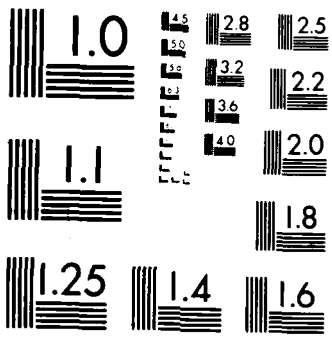
AD-A192 838 ON THE EXEEDANCE RANDOM MEASURES FOR STATIONARY 1/1
PROCESSES(U) NORTH CAROLINA UNIV AT CHAPEL HILL CENTER
FOR STOCHASTIC PROC M R LEADBETTER NOV 87 TR-215
UNCLASSIFIED AFOSR-TR-88-0350 F49620-85-C-0144 F/G 12/3 NL


1.0
2.8
2.5
3.2
2.2
3.6
4.0
2.0
1.1
1.8
1.25
1.4
1.6

DTIC FILE COPY 2

AD-A192 838

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: Unclassified	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release; distribution unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S): Technical Report No. 215	5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 88-0350
6a. NAME OF PERFORMING ORGANIZATION: University of North Carolina / 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION: AFOSR/NM
6c. ADDRESS (City, State, and ZIP Code): Statistics Dept. 321-A Phillips Hall 039-A Chapel Hill, NC 27514	7b. ADDRESS (City, State, and ZIP Code): AFOSR/NM Bldg 410 Bolling AFB DC 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION: AFOSR / 8b. OFFICE SYMBOL (If applicable): NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: ~~AFOSR~~ No. F49620 85C 0144.
8c. ADDRESS (City, State, and ZIP Code): AFOSR/NM Bldg 410 Bolling AFB DC 20332-6448	10. SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO.
61102F	2304	A5	

11. TITLE (Include Security Classification): On the exceedance random measures for stationary processes

12. PERSONAL AUTHOR(S): Leadbetter, M.R.

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
Preprint	FROM 9/87 TO 8/88	Nov. 1987	22

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Key Words & Phrases: N/A

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Two common approaches to extremal theory for stationary processes involve (a) consideration of point processes of upcrossings of high levels and (b) use of the total exceedance time above such levels. The approach (a) yields a greater variety of interesting results regarding the "global" and local maxima, but requires more by way of regularity conditions on the sample paths, than does the approach (b). ~~In~~ this work ~~we~~ combines both approaches by consideration of the "exceedance random measure" thereby obtaining general results under weak conditions on the sample functions. These include previously known results in the case where more sample function regularity is assumed.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION: Unclassified/unlimited
22a. NAME OF RESPONSIBLE INDIVIDUAL: Maj. Brian Woodruff	22b. TELEPHONE (Include Area Code): (202) 767-5026 / 22c. OFFICE SYMBOL: NM

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE: Unclassified/Unlimited

AFOSR-TR- 88-0350

CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

ON THE EXCEEDANCE RANDOM MEASURES

FOR STATIONARY PROCESSES

by

M. R. Leadbetter

Technical Report No. 215

November 1987

141. T. Hsing, On the intensity of crossings by a shot noise process, July 86. Adv. Appl Probab. 19, 1987, 743-745.

142. V. Mandrekar, On a limit theorem and invariance principle for symmetric statistics, July 86.

143. O. Kallenberg, Some new representations in bivariate exchangeability, July 86

144. B.G. Nguyen, Correlation length and its critical exponents for percolation processes, July 86. J. Stat. Phys., to appear.

145. G. Kallianpur and V. Perez-Abreu, Stochastic evolution equations with values on the dual of a countably Hilbert nuclear space, July 86. Appl. Math. Optimization, to appear.

146. B.G. Nguyen, Fourier transform of the percolation free energy, July 86.

147. A.M. Hasofer, Distribution of the maximum of a Gaussian process by Monte Carlo, July 86. J. Sound Vibration, 112, 1987, 283-293.

148. T. Norberg, On the existence of probability measures on continuous semi-lattices, Aug. 86.

149. G. Samorodnitsky, Continuity of Gaussian processes, Aug. 86. Ann. Probability, to appear.

150. T. Hsing, J. Husler, and M.R. Leadbetter, Limits for exceedance point processes, Sept. 86. Prob. Theory and Related Fields, to appear.

151. S. Cambanis, Random filters which preserve the stability of random inputs, Sept. 86 Adv. Appl. Probability, 1987, to appear.

152. O. Kallenberg, On the theory of conditioning in point processes, Sept. 86. Proceedings for the First World Congress of the Bernoulli Society, Tashkent, 1986.

153. G. Samorodnitsky, Local moduli of continuity for some classes of Gaussian processes, Sept. 86.

154. V. Mandrekar, On the validity of Beurling theorems in polydiscs, Sept. 86.

155. R.F. Serfozo, Extreme values of queue lengths in M/G/1 and GI/M/1 systems, Sept. 86

156. B.G. Nguyen, Gap exponents for percolation processes with triangle condition, Sept 86.

157. G. Kallianpur and R. Wolpert, Weak convergence of stochastic neuronal models, Oct. 86. Stochastic Methods in Biology, M. Kimura et al., eds., Lecture Notes in Biomathematics, 70, Springer, 1987, 116-145.

158. G. Kallianpur, Stochastic differential equations in duals of nuclear spaces with some applications, Oct. 86. Inst. of Math. & Its Applications, 1986.

159. G. Kallianpur and R.L. Karandikar, The filtering problem for infinite dimensional stochastic processes, Jan. 87. Proc. Workshop on Stochastic Differential Systems, Stochastic Control Theory & Applications, Springer, to appear.

160. V. Perez-Abreu, Multiple stochastic integrals and their extensions to nuclear space valued Wiener processes, Oct. 86. Appl. Math. Optimization, to appear.

161. R.L. Karandikar, On the Feynman-Kac formula and its applications to filtering theory, Oct. 86. Appl. Math. Optimization, to appear.

162. R.L. Taylor and T.-C. Hu, Strong laws of large numbers for arrays or rowwise independent random elements, Nov. 86.

163. M. O'Sullivan and T.R. Fleming, Statistics for the two-sample survival analysis problem based on product limit estimators of the survival functions, Nov. 86.

164. F. Avram, On bilinear forms in Gaussian random variables, Toeplitz matrices and Parseval's relation, Nov. 86.

165. D.B.H. Cline, Joint stable attraction of two sums of products, Nov. 86. J. Multivariate Anal., to appear.

166. R.J. Wilson, Model fields in crossing theory-a weak convergence perspective, Nov 86

167. D.B.H. Cline, Consistency for least squares regression estimators with infinite variance data, Dec 86

168. L.L. Campbell, Phase distribution in a digital frequency modulation receiver, Nov 86

169. B.G. Nguyen, Typical cluster size for 2-dim percolation processes, Dec 86

170. H. Oodaira, Freidlein-Wentzell type estimates for a class of self-similar processes represented by multiple Wiener integrals, Dec. 86.

