triangle velocities must be altered as the grid rotates and stretches. This process does not prevent the addition or loss of vorticity due to external forces or changes in density at interfaces. Rather it corrects any numerical errors that may arise because the grid has moved. Thus it guarantees conservation of circulation at those vertices where the circulation theorem applies.

The transformation $\mathbf{R}$ is derived by considering the circulation about each vertex. Since triangle velocities are constant over the triangle, the circulation taken about the boundary of the vertex cell can be calculated from Eq. (2.5). The conservation of vorticity then takes the form of the operator $\mathbf{R}$ which preserves the value of the circulation about each vertex as the grid changes.

Conservation of circulation requires that at each timestep, and for each vertex, c ,

$$\sum_{i(c)} \tilde{\mathbf{v}}_{i+1/2}^{1/2,k} \cdot (\mathbf{r}_{i+1}^{n,k} - \mathbf{r}_i^{n,k}) = \sum_{i(c)} \mathbf{v}_{i+1/2}^{1/2} \cdot (\mathbf{r}_{i+1}^o - \mathbf{r}_i^o). \quad (2.17)$$

For convenience in notation, we now drop the superscript 1/2 for the velocities and the iteration superscript k appearing in Eq. (2.17). Since there are two components of velocity on each triangle, but only one constraint at each vertex, the form of the rotator is underdetermined. Fritts and Boris [1] provided the additional constraints by making each term in the circulation integral associated with a given triangle a conserved quantity, and hence the sum in Eq. (2.17) remains unchanged. This means that for each triangle j ,

$$\tilde{\mathbf{v}}_j \cdot (\mathbf{r}_{i+1}^n - \mathbf{r}_i^n) = \mathbf{v}_j \cdot (\mathbf{r}_{i+1}^o - \mathbf{r}_i^o), \quad i = 1, 2, 3. \quad (2.18)$$

Although this approach conserves circulation, the following example shows that it is much too restrictive.

Let us also decompose $\delta \mathbf{v}_{i+1/2}$ into a component, $t_{i+1/2}$, parallel to the side opposite vertex c , and a component, $n_{i+1/2}$, normal to the side opposite vertex c by writing

$$\delta \mathbf{v}_{i+1/2} \equiv n_{i+1/2} \frac{\hat{z} \times (\mathbf{r}_{i+1} - \mathbf{r}_i)}{|\mathbf{r}_{i+1} - \mathbf{r}_i|} + t_{i+1/2} \frac{\mathbf{r}_{i+1} - \mathbf{r}_i}{|\mathbf{r}_{i+1} - \mathbf{r}_i|}. \quad (2.21)$$

With this notation and using the equation for the area, $A_{i+1/2}$, of triangle $i + 1/2$,

$$2A_{i+1/2} = \hat{z} \cdot [(\mathbf{r}_{i+1} - \mathbf{r}_i) \times (\mathbf{r}_c^n - \mathbf{r}_{i+1})], \quad (2.22)$$

Eq. (2.20) becomes

$$\begin{aligned} & \frac{2A_{i+1/2}}{|\mathbf{r}_{i+1} - \mathbf{r}_i|} n_{i+1/2} + \frac{(\mathbf{r}_{i+1} - \mathbf{r}_i) \cdot (\mathbf{r}_c^n - \mathbf{r}_{i+1})}{|\mathbf{r}_{i+1} - \mathbf{r}_i|} t_{i+1/2} \\ & + \frac{-2A_{i-1/2}}{|\mathbf{r}_i - \mathbf{r}_{i-1}|} n_{i-1/2} + \frac{(\mathbf{r}_i - \mathbf{r}_{i-1}) \cdot (\mathbf{r}_{i-1} - \mathbf{r}_c^n)}{|\mathbf{r}_i - \mathbf{r}_{i-1}|} t_{i-1/2} \\ & = (\mathbf{v}_{i-1/2} - \mathbf{v}_{i+1/2}) \cdot \delta \mathbf{r}_c. \end{aligned} \quad (2.23)$$

Let N_c denote the number of triangles (vertices) about an interior vertex c . The N_c equations given by Eq. (2.23) for the $2N_c$ unknowns $\{t_{i+1/2}\}$ and $\{n_{i+1/2}\}$ are linearly dependent. This can be seen by summing the equations, which produces the equation for the change in circulation about vertex c . The equation for the change in circulation at vertex c is a linear combination of the $t_{i+1/2}$'s, which is equal to zero. Since we want the $t_{i+1/2}$'s to be linearly independent, we can set $t_{i+1/2} = 0$ for all i . We still need another equation to determine the normal component for the change in velocities on the triangles.

Let us for the moment write that equation as

$$\sum_{i=1}^{N_c} c_{i+1/2} n_{i+1/2} = b. \quad (2.24)$$

Using Eq. (2.23) with $t_{i+1/2} = 0$ for all i , we can successively eliminate each $n_{i+1/2}$ for $i = 1, \dots, N_c - 1$ in Eq. (2.24) until we arrive at an equation for $n_{N_c+1/2}$. Since the

numbering of the triangles and vertices is arbitrary, this expression is valid for each triangle $i + 1/2$ by replacing $n_{N_c+1/2}$ with $n_{i+1/2}$ and $\mathbf{v}_{N_c+1/2}$ with $\mathbf{v}_{i+1/2}$. The result is that

$$\delta \mathbf{v}_{i+1/2} = \frac{\hat{\mathbf{z}} \times (\mathbf{r}_{i+1} - \mathbf{r}_i)}{2A_{i+1/2}} \left[b - \sum_{k(c)} \frac{c_{k+1/2} |\mathbf{r}_{k+1} - \mathbf{r}_k|}{2A_{k+1/2}} (\mathbf{v}_{k+1/2} - \mathbf{v}_{i+1/2}) \cdot \delta \mathbf{r}_c \right] / \sum_{k(c)} \frac{c_{k+1/2} |\mathbf{r}_{k+1} - \mathbf{r}_k|}{2A_{k+1/2}} . \quad (2.25)$$

Several alternatives are possible for Eq. (2.24). If we conserve divergence about the vertex c , then

$$c_{i+1/2} = |\mathbf{r}_{i+1} - \mathbf{r}_i|, \quad (2.26)$$

$$b = 0 .$$

The transformation $\mathbf{R}$ prescribed by Eq. (2.25) is time-reversible, hence Eqs. (2.10)–(2.14) are also reversible. The entire algorithm advances vertex positions and velocities reversibly while evolving the correct circulation about every interior vertex. This technique is unique for Lagrangian codes, which usually either ignore conservation of circulation completely or conserve circulation through an iteration performed simultaneously with the pressure iteration. With this method the circulation is conserved exactly regardless of whether the pressures have iterated to their final values.

E. Viscous Flows

Viscosity modifies Eq. (2.9), so that now

$$\rho \frac{d\mathbf{v}}{dt} + \nabla p = \mathbf{f}_e + \mu \nabla^2 \mathbf{v}. \quad (2.27)$$

Discretization of the additional term in the momentum equation follows the same approach as the discretization of the other terms. Since the velocity is a triangle-centered

quantity, we need a discrete vertex-centered gradient operator, and a discrete triangle-centered divergence operator. Employing the same techniques as above we have

$$A_c(\nabla f)_c = \frac{1}{2} \sum_{i(c)} f_{i+1/2} (\mathbf{r}_{i+1} - \mathbf{r}_i) \times \hat{\mathbf{z}}, \quad (2.28)$$

and

$$A_j(\nabla \cdot \mathbf{v})_j = \frac{1}{2} \sum_{i(j)} [\mathbf{v}_i \times (\mathbf{r}_{i+1} - \mathbf{r}_{i-1})] \cdot \hat{\mathbf{z}}. \quad (2.29)$$

The Laplacian is found by taking the divergence of the gradient.

The finite difference equations, Eqs. (2.10) and (2.14), can be modified to account for the additional term in the momentum equation by

$$\mathbf{v}_j^{1/2} = \mathbf{v}_j^o - \frac{\delta t}{2\rho_j} (\nabla p)_j^o + \frac{\delta t}{2\rho_j} \mathbf{f}_e + \frac{\mu_j \delta t}{2\rho_j} (\nabla^2 \mathbf{v})_j^o, \quad (2.30)$$

$$\mathbf{v}_j^{n,k} = \tilde{\mathbf{v}}_j^{1/2,k} - \frac{\delta t}{2\rho_j} (\nabla p)_j^{n,k} + \frac{\delta t}{2\rho_j} \mathbf{f}_e + \frac{\mu_j \delta t}{2\rho_j} (\nabla^{n,k} \cdot \nabla^{n,k} \mathbf{v}^{n,k-1})_j. \quad (2.31)$$

These equations are implicit in the velocities, just as the original Eqs. (2.10)–(2.14) are. As in the inviscid case, we solve by iteration.

This algorithm was tested by calculating the spreading of a shear layer of initially zero thickness given by

$$\mathbf{v}(x, y, t = 0) = \begin{cases} (v_x, 0), & \text{for } y > y_0, \\ (0, 0), & \text{for } y = y_0, \\ (-v_x, 0), & \text{for } y < y_0. \end{cases} \quad (2.32)$$

where y_0 is the original location of the vortex sheet. The velocity distribution across this layer evolves as

$$\mathbf{v}(x, y, t) = \hat{x} v_x \operatorname{erf} \left[\frac{(y - y_0)}{(4\nu t)^{1/2}} \right], \quad (2.33)$$

where $\nu = \mu/\rho$. The width Δy of the layer grows as

$$\Delta y \approx 8(\nu t)^{1/2}. \quad (2.34)$$

For the test calculation the grid was initialized to center a vortex sheet in a grid 16 cells wide with an initial layer width of zero. The two opposing streams had initially constant velocity profiles. The evolution of the interface between the streams was governed by the same algorithms as the interior of either fluid, so that no special interface boundary condition was used. The boundary conditions on the sides of the computational region were periodic, and the top and bottom had free-slip boundary conditions.

At the end of the calculation, the layer width agreed to within numerical roundoff with the theory and the layer extended over the whole mesh. The velocity profile for each stream coincided with that given by Eq. (2.33) to within round-off error. The y -components of the velocity remained zero, indicating that the algorithm was working well for the grid distortions presented by the problem.

F. Conservation of Vertex Cell Areas

Equations (2.10)–(2.14) are implicit in the triangle velocities $\{v_j\}$. Because these equations must be solved iteratively to produce a divergence free velocity field, a small residual error may remain. In addition, vertex velocities are derived from the divergence-free triangle velocities. In practice this means that vertex cell areas may not be conserved. Furthermore, as the flow progresses, the triangle sides distort. Yet at any given time we compute using straight triangle sides, which does not produce the equivalent cell area about any given vertex. However, since we know what the triangle area should be, it is possible to at least make a correction to the known error. Our approach, then, is to perform an ad hoc correction step after all the vertices have been advanced in time. This correction step moves the vertices in order to conserve vertex cell area. After this vertex correction step, the rotator is applied to ensure that the circulation has not been changed.

To expand or contract a vertex cell area, we must expand or contract the surrounding triangle areas. Suppose we wish to expand a triangle j with area A_j and vertex coordinates $\mathbf{r}_i$ by an amount δA_j . To do this we will move each vertex $\mathbf{r}_i$ an amount

$$\mathbf{r}_i^{\text{new}} - \mathbf{r}_i = \delta \mathbf{r}_i = d_j [\hat{z} \times (\mathbf{r}_{i-1} - \mathbf{r}_{i+1})]. \quad (2.35)$$

that is, the vertices of the triangle are moved normally to the opposite side a distance prescribed by the triangle expansion factor, d_j . If d_j is positive, the triangle area increases. Using the vector definition for the area of a triangle, we have

$$\begin{aligned} 2\delta A_j &= 2A_j^{\text{new}} - 2A_j \\ &= [(\mathbf{r}_{i+1}^{\text{new}} - \mathbf{r}_i^{\text{new}}) \times (\mathbf{r}_{i-1}^{\text{new}} - \mathbf{r}_{i+1}^{\text{new}})] \cdot \hat{z} - [(\mathbf{r}_{i+1} - \mathbf{r}_i) \times (\mathbf{r}_{i-1} - \mathbf{r}_{i+1})] \cdot \hat{z} \\ &= [(\mathbf{r}_{i+1} - \mathbf{r}_i) \times (\delta \mathbf{r}_{i-1} - \delta \mathbf{r}_{i+1})] \cdot \hat{z} + [(\delta \mathbf{r}_{i+1} - \delta \mathbf{r}_i) \times (\mathbf{r}_{i-1} - \mathbf{r}_{i+1})] \cdot \hat{z} \\ &\quad + [(\delta \mathbf{r}_{i+1} - \delta \mathbf{r}_i) \times (\delta \mathbf{r}_{i-1} - \delta \mathbf{r}_{i+1})] \cdot \hat{z} \\ &= s^2 d_j + 6A_j d_j^2, \end{aligned} \quad (2.36)$$

where s^2 is the sum of the squares of the sides of the triangle. This quadratic in the expansion factor, d_j , can be solved to yield

$$d_j = \frac{-s^2 + \sqrt{s^4 + 48A_j \delta A_j}}{12A_j}. \quad (2.37)$$

The sign in front of the square root was chosen to ensure d_j has the same sign as δA_j .

We relate the change in triangle area, δA_j , to the conservation of vertex cell areas through

$$\delta A_j = \frac{A_j}{3} \sum_{i(j)} \frac{A_i^c - A_i}{A_i} = \sum_{i(j)} \frac{A_j/3}{A_i} \delta A_i, \quad (2.38)$$

where the sum is over the three vertices of the triangle, A_i is the current area about vertex i and A_i^c is the original area about vertex i . Basically, the change in vertex cell areas is apportioned to each contributing triangle according to that triangle's contribution to the vertex cell area.

Although this residual correction is a small numerical effect, we have found that it improves the overall results of a calculation. Because this algorithm expands triangles, it has potential for modelling other physical processes. In a compressible algorithm involving energy release and fluid flows with transit times which are small compared to the energy release times, this algorithm could be used to produce the required expansion of the vertex cells.

G. Grid Restructuring

In Lagrangian calculations the grid may distort to the point where grid restructuring is necessary. The derivation of the reconnection and vertex addition and deletion algorithms are done through the control volume approach and the use of triangle velocities. For all the algorithms used, the area-weighted divergence and curl taken about each vertex are both identically conserved for grid reconnections and vertex addition.

