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CHAPEL HILL CENTER FOR STOCHASTIC PROC.

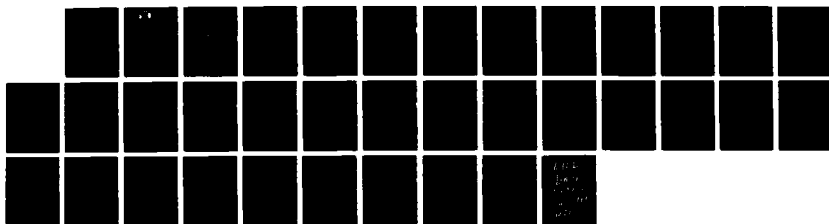
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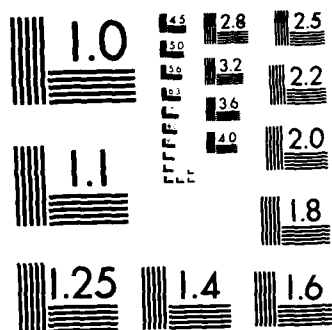
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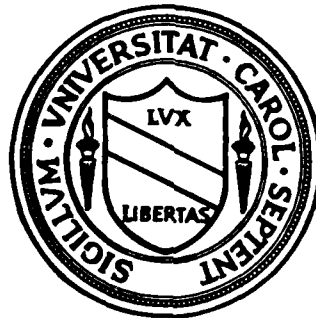
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## TWO CLASSES OF SELF-SIMILAR STABLE PROCESSES WITH STATIONARY INCREMENTS

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# TWO CLASSES OF SELF-SIMILAR STABLE PROCESSES WITH STATIONARY INCREMENTS<sup>1</sup>

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**Abstract** Two disjoint classes of self-similar symmetric stable processes with stationary increments are studied. The first class consists of linear fractional stable processes, which are related to moving average stable processes, and the second class consists of harmonizable fractional stable processes, which are connected to harmonizable stationary stable processes. The domain of attraction of the harmonizable fractional stable processes is also discussed.

AMS 1980 Subject Classification: Primary 60G10, 60G99.

Key words and phrases: Self-similar processes, stable processes, linear and harmonizable fractional processes, domain of attraction.

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## 1. Introduction

A stochastic process  $X = (X(t))_{t \in \mathbb{R}}$  is self-similar with parameter  $H \in \mathbb{R}$  (H-ss) if  $X(c \cdot) \stackrel{d}{=} c^H X(\cdot)$  for all  $c > 0$ , and has stationary increments (si) if  $X(\cdot + b) - X(b) \stackrel{d}{=} X(\cdot) - X(0)$  for all  $b \in \mathbb{R}$ , where  $\stackrel{d}{=}$  means the equality of all finite dimensional distributions. A real-valued stochastic process  $X = (X(t))_{t \in \mathbb{R}}$  is symmetric  $\alpha$ -stable (SoS) if all linear combinations  $\sum_{n=1}^N a_n X(t_n)$  have characteristic functions of the form  $\exp\{-a|\theta|^\alpha\}$  for some  $a > 0$ . Here  $\alpha \in (0, 2]$  and when  $\alpha = 2$ ,  $X$  is Gaussian.

In this paper, we study two disjoint classes of  $H$ -self-similar symmetric  $\alpha$ -stable processes with stationary increments (H-ss si SoS processes). One consists of linear fractional stable processes, which are related to moving average stable processes, and the other consists of harmonizable fractional stable processes, which are connected to harmonizable stationary stable processes.

In Section 2 we give a representation of a SoS process with stationary increments in terms of a stationary SoS process which is either nonanticipating or fully anticipating. The representation is shown for  $1 < \alpha \leq 2$ , but it should in fact be valid for much larger classes than SoS processes.

In Section 3 we introduce the linear fractional stable processes  $\Lambda_{\alpha, H}^{(a, b)} = (\Lambda_{\alpha, H}^{(a, b; t)})_{t \in \mathbb{R}}$  whose corresponding nonanticipating (or fully anticipating) stationary SoS process is a moving average. We show that when  $1 < \alpha < 2$ , to each line through the origin of the parameter plane  $(a, b)$  there corresponds a distinct linear fractional stable process (Theorem 3.1). This result implies that the two linear fractional processes defined independently in [12] and [21] are different.

In Section 4 we introduce the complex harmonizable fractional stable

processes  $\theta_{\alpha,H}(a,b) = (\theta_{\alpha,H}(a,b;t))_{t \in \mathbb{R}}$ , whose corresponding nonanticipating (or fully anticipating) stationary SaS process is harmonizable. When  $1 < \alpha \leq 2$  we show that to each line through the origin of the parameter plane  $(a,b)$  there corresponds a distinct harmonizable fractional stable process (Theorem 4.1), and we study their domain of attraction in Section 6 (Theorems 6.1 and 6.2).

There is only one (distinct) real harmonizable fractional stable process, namely  $\theta_{\alpha,H}(1,1)$ , and it is shown in Section 5 that it is not a linear fractional stable process when  $1 < \alpha < 2$  (Theorem 5.1).

The only reason the results in Sections 3-5 are stated only for  $1 < \alpha \leq 2$  is that the representation in Section 2 is established only for  $1 < \alpha \leq 2$ . If the representation in Section 2 is proved also for  $0 < \alpha \leq 1$ , then the same proofs would establish the results in Sections 3-5 for all  $0 < \alpha < 2$ .

It should also be mentioned that, under mild regularity conditions, the linear and the harmonizable fractional stable processes introduced here seem to be the only self-similar, symmetric stable processes with stationary increments whose corresponding nonanticipating (or fully anticipating) stationary stable processes are moving averages and harmonizable respectively. This characterization result is still under study.

The authors gratefully acknowledge insightful discussions on self-similar stable processes with Murad Taqqu and Florin Avram.

## 2. Representation of si SaS processes

In this section we assume  $1 < \alpha \leq 2$ .

Let  $X = (X(t))_{t \in \mathbb{R}}$  be a continuous in probability si SaS process with  $1 < \alpha \leq 2$ .

Define

$$(2.1) \quad Y(t) = \int_{-\infty}^0 e^u [X(t) - X(t+u)] du = X(t) - \int_{-\infty}^t e^{-(t-u)} X(u) du, \quad t \in \mathbb{R}.$$

where the integrals exist a.s. Clearly  $Y = (Y(t))_{t \in \mathbb{R}}$  is a continuous in probability, stationary SoS process which depends linearly on the past increments (or values) of  $X$ . We call  $Y$  the nonanticipating stationary SoS process corresponding to the si SoS process  $X$ , and a straightforward calculation shows that for all  $s < t$ ,

$$(2.2) \quad X(t) - X(s) = Y(t) - Y(s) - \int_s^t Y(v) dv.$$

One can also introduce the fully anticipating stationary SoS process  $Y$  corresponding to the si SoS process by

$$(2.3) \quad Y(t) = \int_0^\infty e^{-u} [X(t) - X(t+u)] du = X(t) - \int_t^\infty e^{-(v-t)} X(v) dv, \quad t \in \mathbb{R},$$

and derive likewise representation (2.2).