171. J. Nolan, Local properties of index-β stable fields, Dec 86 Ann Probability, to appear.

172. R. Menich and R.F. Serfozo, Optimality of shortest queue routing for dependent service stations, Dec 86

173. F. Avram and M.S. Taqqu, Probability bounds for M-Skorohod oscillations, Dec 86

174. F. Moricz and R.L. Taylor, Strong laws of large numbers for arrays of orthogonal random variables, Dec 86

175. G. Kallianpur and V. Perez-Abreu, Stochastic evolution driven by nuclear space valued martingales, Apr 87

ON THE EXCEEDANCE RANDOM MEASURES

FOR STATIONARY PROCESSES

by

M.R. Leadbetter

University of North Carolina

Summary: Two common approaches to extremal theory for stationary processes involve (a) consideration of point processes of upcrossings of high levels and (b) use of the total exceedance time above such levels. The approach (a) yields a greater variety of interesting results regarding the "global" and local maxima, but requires more by way of regularity conditions on the sample paths, than does the approach (b).

In this work we combine both approaches by consideration of the "exceedance random measure" thereby obtaining general results under weak conditions on the sample functions. These include previously known results in the case where more sample function regularity is assumed.

Research supported by the Air Force Office of Scientific Research under Contract No. F49620 85C 0144.

1. Introduction

Two approaches have been used to obtain theory surrounding the asymptotic distribution of the maxima

$$M(T) = \sup\{\xi_t : 0 \leq t \leq T\} \tag{1.1}$$

of a stationary process $\{\xi_t : t \geq 0\}$, as $T \to \infty$. The first of these involves upcrossings of high levels, using the simple connection

$$\{M(T) \leq u\} \subset \{N_u(T) = 0\} \subset \{M(T) \leq u\} \cup \{\xi(0) > u\}$$

if $N_u(T)$ denotes the number of upcrossings of a level u by ξ_t in $0 \leq t \leq T$. It may be seen from this (cf. [9] and references therein) that the limiting distribution of M(T) is intimately connected with the asymptotic Poisson character of point processes of high level upcrossings. This approach must be modified when the sample functions are so irregular that upcrossings do not form a point process and this may be done by use of the so called ϵ-upcrossings of Pickands [10].

The second approach to extremal theory for M(T), employed by Berman (cf. [1]) uses the exceedance time $L_T(u) = \int_0^T 1_{(\xi_t > u)} dt$, and the immediate equivalence $P\{M(T) \leq u\} = P\{L_T(u) = 0\}$. While the "upcrossing framework" provides a greater variety of associated results (e.g. concerning k^{th} largest local maxima), the use of exceedance times requires very little by way of sample function regularity.

In this paper we explore a simple extension to the notion of exceedance time, namely the exceedance times in arbitrary Borel sets, or "exceedance random measure". This may be defined under the same minimal conditions as $L_T(u)$ but gives new and more detailed results involving upcrossings when the sample functions are more regular. In fact in such cases the limiting random measure represents both the positions of high upcrossing points and the lengths of the high level exceedances thus initiated.

Specifically it will be convenient to consider the random measure (r.m.) ζ_T defined for Borel subsets B of (0,1] as the amount of time in TB for which $\xi_t > u_T$, where $\{u_T: T \geq 0\}$ is a given family of constants, viz.

$$\zeta_T(B) = \int_{T.B} 1_{(\xi_t > u_T)} dt. \tag{1.2}$$

For convenience we assume throughout that the underlying probability space is complete, and that ξ_t has a.s. continuous sample paths (and hence in particular is a measurable process). Clearly $\zeta_t((0,1]) = L_T(u_T)$, the previously defined exceedance time.

Our primary interest *concerns* distributional limits for the random measure $a_T\zeta_T$ as $T \to \infty$, (for suitable constants a_T) when the levels u_T from a "family of normalizers for the maximum M(T)", in the sense that $P\{M(T) \leq u_T\}$ has a non-zero limit. To obtain non-trivial results, it is clearly necessary to restrict the long range dependence in the process to some degree. This will be done by an assumption "$\Delta(u_T)$" of similar type but significantly weaker than strong mixing. This will be discussed in Section 2, and some basic lemmas proved.

Section 3 contains the main results of the paper-characterizing the possible random measure limits for $a_T\zeta_T$ as a class of Lévy Processes of Compound Poisson form (with general type of multiplicity distribution), and giving sufficient conditions for convergence.

Section 4 concerns families of levels $u_T(\tau)$ parametrized by the quantity τ such that $P\{M(T) \leq u_T\} \to e^{-\tau}$, it being shown that convergence of $a_T\zeta_T$ for one such level implies its convergence for all such levels. Finally in Section 5 we interpret the limiting r.m. in terms of high level upcrossings and exceedance times when the sample functions are more regular, and illustrate the theory by obtaining explicit results for stationary normal processes.

In some respects our development parallels that for high level exceedances in discrete time considered in [4], and we have made some technical simpliciations which could also have been used in the discrete time case. But the more essential differences arise from the fact that random measures rather than point processes are considered, with consequent problems of "lack of tightness". In particular a case specifically excluded from [4] where convergence of the so-called "exceedance point process" occurs after multiplication by normalizing factors tending to zero, may be treated using the present methods.

2. Framework and basic lemmas.

The basic dependence condition to be used throughout is an obvious continuous time version of a weak mixing condition used in discrete time (e.g. [4]). Specifically let $\{u_T\}$ be a family of constants and write $B^T_{s,t}=\sigma\{(\xi_v \leq u_T),\ s \leq v \leq t\}$ where $\sigma(\cdot)$ denotes the genrated σ-field. Write also

$$\alpha_{T,\ell} = \sup\{|P(A\cap B)-P(A)P(B)|:\ A \in B^T_{0,s},\ B \in B^T_{s+\ell,T},\ s\geq 0,\ \ell+s\leq T.$$

Then we say that the stationary process ξ_t satisfies the condition $\Delta(u_T)$ if

$$\alpha_{T,\ell_T} \to 0 \quad \text{as } T \to \infty, \text{ for some } \ell_T=o(T). \tag{2.1}$$

The condition Δ is often applied through the following lemma essentially given in [11].

Lemma 2.1 Let $\beta_{T,\ell} = \sup\{|\mathcal{E}YZ - \mathcal{E}Y\mathcal{E}Z|: Y,Z\ B^T_{0,s}\text{-},\ B^T_{s+\ell,T}\text{- measurable}$ respectively, $|Y|, |Z| \le 1,\ \ell,s \ge 0,\ \ell+s\le T\}$. Then $\alpha_{T,\ell} \le \beta_{T,\ell} \le 4\alpha_{T,\ell}$ so that $\Delta(u_T)$ holds iff $\beta_{T,\ell_T} \to 0$ for some $\ell_T = o(T)$. □

The following result show how the Δ condition implies approximate independence of the Laplace transform of ζ_T in appropriately chosen disjoint intervals. Results of this type have been used in various forms (cf. [7]) and the present statement corresponds closely to the general discrete time version by Hsing ([4]). A proof will be given since a slightly more general statement is given than is covered by a direct transposition of that in [4], and some notational simplification is possible in the continuous time context. A corresponding result for the maxima will be obtained as a corollary. Here and throughout $m(\cdot)$ will denote Lebesgue measure. Use will be made at various points of the inequality

$$(2.2)\qquad \left|\prod_1^k y_i - \prod_1^k x_i\right| \le \sum_1^k |y_i - x_i|,\ 0 \le x_i,\ y_i \le 1.$$