The accuracy of a general triangular mesh is diminished by large obtuse angles within triangles. With reconnections, accuracy can be recovered by ensuring that large obtuse angles are preferentially eliminated. There are many ways of formulating a reconnection algorithm. The one we have chosen is based on requirements for solving the pressure Poisson equation. The pressure Poisson equation is formally equivalent to that obtained by a piece-wise linear Rayleigh-Ritz-Galerkin finite element procedure on a triangular grid. (See, for example [15].) Since we solve the equation by iteration, we want the iteration to converge as rapidly as possible. Mathematically, convergence is assured if the finite difference equation has a maximum principle: that is, all the off-diagonal terms are negative, the diagonal term is positive and greater than or equal to the absolute value of the sum of the off-diagonal terms, with strict inequality for at least one equation. (That one equation typically involves boundary conditions. Our boundary condition prescribes the integrated pressure along the upper boundary.)

To see how large angles affect the maximum principle, consider the difference equation for vertex l of Fig. 3a. The off-diagonal coefficient relating vertex l to vertex j is

$$a = -\frac{1}{2}(\cot \theta^- + \cot \theta^+) \quad (2.39)$$

where θ^+ and θ^- are the angles opposite the line from the vertex j to the vertex l as shown in Fig. 3a. The other off-diagonal terms are determined in a similar manner from the remaining edges emanating from vertex l . The diagonal coefficient is the negative of the sum of the off-diagonal terms. For positive area triangles, θ^+ and θ^- are both between 0° and 180° . Hence, each term in Eq. (2.39) is negative only when $\theta^+ + \theta^- > 180^\circ$, since

$$a = \frac{\sin(\theta^+ + \theta^-)}{2 \sin \theta^+ \sin \theta^-} \quad (2.40)$$

The reconnection algorithm ensures that the angles subtended by any given edge sum to no more than 180° . If $\theta^+ + \theta^-$ is greater than 180° , the grid line is reconnected as shown in Fig. 3b. The new angles, θ'^+ and θ'^- , must sum to less than 180° since $(\theta^+ + \theta^- + \theta'^+ + \theta'^-)$ is the sum of the interior quadrilateral angles, which must be 360° . By choosing the diagonal which divides the largest opposing angles, the reconnection algorithm preferentially eliminates large angles in triangles.

Interface sides are never allowed to reconnect. In such cases vertex addition algorithms are needed. Vertex addition algorithms are also needed where the flow naturally depletes vertices. For vertex addition, satisfaction of conservation integrals is particularly simple. The vertex added at the centroid of a triangle subdivides that triangle into three smaller triangles. A vertex added to the midpoint of a side subdivides the two adjacent triangles into four smaller triangles. If the new triangle velocities are all the same as the velocity of the subdivided triangles, all conservation laws are satisfied. Since the reconnection algorithm is also conservative, subsequent reconnections to other vertices ensure that the only effect of the addition is an increase in resolution.

The case is not as obvious for vertex deletion. Reconnections can be used to surround any interior vertex within a triangle. The vertex is then removed and the new larger triangle given a velocity which is the area-weighted sum of the old velocities,

$$A_l \mathbf{v}_l = A_i \mathbf{v}_i + A_j \mathbf{v}_j + A_k \mathbf{v}_k. \quad (2.41)$$

Such a substitution redistributes circulation in accordance with area coordinates. Figure 4 illustrates the triangles before and after vertex removal. If ζ_4 is the vorticity about vertex 4 before it is removed, then the vorticity about each of the other three vertices is increased by an amount ζ'_i given by

$$\begin{aligned} \zeta'_1 &= A_j \zeta_4 / A_l \\ \zeta'_2 &= A_k \zeta_4 / A_l, \\ \zeta'_3 &= A_i \zeta_4 / A_l \end{aligned} \quad (2.42)$$

where

$$\zeta'_1 + \zeta'_2 + \zeta'_3 = \zeta_4$$

since

$$A_i + A_j + A_k = A_l.$$

Therefore, total vorticity is conserved and redistributed in a reasonable and natural manner.

III. Surface Tension

A. The Algorithm

The surface tension at an interface between two materials depends on the curvature of the interface. In the conventional numerical representation of surface tension, it is cast into a finite-difference form by fitting vertices on the material interface to some parametric function. This function is then used to find an estimate of local curvature. Once the curvature is known, a surface tension force is evaluated and used to accelerate interface vertices.

This scheme fails in SPLISH for two reasons. First, the interface vertices are accelerated directly by surface tension forces evaluated on the vertices. Since velocities are centered on triangles in SPLISH, the velocity field sees the effect of the acceleration a half-timestep later, unless a secondary calculation is made. As a result, the pressure calculated within the droplet is inconsistent with that found from the surface tension formula. Second, since the pressure gradient forces and surface tension forces are not calculated in the same manner, numerical errors result which grow with each timestep.

Both of these problems are eliminated by a different formulation of surface tension, in which a surface tension potential is used to generate the forces. The surface tension force is formulated as a gradient of a potential present only at the surfaces. With this method, the pressure gradient forces are calculated in the same manner and on the same grid as the forces derived from the surface tension potential. Therefore both the surface tension potential and the pressure are dynamically similar, and the physical pressure drop across the interface must exactly cancel the surface tension forces. Preliminary aspects of this work were described by Fritts et al. [16, 17].

The finite-difference algorithms for surface tension are straightforward. The surface tension forces are included through Laplace's formula for the pressure jump across an interface [18],

$$p_i - p_o = \sigma/R, \quad (3.1)$$

where p_i is the pressure just inside the droplet at the interface, p_o is the pressure just outside the droplet at the interface, σ is the surface tension coefficient associated with the two media which define the interface, and R is the radius of curvature in the two-dimensional plane. The radius of curvature is positive at points on the interface where the droplet surface is convex (a circle is convex everywhere) and negative when the surface is concave. These pressure jumps are included in the Poisson equation for the pressure. The average pressure, $(p_i + p_o)/2$, is computed at the interface vertices. From the average pressure and the pressure jump, we can compute a pressure gradient centered on triangles, both inside and outside the surface. This pressure gradient is used in the momentum equation.

The radius of curvature is computed from a parametric cubic spline interpolant to the interface vertices. Past calculations of droplets oscillating due to surface tension forces [19, 20] also use cubic spline interpolation. However, they divided the surface into at least four segments (the top, bottom, right and left sides of the droplet) to produce an interpolant on each segment. Each interpolant was matched at the joints to produce an overall curve. The parametric interpolant used here does not require this special matching.

The parametric spline is produced in the following manner. Denote the interface vertices by $\mathbf{r}_i = (x_i, y_i)$, $i = 1, \dots, N$, with $\mathbf{r}_N = \mathbf{r}_1$. Also define a pseudo arc length parameter, s , such that the spline knots occur at the points

$$\begin{aligned} s_1 &= 0, \\ s_i &= s_{i-1} + |\mathbf{r}_i - \mathbf{r}_{i-1}|, \quad i = 2, \dots, N. \end{aligned} \tag{3.2}$$

We generate the twice differentiable periodic spline interpolants, $\mathbf{r}(s) = (x(s), y(s))$ from the data $\{s_i\}$, and $\{\mathbf{r}_i\}$, $i = 1, \dots, N$, as prescribed by DeBoor [21]. The curvature is then given by

$$K \equiv R^{-1} = \frac{|\mathbf{r}'' \times \mathbf{r}'|}{|\mathbf{r}'|^3} \tag{3.3}$$

where the prime indicates differentiation with respect to the parameter s . The sign of R at an interface vertex, $\mathbf{r}_i$, is given by the sign of $\hat{\mathbf{z}} \cdot [(\mathbf{r}_{i+1} - \mathbf{r}_i) \times (\mathbf{r}_{i-1} - \mathbf{r}_i)]$.

We can iterate the process if necessary. From the spline fit we can generate new values for the $\{s_i\}$ by integrating the expression for arc length along a parametrically prescribed curve. For symmetrically placed vertices on a symmetric droplet, however, we have found the iteration on arc length parameter is unnecessary.

The parametric spline fit is also used for regridding. When the regridding algorithm calls for the bisection of a triangle side which borders the two media, a new vertex is added on the spline interpolant between the vertices. This is done rather than bisecting the straight-line segment, since a straight-line bisection introduces spurious interface oscillations. Bisecting the spline maintains a better overall shape for the interface.

B. Test Results

We tested the algorithm for surface tension in SPLISH using two test problems. The first test problem consists of internal capillary waves. In the second test problem we calculated the oscillation of a droplet due to surface tension. For completeness we also present calculations of internal gravity waves as a test of the overall hydrodynamic algorithms in SPLISH.

1. Internal Gravity and Capillary Waves

The linear theory for the small amplitude oscillation of an interface between two fluids, bounded above and below by solid walls, gives the frequency ω as a function of wavenumber k ,

$$\omega^2 = \frac{(\rho - \rho')gk + \sigma k^3}{\rho \coth kh + \rho' \coth kh'}. \quad (3.1)$$

Here the upper fluid is of depth h' and density ρ' , the lower fluid is of depth h and density ρ . g is the acceleration due to gravity and σ is the coefficient of surface tension for the two media. Following the free-surface wave calculations of Fritts and Boris [1],

we take $k = 2\pi/\lambda$, $\lambda = 2.5$ cm, $h = h' = 1.0$ cm, $\rho = 2$ g/cc, and $\rho' = 1$ g/cc. For an internal gravity wave, we have $g = 980$ cm²/s and $\sigma = 0$ dynes/cm. For an internal capillary wave, we have $g = 0$ cm²/sec and $\sigma = 30$ dynes/cm. These values give a period $\tau = 2\pi/\omega = 0.22073$ s for the internal gravity wave and $\tau = 0.50196$ s for the internal capillary wave. The amplitude of the oscillation is taken as $A = 0.0672h$. For this amplitude the free-surface oscillations of Fritts and Boris [1] showed negligible non-linear effects. Figure 5 shows the initial grid for the mesh size $\delta s = 0.125$ cm.

Figure 6 shows the wave period as a function of mesh size for the internal gravity wave problem. The ratio of timesteps for any two calculations was the same as that for the mesh sizes. Each data point on the curve is an average over several periods and is accurate to three digits. If we extrapolate to zero mesh size using a parabolic least-squares fit, $\tau = \tau_0 + b\delta s + a(\delta s)^2$ to the data points, we obtain $\tau_0 = 0.2214$, $b = 0.0726$, and $a = 0.1549$ for this problem. The extrapolated value, τ_0 , is accurate to 0.3%. The finite-difference derivatives given in section II are accurate to second order in the mesh size for triangular grids in which the centroid of a vertex cell is the vertex itself. The truncation error is linear in the distance between the vertex and the centroid of the vertex cell. This truncation error can occur in this problem for vertex cells near the interface in our discretization and hence the linear term in δs in the above quadratic expression. This linear term has a coefficient on the order of the wave amplitude which is the approximate distortion of the grid. The order of convergence for the algorithm is essentially quadratic with a small linear contribution.

Figure 7 shows the wave period as a function of mesh size for the internal capillary wave problem. Here the least-squares fit to the data gives $\tau_0 = 0.4995$, $b = 0.2198$, and $a = 0.0640$. The extrapolated period is accurate to 0.5%. With surface tension included, the convergence is primarily linear in the mesh size. The reduction in rate of convergence is due to the use of cubic splines to calculate the curvatures. The cubic spline curve itself is fourth-order accurate, and theorems exist showing the second-order

accuracy of its second derivatives. However, we know of no theorem giving the accuracy of the combination of derivatives needed to produce the curvature in Eq. (3.3).

2. Droplet Oscillation.

As a further test of the algorithm for surface tension in SPLISH, we calculated the oscillation of a droplet due to surface tension. Rayleigh [22] derived a linear theory for small amplitude oscillations on cylindrical jets that applies to the cylindrical droplets we are discussing. He concluded that when the perturbation is totally in the plane perpendicular to the axis of the cylinder, the frequency, ω , for the oscillation is given by

$$\omega_n^2 = (n^3 - n) \frac{\sigma}{\rho a^3}, \quad (3.4)$$

where the surface of the droplet is given in polar coordinates by

$$r = a + \epsilon \cos(n\theta), \quad (3.5)$$

where ρ is the density of the jet, a is the unperturbed radius of the jet, and n prescribes the mode of oscillation in the plane with amplitude ϵ . For large amplitude oscillations, Rayleigh found that the experimental frequency diverged from that predicated by the linear theory, and he attributed these differences to nonlinear effects.

We have extended Rayleigh's theory to include the presence of an external fluid. Equation (3.4) then becomes

$$\omega_n^2 = (n^3 - n) \frac{\sigma}{(\rho_d + \rho_e) a^3}, \quad (3.6)$$

where ρ_d is the droplet density and ρ_e is the density of the external fluid.

The tests of the surface tension algorithm consisted of a series of calculations of oscillations initiated in the lowest oscillating mode, $n = 2$ in Eq. (3.6). Also, we have chosen

$$a = 0.0125 \text{ cm}$$

$$\sigma = 30 \text{ dynes/cm.}$$

values which are typical for many practical droplet problems. We discuss results for two different sets of conditions. First we consider a droplet density of 2 g/cc in a background external fluid density of 1 g/cc. If we use the definition of the period as $2\pi/\omega$, Eq. (3.6) gives a period

$$\tau = 1.13 \times 10^{-3} \text{ s.}$$

The second set of conditions are for a kerosene droplet, with density 0.82 g/cc, in a background of air, with density 0.0013 g/cc. This second case, with the 650:1 density ratio, is a stringent test of the numerical approximations.