Of course the increments of a stationary process  $Y$  define a si process  $X$  via  $X(t) - X(s) = Y(t) - Y(s)$ , and so does the indefinite integral of a stationary process  $Y$  with a.s. locally integrable paths via  $X(t) - X(s) = \int_s^t Y(v) dv$ . In fact, a si process is the indefinite integral of a stationary process if and only if its paths are a.s. locally absolutely continuous; and is both the increment of some stationary process as well as the indefinite integral of some (other) stationary process if and only if its paths are a.s. locally absolutely continuous and its derivative process is the derivative of a stationary process. These two simple classes of si processes are distinct (with nonempty intersection) but their union is not broad enough to encompass all si processes (as is easily shown via examples). Representation (2.2) is therefore the only one generally available for all si processes.

Representation (2.2) follows also from Masani's representation of helices in Banach spaces ([15]), by viewing  $X$  as a helix in  $L_p(\Omega, \mathcal{F}, P)$  for  $1 \leq p < \infty$ . It should be pointed out that (2.2) defines a si SoS process  $X$  for each stationary SoS process  $Y$  even when  $0 < \alpha \leq 1$ , provided the integral in (2.2) is well defined:



for necessary and sufficient conditions see [6], Theorem 4.1. The conjecture here is that (2.2) would be valid for  $0 < \alpha \leq 1$  as well, but there may be technical difficulties in its derivation. For the sake of completeness we indicate the simple proofs of (2.1) and (2.2) when  $1 < \alpha \leq 2$ .

Proof of (2.1). The integral in the middle expression in (2.1) will exist a.s. if

$$\int_{-\infty}^0 e^u E|X(t) - X(t+u)| du < \infty.$$

(In fact this is a necessary and sufficient condition in this SoS case ([5]).) But the stationarity of the increments of  $X$  implies that  $E|X(0) - X(u)|$  grows linearly in  $|u|$ . The equality of the two expressions a.s. is shown likewise.  $\square$

Proof of (2.2). From (2.1) we obtain

$$\begin{aligned} \int_s^t Y(v) dv &= \int_s^t X(v) dv - \int_s^t dv \int_{-\infty}^v e^{-(v-u)} X(u) du \\ &= \int_s^t X(v) dv - \int_{-\infty}^t du X(u) \int_s^t v u e^{-(v-u)} dv \\ &= \int_s^t X(v) dv - (e^{-s} - e^{-t}) \int_{-\infty}^s e^u X(u) du - \int_s^t (1 - e^{u-t}) X(u) du \\ &= -(e^{-s} - e^{-t}) \int_{-\infty}^s e^u X(u) du + \int_s^t e^{u-t} X(u) du \\ &= \int_{-\infty}^t e^{-(t-u)} X(u) du - \int_{-\infty}^s e^{-(s-u)} X(u) du \\ &= [Y(t) - X(t)] - [Y(s) - X(s)], \end{aligned}$$

i.e. (2.2). The interchange of the order of integration is justified by Fubini's theorem as for (2.1).  $\square$

The nonanticipating or fully anticipating stationary SoS process  $Y$  corresponding to the si SoS process  $X$  via the representation (2.2) is continuous in probability. Every continuous in probability stationary SoS process  $Y$  ( $0 < \alpha \leq 2$ ) has a version of the form

$$(2.4) \quad Y(t) = \int_{-\infty}^{\infty} a(t, u) dZ(u), \quad t \in \mathbb{R},$$

where  $Z$  has independent SoS increments and control measure  $\mu$ , i.e.,

$$(2.5) \quad E \exp(i r \int f dZ) = \exp(-|r|^{\alpha} \int |f|^{\alpha} d\mu)$$

for  $f \in L_{\alpha}(\mu)$ .  $(V(t))_{t \in \mathbb{R}}$  is a strongly continuous group of isometries in  $L_{\alpha}(\mu)$  and  $a(t, \cdot) = V(t)\{a(0, \cdot)\}$  ([8]). Since  $Y$  has a measurable modification (being continuous in probability), the kernel  $a(t, u)$  has a version jointly measurable in  $(t, u)$  ([18]), and we obtain from (2.2) and (2.4), provided all integrals are well defined,

$$\begin{aligned} X(t) - X(s) &= \int_{-\infty}^{\infty} [a(t, u) - a(s, u) - \int_s^t a(v, u) dv] dZ(u) \\ &= \int_{-\infty}^{\infty} [b(t, u) - b(s, u)] dZ(u), \end{aligned}$$

where

$$\begin{aligned} b(t, u) - b(s, u) &= a(t, u) - a(s, u) - \int_s^t a(v, u) dv \\ &= [V(t) - V(s) - \int_s^t V(v) dv]\{a(0, \cdot)\}(u). \end{aligned}$$

The most important examples of stationary SoS processes are moving averages and harmonizable processes.  $Y$  is a SoS moving average process ( $0 < \alpha \leq 2$ ) if

$$Y(t) = \int_{-\infty}^{\infty} h(t-s) dM(s), \quad t \in \mathbb{R},$$

where  $M = (M(s))_{s \in \mathbb{R}}$  is a SoS motion (i.e., has stationary, independent SoS increments and Lebesgue control measure) and  $h \in L_{\alpha} := L_{\alpha}(\text{Leb})$ .

Up to this point we have only considered real-valued SoS processes. When dealing with harmonizable processes it is natural and convenient to consider complex-valued SoS processes, indeed radially SoS complex random variables will suffice. A complex r.v.  $X = X_1 + iX_2$  is radially SoS if  $X_1$  and  $X_2$  are jointly SoS with radially symmetric distribution, i.e. with  $z = z_1 + iz_2$ ,  $E \exp\{i \Re z X\} =$

$E \exp\{i(z_1 X_1 + z_2 X_2)\} = \exp\{-c|z|^\alpha\}$  for some  $c > 0$ . A complex-valued process  $X = (X(t))_{t \in \mathbb{R}}$  is radially SoS if all complex linear combinations  $\sum_{n=1}^N z_n X(t_n)$  are complex radially SoS r.v.'s.  $Y$  is a harmonizable stationary SoS process ( $0 < \alpha \leq 2$ ) if

$$(2.6) \quad Y(t) = \int_{-\infty}^{\infty} e^{it\lambda} dZ(\lambda), \quad t \in \mathbb{R},$$

where  $Z$  has complex, independent, radially SoS increments and finite spectral (control) measure  $\mu$ , i.e.

$$(2.7) \quad E \exp(i \int_{\mathbb{R}} z f dZ) = \exp(-|z|^\alpha \int |f|^\alpha d\mu)$$

for all complex numbers  $z$  and functions  $f$  in  $L_\alpha(\mu)$  ([4]). (Unless the stable distribution of  $Z$  is radially symmetric,  $Y$  is not stationary ([23]).) When  $\mu$  is absolutely continuous with respect to Lebesgue measure,  $\varphi(\lambda) = d\mu(\lambda)/d\lambda$  is called the spectral density of  $Y$ . When  $\mu$  is Lebesgue measure,  $Z$  is called a complex SoS motion and is denoted by  $\tilde{M}$ . A real-valued SoS process  $Y$  is harmonizable if

$$Y(t) = \Re \int_{-\infty}^{\infty} e^{it\lambda} dZ(\lambda), \quad t \in \mathbb{R},$$

where  $Z$  is as above necessarily complex. For simplicity here we will consider complex harmonizable processes, while moving average processes will always be considered real.