Lemma 2.2 Let $\Delta(u_T)$ hold and $\{k_T\}$ be integers such that

$$(2.3)\qquad k_T\ell_T/T \to 0 \qquad k_T\alpha_{T,\ell_T} \to 0$$

where ℓ_T is as in (2.1). Let $J_i = (J_{T,i})\ 1 \le i \le k_T$, be disjoint subintervals of $(0,1)$ with $J(=J(T)) = \bigcup_1^{k_T} J_i$, and $\{a_T\}$ positive constants. Let f be a bounded non-negative measurable function such that $f(x) \ge \alpha > 0$ on a non-degenerate interval $I \subset (0,1]$ and suppose that $Tm(I \cap J)/(k_T\ell_T) \to \infty$. Then

$$(2.4)\qquad \gamma_T = \mathcal{E}\exp(-a_T\textstyle\int_J f d\zeta_T) - \prod_1^{k_T} \mathcal{E}\exp(-a_T\int_{J_i} f d\zeta_T) \to 0 \quad \text{as } T \to \infty.$$

Proof: It is sufficient to show that any convergent subsequence $\{\tau_T: T \in S\}$ of $\{\tau_T\}$ has limit zero. Write

$$G\ (=G_T) = \{i,\ 1 \le i \le k_T:\ m(J_i) > \ell_T/T\}$$
$$H\ (=H_T) = \{i,\ 1 \le i \le k_T:\ m(J_i) \le \ell_T/T\}$$

The intervals J_i may be open or closed at either endpoint, but for definiteness we shall regard them as semiclosed and write $J_i=(\alpha_i,\beta_i]$, and for $i \in G$ define $I_i = (\alpha_i,\ \beta_i-\ell_i/T]$, $I_0^* = (0,\ell_T/T]$. Note that by stationarity $\zeta_T(I_i^*)$ has the same distribution as $\zeta_T(I_0^*)$ for each $i \in G$.

Let A be an upper bound for f and suppose first that

(a) $\mathcal{E}^{k_T}\exp(-a_T A\zeta_T(I_0^*)) \to 1$ as $T \to \infty$ through S.

Taking limits through S, and using stationarity and (2.2),

$$(2.5)\qquad |\mathcal{E}\exp(-a_T \sum_{i\in G}\int_{J_i} f d\zeta_T) - \mathcal{E}\exp(-a_T\int_J f d\zeta_T)|$$
$$\le \sum_{j\in H}\mathcal{E}(1-\exp(-a_T\int_{J_i} f d\zeta_T))$$
$$\le k_T\ \mathcal{E}\{1-\exp(-a_T A\zeta_T(I_0^*))\}$$

This expression tends to zero by taking logs in assumption (a).

It follows in a very similar way that

$$(2.6)\qquad \mathcal{E}\exp(-a_T \sum_{i\in G}\int_{I_i} f d\zeta_T) - \mathcal{E}\exp(-a_T \sum_{i\in G}\int_{J_i} f d\zeta_T) \to 0.$$

Also by an obvious induction from Lemma 2.1,

$$(2.7)\qquad |\mathcal{E}\exp(-a_{T_i} \sum_{i\in G}\int_{I_i} f d\zeta_T) - \prod_{i\in G}\mathcal{E}\exp(-a_T\int_{I_i} f d\zeta_T)| \le 4k_T\alpha_{T,\ell_T} \to 0$$

and by (2.2)

$$(2.8)\qquad \left|\prod_{i\in G} \mathcal{E}\exp(-a_T\textstyle\int_{I_i} fd\zeta_T) - \prod_{i\in G}\mathcal{E}\exp(-a_T\int_{J_i} fd\zeta_T)\right| \le \sum_{i\in G} \mathcal{E}(1-\exp(-a_T\textstyle\int_{I_i^*} fd\zeta_T)) \to 0$$

exactly as in (2.5). Finally

$$(2.9)\qquad \left|\prod_{i\in G} \mathcal{E}\exp(-a_T\textstyle\int_{J_i} fd\zeta_T) - \prod_{i=1}^{k_T}\mathcal{E}\exp(-a_T\int_{J_i} fd\zeta_T)\right| \le \sum_{i\in H} \mathcal{E}\{1-\exp(-a_T\textstyle\int_{J_i}\zeta_T)\}$$

which again tends to zero as in (2.5).

It thus follows by combining (2.5)-(2.9) that if (a) holds then $\gamma_T \to 0$ as $T \to \infty$ through S.

(b) If (a) does not hold there is a subsequence S' of S such that $\mathcal{E}^{k_T}\exp(-a_T A\zeta_T(I_0^*)) \to c < 1$ as $T \to \infty$ through S'.

We have $f(x) \ge \alpha > 0$ for $x \in I$. Choose θ_T such that $\theta_T/k_T \to \infty$, $\theta_T\alpha_{T,\ell_T} \to 0$, $Tm(I\cap J)/(\theta_T\ell_T) \to \infty$, and write (with $[\cdot]$ denoting integer part)

$$\theta_{T,i} = [\theta_T m(J_i \cap I)/m(J\cap I)]$$

Clearly, since $k_T = o(\theta_T)$, $\sum_i \theta_{T,i} \sim \theta_T$ and $\theta_{T,i}(2\ell_T)/(Tm(J_i\cap I)) = 2\ell_T\theta_T/(Tm(I\cap J)) \to 0$ uniformly for all J_i intersecting I. Hence $\theta_{T,i}$ (≥ 0) subintervals E_{ij} of $(J_i \cap I)$ may be chosen of length ℓ_T/T, and mutually separated by at least ℓ_T/T, giving

$$\begin{aligned}\mathcal{E}\exp(-a_T\textstyle\int fd\zeta_T) &\le \mathcal{E}\exp(-a_T\displaystyle\sum_{i,j}\int_{E_{ij}} fd\zeta_t) \\ &\le \mathcal{E}^{\theta_T}\exp(-a_T\alpha\zeta_T(I_0^*)) + 4\theta_T\alpha_{T,\ell_T}\end{aligned}$$

which, using Hölder's Inequality, (noting $\alpha/A \le 1$) does not exceed

$$\mathcal{E}^{\theta_t \alpha/A} \exp(-a_T A \zeta_T(I_0^*)) + o(1) = \{\mathcal{E}^{k_T} \exp(-a_T A \zeta_T(I_0^*))\}^{\theta_T \alpha/(k_T A)} \to 0$$

since the inside term tends to $c < 1$ and $\theta_T/k_T \to \infty$. Hence the first term in γ_T tends to zero. Similarly the second (product) terms does not exceed $\mathcal{E}^{\theta_T} \exp(-a_T \alpha \zeta_T(I_0^*)) + o(1)$ which tends to zero as above so that $\gamma_T \to 0$ as $T \to \infty$ through S' and hence through S, as required. □

Remark: The result still holds if the function f changes with T, i.e. $f = f_T$, for example, provided each f_T is bounded above and the same lower bound constant α applies to all f_T (though the interval I can depend on T).