Figure 8 is a composite of frames from a calculation in which $\epsilon = 0.2a = 0.0025 \text{ cm}$ for the 2:1 density ratio case. In this calculation there are 17 vertices in each direction along the exterior boundaries, 12 vertices on the droplet interface and a total of 313 vertices initially in the calculation. The computational domain is 0.1 cm on a side. The left and right boundaries are periodic while the top and bottom boundaries are solid walls. The timestep is $\delta t = 2.5 \times 10^{-5} \text{ s}$. The figures shows four and a half oscillations of the droplet. We can see that as the calculation proceeds, no new vertices have been added, but in fact some have been subtracted. This was the case because the initial gridding was adequate to represent the droplet shape. From these calculations, the period of oscillation is

$$\tau_{12} = 1.35 \times 10^{-3} \text{ s.}$$

Similar calculations with 20 vertices surrounding the droplet (a 21x21 grid) show a period of

$$\tau_{20} = 1.33 \times 10^{-3} \text{ s.}$$

for 24 vertices surrounding the droplet (a 25x25 grid) we have a period of

$$\tau_{24} = 1.31 \times 10^{-3} \text{ s.}$$

and for 28 vertices surrounding the droplet (a 33x33 grid) the period is

$$\tau_{28} = 1.27 \times 10^{-3} \text{ s.}$$

In each case, the period does not change during the calculation. Figures 9 and 10 show the initial oscillation for the more resolved cases. For these calculations, it was necessary to decrease the timestep, as discussed below. The time step for the calculation with 12 vertices surrounding the droplet is such that the period cannot be resolved to better than two digits. It appears that the calculations are not converging to the theoretical value, but to a value of $1.19 \pm .06\text{s}$, based on the graph of the computed period as a function of mesh size shown in fig. 11. The convergence is essentially linear as it was in the internal capillary wave test problem, but with a numerical error of about 5.5% for this calculation.

Since the internal wave tests show much better convergence properties for the algorithm, as do previous free-surface wave calculations [1, 2], than the droplet oscillation test problem, we performed several other numerical tests on the droplet oscillation problem to determine if the poorer convergence properties were due to other numerical parameters.

Firstly, we tested whether the presence of boundaries a finite distance away could alter the calculated period by performing calculations in a larger domain of length 0.2 cm. Here there were twice as many vertices on the boundary, but still only 12 vertices surrounding the droplet which was the same size as the droplets in the tests described above. These calculations showed no change in period, so we conclude that the effects of periodic boundaries and reflecting walls are negligible.

It was also important to evaluate the possible effects of nonlinearity in the solution. The theoretical value is from a linear analysis, and the calculation is a full nonlinear calculation. It is possible that this could account for part of the discrepancy. To test this, we performed calculations with smaller amplitudes, ϵ , to see if there was any difference in calculated period. The result was that the numerical value of the period

was the same for $\epsilon = 0.01a = 0.000125$ cm over the course of two oscillations as it was for $\epsilon = 0.2a$. Our conclusion is that the calculations were in a range in which the linear theory is valid.

We used two diagnostics to determine the period of the computed droplet oscillation. One is the time history of the position of the rightmost vertex on the droplet interface, denoted by x_r . The other diagnostic is the quadratic moment, defined by

$$\langle x^2 \rangle = \int x^2 dx dy, \quad (3.7)$$

where the integral is performed over the triangles which define the droplet. Tables I - IV give the values of $\langle x^2 \rangle$ and x_r as a function of time for the resolutions of 12 and 24 vertices surrounding the droplet. From the maxima and minima in Tables I and III, we can determine the period of oscillation. It is less well defined from the values of x_r in Tables II and IV. However, it never differs by more than two timesteps from that given by the moments.

Finally, we examined the oscillations of a kerosene droplet in air. This calculation tests the effects of the external fluid density on the numerical convergence of the pressure algorithm as well as any role the external fluid may have in introducing higher frequency modes. Here the theoretical value of the period is 5.9×10^{-4} s. Using a resolution of 12 vertices around the droplet surface, we find a computed value of 7.1×10^{-4} s. The ratio between the theoretical and numerical results is 0.83, compared to a ratio of 0.84 for the 2:1 density ratio calculation at the same resolution. Since changing the density ratio from 2:1 to 650:1 did not alter the relative error, we conclude that only minor errors arise by including of the external fluid in the calculation.

C. Some Difficulties and Limitations

We now believe that the inability of the method to produce as accurate a solution for the droplet oscillation test problem as for the internal wave test problems is a combination of the physical problem itself and the spline approximations to curvature.

The surface tension algorithm discussed above suffers a basic problem in curve fitting. We are trying to approximate an unknown continuous function by a known curve through a finite number of points or computational cells. For example, we are trying to represent the droplet interface or capillary wave interface by a spline fit to a finite number of points. Whereas an accurate interpolant can be found that goes through a set of points, it is not always clear that the other properties of the curve calculated at the points, e.g., the curvature, are well represented by this interpolant. Splines are notorious for introducing spurious oscillations between the points defining them initially.

Figure 12 shows the curvature at each vertex around the droplet. The exact curvature for the initial drop is compared to the curvature produced by the spline interpolant and to curvatures produced after one oscillation is completed. The initial curvature, defined by splines on the interface vertices, is reasonable. However, by the end of a cycle, there are spurious oscillations even though the curvature has the same basic shape.

In the internal capillary wave problem, the range of values for the interface curvature was a factor of 15 smaller than for the droplet oscillation problem. As a result the interface curvature for the internal capillary wave is determined with greater accuracy. In the droplet oscillation problem where the interface "bends" more sharply, the spline has a greater difficulty approximating the curvatures accurately.

Interpolations can cause other problems in the calculations. Our calculations have shown that the final result can be affected by the location of additional vertices used to obtain a better initial approximation of the droplet interface. The grid initialization procedure involves two phases: a first phase to generate a course grid, and a second

phase which refines the grid produced in the first phase. During the refinement phase, we have two choices for the location of new interface vertices. The initial grid produced in Fig. 8 placed new vertices on the droplet defined by the Rayleigh oscillation mode. We could also add the vertex on the existing spline interpolant. Figure 13 shows $\langle x^2 \rangle$ as a function of time for the two types of initialization. The curve labelled 1 is the calculation in which the additional vertices were placed on the Rayleigh drop. The curve labelled 2 placed the additional vertices on the spline fit. After one oscillation, the value of $\langle x^2 \rangle$ differs by 8%. After one oscillation the value of $\langle x^2 \rangle$ on the curve with label 1 is lower than the initial value of $\langle x^2 \rangle$ and the value of $\langle x^2 \rangle$ on the curve with label 2 is higher than the initial value of $\langle x^2 \rangle$. Notice also that the period, as well as the amplitude, is affected by the type of initialization.

In Fig. 8 we see that the amplitude of the droplet oscillation decays as a function of time even though the period is not changing. The damping rate is about 18% per oscillation. The shape of the droplet at the end of the calculation is notably different than it was at that same place in an earlier oscillation cycle. In the ideal case, this would not occur.

The decay of the oscillation is also apparent in the moment $\langle x^2 \rangle$ and the variation in the location of x_r from oscillation to oscillation. It is apparent from Fig. 14 that the $\langle x^2 \rangle$ moment is dissipating, and it is apparent from Fig. 15 that the overall shape of the droplet is changing. Energy associated with this lowest-mode oscillation is going into other modes, which is reflected in the reduction of the timestep required to keep the computations stable. In general, to carry out these droplet oscillation calculations it was necessary to reduce the timestep to the point where we could resolve the highest mode of oscillation the droplet could support at a given resolution. When we doubled the resolution around the droplet, we found that the timestep had to be decreased by a factor of about 2.8. This is consistent with the analysis which says that since the period is inversely proportional to $n^{3/2}$, where n is the mode of

oscillation. Increasing the resolution of the droplet interface by a factor of 2 means that the timestep must decrease by a factor of $2^{3/2} \approx 2.8$.

A physical mechanism for the observed decay in the $n = 2$ normal mode oscillation is the existence of a resonance between the $n = 2$ and the $n = 3$ normal modes; that is $\omega_3 = \pm 2\omega_2$. A similar behavior in three dimensions has been analyzed by Natarajan and Brown [23]. In their three dimensional analysis, significant energy can be transferred from one resonant mode to another within ten oscillations.

In summary, the total damping rate for the droplet oscillation calculation is roughly 18% for the $n = 2$ mode. Much of the energy from this mode is transferred to higher harmonics, as evidenced by the calculated droplet shapes, curvatures and the numerical timestep limitations. The difference in initialization procedures alone produced an 8% change in amplitude. Since the total numerical error is 5.5%, we conclude that the majority of this error arises from the inability of the spline fit to approximate large curvatures accurately. This error is large enough to mask other error contributions, so that we cannot evaluate additional error terms other than to indicate that they are apparently much smaller than that due to the spline fit.

However, despite all the problems with spline fits, we found that they provided a good way to calculate curvature. In the search for better curvatures, we have also tried other methods, none of which produced better results. We enumerate these attempts both for completeness and to emphasize that better numerical approximations for curvature are still needed to permit more accurate calculations of surface tension.

(1) We averaged the curvature between the vertices, such that

$$K_i = \frac{1}{s_{i+1} - s_{i-1}} \int_{s_{i-1}}^{s_{i+1}} K(s) ds.$$

The results were found to depend sensitively how the integral was actually performed. However, integration produced results which were no better than the pointwise curvatures discussed above.

- (2) We smoothed the curvatures K_i with a least squares linear spline. This method worked well for one period, but the method failed on subsequent oscillations.
- (3) We used a circular arc to calculate the curvatures. A circle was placed through the three adjoining vertices. The radius of that circle was used as the radius of curvature for the interface at the center vertex of the three vertices. This method did not work at all. The droplet interface distorted wildly within the first oscillation.
- (4) We used splines under tension. This approach introduced a free parameter which could not be consistently determined.
- (5) Based on the experiences of Foote [20], we tried producing an interpolant through every other vertex and averaging the result. The motivation was that fewer points could introduce fewer oscillations, and that averaging the interpolants could damp the oscillations. This produced poor results. The calculation is really the average of two calculations with half the original accuracy.
- (6) We considered but did not implement nonlinear splines [24]. Although these splines produce differentiable curvatures, there is no guarantee that there exists such a spline through a given set of points and, if such a spline does exist, there is no guarantee that it is unique.
- (7) We considered several methods for calculating an interpolant based on the Rayleigh modes. The high mode oscillations could then be eliminated. None of the schemes we considered gave better results than the spline interpolant, and all introduced arbitrary parameters into the calculation. These parameters could be well-determined for a particular known shape, but could not be determined for a general unknown shape.

IV. Incompressible Flow about a Droplet

In this section we present some preliminary calculations of forced, asymmetric drop oscillations induced by flow around a droplet. These calculations include both the effects of viscosity and surface tension. The capability of studying such flows for highly viscous droplets in shear flows, in two and eventually in three dimensions, is the motivation for developing the viscosity and surface tension algorithms.

The initial conditions we used specified an initially steady-state potential flow about a periodic series of cylinders. Again, the boundary conditions on the left and right sides are periodic, and the upper and lower boundary conditions are reflecting walls. Initially, a perfectly circular droplet is at rest in a background flow. A physical situation modelled by such an initialization might occur if the flow velocity were ramped up to its final value before any significant structure could develop in the flow, and before the droplet could pick up any substantial velocity. Basically, it is a smooth start for the calculation. Previously we had performed calculations which began with an impulsive start, but found that as a result there was a large amount of momentum transferred across the droplet interface early in the calculation.

The calculations presented here model the forced fluid flow due to a fast air stream about an initially stationary kerosene droplet. The physical parameters, given in Table V, are appropriate for a combustor environment. A total of 309 vertices were used to initialize the problem, with 12 vertices at the droplet interface. Figure 16 follows the evolution of pathlines in the internal and external flow fields through a series of timesteps. For an air velocity of 100 m/s and a droplet radius of 125 microns, the corresponding Reynolds number is roughly 1600. The pathlines are defined by the paths of vertices over five timesteps. By the last frame of Fig. 16, the fluid originally to the left of the droplet has progressed through the mesh and interacted with the face of the (next) droplet.

The first clear indication of the development of the recirculation region is seen in the fourth frame of Fig. 16, which shows a pair of counter-rotating vortices. The

recirculation zone continues to develop throughout the calculation, although at times the vortex pair is not as evident due to the deletion and addition of vertices, which interrupts the continuity of the pathlines. By the last frame, another pair of vortices is forming near the droplet, and the original pair has been shed. The leading face of the droplet is now quite distorted, and the droplet is about to enter the wake of the preceding droplet.

Distortions in the face of the droplet are evident in at least the seventh frame. These distortions occur because the curvature has increased and the streamlines of the streamlines in the external flow are condensed by the approaching wake. The internal velocities are small compared to the external flow rates and therefore cannot be distinguished as pathlines. However, indication of the (small) internal recirculation may be obtained by comparing internal vertex positions at various timesteps.

Figure 17 shows the grid at times in the calculation corresponding to those in Fig 16. During the course of the calculation, a great deal of vertex addition and deletion has occurred. Vertex addition, however, is most noticeable in the wake of the droplet and around the droplet interface. Whereas there were 300 vertices at the beginning of the calculation, there are 450 at the end.

Figure 18 shows the pathlines for a simulation with the air speed increased to 120 m/s, corresponding to a Reynolds number of 2000. The fluid now completely passes through the mesh. The fluid initially near the droplet has completely passed the next droplet by the time the calculation was terminated. The initial flow about the droplet is similar to that shown above, except for a more pronounced flattening at the face of the droplet due to the higher flow speed. The wake develops in much the same manner, but it now interacts strongly with the flow at the forward stagnation point on the droplet. Oscillations in the flow due to the wake are transmitted to the forward face of the droplet and give rise to fairly large perturbations.

As seen in Fig. 19, the computational grid needs further refinement at this time because the perturbations cannot be resolved by the limits set on minimum triangle

size originally chosen for the calculation. A sign that the calculation is under-resolved is that one of the crests of the surface wave is spanned by a single triangle, a situation which allows no communication of that surface fluid with the interior of the droplet. In order to continue the simulation, better resolution must be obtained about the droplet surface. Another algorithm is currently being included to allow higher resolution near points of large curvature at material interfaces.

V. Summary and Conclusions

This paper presented the current algorithms included in the code SPLISH, a two-dimensional Cartesian Lagrangian treatment of incompressible flows with a dynamically restructuring grid. The new rotator algorithm is an improvement on the one previously used for conserving circulation. The residual algorithm ensures conservation of the area of cells. These algorithms together with the original SPLISH framework constitute an extremely flexible code for calculating incompressible flows in highly distorted geometries or with obstacles in the flow.