### 3. The linear fractional stable processes

In this section we consider the linear fractional stable processes. Let  $0 < H < 1$ ,  $a, b \in \mathbb{R}$ , and  $M$  be a SoS motion defined in Section 2. For  $0 < \alpha \leq 2$ ,  $H \neq 1/\alpha$ , define

$$(3.1) \quad \Delta_{\alpha, H}(a, b; t) = \int_{-\infty}^{\infty} \{ a[(t-u)_+^{H-1/\alpha} - (-u)_+^{H-1/\alpha}] + b[(t-u)_-^{H-1/\alpha} - (-u)_-^{H-1/\alpha}] \} dM(u), \quad t \in \mathbb{R},$$

with the convention  $0^\gamma = 0$  even for  $\gamma < 0$ ; and for  $1 < \alpha \leq 2$ ,  $H = 1/\alpha$ .

$$(3.2) \quad \Delta_{\alpha, 1/\alpha}(a, b; t) = a\Lambda(t) + bM(t), \quad t \in \mathbb{R},$$

where

$$(3.3) \quad \Lambda(t) = \int_{-\infty}^{\infty} \{ \log|t-u| - \log|u| \} dM(u), \quad t \in \mathbb{R}.$$

The process  $\Delta_{\alpha, H}(1, 0)$  was introduced in [21] and called fractional Lévy motion, and the process  $\Delta_{\alpha, H}(1, 1)$  was introduced in [12], and called fractional stable process.  $\Lambda$  in (3.3) is the log-fractional stable process defined in [11]. The process  $\Delta_{\alpha, H}$ ,  $H \neq 1/\alpha$ , can be defined for any  $\alpha$ ,  $0 < \alpha \leq 2$ , but the process  $\Lambda(t)$  can be defined only in the case  $1 < \alpha \leq 2$ . It is easy to see that the processes  $(\Delta_{\alpha, H}(a, b; t))_{t \in \mathbb{R}}$  as in (3.1) and (3.2) are H-ss si SoS. In the rest of this section, we assume  $1 < \alpha \leq 2$ .

The nonanticipating stationary SoS process  $Y_{\alpha, H}(a, b; \cdot)$  corresponding to the ss si SoS process  $\Delta_{\alpha, H}(a, b; \cdot)$  of (3.1) is determined via (2.1) as follows: For  $t \in \mathbb{R}$

$$\begin{aligned} Y_{\alpha, H}(a, b; t) &= \int_{-\infty}^0 e^u \left( \int_{-\infty}^{\infty} \{ a[(t-v)_+^{H-1/\alpha} - (t+u-v)_+^{H-1/\alpha}] + b[(t-v)_-^{H-1/\alpha} - (t+u-v)_-^{H-1/\alpha}] \} dM(v) \right) du \\ &= \int_{-\infty}^0 \left( \int_{-\infty}^0 e^u \{ a[(t-v)_+^{H-1/\alpha} - (t+u-v)_+^{H-1/\alpha}] + b[(t-v)_-^{H-1/\alpha} - (t+u-v)_-^{H-1/\alpha}] \} du \right) dM(v) \\ (3.4) \quad &= \int_{-\infty}^0 h_{a, b}(t-v) dM(v), \end{aligned}$$

where

$$\begin{aligned} h_{a, b}(x) &= \int_{-\infty}^0 e^u \{ a[x_+^{H-1/\alpha} - (x+u)_+^{H-1/\alpha}] + b[x_-^{H-1/\alpha} - (x+u)_-^{H-1/\alpha}] \} du \\ (3.5) \quad &= ah_1(x) + bh_2(x), \end{aligned}$$

$$(3.6) \quad h_1(x) = x_+^{H-1/\alpha} - e^{-x} \int_0^x e^v v^{H-1/\alpha} dv.$$

$$(3.7) \quad h_2(x) = x_-^{H-1/\alpha} - e^{-x} \int_x^\infty e^{-v} v^{H-1/\alpha} dv.$$

Hence  $Y_{\alpha,H}(a,b)$  is a moving average of the SoS motion  $M$ , i.e. a linear process, and we call  $\Delta_{\alpha,H}(a,b)$  a linear fractional stable process.  $Y_{\alpha,H}(a,b)$  is a nonanticipating moving average of  $M$  only if  $h_{a,b}(x) = 0$  for  $x < 0$ , and since for  $x < 0$ ,  $h_1(x) = 0$  but  $h_2(x) \neq 0$ , this is true only when  $b=0$ .

As in (3.4), the nonanticipating stationary SoS process  $Y_{\alpha,1/\alpha}(a,b)$  corresponding to the  $(1/\alpha)$ -ss si SoS process  $\Delta_{\alpha,1/\alpha}(a,b;\cdot) = a\Lambda(\cdot) + bM(\cdot)$  is determined via (2.1) as the following moving average

$$(3.8) \quad Y_{\alpha,1/\alpha}(a,b;t) = \int_{-\infty}^{\infty} g_{a,b}(t-u) dM(u),$$

where

$$(3.9) \quad g_{a,b}(x) = ag_1(x) + bg_2(x),$$

$$g_1(x) = \log|x| - e^{-x} \int_{-\infty}^x e^v \log|v| dv.$$

$$g_2(x) = e^{-x} \quad \text{for } x > 0, \quad = 0 \quad \text{for } x < 0.$$

Therefore,  $Y_{\alpha,1/\alpha}(a,b)$  is a nonanticipating moving average of  $M$  only when  $a=0$ .

The linear fractional stable processes  $\Delta_{\alpha,H}(a,b)$  are indexed by the parameters  $(a,b) \in \mathbb{R}^2$ . When  $\alpha=2$ , the processes  $\Delta_{2,H}(a,b)$  are Gaussian, called fractional Brownian motion, and by comparing their covariances we see that for all  $a$  and  $b$  with  $|a|+|b| \neq 0$ , the processes  $\Delta_{2,H}(a,b)$  are multiples of the same Gaussian processes in distribution. The main purpose of this section is to show that if  $1 < \alpha < 2$  each line through the origin of the parameter plane  $(a,b)$  determines a distinct process in distribution (up to scaling of course).

Theorem 3.1. Let  $0 < H < 1$ ,  $1 < \alpha < 2$ ,  $|a|+|b| \neq 0$ , and  $\Delta_{\alpha,H}(a,b;\cdot)$  be given by (3.1) and (3.2). Then we have

$$C_{\alpha,H}^{-1}(a,b) \Delta_{\alpha,H}(a,b;\cdot) \stackrel{d}{=} C_{\alpha,H}^{-1}(a',b') \Delta_{\alpha,H}(a',b';\cdot),$$

where  $C_{\alpha,H}(a,b)$  is defined by (3.10) and (3.14) below, if and only if

- (i)  $a = a' = 0$  or
- (ii)  $b = b' = 0$  or
- (iii)  $aa'bb' \neq 0$  and  $a/b = a'/b'$ .

Proof. Recall that for two SoS random variables  $X$  and  $Y$ , by writing  $E \exp\{irX\} = \exp\{-|r|^\alpha ||X||_\alpha^\alpha\}$ , it follows that  $X$  and  $Y$  have the same distribution,  $X \stackrel{d}{=} Y$ , if and only if they have the same scale parameters,  $||X||_\alpha^\alpha = ||Y||_\alpha^\alpha$ . Thus to compare the marginal distributions of  $\Delta_{\alpha,H}(a,b;t)$  and  $\Delta_{\alpha,H}(a',b';t)$  we need to compute their scale parameters.