Lemma 2.2 is often applied in the following form

Lemma 2.3 Let the assumptions of Lemma 2.2 hold and suppose $J = (\alpha, \beta_T)$ where $\beta_T \uparrow \beta$ $(0 \leq \alpha \leq \beta \leq 1)$. Suppose also that $I \cap (\alpha, \beta) \neq \phi$ (which guarantees also the last assumption of Lemma 2.2). Then

$$\mathcal{E} \exp(-a_T \int_\alpha^\beta f d\zeta_T) - \prod_1^{k_T} \mathcal{E} \exp(-a_T \int_{J_i} f d\zeta_T) \to 0 \text{ as } T \to \infty$$

Proof: By Lemma 2.2 it is sufficient to show that

$$\gamma_T' = \mathcal{E} \exp(-a_T \int_J f d\zeta_T) - \mathcal{E} \exp(-a_T \int_\alpha^\beta f d\zeta_T) \to 0$$

as $T \to \infty$ through a sequence S such that γ_T' has a limit as $T \to \infty$ through S. Since $0 \leq \gamma_T' \leq 1 - \exp(-a_T A \zeta_T(J^*))$ by familiar arguments, where $J^* = (\beta_T, \beta]$ the result follows if $\mathcal{E} \exp(-a_T A \zeta_T(J^*)) \to 1$ as $T \to \infty$ through S. Otherwise there is a subsequence $S' \subset S$ such that as $T \to \infty$ through S',

(2.10) $$\mathcal{E} \exp(-a_T A \zeta_T(J^*)) \to c < 1$$

Now since $m(I \cap J) \to m(I \cap (\alpha,\beta)) > 0$, $m(J^*) \to 0$, $\ell_T/T \to 0$ it follows that $m(I \cap J)/(\ell_T/T + m(J^*)) \to \infty$ and hence we can find in $I \cap J$, $\theta_T \to \infty$ copies $E_1 \ldots E_{\theta_T}$ of J^*, mutually separated by at least ℓ_T/T. Hence

$$\begin{aligned}\mathcal{E}\exp(-a_T\textstyle\int_J f d\zeta_T) &\leq \mathcal{E}\exp(-a_T\alpha\zeta_T(I \cap J)) \\ &\leq \mathcal{E}^{\theta_T}\exp(-a_T\alpha\zeta_T(J^*)) + 4\theta_T\alpha_{T,\ell_T} \\ &\leq \mathcal{E}^{\alpha\theta_T/A}\exp(-a_TA\zeta_T(J^*)) + o(1)\end{aligned}$$

(choosing θ_T so that $\theta_T\alpha_{T,\ell_T} \to 0$). But this last expression tends to zero by (2.10) since $\theta_T \to \infty$. Hence the first term of γ_T', tends to zero as $T \to \infty$ through S'. But γ_T' is dominated by this term and hence itself tends to zero, completing the proof. □

The following result showing approximate independence of maxima in disjoint intervals follows simply. In this and throughout, $M(E)$ will denote $\sup(\xi_t : t \in E)$ for sets $E \subset (0,T]$ (so that $M((0,T]) = M(T)$ as previously defined). Note the slight asymmetry of notation in that $M(E)$ is defined for subsets of $(0,T]$ whereas $\zeta_T(B)$ is defined for subsets $B \subset (0,1]$, and the equivalence $\{\zeta_T(I) = 0\} = \{M(T.I) \leq u_T\}$ for an interval $I \subset (0,1]$.

<u>Lemma 2.4</u> Let $\Delta(u_T)$ hold and $\{k_T\}$ be integers satisfying (2.3). Let J_i $(=J_{T,i})$, $1 \leq i \leq k_T$, be disjoint subintervals of $[0,1]$, $J\ (=J(T)) = \bigcup_1^{k_T} J_i$. Then

$$P\{M(T.J) \leq u_T\} = \prod_1^{k_T} P\{M(T.J_i) \leq u_T\} \to 0 \text{ as } T \to \infty$$

Proof: Putting $f \equiv 1$ in Lemma 2.2 gives

$$\mathcal{E}\exp(-a_T\zeta_T((0,1])) - \prod_1^{k_T} \mathcal{E}\exp(-a_T\zeta_T(J_i)) \to 0.$$

Now write G_T for the d.f. of $\zeta_T((0,1)]$, $G_{T,i}$ for that of $\zeta_T(J_i)$. Clearly constants $b_T > 0$ may be chosen so that

$$G_T(b_T) - G_T(0) \to 0, \qquad \sum_{i=1}^{k_T} (G_{T,i}(b_T) - G_{T,i}(0)) \to 0 \quad \text{as} \quad T \to \infty.$$

Further choose a_T such that $k_T\exp(-a_Tb_T) \to 0$. Then

$$\begin{aligned} 0 &\le \mathcal{E}\exp(-a_T\zeta_T((0,1])) - P\{\zeta_T((0,1]) = 0\} \\ &= \int_{(0,\infty)} \exp(-a_Tx)dG_T(x) \le G_T(b_T) - G_T(0) + \exp(-a_Tb_T) \end{aligned}$$

which tends to zero. The same inequality holds for $\zeta_T(J_i)$ with $G_{T,i}$ replacing G_T, and thus by (2.2),

$$\left| \prod_1^{k_T} \mathcal{E}\exp(-a_T\zeta_T(J_1)) - \prod_1^{k_T} P\{\zeta_T(J_i) = 0\} \right| \le \sum_1^{k}(G_{T,i}(b_T) - G_{T,i}(0)) + k_T\exp(-a_Tb_T)$$

which also tends to zero by choice of a_T and b_T. The result thus follows by identifying $\{\zeta_T(B) = 0\}$ with $M\{(T.B) \le u_T\}$ for $B = J$, J_i. □

The following analog of Lemma 2.3 follows simply.

Lemma 2.5 Let the assumptions of Lemma 2.3 hold and $J = \cup J_i \subset (\alpha,\beta)$, $0<\alpha<\beta\le 1$, with $m(J) \to \beta-\alpha$. Then

$$P\{M((T\alpha,T\beta)) \le u_T\} - \prod_1^{k_T} P\{M(T.J_i) \le u_T\} \to 0 \text{ as } T \to \infty.$$

Proof: This follows from Lemma 2.3, or may be similarly proved. First note that the J_i may be replaced by abutting intervals of the same length without affecting either term (using stationarity) so that J becomes an interval (α, β_T) with $\beta_T \to \beta$.

3. Convergence of ζ_T

It is straightforward to characterize the class of possible limits in distribution for $a_T\zeta_T$ where $a_T\zeta_T$ is any family of positive constants. Specifically if $a_T\zeta_T$ converges in distribution to a random measure ζ, then ζ may be shown to be stationary, to have no fixed atoms and to have independent increments and hence (along the same lines as Lemma 3.1 of [4]) to have Laplace Transform L_ζ satisfying

$$(3.1) \qquad -\log L_\zeta(f) = \alpha\int_0^1 f dx + \int_0^1 \int_0^\infty (1-e^{-yf(x)})d\nu(y)dx$$

where $\alpha \geq 0$ and the (Lévy) measure ν on $(0,\infty)$ satisfies

$$(3.2) \qquad \int_0^\infty (1-e^{-y})d\nu(y) < \infty.$$

In fact this result may be strengthened to replace weak convergence of the random measures $a_T\zeta_T$ by just weak convergence of the random variables $a_T\zeta_T(I)$ for one fixed subinterval of (0,1]. Further an elementary proof may be given as will now be indicated.

Theorem 3.1 Let $\Delta(u_T)$ hold for the stationary process $\{\xi_T\}$, assumed to have continuous sample paths and write ζ_T for the exceedance random measure corresponding to $\{u_T\}$. Suppose that for some non empty subinterval $I \subset (0,1]$ and a family $\{a_T>0\}$ of constants, that $a_T\zeta_T(I)$ converges in distribution to a r.v. ζ_0. Then $a_T\zeta_T \xrightarrow{d} \zeta$ where ζ is a random measure with Laplace Transform given by (3.1).