New algorithms for modelling the physical effects of viscosity and surface tension have been added. Whereas adding the viscosity algorithm was relatively straightforward, adding surface tension caused a number of numerical problems. Detailed benchmarks of the final algorithm selected were presented using internal capillary waves and a Rayleigh oscillating droplet as test problems. The surface tension algorithm, based on spline fits to determine curvature, allowed the droplet to oscillate many times and still maintain a constant period. The numerical tests on the internal capillary waves indicate that the surface tension algorithm produced a convergence rate which is linear in the mesh size, whereas the basic hydrodynamic algorithms are quadratic in the mesh size for ideal meshes. The droplet oscillation test problem however indicates some difficulties with the spline fits for curvature when the interface becomes highly distorted.

Previous numerical calculations of oscillating spherical droplets with surface tension and viscosity using a marker-and-cell method showed only one oscillation of a water droplet in air [20], and thus did not give any information about the subsequent behavior of the mode amplitudes. These calculations used 2.5 times the resolution of our most resolved calculations. Their calculated period differed from the theoretical period by 6% compared to our 12% for a similar initial deformation. Their viscous calculations failed to damp as quickly as required by theory which may indicate that they suffer from a similar problem of approximating curvatures.

We presented calculations showing how a kerosene droplet deforms and sheds vortices in the wake of a shear flow. Calculations of fluid flow in and around fuel droplets are important in the study of spray combustors. The flow patterns influence droplet breakup, evaporation and burning rates.

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Table I

16 × 16 grid

time	last period	$\langle x^2 \rangle$
0.0000		0.3061E-07
0.6500E-03		0.1426E-07
0.1300E-02	.0013	0.2929E-07
0.1975E-02		0.1497E-07
0.2600E-02	.0013	0.2821E-07
0.3250E-02		0.1563E-07
0.3900E-02	.0013	0.2711E-07
0.4550E-02		0.1626E-07
0.5200E-02	.0013	0.2627E-07
0.5850E-02		0.1677E-07
0.6500E-02	.0013	0.2556E-07
0.7150E-02		0.1718E-07
0.7775E-02	.001275	0.2493E-07
0.8425E-02		0.1757E-07
0.9075E-02	.0013	0.2435E-07
0.9725E-02		0.1793E-07
0.1037E-01	.0013	0.2387E-07
0.1100E-01		0.1827E-07
0.1165E-01	.001275	0.2349E-07

Table II

16 $\times$ 16 grid

time	last period	x_r
0.0000		0.1500E-01
0.6250E-03		0.9974E-02
0.1325E-02	.001325	0.1486E-01
0.1975E-02		0.1046E-01
0.2600E-02	.001275	0.1463E-01
0.3275E-02		0.1064E-01
0.3900E-02	.0013	0.1423E-01
0.4550E-02		0.1078E-01
0.5225E-02	.001325	0.1400E-01
0.5825E-02		0.1104E-01
0.6525E-02	.0013	0.1392E-01
0.7150E-02		0.1131E-01
0.7775E-02	.00125	0.1378E-01
0.8450E-02		0.1140E-01
0.9050E-02	.001275	0.1355E-01
0.9725E-02		0.1146E-01
0.1037E-01	.001325	0.1335E-01
0.1100E-01		0.1159E-01
0.1167E-01	.0013	0.1332E-01

Table III

24 × 24 grid

time	last period	$\langle x^2 \rangle$
0.0000		0.3513E-07
0.6594E-03		0.1628E-07
0.1306E-02	.00131	0.3420E-07
0.1966E-02		0.1637E-07

Table IV

24 × 24 grid

time	last period	x_r
0.0000		0.1500E-01
0.6594E-03		0.1038E-01
0.1250E-02	.00125	0.1503E-01
0.1931E-02		0.1015E-01

Table V

density of kerosene	0.82 g/cc
density of air	0.0013 g/cc
surface tension (STP)	30 dynes/cm
viscosity of kerosene	1.8 centipoise
viscosity of air	0.018 centipoise
air velocity	100 or 120 m/s
initial droplet velocity	0.0 m/s
droplet radius	125 microns

Figure Captions

- Figure 1. A section of a triangular grid showing a) a material interface, b) a vertex-cell.
- Figure 2. A test problem for conservation of circulation. a) The initial flow pattern. b) The velocities after a half-time step. c) The velocities after the old rotator operator is applied.
- Figure 3. a) Definition of the angles θ^+ and θ^- for the diagonal line drawn from j to l . b) The angles θ'^+ and θ'^- formed by connecting the other quadrilateral diagonal.
- Figure 4. a) Vertex 4 isolated within a larger triangle before its removal. b) The larger triangle remaining after deletion of vertex 4 and three associated sides and triangles.
- Figure 5. The initial grid for the internal wave test problems.
- Figure 6. The period τ as a function of mesh size for the internal gravity wave test problem.
- Figure 7. The period τ as a function of mesh size for the internal capillary wave test problem.
- Figure 8. A composite of frames from a calculation of an $n = 2$ normal mode droplet oscillation with 12 vertices around the droplet: $\rho_e = 1$ g/cc, $\rho_d = 2$ g/cc, $\sigma = 30$ dynes/cm, $a = 0.0125$ cm. Each frame is 0.1×0.1 cm².
- Figure 9. A composite of frames from a calculation of an $n = 2$ normal mode droplet oscillation with 24 vertices around the droplet. Same conditions as in Fig. 7.
- Figure 10. A composite of frames from a calculation of an $n = 2$ normal mode droplet oscillation with 28 vertices around the droplet. Same conditions as in Fig. 7.
- Figure 11. The period as a function of mesh size for the droplet oscillation problem.
- Figure 12. Curvature as a function of vertex index around the drop in the 16×16 calculation. (1) Exact solution; (2) initial spline; (3) after one oscillation.

Figure 13. The moment $\langle x^2 \rangle$ as a function of time for two initializations: (1) all initial vertices on the Rayleigh drop; (2) initial refining vertices on the spline interpolant.

Figure 14. The moment $\langle x^2 \rangle$ as a function of time in the 16×16 calculation.

Figure 15. The position of the rightmost vertex, x_r , as a function of time in the 16×16 calculation.

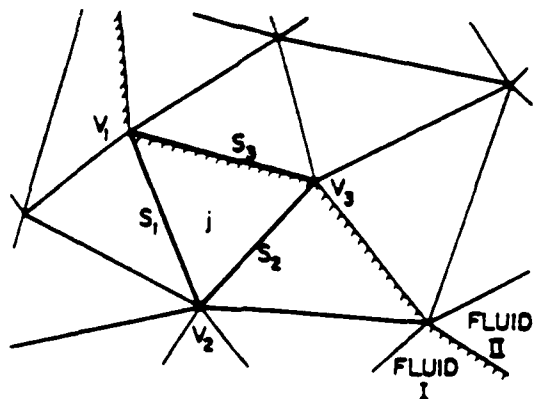
Figure 16. Pathlines from a calculation of air flowing past a deforming, viscous kerosene droplet. Surface tension forces are included at the material interface. Heads of pathlines are the current vertex positions and the tails are made up of the previous five positions. The flow speed is 100 m/s and $Re \approx 1600$.

Figure 17. Frames showing the triangular grid at the same times as shown for the pathlines in Fig. 14.

Figure 18. Pathlines from a calculation similar to that shown in Fig. 14, but with a flow velocity of 120 m/s and $Re \approx 2000$.

Figure 19. Frames showing the triangular grid at the same times as shown for the pathlines in Fig. 16.

a)



b)

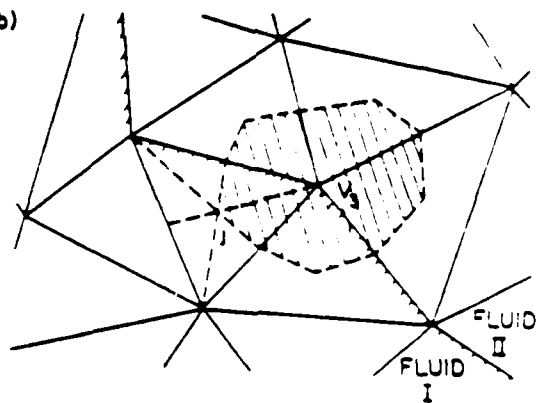


Figure 1

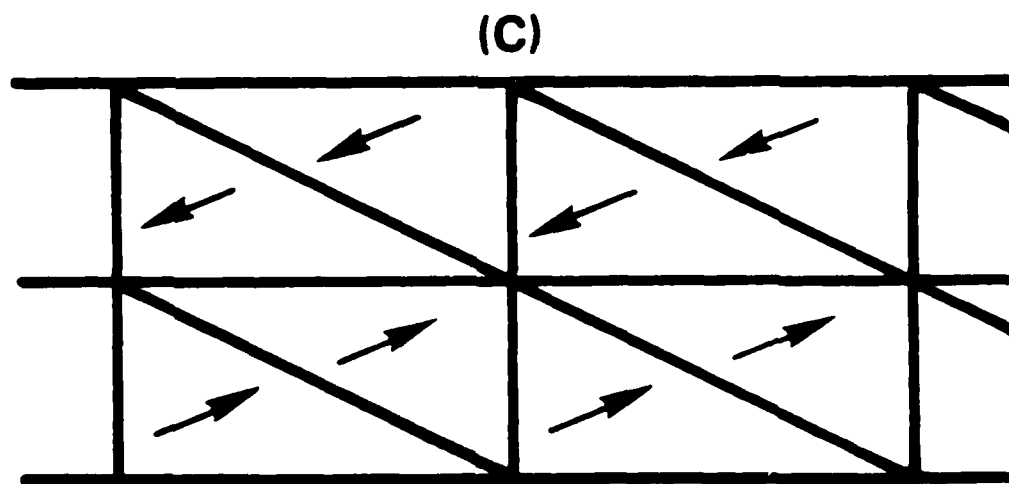
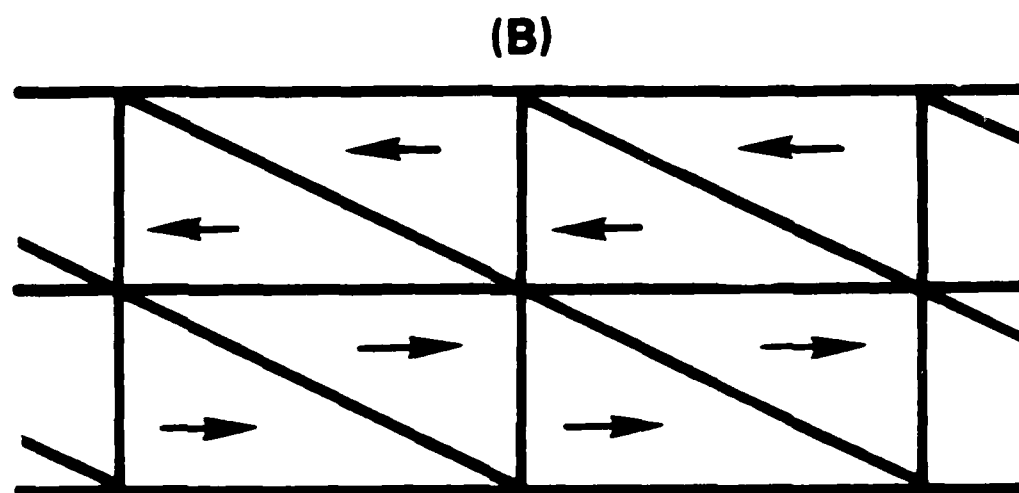
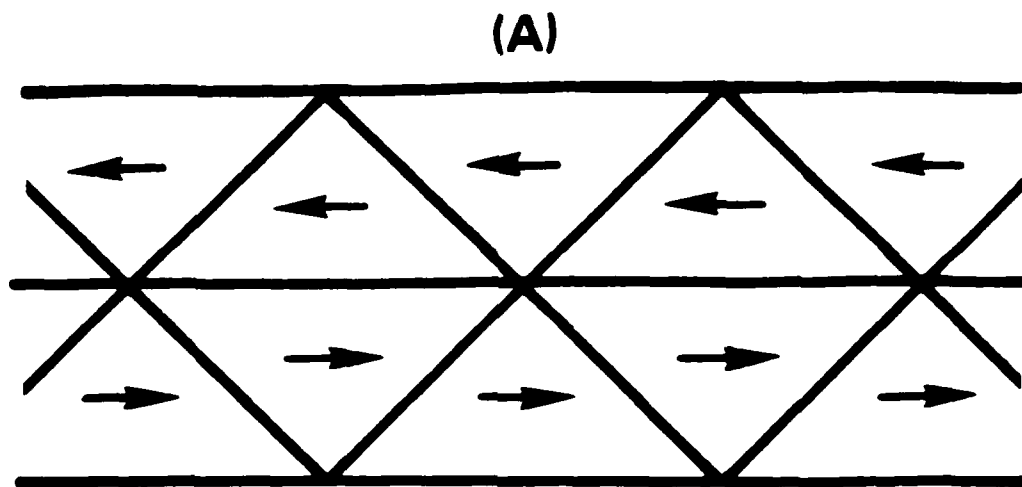


Figure 2

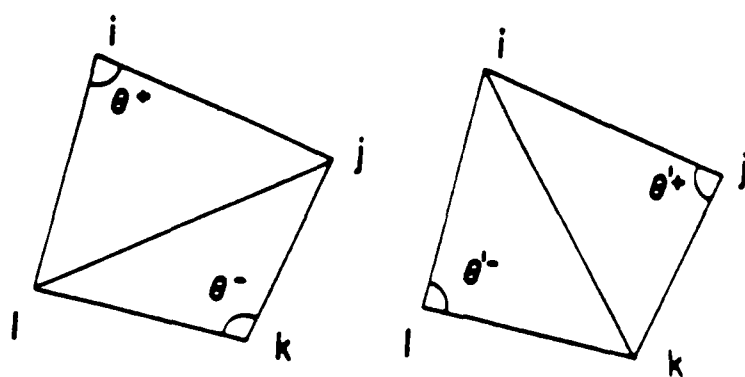


Figure 3

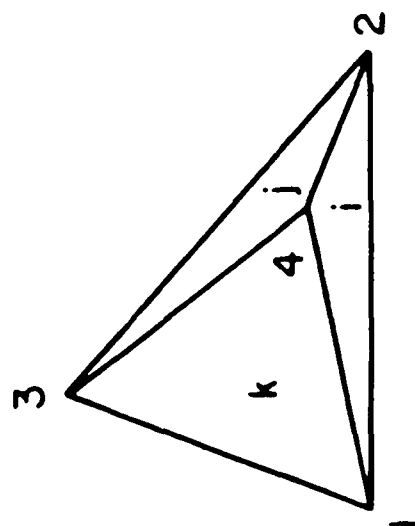
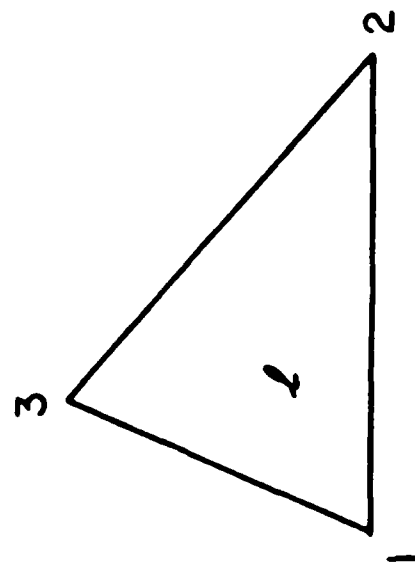