First assume  $H \neq 1/\alpha$ . We find from (3.1) and (2.5) that

$$\begin{aligned} ||\Delta_{\alpha,H}(a,b;t)||_\alpha^\alpha &= |t|^{\alpha H} \{ (|a|^\alpha + |b|^\alpha) \int_1^\infty ||1-v|^{H-1/\alpha} - |v|^{H-1/\alpha}|^\alpha dv \\ &\quad + \int_0^1 |a|1-v|^{H-1/\alpha} - b|v|^{H-1/\alpha}|^\alpha dv \} \\ (3.10) \quad &=: |t|^{\alpha H} C_{\alpha,H}^\alpha(a,b). \end{aligned}$$

For simplicity we drop the subscripts of  $\alpha, H$ . It follows that for each fixed  $t \geq 0$ ,

$$C^{-1}(a,b) \Delta(a,b;t) \stackrel{d}{=} C^{-1}(a',b') \Delta(a',b';t),$$

where we are assuming of course that  $|a| + |b| \neq 0 \neq |a'| + |b'|$ .

In view of the one-to-one relationship between the si SoS process  $\Delta(a,b;\cdot)$  of (3.1) and its corresponding nonanticipatory stationary SoS process  $Y(a,b;\cdot)$  of (3.4),

$$(3.11) \quad C^{-1}(a,b) \Delta(a,b;\cdot) \stackrel{d}{=} C^{-1}(a',b') \Delta(a',b';\cdot)$$

is equivalent to

$$C^{-1}(a,b)Y(a,b;\cdot) \stackrel{d}{=} C^{-1}(a',b')Y(a',b';\cdot),$$

which, in view of (3.4) and Kanter's theorem [9], is equivalent to

$$C^{-1}(a,b)h_{a,b}(t) = \epsilon C^{-1}(a',b')h_{a',b'}(t-\tau) \quad \text{a.e.}(t)$$

for some  $\epsilon \in \{\pm 1\}$  and  $\tau \in \mathbb{R}$  (depending on  $a,b,a',b'$ ); and this in turn is equivalent to

$$C^{-1}(a,b)\{ah_1(t) + bh_2(t)\} = \epsilon C^{-1}(a',b')\{a'h_1(t-\tau) + b'h_2(t-\tau)\} \quad \text{a.e.}(t)$$

in view of (3.5), which is equivalent to

$$(3.12) \quad \left\{ \frac{a}{C(a,b)} = \epsilon \frac{a'}{C(a',b')}, \quad \frac{b}{C(a,b)} = \epsilon \frac{b'}{C(a',b')} \right\}$$

To obtain the necessity of the last condition, take  $t < \min(0,\tau)$ , so that  $h_1(t) = 0 = h_1(t-\tau)$  and differentiate  $C^{-1}(a,b)bh_2(t) = \epsilon C^{-1}(a',b')b'h_2(t-\tau)$  to obtain

$$bC^{-1}(a,b)(-t)^{H-1/\alpha-1} = \epsilon b'C^{-1}(a',b')(\tau-t)^{H-1/\alpha-1} \quad \text{a.e. } t < \min(0,\tau)$$

from which  $\tau = 0$  and  $bC^{-1}(a,b) = \epsilon b'C^{-1}(a',b')$  follow, and thus also  $aC^{-1}(a,b) = \epsilon a'C^{-1}(a',b')$ .

If  $a = 0$ , by (3.12),  $a' = 0$  and (3.12) is satisfied with  $\epsilon = \text{sgn}(bb')$ , since by (3.10)

$$C(0,b) = |b|\left\{ \int_1^\infty |1-v|^{H-1/\alpha} - |v|^{H-1/\alpha} dv + \int_0^1 |v|^{H-1/\alpha} dv \right\}.$$

Hence if  $a = a' = 0$  or if  $b = b' = 0$ , then (3.11) is satisfied.

Now assume  $aa'bb' \neq 0$ . Then (3.12) is equivalent to

$$(3.13) \quad \left\{ \frac{a}{b} = \frac{a'}{b'}, \quad \frac{C^\alpha(a,b)}{|a|^\alpha} = \frac{C^\alpha(a',b')}{|a'|^\alpha}, \quad \frac{C^\alpha(a,b)}{|b|^\alpha} = \frac{C^\alpha(a',b')}{|b'|^\alpha} \right\}.$$

Putting

$$A = \int_1^\infty ||1-v|^{H-1/\alpha} - |v|^{H-1/\alpha}|^\alpha dv,$$

$$f(x) = \int_0^1 ||1-v|^{H-1/\alpha} - x|v|^{H-1/\alpha}|^\alpha dv,$$

$$g(x) = (1+|x|^\alpha)A + f(x),$$

we have from (3.10),  $C^\alpha(a,b) = |a|^\alpha g(b/a) = |b|^\alpha g(a/b)$  and thus (3.13) is equivalent to

$$\left\{ \frac{a}{b} = \frac{a'}{b'}, \quad g\left(\frac{b}{a}\right) = g\left(\frac{b'}{a'}\right), \quad g\left(\frac{a}{b}\right) = g\left(\frac{a'}{b'}\right) \right\}$$

which is equivalent to  $a/b = a'/b'$  since  $g(1/x) = g(x)/|x|^\alpha$ . This completes the proof when  $H \neq 1/\alpha$ .

When  $H = 1/\alpha$ , we find  $||\Delta_{1/\alpha}(a,b;t)||_\alpha^\alpha = |t| C_{1/\alpha}^\alpha(a,b)$  with

$$\begin{aligned} (3.14) \quad C_{\alpha, 1/\alpha}^\alpha(a,b) &= 2|a|^\alpha \int_1^\infty |\log(v-1) - \log v|^\alpha dv \\ &\quad + \int_0^1 |a[\log(1-v) - \log v] + b|^\alpha dv \\ &=: |a|^\alpha \{C + f(b/a)\}, \end{aligned}$$

where

$$f(x) = \int_0^1 |\log(1-v) - \log v + x|^\alpha dv.$$

The rest of the argument is similar to the case  $H \neq 1/\alpha$ , using instead expressions (3.8) and (3.9). □

The linear fractional stable processes have the following time domain symmetry

$$(3.15) \quad \Delta_{\alpha, H}(a, b; \cdot) \stackrel{d}{=} \Delta_{\alpha, H}(a, b; -\cdot).$$

In general, if a self-similar process  $X = (X(t))_{t \in \mathbb{R}}$  has a version of the form

$$X(t) = \int_{-\infty}^\infty [f(t-u) - f(-u)] dM(u), \quad t \in \mathbb{R},$$



where  $M = (M(u))_{u \in \mathbb{R}}$  has stationary and symmetrically distributed increments, then  $X$  has the property (3.15). Indeed, by using the si property of  $M$  and the symmetry of the distribution of  $X$ , we have

$$\begin{aligned} X(\cdot) &= \int_{-\infty}^{\infty} [f(\cdot - u) - f(-u)] dM(u) \\ &\stackrel{d}{=} \int_{-\infty}^{\infty} [f(-v) - f(-\cdot - v)] dM(v) \\ &= -X(-\cdot) \stackrel{d}{=} X(-\cdot). \end{aligned}$$