Proof: It will be notationally convenient to take $I = (0,1]$. With the notation of $\Delta(u_T)$, choose integers $k_T \to \infty$, satisfying (2.3). If ζ_0 has Laplace Transform $\psi(s) = \mathcal{E}\exp(-s\zeta_0)$ we have $\psi(s) = \lim \mathcal{E}\exp(-s\zeta_T(0,1])$ and it follows from Lemma 2.2 with $f(x) \equiv s$, and $J_i = ((i-1)/k_T, i/k_T]$, that $\mathcal{E}^{k_T}\exp(-s\zeta_T(J_1)) \to \psi(s)$. Again by Lemma 2.3 by obvious calculations for any interval $I = (\alpha,\beta] \subset (0,1]$ it follows that $\mathcal{E}\exp(-s\zeta_T(I)) = \mathcal{E}^{n_T}\exp(-s\zeta_T(J_1))(1+o(1))$ where $n_T = [k_T(\beta-\alpha)]$ from which it is simply shown that

$$\mathcal{E}\exp(-s\zeta_T(I)) \to \psi(s)^{m(I)} \quad \text{as} \quad T \to \infty.$$

In particular if $I = (0, 1/k]$ for a fixed integer k it follows that $\psi(s)^{1/k} = \lim_{T\to\infty} \mathcal{E}\exp(-s\zeta_T(I))$ is a Laplace Tranform so that since $\psi(s) = ((\psi(s))^{1/k})^k$, ζ_o is infinitely divisible and hence

$$-\log \psi(s) = \alpha s + \int_0^\infty (1-e^{-sy})d\nu(y) \tag{3.3}$$

for some constant α, and measure ν on $(0,\infty)$ satisfying (3.2).

A further application of Lemma 2.2 shows that if $I_1,\ldots,I_k$ are disjoint semiclosed subintervals of (0,1] (and $f(x) = s_j$ on I_j, then $\zeta_T(I_1)\ldots\zeta_T(I_k)$ have the joint Laplace Transform

$$\mathcal{E}\exp(-\sum_i^k s_i\zeta_T(I_i)) \to \prod_1^k \psi(s_i)^{m(I_i)}$$

so that

$$(\zeta_T(I_1),\ldots\zeta_T(I_k)) \xrightarrow{d} (\zeta_1,\ldots,\zeta_k)$$

where ζ_i are independent and $-\log \mathcal{E}\exp(-s\zeta_i) = m(I_i)[\alpha s + \int_0^\infty(1-e^{-sy})d\nu(y)]$ may thus be recognized as having the distribution of $(\zeta(I_1),\ldots,\zeta(I_k))$ where ζ is a random measure with Laplace Transform (3.1) so that $\zeta_T \to \zeta$ (e.g. [6], Theorem

4.2). □

Corollary 3.2 If the convergence of $a_T\zeta_T(I) \xrightarrow{d} \zeta_0$ in the above statement is replaced by convergence in distribution of $a_T\zeta_T$ to a random measure ζ, then ζ has Laplace Transform given by (3.1).

Proof: The stationarity of ζ_T may be used to show that ζ has no fixed atoms and hence $\zeta(\{0\}) = \zeta(\{1\}) = 0$ a.s. giving $\zeta_T((0,1]) \xrightarrow{d} \zeta((0,1])$ so that the theorem applies. □

A random measure ζ satisfying (3.1) also has the "cluster" representation

$$\zeta(\cdot) = \alpha m(\cdot) + \int_{x=0}^{1} \int_{y=0}^{\infty} y\delta_x(\cdot)d\eta(x,y) \tag{3.4}$$

where δ_x denotes unit mass at x and η is a Poisson Process on $(0,1]\times(0,\infty)$ with intensity $m \times \nu$. Thus ζ has a uniform mass on $(0,1]$ together with a sequence of point masses y_i at points x_i where (x_i,y_i) are the points of η. In general there may be infinitely many of the atoms x_i in $(0,1]$ (though their total mass is finite) so that this component is then an atomic random measure which is not a point process. However if ν is finite the x_i do form a point process - indeed a stationary Poisson Process on $(0,1]$ with intensity parameter $\nu(0,\infty)$. In any case the points x_i for which $y_i>a$ form a Poisson Process with intensity parameter $\nu(a,\infty)$, for any $a>0$. It is also readily seen that $P\{\zeta(0,1) = 0\} > 0$ if and only if $\alpha = 0$ and $\nu(0,\infty) < \infty$ so that if α or $\nu(0,\infty) = \infty$ the interval $(0,1]$ (and in fact every interval) contains ζ-mass with probability one.

In the case when $\alpha=0$ and $\nu(0,\infty) < \infty$, $\pi(\cdot)=\nu(\cdot)/\nu(0,\infty)$ is a probability distribution on $(0,\infty)$ with Laplace Transform $\phi(s) = \int_0^\infty e^{-sx}d\pi(x)$. Then from (3.1), writing $\nu(0,\infty) = \nu$,

$$(3.5) \qquad -\log L_\zeta(f) = \nu\int_0^1[1 - \phi(f(x))]dx$$

which shows that ζ is a Compound Poisson Process (with not necessarily integer valued multiplicities) based on a Poisson Process with rate ν, and multiplicity distribution π.

As might be anticipated, the case where every interval contains ζ-mass with probability one arises when the level u_T is low in comparison with the values of the process i.e. when $P\{M(T) \le u_T\}$ is small. Specifically the following result holds.

Theorem 3.3 Suppose that the conditions of Theorem 3.1 hold and that $P\{M(T) \le u_T\} \not\to 0$ as $T \to \infty$. Then $\alpha=0$ and $\nu(0,\infty) < \infty$ in (3.1).

Proof: If $P\{M(T) \le u_T\} \not\to 0$, lim sup $P\{M(T) \le u_T\} > 0$ so that since $a_T\zeta_T(0,1) \xrightarrow{d} \zeta_0$,

$$\begin{aligned} P\{\zeta_0 = 0\} &\ge \limsup P\{a_T\zeta_T(0,1) = 0\} \\ &= \limsup P\{M(T) \le u_T\} > 0. \end{aligned}$$

But from (3.3), $P\{\zeta_0 = 0\} = \lim_{s\to\infty} \mathcal{E}\exp(-s\zeta_0) = 0$ if either $\alpha > 0$ or $\nu(0,\infty) = \infty$, so that $\alpha = 0$ and $\nu(0,\infty) < \infty$ both follow, as required. □

It will be convenient to refer to the set of points (if any) in an interval $J_i = ((i-1)/k_T, i/k_T]$ for which $\xi_t > u_T$ as an *excursion* of ξ_t above u_T. An excursion may consist of disconnected segments within one J_i, and points in successive intervals at which $\xi_t > u_T$ are regarded as belonging to different *excursions*. The conditional distributions π_T of $a_T\zeta_T(J_i)$ given $\zeta_T(J_i)>0$ will be termed *excursion length distributions*. It will be seen below that these distributions (on $(0,\infty)$) converge weakly to a probability distribution π on

$[0,\infty)$ under the conditions of Theorem 3.1, though the limit may have mass at zero. A more definitive result is possible if it is assumed that the distribution π_T are *tight at zero* in the sense that $\lim_{\epsilon\to 0} \liminf_{T\to\infty} \pi_T((0,\epsilon)) = 0$.