Figure 4

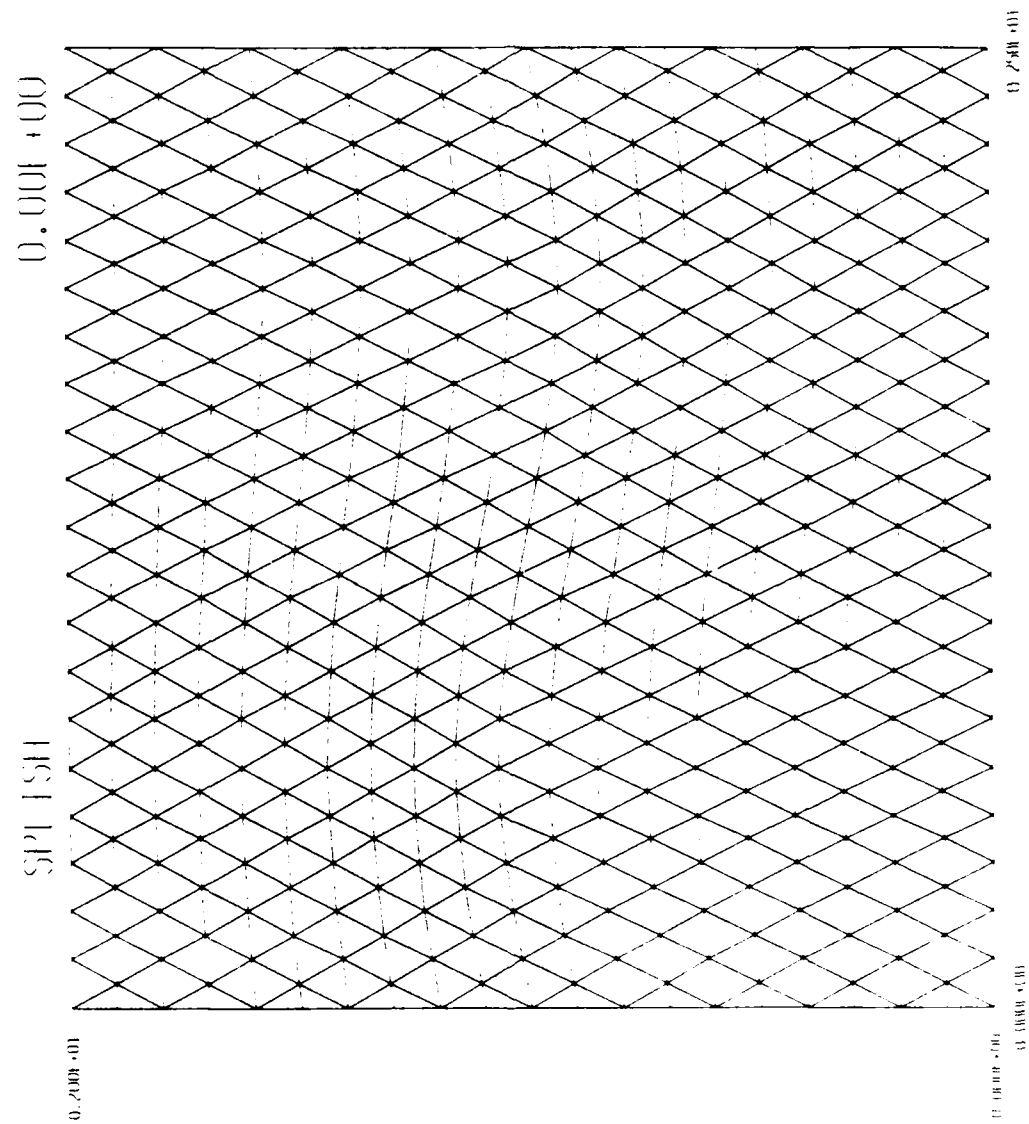


Figure 5.

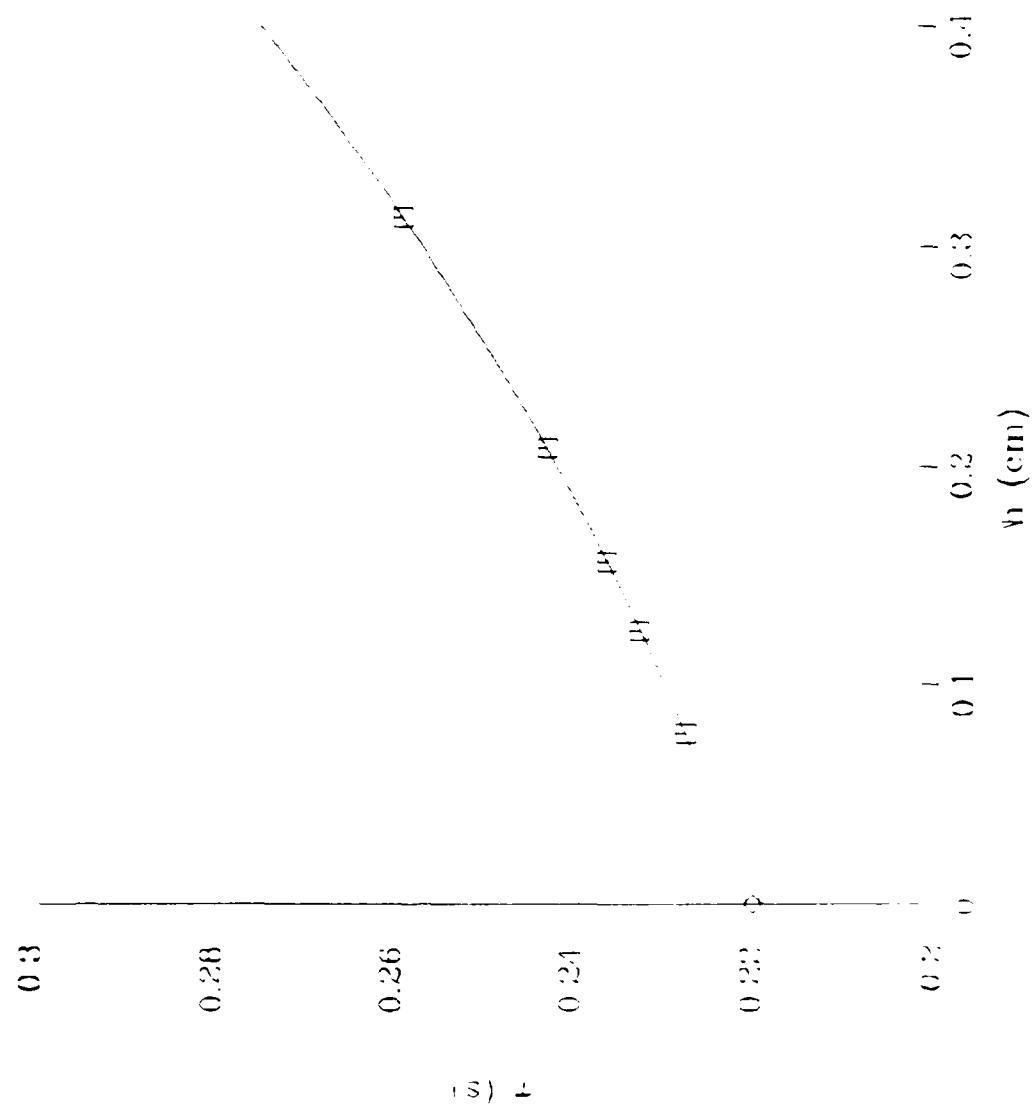


Figure 6.

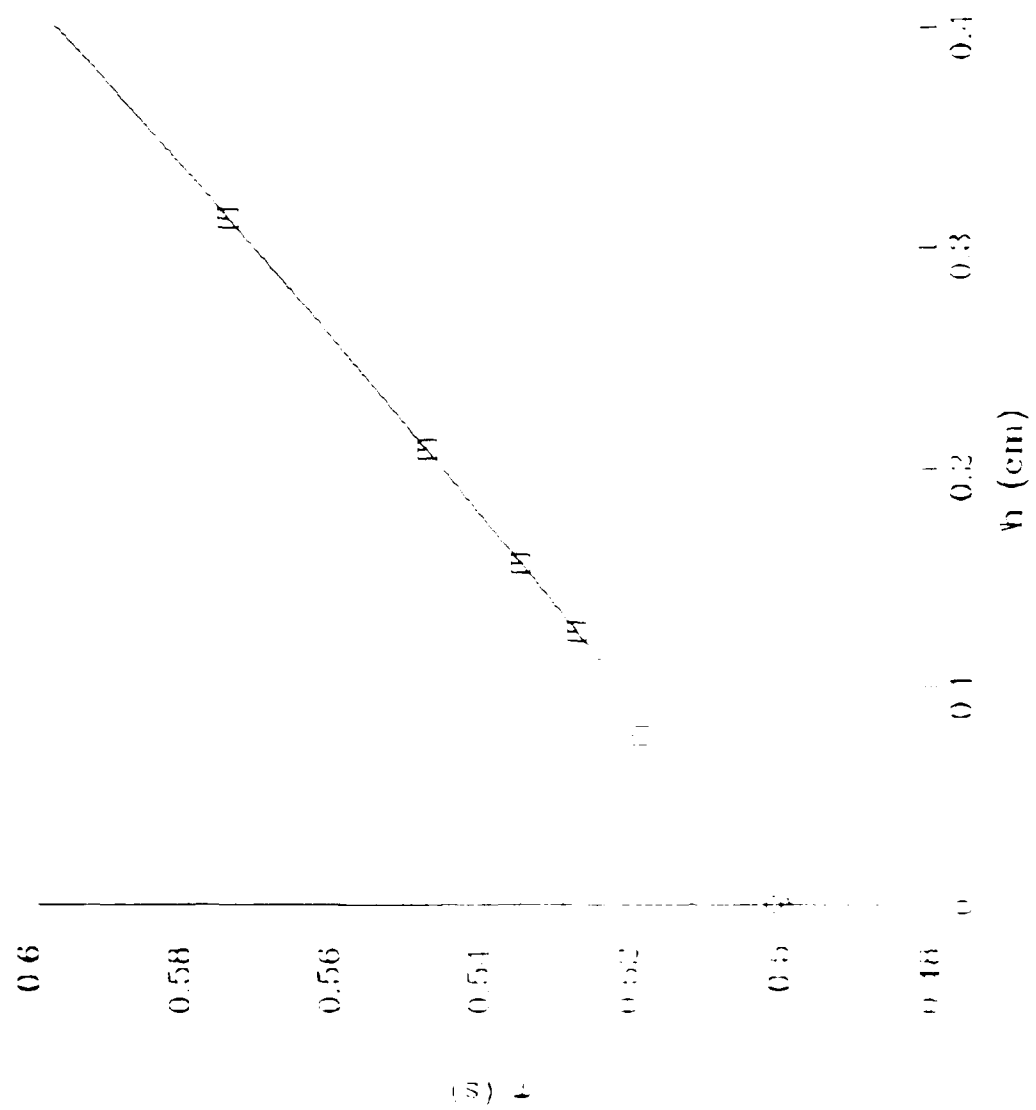


Figure 7.

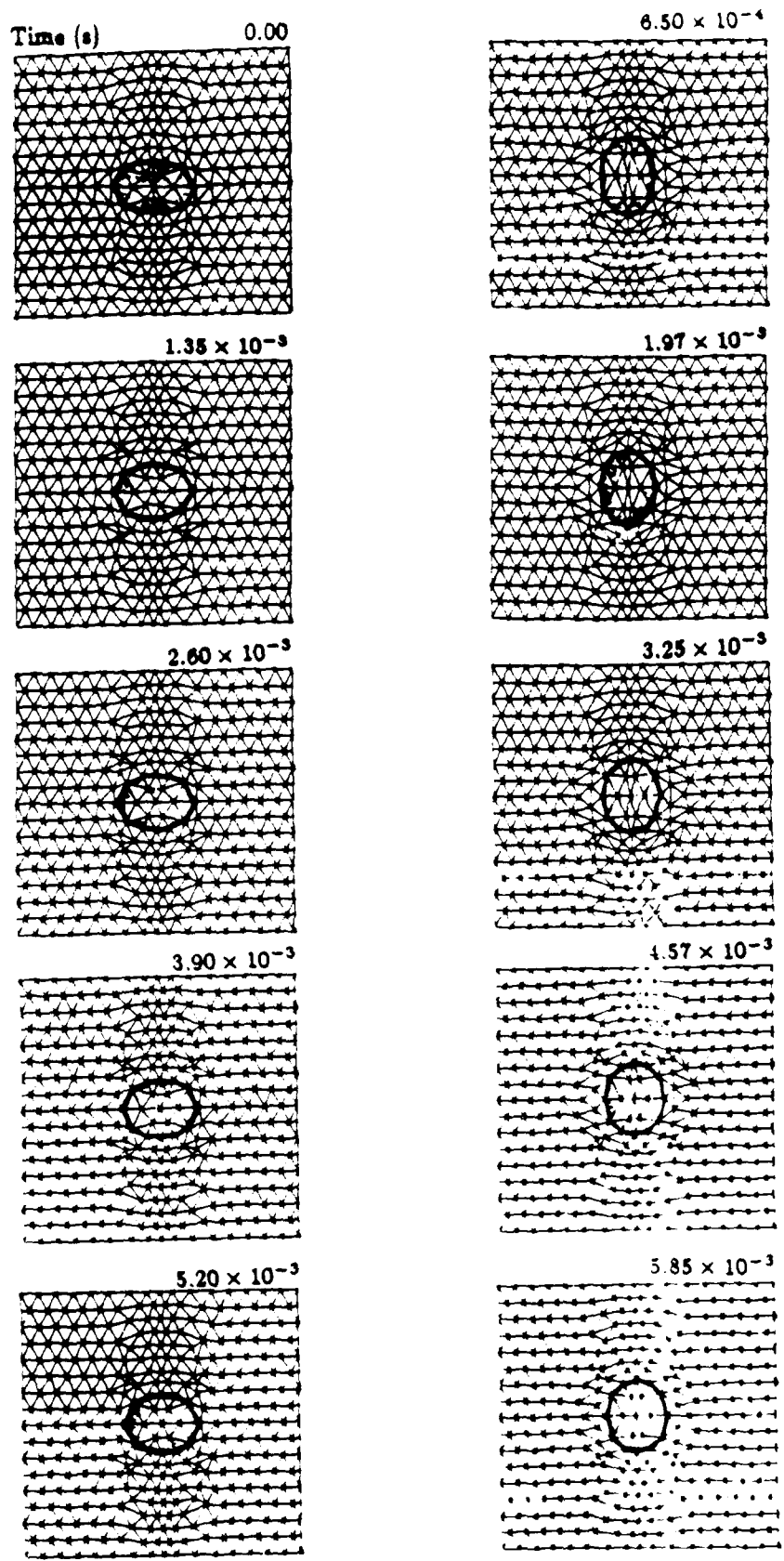


Figure 8.