We now comment on the sample path properties of the linear fractional stable processes  $\Delta_{\alpha, H}(a, b; t)$  with  $0 < \alpha < 2$ . The fractional Brownian motion  $\Delta_{2, H}(a, b; t)$  has always a sample continuous version. Kolmogorov's moment criterion implies that if  $X(t)$  is  $H$ -ss si with  $0 < H \leq 1$  and  $E|X(t)|^p < \infty$  for some  $p > 1/H$ , then  $X(t)$  has a sample continuous version. Since  $\Delta_{\alpha, H}(a, b; t)$  is SaS,  $E|\Delta_{\alpha, H}(a, b; t)|^p < \infty$  for any  $0 < p < \alpha$ . Thus, if  $1/\alpha < H < 1$ , then there exists  $p$  such that  $1/H < p < \alpha$ , and hence  $\Delta_{\alpha, H}(a, b; t)$  has a sample continuous version; still the paths have a.s. nowhere bounded variation, since  $H < 1$  ([24], Theorem 3.3). When  $0 < H \leq 1/\alpha$ , the kernels of the stable integrals defining  $\Delta_{\alpha, H}(a, b; t)$  have singular points, which implies that their sample paths are nowhere bounded (see [17] and [19], and for a special case [13]).

#### 4. The harmonizable fractional stable processes

In this section, we introduce a new class of complex ss si SaS processes. Let  $0 < \alpha \leq 2$ ,  $0 < H < 1$ ,  $a \geq 0$ ,  $b \geq 0$ ,  $a+b > 0$  and  $\tilde{M}$  be a complex SaS motion (introduced in Section 2). Define

$$(4.1) \quad \Theta_{\alpha, H}(a, b; t) = \int_{-\infty}^{\infty} \frac{e^{it\lambda} - 1}{i\lambda} (a\lambda_+^{1-H-1/\alpha} + b\lambda_-^{1-H-1/\alpha}) d\tilde{M}(\lambda), \quad t \in \mathbb{R}.$$

It is easy to check that  $\theta_{\alpha,H}(a,b;\cdot)$  are H-ss si SoS. When  $\alpha=2$ ,  $\Re \theta_{2,H}(a,b;\cdot)$  is the same process as the fractional Brownian motion, which is the linear fractional stable process with  $\alpha=2$ . This can easily be checked by calculating the covariance functions of those processes. However, when  $\alpha < 2$ ,  $\theta_{\alpha,H}(a,b;\cdot)$  is a new class of H-ss si radially SoS processes, as will be shown in the next section for the case  $1 < \alpha < 2$ . In what follows, we assume  $1 < \alpha \leq 2$  for technical reasons.

The nonanticipating stationary SoS process corresponding to  $\theta_{\alpha,H}(a,b;\cdot)$  given by (2.1) is

$$(4.2) \quad Y_{\alpha,H}(a,b;t) = \int_{-\infty}^{\infty} e^{it\lambda} \frac{1}{-i\lambda-1} (a\lambda_+^{1-H-1/\alpha} + b\lambda_-^{1-H-1/\alpha}) d\tilde{M}(\lambda), \quad t \in \mathbb{R},$$

and is thus harmonizable of the form (2.6) with

$$Z(\lambda) = \int_0^\lambda \frac{1}{-iu-1} (au_+^{1-H-1/\alpha} + bu_-^{1-H-1/\alpha}) d\tilde{M}(u), \quad \lambda \in \mathbb{R}.$$

In view of the harmonizability of its nonanticipating stationary process we call the processes  $\theta_{\alpha,H}$  harmonizable fractional stable processes. It is interesting to note that the fully anticipating stationary SoS process corresponding to  $\theta_{\alpha,H}(a,b;\cdot)$  given by (2.3) is

$$(4.3) \quad \int_{-\infty}^{\infty} e^{it\lambda} \frac{1}{i\lambda-1} (a\lambda_+^{1-H-1/\alpha} + b\lambda_-^{1-H-1/\alpha}) d\tilde{M}(\lambda), \quad t \in \mathbb{R}.$$

Both stationary processes in (4.2) and (4.3) are (H-1)-ss, harmonizable, and they are identically distributed.

The complex harmonizable fractional stable processes  $\theta_{\alpha,H}(a,b;\cdot)$  of (4.1) are indexed by the parameters  $(a,b) \in \mathbb{R}_+^2$ . We now show that each ray through the origin of the parameter space  $\mathbb{R}_+^2$  determines a distinct complex process in distribution (up to scaling of course). We also show that the real harmonizable fractional stable processes  $\Re \theta_{\alpha,H}(a,b;\cdot)$  are all multiples of each other in distribution, namely each  $\Re \theta_{\alpha,H}(a,b;\cdot)$  is a multiple of the process

$$\begin{aligned}
 (4.4) \quad \psi_{\alpha,H}(t) &:= \Re \theta_{\alpha,H}(1,1;t) = \Re \int_{-\infty}^{\infty} \frac{e^{it\lambda}-1}{i\lambda} |\lambda|^{1-H-1/\alpha} d\tilde{M}(\lambda), \quad t \in \mathbb{R}, \\
 &= \int_{-\infty}^{\infty} \frac{\sin t\lambda}{\lambda} |\lambda|^{1-H-1/\alpha} dM_1(\lambda) - \int_{-\infty}^{\infty} \frac{\cos t\lambda-1}{\lambda} |\lambda|^{1-H-1/\alpha} dM_2(\lambda),
 \end{aligned}$$

where  $M_1$  and  $M_2$  are the real and imaginary parts of the complex SoS motion  $\tilde{M}$ . It follows by [16], Theorem 7.2, that  $M_1$  and  $M_2$  are real SoS motions, whose increments are independent at distinct points  $\lambda_1 \neq \lambda_2$ :  $dM_1(\lambda_1) \perp dM_2(\lambda_2)$ , and are dependent at the same point  $\lambda$  in the following manner:

$$(dM_1(\lambda), dM_2(\lambda)) \stackrel{d}{=} 2^{1/2} (d\lambda)^{1/\alpha} R^{1/2} (G_1, G_2)$$

where  $R$  is positive  $(\alpha/2)$ -stable with  $E \exp(-rR) = \exp(-r^{\alpha/2})$ ,  $r \geq 0$ ,  $G_1$  and  $G_2$  are standard normal, and  $R, G_1, G_2$  are independent. It should be mentioned here that  $\theta_{\alpha,H}(1,1;t)$  is also mentioned in the recent paper [20].

**Theorem 4.1** Let  $1 < \alpha \leq 2$ ,  $0 < H < 1$ ,  $a \geq 0$ ,  $b \geq 0$ ,  $a+b > 0$ , and  $\theta_{\alpha,H}(a,b;\cdot)$  be given by

(4.1). Then we have

$$(4.5) \quad (a^{\alpha} + b^{\alpha})^{-1/\alpha} \theta_{\alpha,H}(a,b;\cdot) \stackrel{d}{=} (a'^{\alpha} + b'^{\alpha})^{-1/\alpha} \theta_{\alpha,H}(a',b';\cdot)$$

if and only if

- (i)  $a = a' = 0$  or
- (ii)  $b = b' = 0$  or
- (iii)  $aa'bb' \neq 0$  and  $a/b = a'/b'$ .