Theorem 3.4 Let the conditions of Theorem 3.1 hold and suppose $P\{M(T) \le u_T\} \nrightarrow 0$. Then the representation (3.5) holds, and if $P\{M(T) \le u_T\} \to e^{-\upsilon_0}$ $0 \le \upsilon_0 \le \infty$ as $T \to \infty$ through some sequence S, then $\upsilon_0 \ge \upsilon$ and

$$\pi_T \xrightarrow{w} (1-\upsilon/\upsilon_0)\delta_0 + (\upsilon/\upsilon_0)\pi \tag{3.6}$$

as $T \to \infty$ through S, δ_0 being unit mass at zero. In particular if $P\{M(T) \le u_T\}$ has a limit $e^{-\upsilon_0}$ as $T \to \infty$ then (3.6) holds as $T \to \infty$.

Proof: By Theorem 3.3 it follows that $\alpha = 0$ and $\upsilon(0,\infty) < \infty$, giving the representation (3.5). Writing $J_1 = (0,1/k_T]$ again and $r_T = T/k_T$, we have by Lemma 2.2,

$$\begin{aligned}
\mathcal{E} \exp(-sa_T\zeta_T((0,1])) &= \mathcal{E}^{k_T}\exp(-sa_T\zeta_T(J_1)) + o(1)\\
&= [P\{M(r_T) \le u_T\} + P\{M(r_T) > u_T\} \textstyle\int_0^\infty e^{-sx}d\pi_T(x)]^{k_T} + o(1)\\
&= [1 - P\{M(r_T) > u_T\} \textstyle\int_0^\infty (1-e^{-sx})d\pi_T(x)]^{k_T} + o(1),
\end{aligned}$$

so that

$$\begin{aligned}
k_T P\{M(T) \le u_T\} \textstyle\int_0^\infty (1-e^{-sx})d\pi_T(x) &\to -\log \mathcal{E}e^{-s\zeta((0,1])}\\
&= \upsilon \textstyle\int_0^\infty (1-e^{-sy})d\pi(y)
\end{aligned}$$

by (3.5). If $P\{M(T) \le u_T\} \to e^{-\upsilon_0}$ $0 \le \upsilon_0 \le \infty$ as $T \to \infty$ through a sequence S, then Lemma 2.4 shows that $k_T P\{M(r_T) > u_T\} \to \upsilon_0$ $(\le \infty)$ and hence

$$\int_0^\infty (1-e^{-sx})d\pi_T(x) \to (\nu/\nu_0) \int_0^\infty (1-e^{-sy})d\pi(y) \quad (\geq 0).$$

That is, as $T \to \infty$ through S,

$$\int_0^\infty e^{-sx}d\pi_T(x) \to (1-\nu/\nu_0) + (\nu/\nu_0) \int_0^\infty e^{-sy}d\pi(y)$$

from which (3.6) follows by the continuity theorem for Laplace Transforms. Finally by (3.6),

$$1 = \liminf \pi_T(0,\infty) \geq \{(1-\nu/\nu_0)\delta_0 + (\nu/\nu_0)\pi\}(0,\infty) = \nu/\nu_0$$

so that $\nu \leq \nu_0$ as required. □

<u>Theorem 3.5.</u> Suppose that the conditions of Theorem 3.1 hold and that the family $\{\pi_T\}$ of excursion distributions is tight as zero in the sense defined above. Assume that $P\{M(T) \leq u_T\} \not\to 0$ as $T \to \infty$. Then the representation (3.5) holds, $P\{M(T) \leq u_T\} \to e^{-\nu}$ and $\pi_T \xrightarrow{w} \pi$.

Proof: The representation (3.5) holds by Theorem 3.4. Let S be a sequence through which $P(M(T) \leq u_T\}$ converges to some limit $e^{-\nu_0}$ $0 \leq \nu_0 \leq \infty$. Then from (3.6) for $\epsilon > 0$,

$$\liminf_{T \in S} \pi_T([0,\epsilon)) \geq \frac{\nu}{\nu_0} \pi([0,\epsilon)) + 1 - \frac{\nu}{\nu_0} \geq 0 \quad .$$

But by tightness at zero the left hand side has zero limit as $\epsilon \to 0$. This rules out the case $\nu_0 = \infty$ and further shows that $\nu_0 = \nu$. Thus $P\{M(T) \leq u_T\}$ has the limit $e^{-\nu}$ as $T \to \infty$ through any sequence S for which there is convergence so that $P\{M(T) \leq u_T\} \to e^{-\nu}$ as $T \to \infty$. Hence also (3.6) gives $\pi_T \xrightarrow{w} \pi$ as $T \to \infty$, completing the proof. □

The final result of this section gives sufficient conditions for convergence of $a_T\zeta_T$. The same notation as above will be used - in particular $k_T \to \infty$ will be chosen to satisfy (2.3), and $r_T = T/k_T$.

Theorem 3.6 Let $\{u_T\}$ be a family of constants such that $P\{M(T) \le u_T\} \to e^{-\upsilon}$ for some $0 \le \upsilon < \infty$ and such that $\Delta(u_T)$ holds. Suppose that for some family $\{a_T\}$ of positive constants, the excursion length distributions π_T converge weakly to a probability distribution π on $(0,\infty)$. Then $a_T\zeta_T \xrightarrow{d} \zeta$, where ζ is a r.m. with Laplace Transform satisfying (3.5).

Proof: It follows as in the proof of Theorem 3.4 that

$$\mathcal{E}\exp(-sa_T\zeta_T((0,1])) = \{1-P\{M(r_T) > u_T\} \int_0^1 (1-e^{-sx})d\pi_T(x)\}^{k_T} + o(1)$$

which converges to $\exp\{-\upsilon\int_0^1(1-e^{-sx})d\pi(x)\}$ as $T \to \infty$ since $P\{M(r_T) > u_T\} \sim \upsilon/k_T$ by Lemma 2.4 and $\pi_T \xrightarrow{w} \pi$. This shows that $a_T\zeta_T(0,1)$ converges in distribution so that the conditions of Theorem 3.1 hold. Since the assumed weak convergence of π_T clearly implies its "tightness at zero" the conditions of Theorem 3.5 thus hold, so that (3.5) holds with the given υ,π. □

4. Families of levels.

Suppose that For each $\tau > 0$ there exists $u_T(\tau)$ such that

(4.1) $$P\{M(T) \le u_T(\tau)) \to e^{-\tau}$$

This will be the case (cf. [3]) if there are normalizing functions (not necessarily linear) $\upsilon_T(x)$ for the maximum such that $P\{M(T) \le \upsilon_T(x)\} \to G(x)$ where G is a continuous d.f. (For if $\tau > 0$, choose x such that $G(x) = e^{-\tau}$ and $u_T(\tau) = v_T(x)$). If $\Delta(u_T)$ holds for each $\tau > 0$ then it may be shown (cf. [3]) from Lemma 2.2 that $P\{M(T) \le u_{T/\tau}(1)\} \to e^{-\tau}$ so that (4.1) still holds if $u_T(\tau)$ is replaced by $u_{T/\tau}(1)$. That is $u_T(\tau)$ may be replaced by a new version satisfying (4.1) and such that

$$u_{\sigma T}(\sigma\tau) = u_T(\tau) \tag{4.2}$$

for each $\sigma > 0$. For simplicity we assume in what follows that $u_T(\tau)$ satisfies (4.2) as well as (4.1). The exceedance random measures for the levels $u_T(\tau)$ will be denoted by $\zeta_T^{(\tau)}$. The first result shows that convergence in distribution of (a normalized version of) $\zeta_T^{(\tau)}$ for some $\tau > 0$ implies such convergence for all $\tau > 0$.