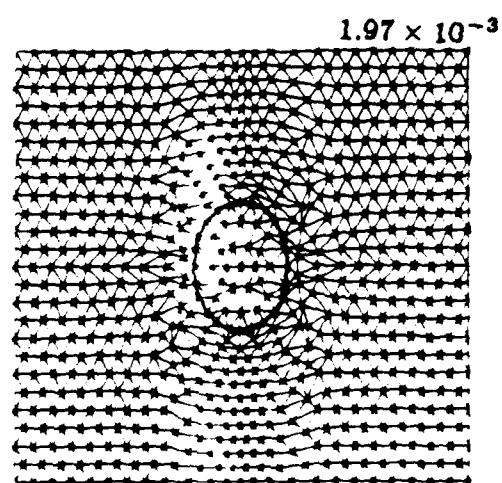
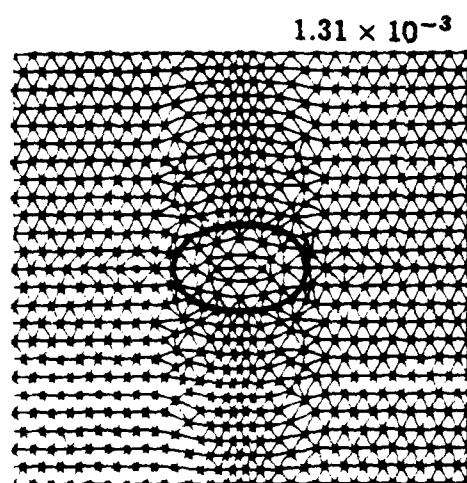
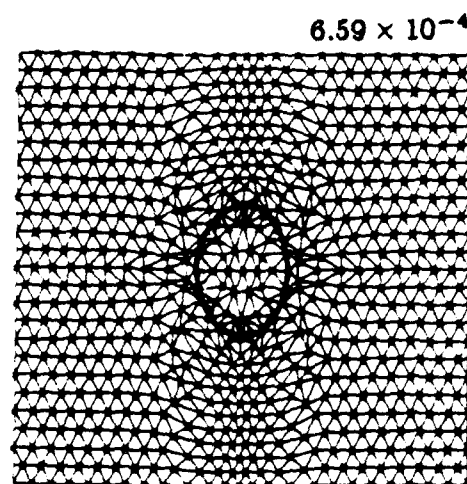
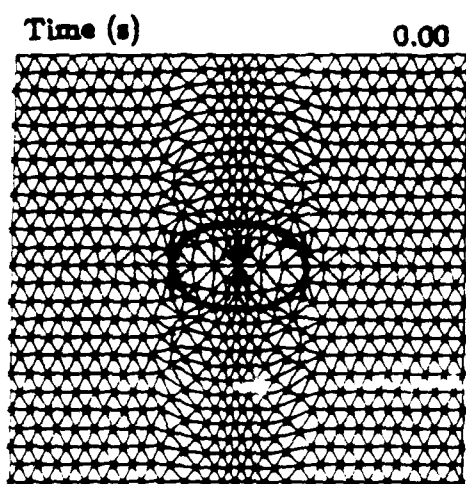
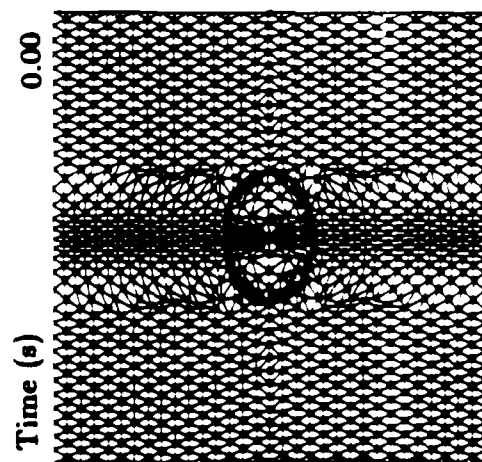
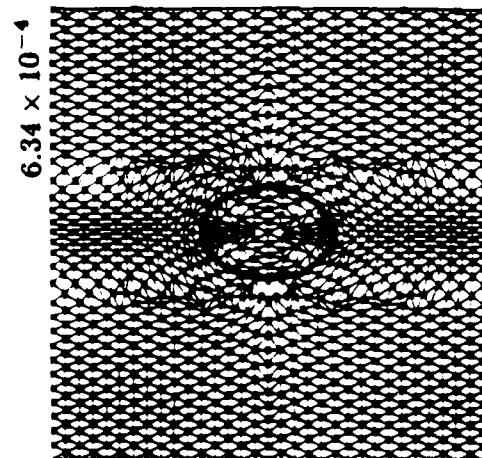
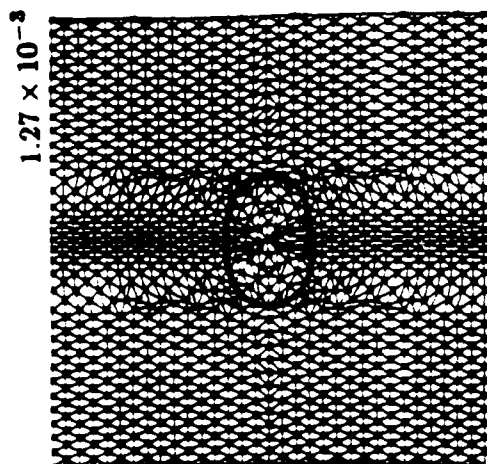


Figure 9.



Time (s)

Figure 10.

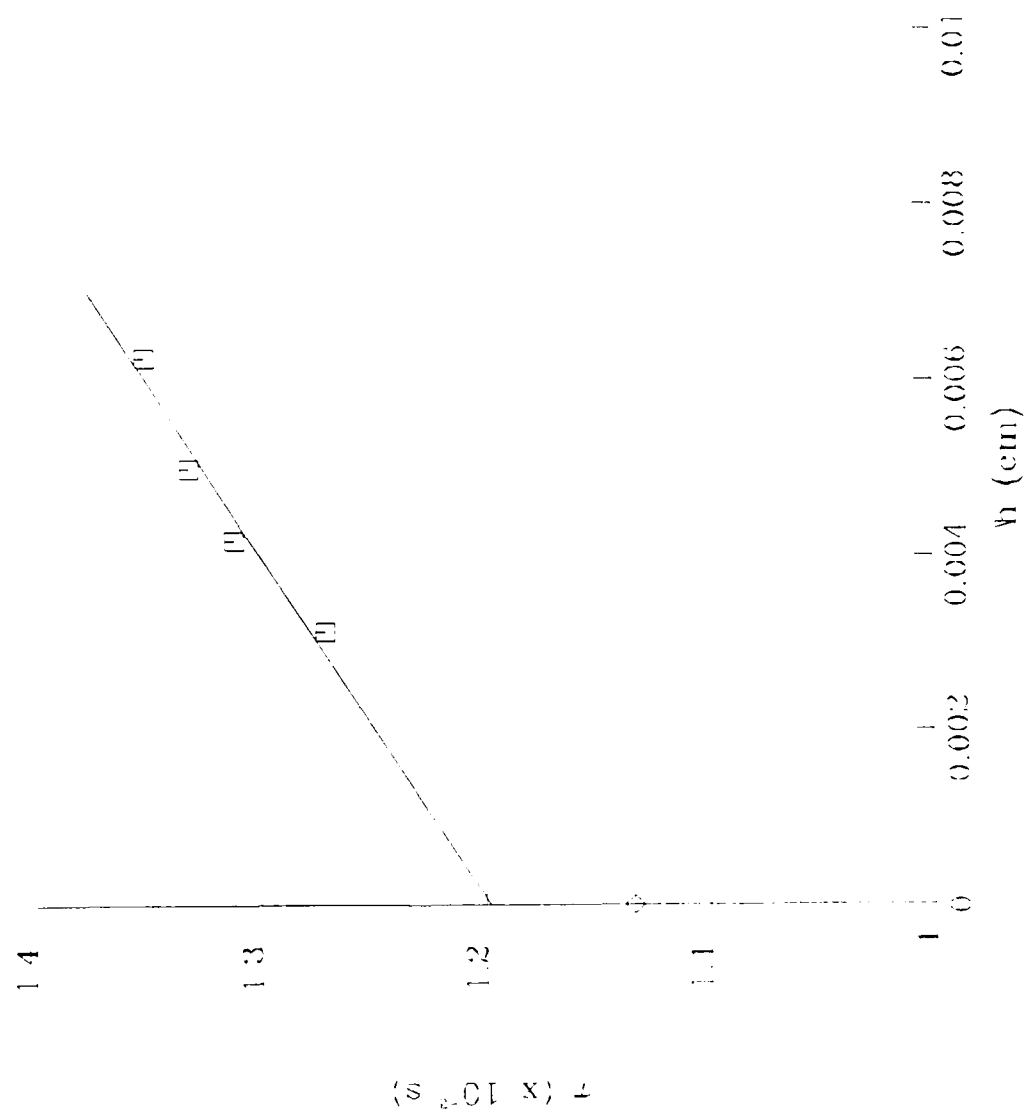


Figure 11.

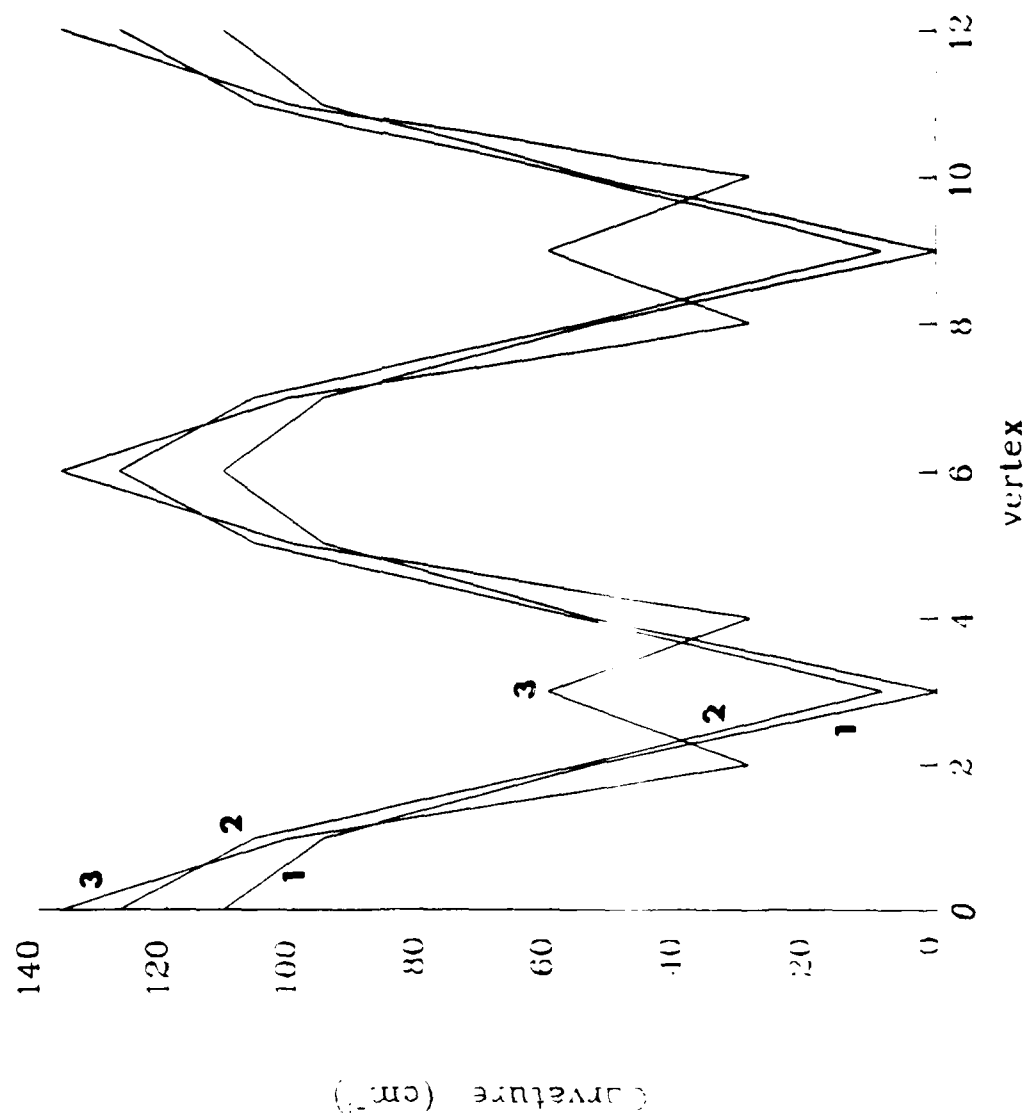


Figure 12.