Also for all  $a, a', b, b'$  we have

$$(4.6) \quad (a^{\alpha} + b^{\alpha})^{-1/\alpha} \Re \theta_{\alpha,H}(a,b;\cdot) \stackrel{d}{=} (a'^{\alpha} + b'^{\alpha})^{-1/\alpha} \Re \theta_{\alpha,H}(a',b';\cdot).$$

**Proof.** In view of (2.7) the complex radially SoS r.v.'s  $\int f dZ$  and  $\int g dZ$  have the same distribution if and only if their scale parameters  $||\int f dZ||_{\alpha}^{\alpha} = \int |f|^{\alpha} d\mu$  and  $||\int g dZ||_{\alpha}^{\alpha} = \int |g|^{\alpha} d\mu$  are equal. Thus to compare the marginal distributions of  $\theta_{\alpha,H}(a,b;t)$  and  $\theta_{\alpha,H}(a',b';t)$  we need to compute their scale parameter. For

simplicity we drop the subscripts  $\alpha, H$  in  $\theta$  and  $Y$ . We have

$$\begin{aligned}
 ||\theta(a, b; t)||_{\alpha}^{\alpha} &= \int_{-\infty}^{\infty} \left| \frac{e^{it\lambda} - 1}{i\lambda} (a\lambda_+^{1-H-1/\alpha} + b\lambda_-^{1-H-1/\alpha}) \right|^{\alpha} d\lambda \\
 &= \int_{-\infty}^{\infty} \left| \frac{2}{\lambda} \sin\left(\frac{t\lambda}{2}\right) \right|^{\alpha} (a\lambda_+^{\alpha(1-H)-1} + b\lambda_-^{\alpha(1-H)-1}) d\lambda \\
 &= (a^{\alpha} + b^{\alpha}) 2^{\alpha} \int_0^{\infty} \left| \sin\left(\frac{t\lambda}{2}\right) \right|^{\alpha} \lambda^{-(\alpha H + 1)} d\lambda \\
 &= |t|^{\alpha H} (a^{\alpha} + b^{\alpha}) 2^{\alpha(1-H)} \int_0^{\infty} |\sin \lambda|^{\alpha} \lambda^{-(\alpha H + 1)} d\lambda \\
 &=: |t|^{\alpha H} (a^{\alpha} + b^{\alpha}) D_{\alpha, H}.
 \end{aligned}$$

Hence it follows that for each fixed  $t > 0$ ,

$$(a^{\alpha} + b^{\alpha})^{-1/\alpha} \theta(a, b; t) \stackrel{d}{=} (a'^{\alpha} + b'^{\alpha})^{-1/\alpha} \theta(a', b'; t).$$

In view of the one-to-one relationship between the si SaS process  $\theta(a, b; \cdot)$  of (4.1) and its corresponding nonanticipating stationary SaS process  $Y(a, b; \cdot)$  of (4.2), (4.5) is equivalent to

$$(a^{\alpha} + b^{\alpha})^{-1/\alpha} Y(a, b; \cdot) \stackrel{d}{=} (a'^{\alpha} + b'^{\alpha})^{-1/\alpha} Y(a', b'; \cdot)$$

which, because of (2.7), is equivalent to

$$(4.8) \quad \left| \left| \sum_{n=1}^N \bar{z}_n (a^{\alpha} + b^{\alpha})^{-1/\alpha} Y(a, b; t_n) \right| \right|_{\alpha}^{\alpha} = \left| \left| \sum_{n=1}^N \bar{z}_n (a'^{\alpha} + b'^{\alpha})^{-1/\alpha} Y(a', b'; t_n) \right| \right|_{\alpha}^{\alpha}.$$

But

$$\begin{aligned}
 (4.9) \quad & \left| \left| \sum_{n=1}^N \bar{z}_n (a^{\alpha} + b^{\alpha})^{-1/\alpha} Y(a, b; t_n) \right| \right|_{\alpha}^{\alpha} \\
 &= (a^{\alpha} + b^{\alpha})^{-1} \int_{-\infty}^{\infty} \left| \sum_{n=1}^N \bar{z}_n e^{it_n \lambda} \right|^{\alpha} (a\lambda_+^{\alpha(1-H)-1} + b\lambda_-^{\alpha(1-H)-1}) (1+\lambda^2)^{-\alpha/2} d\lambda \\
 &= (a^{\alpha} + b^{\alpha})^{-1} \int_0^{\infty} \left\{ a^{\alpha} \left| \sum_{n=1}^N \bar{z}_n e^{it_n \lambda} \right|^{\alpha} + b^{\alpha} \left| \sum_{n=1}^N \bar{z}_n e^{-it_n \lambda} \right|^{\alpha} \right\} \lambda^{\alpha(1-H)-1} (1+\lambda^2)^{-\alpha/2} d\lambda.
 \end{aligned}$$

Hence if one of conditions (i), (ii), (iii) holds then (4.8) holds, and so does (4.5). This is the "if" part.

We next prove the "only if" part. Since (i) and (ii) are easy cases, we only show that (4.8) implies (iii). As usual, we consider (4.9), with  $N=2$ . Putting  $c = b/a$  and taking  $z_1=1$ ,  $z_2=-1$ ,  $t_1=0$ ,  $t_2=t \geq 0$ , (4.9) becomes

$$\frac{1}{1+c^\alpha} \int_0^\infty \{ |1+ie^{it\lambda}|^\alpha + c^\alpha |1+ie^{-it\lambda}|^\alpha \} \lambda^{\alpha(1-H)-1} (1+\lambda^2)^{-\alpha/2} d\lambda.$$

Therefore (4.8) becomes, with  $c' = b'/a'$ ,

$$\begin{aligned} (4.10) \quad & \frac{(c'^\alpha - c^\alpha)}{(1+c^\alpha)(1-c'^\alpha)} \int_0^\infty \{ |1+ie^{it\lambda}|^\alpha - |1+ie^{-it\lambda}|^\alpha \} \lambda^{\alpha(1-H)-1} (1+\lambda^2)^{-\alpha/2} d\lambda \\ &= \frac{2^{\alpha/2}(c'^\alpha - c^\alpha)}{(1+c^\alpha)(1-c'^\alpha)} \int_0^\infty \{ (1-\sin t\lambda)^{\alpha/2} - (1+\sin t\lambda)^{\alpha/2} \} \lambda^{\alpha(1-H)-1} (1+\lambda^2)^{-\alpha/2} d\lambda = 0 \end{aligned}$$

for all  $t \geq 0$ . We now show that (4.10) is possible only when  $c = c'$ . To this end, it is enough to show that

$$f(t) := \int_0^\infty \{ (1+\sin t\lambda)^{\alpha/2} - (1-\sin t\lambda)^{\alpha/2} \} \lambda^{\alpha(1-H)-1} (1+\lambda^2)^{-\alpha/2} d\lambda \neq 0$$

for some  $t > 0$ .