Theorem 4.1 Suppose $\Delta(u_T(\tau))$ holds for each $\tau > 0$ where levels $u_T(\tau)$ are defined as above satisfying (4.1) and (4.2). Suppose that for some $\tau_1 > 0$ and some constants $a_T > 0$, $a_T\zeta_T^{(\tau_1)}$ converges in distribution to a random measure $\zeta^{(\tau)}$. Then

$$a_T^{(\tau)}\zeta_T^{(\tau)} \xrightarrow{d} \zeta^{(\tau)}$$

where $a_T^{(\tau)} = a_{T\tau_1/\tau}$ and $\zeta^{(\tau)}$ has Laplace Transform given by

$$-\log \mathcal{E}E^{-\int f d\zeta^{(\tau)}} = \tau \int_0^1 (1 - \phi(f(x)))dx$$

in which $\phi(s) = \int_0^\infty e^{-sx} d\pi(x)$ is the Laplace Transform of a distribution π on $(0,\infty)$ which is independent of τ.

Proof: Fix $\tau > 0$, choose $\epsilon > 0$ such that $\epsilon\tau/\tau_1 < 1$ and write $I = (0,\epsilon)$. Then

$$a_T^{(\tau)}\zeta_T^{(\tau)}(I) = a_T^{(\tau)}\int_0^{T\epsilon} 1_{\{\xi_t > u_T(\tau)\}}dt$$

$$= a_{T\tau_1/\tau}\int_0^{(T\tau_1/\tau)(\tau\epsilon/\tau_1)} 1_{\{\xi_t > u_{T\tau_1/\tau}(\tau_1)\}}dt$$

$$= a_{T\tau_1/\tau}\,\zeta^{(\tau_1)}_{T\tau_1/\tau}(\frac{\tau}{\tau_1}I) \xrightarrow{d} \zeta^{(\tau_1)}(\frac{\tau}{\tau_1}I)$$

Hence the convergence of $a_T^{(\tau)}\zeta_T^{(\tau)}$ to a r.m. $\zeta^{(\tau)}$ with the desired Laplace Transform follows from Theorem 3.1. The fact that π does not depend on τ follows since with the above notation,

$$\mathscr{E}\exp(-sa_T^{(\tau)}\zeta_T^{(\tau)}(I) \to \mathscr{E}\exp(-s\zeta^{(\tau_1)}(\frac{\tau}{\tau_1}I))$$

$$= \exp(-\tau_1(\frac{\tau}{\tau_1})m(I)(1-\phi(s)))$$

$$= \exp(-\tau m(I)(1 - \phi(s)))$$

so that $\phi(s)$ (and hence π) is the same for all values of τ. □

If $\tau_1 < \tau_2$, $u_T(\tau_2)$ will typically be less than $u_T(\tau_1)$ (and indeed may be assumed so if desired) giving $\zeta_T^{(\tau_1)}(B) \le \zeta_T^{(\tau_2)}(B)$ for Borel subsets B of (0,1], so that $\zeta_T^{(\tau_1)}$ is a "thinned version" of $\zeta_T^{(\tau_2)}$. Thus one expects $\zeta^{(\tau_1)}$ to be a thinned version of $\zeta^{(\tau_2)}$. While the thinning process may be complicated it is readily seen that the probability that an event in $\zeta^{(\tau_2)}$ is totally eliminated in $\zeta^{(\tau_1)}$ with probability $1-\tau_1/\tau_2$ and, if not eliminated, retains the same marginal multiplicity distribution (π) as for $\zeta^{(\tau_2)}$. A detailed discussion of these and "multilevel" cases is planned for [5].

5. "Regular" sample functions and stationary normal processes.

Suppose now that ξ_t is stationary with continuous sample functions and that the mean number $\mu(u)$ of upcrossings (cf. [9, Chapter 7]) of u per unit time is finite for each u. Suppose also that $\Delta(u_T)$ holds where $\{u_T\}$ is a family of levels such that $T\mu(u_T) \to \nu$. Then it may be shown under natural further conditions (cf. the Condition C' of [9] Section 13.2) that $P\{M(T) \le u_T\} \to e^{-\nu}$.

(In the following it will be simply assumed that this limit holds.) Write $\tilde{N}_T$ for the point process of upcrossings of u_T, defined on the unit interval by writing $\tilde{N}_T(B)$ to be the number of upcrossings of u_T by ξ_t for $t \in T.B$, for each Borel subset of $(0,1]$. The following result is then readily proved.

Theorem 5.1 Suppose that ξ_t satisfies the above general conditions and $P\{M(T) \le u_T\} \to e^{-\upsilon}$, where $T\mu(u_T) \to \upsilon$. Then $\tilde{N}_T \xrightarrow{d} N$, a Poisson Process on $(0,1]$ with intensity υ.

Proof: This follows simply by standard arguments from a theorem of Kallenberg ([6], Theorem 4.7). □

This result and Theorem 3.5 suggest that one may regard the upcrossings asymptotically as forming the underlying Poisson Process in the Compound Poisson limit for the normalized exceedance r.m. $a_T\zeta_T$. A further natural question is whether the excursion length distribution π_T as defined prior to Theorem 3.4 is equivalent to the distribution of time from an upcrossing to the next downcrossing, after normalization. The affirmative answer to this question stated below is obtained from Prop. 4.5 of [8]. Specifically we write $\tilde{\pi}_T$ for the conditional distribution of the time to the first downcrossing of u_T after $t = 0$, given an upcrossing occurred at $t = 0$ (in the Palm or "horizontal window" sense). $\tilde{\pi}_T$ may be evaluated by

$$\tilde{\pi}_T(x) = \mu_x(u_T)/\mu(u_T)$$

where $\mu_x(u)$ is the mean number of upcrossing of u per unit time such that the next downcrossing occurs within a further time x.

If $\Delta(u_T)$ holds write π_T' for the conditional distribution of $\zeta_T(J_1)$ given $\zeta_T(J_1) > 0$ and $J_1 = (0,k_T^{-1}]$ with k_T satisfying (2.3). (Thus $\pi_T'(x)$ is the non-normalized excursion length distribution and $\pi_T(x) = \pi_T'(a_T x)$. Proposition 4.5 of [8] then gives

<u>Theorem 5.2</u> Let $\Delta(u_T)$ hold where $T\mu(u_T) \to \upsilon$ and $P\{M(T) \leq u_T\} \to e^{-\upsilon}$ for some $\upsilon \geq 0$, and $\tilde{\pi}_T$, π_T' be defined as above. Then $\tilde{\pi}_T(x) - \pi_T'(x) \to 0$ uniformly in x as $T \to \infty$. □

It thus follows from this result that $\tilde{\pi}_T(a_T^{-1}x)$ may be used to replace $\pi_T(x)$ to give the multiplicity distribution π in the Compound Poisson limit (e.g. Theorem 3.6).

In these results we see that the underlying Poisson event for the limiting Compound Poisson process for $a_T\zeta_T$ may be identified with high level upcrossings and the event multiplicities with lengths of excursions above the level following upcrossings. A closer identification may be obtained by showing that the point process of upcrossings "marked" with the immediately following excursion lengths, has the same compound Poisson limit as does the exceedance r.m., but the details of this will not be pursed here.