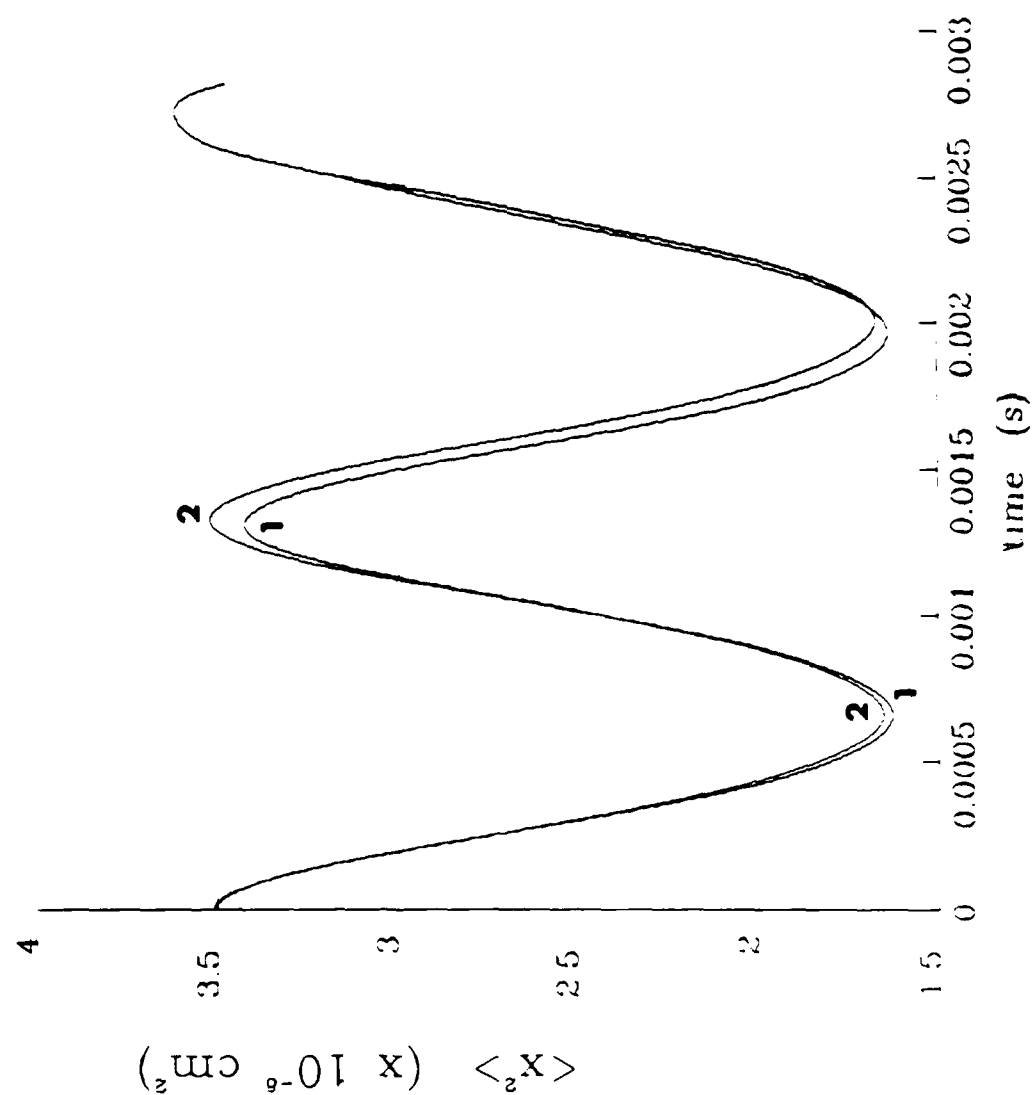


Figure 13.

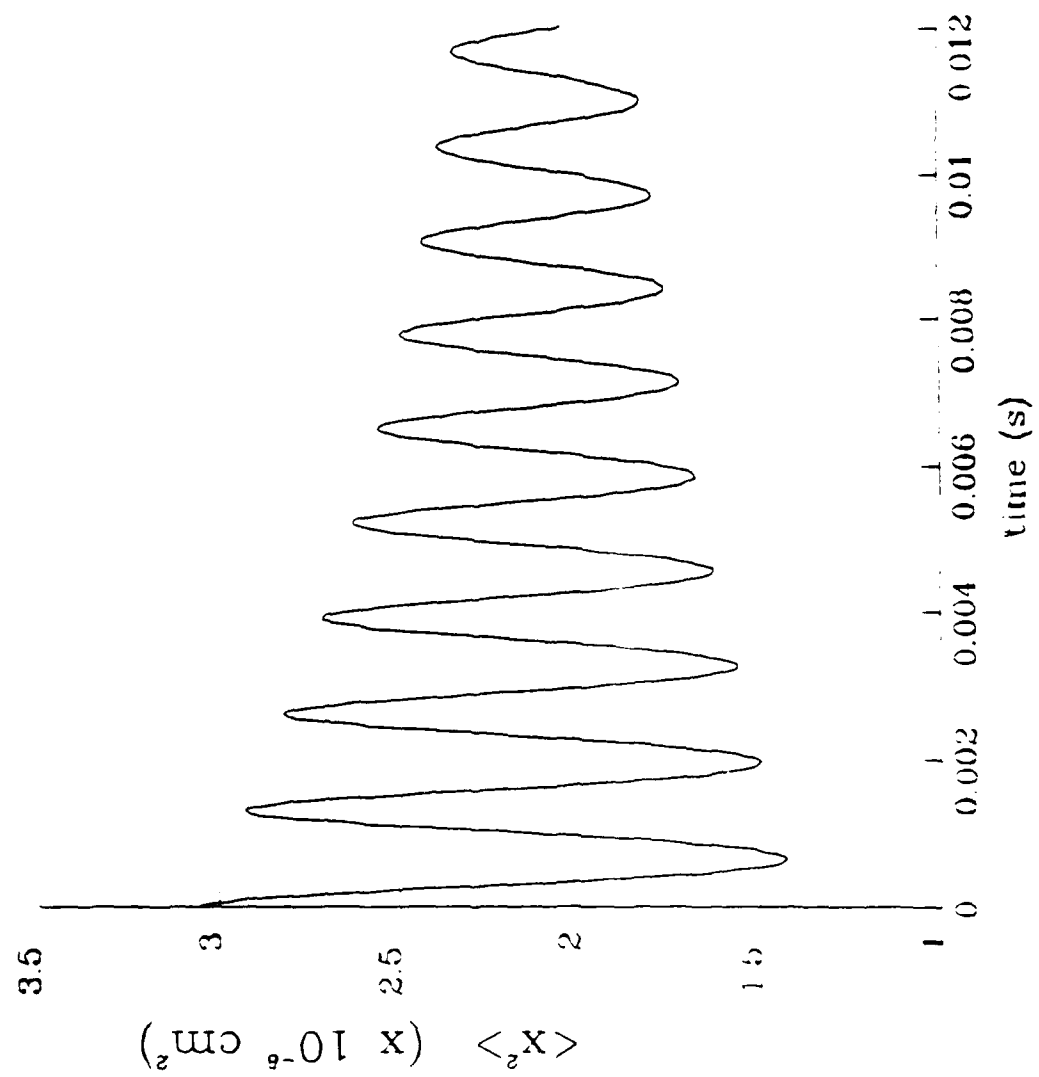


Figure 14.

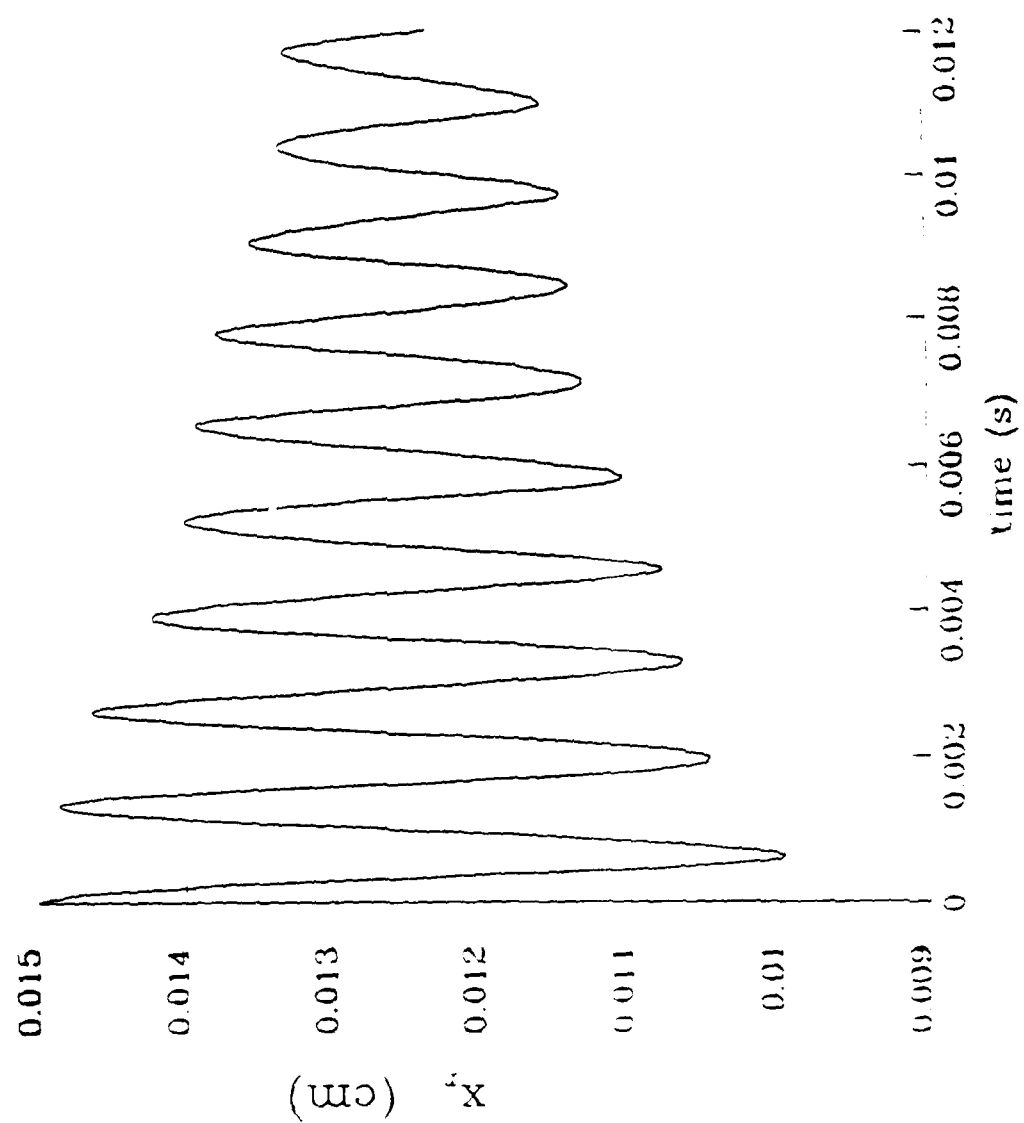


Figure 15.

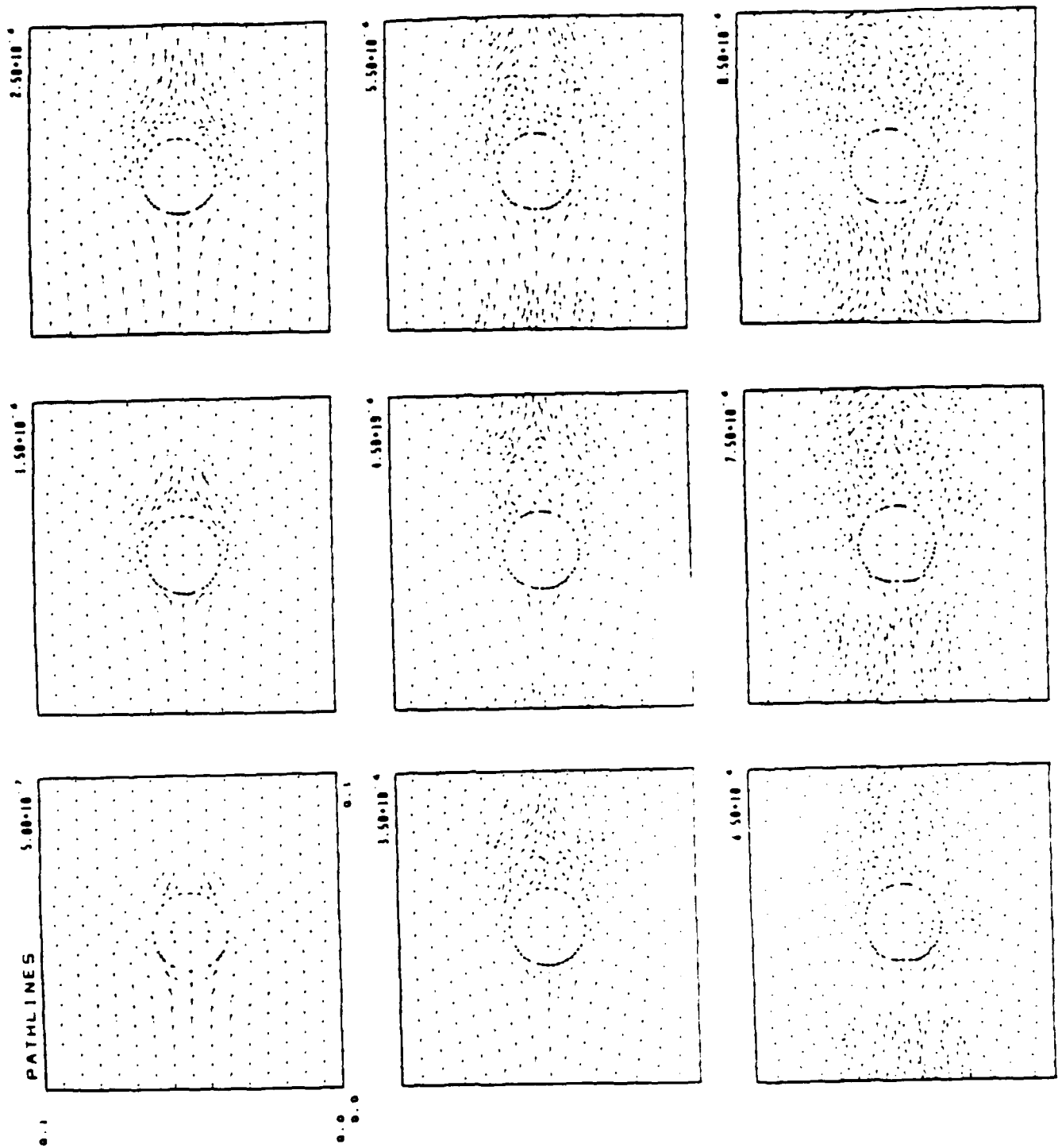


Figure 16.