We first consider the case  $1 \leq \alpha H < 2$ . We have

$$\begin{aligned} f(t) &= \left( \int_0^{\pi/2t} + \int_{\pi/2t}^\infty \right) \{ (1+\sin t\lambda)^{\alpha/2} - (1-\sin t\lambda)^{\alpha/2} \} \lambda^{\alpha(1-H)-1} (1+\lambda^2)^{-\alpha/2} d\lambda \\ &= \int_0^{\pi/2t} \{ (1+\frac{2}{\pi}t\lambda)^{\alpha/2} - (1-\frac{2}{\pi}t\lambda)^{\alpha/2} \} \lambda^{\alpha(1-H)-1} (1+\lambda^2)^{-\alpha/2} d\lambda \\ &\quad - 2 \int_{\pi/2t}^\infty \lambda^{\alpha(1-H)-1} (1+\lambda^2)^{-\alpha/2} d\lambda \\ &> \frac{4}{\pi} t \int_0^{\pi/2t} \lambda^{\alpha(1-H)} (1+\lambda^2)^{-\alpha/2} d\lambda - 2 \int_{\pi/2t}^\infty \lambda^{-\alpha H-1} d\lambda. \end{aligned}$$

In the following,  $c_j(\cdot, \cdot)$  will denote positive constants depending only on the parameters in the parentheses. If  $1 < \alpha H < 2$ , for sufficiently small  $t > 0$  we have

$$f(t) > c_1(\alpha, H) t - c_2(\alpha, H) t^{\alpha H} > 0.$$

If  $\alpha H = 1$ , also for sufficiently small  $t > 0$  we have

$$f(t) > c_3(\alpha) t \log \frac{1}{t} - c_4 t > 0.$$

We next consider the case  $0 < \alpha H < 1$ . By changing variables, we have

$$\begin{aligned} f(t) &= t^{\alpha H} \int_0^\infty \{ (1+\sin x)^{\alpha/2} - (1-\sin x)^{\alpha/2} \} x^{\alpha H-1} (t^2+x^2)^{-\alpha/2} dx \\ &= t^{\alpha H} \int_0^\infty \{ (1+\sin x)^{\alpha/2} - (1-\sin x)^{\alpha/2} \} x^{\alpha H-1} dx \\ &\quad + t^{\alpha H} \int_0^\infty \{ (1+\sin x)^{\alpha/2} - (1-\sin x)^{\alpha/2} \} \{ (1+t^2/x^2)^{-\alpha/2} - 1 \} x^{\alpha H-1} dx \\ (4.12) \quad &=: t^{\alpha H} \{ c_5(\alpha, H) + g(t) \}. \end{aligned}$$

Since  $x^{-\alpha H-1}$  is strictly decreasing on  $(0, \infty)$ , we have  $c_5(\alpha, H) > 0$ . Choosing  $\epsilon > 0$  and  $t > 0$  sufficiently small we obtain

$$\begin{aligned} |g(t)| &\leq \int_0^\infty | (1+\sin x)^{\alpha/2} - (1-\sin x)^{\alpha/2} | \{ 1 - (1+t^2/x^2)^{-\alpha/2} \} x^{\alpha H-1} dx \\ &\leq 2 \int_0^\epsilon x^{-\alpha H} dx + 2 \int_\epsilon^\infty \frac{t^2}{x^2} x^{\alpha H-1} dx \\ &< \frac{2\epsilon^{1-\alpha H}}{1-\alpha H} + \frac{2t^2}{(\alpha H+2)\epsilon^{\alpha H+2}} < c_5(\alpha, H). \end{aligned}$$

Hence by (4.12), we have  $f(t) > 0$  for sufficiently small  $t > 0$ . This concludes the proof of the "only if" part.

The case of the real processes  $\Re \theta(a, b; \cdot)$  is easier to handle, because their scale parameter is as in (4.7), and in (4.8) we need to consider only real coefficients  $r_n$ , in which case (4.9) simplifies to

$$\left| \left| \sum_{n=1}^N r_n (a^\alpha + b^\alpha)^{-1/\alpha} \Re Y(a, b; t_n) \right| \right|_\alpha^\alpha = \int_0^\infty \left| \sum_{n=1}^N r_n e^{it_n \lambda} \right|_\alpha^\alpha \lambda^{\alpha(1-H)-1} (1+\lambda^2)^{-\alpha/2} d\lambda$$

and shows that  $(a^\alpha + b^\alpha)^{-1/\alpha} \Re Y(a, b; \cdot) \stackrel{d}{=} (a'^\alpha + b'^\alpha)^{-1/\alpha} \Re Y(a', b'; \cdot)$  for all

$a, b, a', b'$ , and hence (4.6). □

By a reasoning analogous to that at the end of Section 3, if  $1/\alpha < H < 1$ , then  $\theta_{\alpha, H}(a, b; t)$  has a sample continuous version, whose paths have nowhere bounded variation. (When  $0 < H \leq 1/\alpha$  the paths of  $\theta_{\alpha, H}(a, b; t)$  are expected to be nowhere bounded.)

### 5. The harmonizable versus the linear fractional stable processes

In this section we show that, when  $1 < \alpha < 2$ , the real harmonizable fractional stable process introduced in Section 4 is different from the linear fractional stable processes discussed in Section 3.

**Theorem 5.1** Let  $1 < \alpha < 2$ . Then the law of the real harmonizable fractional stable process  $\psi_{\alpha, H}(\cdot)$  of (4.4) is distinct from the laws of the linear fractional stable processes  $\Delta_{\alpha, H}(a, b; \cdot)$  of (3.1) when  $H \neq 1/\alpha$ , and  $\Lambda(\cdot)$  of (3.3) when  $H = 1/\alpha$ .

**Proof.** We first consider the case  $H \neq 1/\alpha$ . Recall from (4.6) that  $\|\psi_{\alpha, H}(t)\|_{\alpha} = |t|^{H-1/\alpha} D_{\alpha, H}^{1/\alpha} =: |t|^H \gamma_{\alpha, H}$  and from (3.10) that  $\|\Delta_{\alpha, H}(a, b; t)\|_{\alpha} = |t|^H C_{\alpha, H}(a, b) =: |t|^H \delta_{\alpha, H}(a, b)$ . For simplicity we delete the subscripts  $\alpha, H$ . We will assume

$$\gamma^{-1} \psi(\cdot) \stackrel{d}{=} \delta^{-1} \Delta(a, b; \cdot)$$

and we will reach a contradiction. This implies the equality in distribution of the corresponding nonanticipating stationary SoS processes given in (4.2) and (3.4):

$$\{\gamma^{-1} \int_{-\infty}^{\infty} e^{it\lambda} \frac{|\lambda|^{1-H-1/\alpha}}{-i\lambda-1} dM(\lambda), t \in \mathbb{R}\} \stackrel{d}{=} \{\delta^{-1} \int_{-\infty}^{\infty} h_{a,b}(t-s) dM(s), t \in \mathbb{R}\}.$$

Introducing the independent radially SoS increments process  $N$ :

$$N(\lambda) = \frac{\delta}{\gamma} \int_0^\lambda \frac{|u|^{1-H-1/\alpha}}{(1+u^2)^{1/2}} d\tilde{M}(u), \quad \lambda \in \mathbb{R},$$

with spectral measure  $\nu$ :  $d\nu(\lambda)/d\lambda = (\delta/\gamma) |\lambda|^{1-H-1/\alpha} (1+\lambda^2)^{-1/2}$ , it follows that

$$\left\{ \Re \int_{-\infty}^{\infty} e^{it\lambda} dN(\lambda), t \in \mathbb{R} \right\} \stackrel{d}{=} \left\{ \int_{-\infty}^{\infty} h_{a,b}(t-s) dM(s), t \in \mathbb{R} \right\}.$$