As a specific case consider a stationary normal process ξ_t with zero mean, unit variance and covariance function $r(\tau)$ which is twice differentiable at $\tau = 0$, $-r''(0) = 1$. Then Rice's Formula gives the upcrossing intensity $\mu(u) = (2\pi)^{-1}e^{-u^2/2}$. It is known ([2], Section 12.5) that if $\theta_T=(\mu(u_T))^{-1}P\{\xi(0) > u_T\}$ where $T\mu(\mu_T) \to \upsilon$ (hence $\theta_T \sim (\pi/\log T)^{1/2}$) then

$$\tilde{\pi}_T(\theta_T x) \to 1 - \exp(-(\pi/4)x^2) \quad \text{as} \quad T \to \infty$$

A convenient non-degenerate limit for π_T is thus obtained by taking $a_T=\sqrt{\pi}/\theta_T \sim (\log T)^{1/2}$ so that $\tilde{\pi}_T(a_T^{-1}x) \to 1-e^{-x^2/4}$. The condition $\Delta(u_T)$ certainly holds under reasonable conditions (e.g. strong mixing and it seems likely to hold even under the weaker condition $r(t)\log t \to 0$ as $t \to \infty$, though we have not attempted to verify this). Hence for such processes the limiting Compound Poisson Process for $a_T\zeta_T$ has underlying Poisson intensity $\nu = \lim T\mu(u_T)$ and multiplicity distribution function $1-e^{-x^2/4}$.

Finally we note that similar results will apply under appropriately modified mixing conditions to other functionals associated with high level exceedances, or excursions into other "rare sets". Indeed a discussion could be carried out as a study of a class of random measures on the real line without reference to a real valued process ξ_t at all. Here we prefer to use the more specific framework within the context of extremal theory.

Acknowledgments

This research was initiated during a visit to the Catholic University of Leuven and it is a pleasure to record my gratitutde to the University and Professor Jozef Teugels for this hospitality. I am also very grateful to Professor Tailen Hsing for many stimulating conversations on this and related subjects.

References

[1] Berman, S.M. (1982). Sojourns and extremes of stationary processes, Ann. Probab. 10, 1-46.

[2] Cramér, H. and Leadbetter, M.R. (1967). Stationary and related stochastic processes. Wiley, N.Y.

[3] Hsing, T. (1987). On the extreme order statistics for a stationary sequence. Preprint, Statistics Department University of Illinois.

[4] Hsing, T., Hüsler, J., and Leadbetter, M.R. (1988). On the exceedance point process for a stationary sequence, Prob. Theor. and Rel. Fields., to appear.

[5] Hsing, T., Leadbetter, M.R. (1988). Multilevel and "complete" convergence theorems for high level exceedance by stationary processes, in preparation.

[6] Kallenberg, O. (1976). Random Measures. Academic Press, N.Y.

[7] Leadbetter, M.R. (1983). Extremes and local dependence in stationary sequences, Z. Wahrsch. verw. Gebiete 65, 291-306.

[8] Leadbetter, M.R. and Nandagopalan, S., On exceedance random measures for stationary processes with "regular" sample paths, in preparation.

[9] Leadbetter, M.R., Lindgren, G. and Rootzén, H. (1983). Extremes and related properties of random sequences and processes. Springer Series in Statistics.

[10] Pickands, J. III (1969). Upcrossing probabilities for stationary Gaussian processes. Trans. Amer. Math. Soc. 145, 51-73.

[11] Volkonski, A.A. and Rozanov, Yu V. (1958). Some limit theorems for random functions I. Theor. Prob. Appl. 4, 178-197.

176 E. Merzbach, Point processes in the plane, Feb 87.

177. Y. Kasahara, M. Maejima and W. Vervaat, Log fractional stable processes, March 87.

178 G. Kallianpur, A.G. Miamee and H. Niemi, On the prediction theory of two parameter stationary random fields, March 87.

179 R. Brigola, Remark on the multiple Wiener integral, Mar. 87.

180. R. Brigola, Stochastic filtering solutions for ill-posed linear problems and their extension to measurable transformations, Mar. 87.

181. G. Samorodnitsky, Maxima of symmetric stable processes, Mar 87.

182. H.L. Hurd, Representation of harmonizable periodically correlated processes and their covariance, Apr. 87.

183. H.L. Hurd, Nonparametric time series analysis for periodically correlated processes, Apr. 87.

184. T. Mori and H. Oodaira, Freidlin-Wentzell estimates and the law of the iterated logarithm for a class of stochastic processes related to symmetric statistics, May 87.

185. R.F. Serfozo, Point processes, May 87. Operations Research Handbook on Stochastic Processes, to appear.

186 Z.D. Bai, W.Q. Liang and W. Vervaat, Strong representation of weak convergence, June 87.

187. O. Kallenberg, Decoupling identities and predictable transformations in exchangeability, June, 87.

188. O. Kallenberg, An elementary approach to the Daniell-Kolmogorov theorem and some related results, June 87. Math. Nachr., to appear.

189. G. Samorodnitsky, Extrema of skewed stable processes, June 87.

190. D. Nualart, M. Sanz and M. Zakai, On the relations between increasing functions associated with two-parameter continuous martingales, June 87.

191. F. Avram and M. Taqqu, Weak convergence of sums of moving averages in the α-stable domain of attraction, June 87.

192 M.R. Leadbetter, Harald Cramér (1893-1985), July 87. ISI Review, to appear.

193 R. LePage, Predicting transforms of stable noise, July 87.

194 R. LePage and B.M. Schreiber, Strategies based on maximizing expected log, July 87.

195 J. Rosinski, Series representations of infinitely divisible random vectors and a generalized shot noise in Banach spaces, July 87.

196. J. Szulga, On hypercontractivity of α-stable random variables, $0<\alpha<2$, July 87.

197. I. Kuznezova-Sholpo and S.T. Rachev, Explicit solutions of moment problems I, July 87.

198. T. Hsing, On the extreme order statistics for a stationary sequence, July 87.

199 T. Hsing, On the characterization of certain point processes, Aug. 87.

200. J.P. Nolan, Continuity of symmetric stable processes, Aug. 87.

201. M. Marques and S. Cambanis, Admissible and singular translates of stable processes, Aug. 87.

202. O. Kallenberg, One-dimensional uniqueness and convergence results for exchangeable processes, Aug. 87.

203. R.J. Adler, S. Cambanis and G. Samorodnitsky, On stable Markov processes, Sept. 87.

204. G. Kallianpur and V. Perez-Abreu, Stochastic evolution equations driven by nuclear space valued martingales, Sept. 87.

205 R.L. Smith, Approximations in extreme value theory, Sept. 87.

206. E. Willekens, Estimation of convolution tails, Sept. 87.

207. J. Rosinski, On path properties of certain infinitely divisible processes, Sept. 87.

208 A.H. Korezlioglu, On the computation of nonlinear filters, Sept. 87.

209 J. Bather, Stopping rules and observed significance levels, Sept. 87.

210 S.T. Rachev and J.E. Yukich, Convolution metrics and rates of convergence in the central limit theorem, Sept. 87.

211 T. Fujisaki, Normed Bellman equation with degenerate diffusion coefficients and its applications to differential equations, Oct. 87.

212 G. Simons, Y.C. Yao and X. Wu, Sequential tests for the drift of a Wiener process with a smooth prior, and the heat equation, Oct. 87.

213 R.L. Smith, Extreme value theory for dependent sequences via the Stein-Chen method of Poisson approximation, Oct. 87.

214 C. Houdré, A note on vector bimeasures, Nov 87.

215 M.R. Leadbetter, On the exceedance random measures for stationary processes, Nov. 87.

END
DATE
FILMED
6-88
DTIC