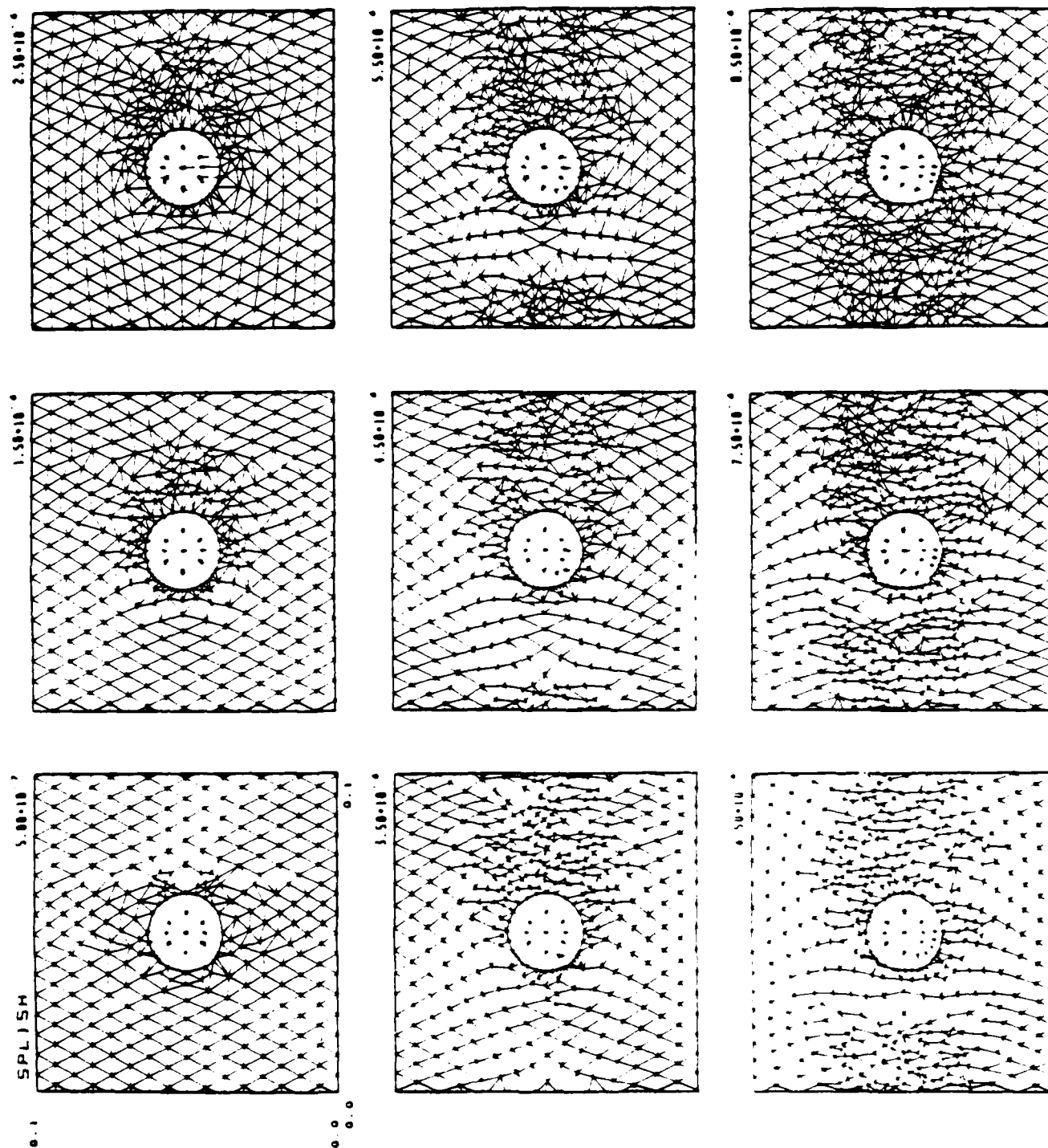


Figure 17.

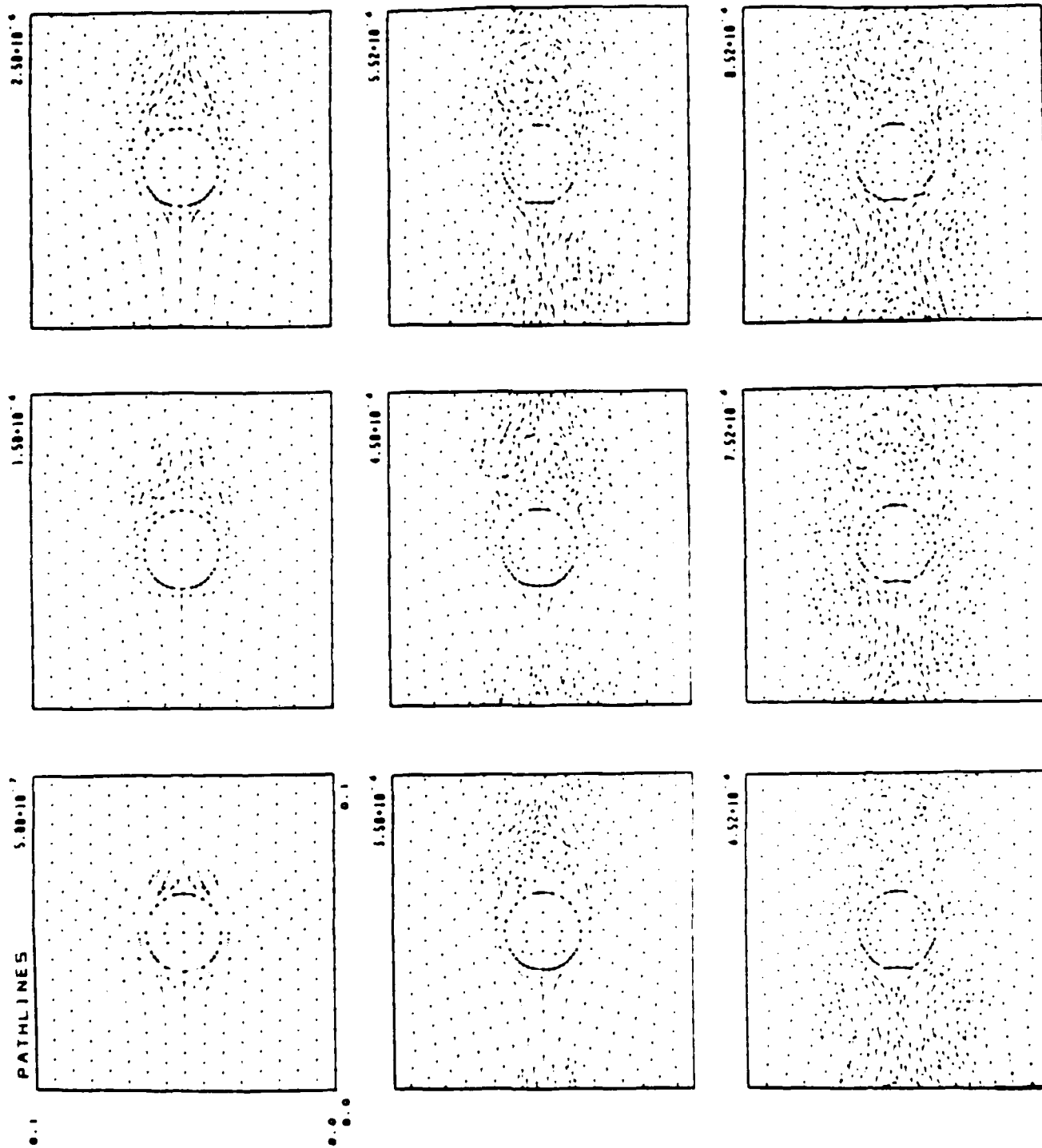


Figure 18.

Consider an inviscid shear flow on the grid shown in Fig. 2a. Triangles above $y = 0$ have a velocity $v_x \approx -1$, and those below have a velocity $v_x \approx +1$. If after one step the vertices have moved as in Fig. 2b, conservation of circulation through Eq. (2.18) imparts a y -component to the velocities for those triangles bordering the shear. Although the circulation integral about each vertex in the grid is conserved, the flow is now no longer independent of y .

To obtain a better formulation of the transformation $\mathbf{R}$ we must consider Eq. (2.17) more carefully. Since Eq. (2.17) is linear in the unknowns $\{\tilde{\mathbf{v}}_i\}$, we can obtain the change in triangle velocities by considering the change produced by the movement of a single vertex c , with coordinates $\mathbf{r}_c$, and sum the resultant expression over all vertices. It is reasonable to assume that the rotator should change only the velocities of the triangles which have c as a vertex. As a result, conservation of circulation gives

$$\tilde{\mathbf{v}}_{i+1/2} \cdot (\mathbf{r}_c^n - \mathbf{r}_{i+1}) + \tilde{\mathbf{v}}_{i-1/2} \cdot (\mathbf{r}_{i-1} - \mathbf{r}_c^n) = \mathbf{v}_{i+1/2} \cdot (\mathbf{r}_c^o - \mathbf{r}_{i+1}) + \mathbf{v}_{i-1/2} \cdot (\mathbf{r}_{i-1} - \mathbf{r}_c^o) \quad (2.19)$$

for each vertex i about c . We have used $\mathbf{r}_i = \mathbf{r}_i^n = \mathbf{r}_i^o$ for those vertices which are stationary. If only vertex c moves, the cell area at vertex c is constant, so that vorticity is conserved about vertex c as well. However, at all neighboring vertices, circulation, not vorticity, is conserved. By introducing the notation

$$\delta \mathbf{v}_{i+1/2} \equiv \tilde{\mathbf{v}}_{i+1/2} - \mathbf{v}_{i+1/2}$$

and

$$\delta \mathbf{r}_c \equiv \mathbf{r}_c^n - \mathbf{r}_c^o,$$

Eq. (2.19) may be rewritten as

$$\delta \mathbf{v}_{i+1/2} \cdot (\mathbf{r}_c^n - \mathbf{r}_{i+1}) + \delta \mathbf{v}_{i-1/2} \cdot (\mathbf{r}_{i-1} - \mathbf{r}_c^n) = (\mathbf{v}_{i-1/2} - \mathbf{v}_{i+1/2}) \cdot \delta \mathbf{r}_c. \quad (2.20)$$

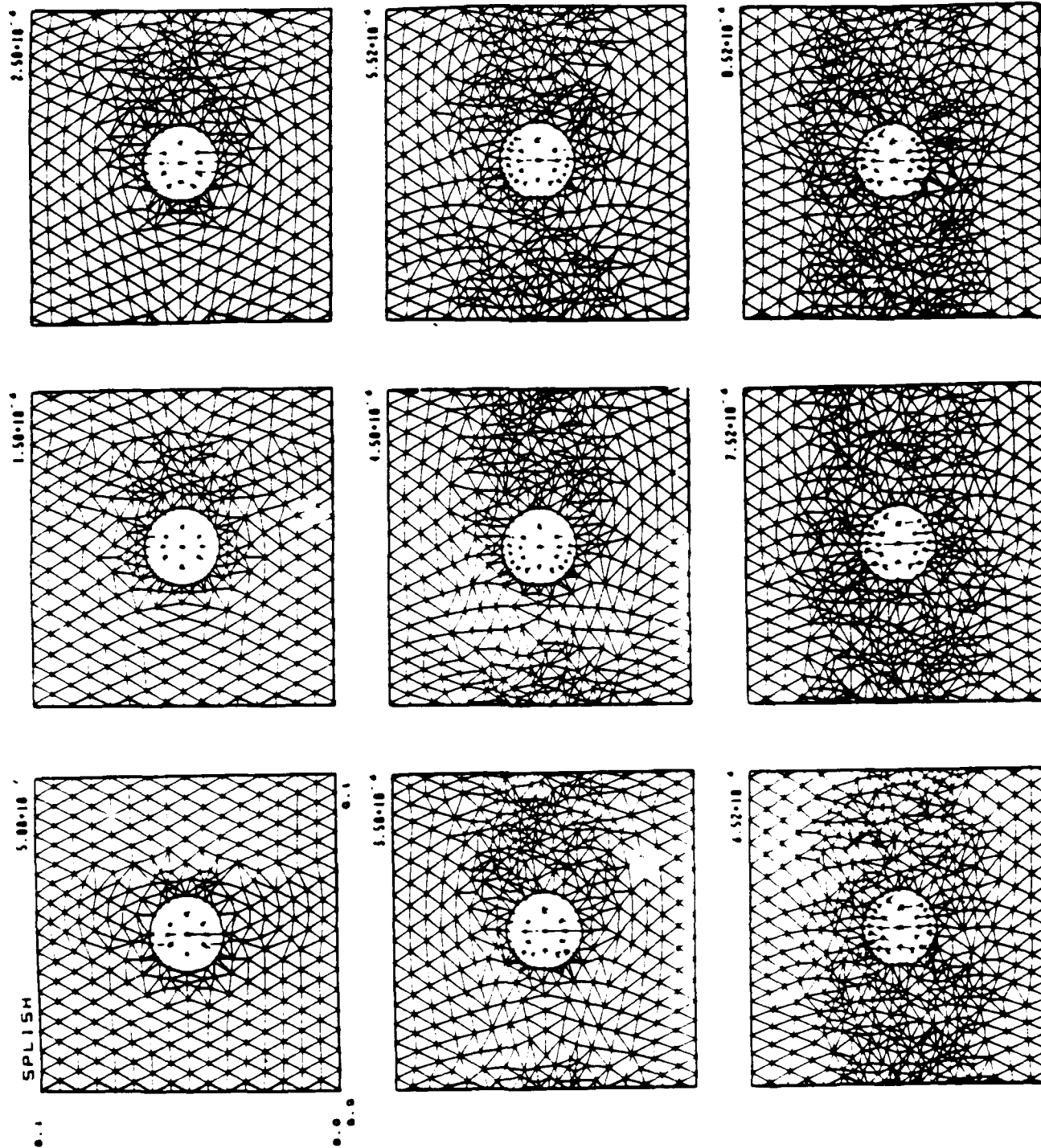


Figure 19.

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