Since the spectral measure  $N$  has no atoms, the usual inversion relationship gives with  $I=(x,y)$ ,  $0 \leq x < y$ ,

$$\begin{aligned} N_s(I) &:= \frac{1}{2} \{N(I) + \bar{N}(-I)\} = \frac{1}{2} \{N(y) - N(x) + \bar{N}(-x) - \bar{N}(-y)\} \\ &= \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-iyt} - e^{-ixt}}{-it} \Re \left[ \int_{-\infty}^{\infty} e^{it\lambda} dN(\lambda) \right] dt, \end{aligned}$$

where the limit is in probability or in  $\|\cdot\|_\alpha$ -norm. For simplicity we put

$$\frac{e^{-iyt} - e^{-ixt}}{-it} = \int_I e^{-iut} du =: f_I(t)$$

and drop the subscripts in  $h_{a,b}$ . It then follows that

$$\begin{aligned} N_s(I) &\stackrel{d}{=} \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T f_I(t) \left\{ \int_{-\infty}^{\infty} h(t-s) dM(s) \right\} dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-T}^T f_I(t) h(t-s) dt \right\} dM(s). \end{aligned}$$

Now  $\|\int f dZ\|_\alpha = \|f\|_\alpha$  implies that  $\int f_n dZ$  converges in  $\|\cdot\|_\alpha$ -norm if and only if  $f_n$  converges in  $L_\alpha$ . It follows from the above convergence in  $\|\cdot\|_\alpha$  norm as  $T \rightarrow \infty$ , that the integrand  $\int_{-T}^T f_I(t) h(t-s) dt$  converges in  $L_\alpha$  as  $T \rightarrow \infty$ , and hence for a.e.(s) along some subsequence  $T_n \rightarrow \infty$ . But

$$\lim_{T \rightarrow \infty} \int_{-T}^T f_I(t) h(t-s) dt = \int_{-\infty}^{\infty} f_I(t) h(t-s) dt \quad \text{for a.e.}(s)$$

by Lebesgue dominated convergence, since

$$\int_{-\infty}^{\infty} |f_I(t) h(t-s)| dt \leq \left[ \int_{-\infty}^{\infty} \left| \frac{e^{-iyt} - e^{-ixt}}{-it} \right|^{\alpha'} dt \right]^{1/\alpha'} \left[ \int_{-\infty}^{\infty} |h(t-s)|^\alpha dt \right]^{1/\alpha}$$



because  $\alpha' > 1$  ( $1/\alpha + 1/\alpha' = 1$ ). We thus obtain

$$N_s(I) = \frac{d}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f_I(t) h(t-s) dt \right\} dM(s).$$

It follows likewise that the joint distribution of  $N_s(I_1), N_s(I_2)$  is the same as that of

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f_{I_1}(t) h(t-s) dt \right\} dM(s), \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f_{I_2}(t) h(t-s) dt \right\} dM(s).$$

For disjoint intervals  $I_1$  and  $I_2$  in  $\mathbb{R}_+$ , the independence of the former pair of random variables implies that of the latter. But with  $1 < \alpha < 2$ ,  $\int g_1 dZ$  and  $\int g_2 dZ$  are independent if and only if  $g_1 g_2 = 0$  a.e. (in this complex case see [4]). Therefore with  $I_1 = (0, x)$ ,  $I_2 = (a, b)$ ,  $0 < x < a < b$ , we obtain

$$\int_{-\infty}^{\infty} \frac{e^{-ixt} - 1}{-it} h(t-s) dt \cdot \int_{-\infty}^{\infty} \frac{e^{-ibt} - e^{-iat}}{-it} h(t-s) dt = 0 \quad \text{a.e.}(s).$$

The same argument used to show integrability of  $f_I(\cdot)h(\cdot-s)$ , establishes the continuity of the integral  $\int_{-\infty}^{\infty} f_I(t)h(t-s)ds$  as a function of  $s$ , since the map  $\mathbb{R} \ni s \rightarrow h(\cdot-s) \in L_{\alpha}$  is continuous. Thus the above equality holds for all  $s$ . Putting  $s=0$ , we obtain

$$(5.1) \quad \int_{-\infty}^{\infty} \frac{e^{-ixt} - 1}{-it} h(t) dt \cdot \int_{-\infty}^{\infty} \frac{e^{-i(b-a)t} - 1}{-it} e^{-iat} h(t) dt = 0.$$

But for  $h \in L_{\alpha}$  with Fourier transform  $H \in L_{\alpha}$ , we have (cf. [22], Theorem 74)

$$H(x) = \frac{d}{dx} \int_{-\infty}^{\infty} h(t) \frac{e^{-ixt} - 1}{-it} dt \quad \text{a.e.}(x)$$

and

$$H(y+a) = \frac{d}{dy} \int_{-\infty}^{\infty} \frac{e^{-iyt} - 1}{-it} e^{-iat} h(t) dt$$

(which follows by exactly the same proof as for Theorem 74 in [22]). Thus differentiating (5.1) with respect to  $x$  and  $y=b-a$  we obtain  $H(x)H(b) = 0$  for all  $0 < x < b$ , where  $H$  is the Fourier transform of  $h$ . It follows that  $H = 0$  a.e. on  $\mathbb{R}_+$ :

and likewise on  $\mathbb{R}_-$ . Thus  $H = 0$  which implies  $h = 0$  a.e. Because of the linear independence of  $h_1$  and  $h_2$  in (3.6) and (3.7),  $h_{a,b} = 0$  implies  $a = 0 = b$  which contradicts  $a + b \neq 0$ .

The case  $H = 1/\alpha$  can be treated similarly. □

Now consider for  $1 < \alpha \leq 2$  the real harmonizable fractional stable process

$$\psi_{\alpha, 1/\alpha}(t) = \Re e \int_{-\infty}^{\infty} \frac{e^{it\lambda} - 1}{i\lambda} |\lambda|^{1-2/\alpha} d\tilde{M}(\lambda), \quad t \in \mathbb{R},$$

which is  $(1/\alpha)$ -ss si SoS process. It is distinct from SoS motion (the simplest  $(1/\alpha)$ -ss si SoS process), as it does not have independent increments. As we have seen in Theorem 4.1 it is also distinct from the log-fractional stable process  $\Lambda(\cdot)$ , introduced in [11] as a new  $(1/\alpha)$ -ss in SoS process. Thus  $\psi_{\alpha, 1/\alpha}(\cdot)$  is a new example of a  $(1/\alpha)$ -ss si SoS process.

In [7], Theorem 3.1, it was shown (using Beurling's theorem) that the class of complex nonanticipating, invertible, SoS moving average processes is disjoint from that of the regular, harmonizable, stationary SoS processes. The proof of Theorem 5.1 shows that the two classes of processes,  $\{\Re e \int_{-\infty}^{\infty} e^{it\lambda} dZ(\lambda), t \in \mathbb{R}\}$  and  $\{\int_{-\infty}^{\infty} h(t-s) dM(s), t \in \mathbb{R}\}$ , are disjoint. Namely, the entire class of real harmonizable stationary SoS processes is disjoint from that of real SoS moving average processes.

#### 6. The domain of attraction of the harmonic fractional stable process

The domain of attraction of the linear fractional and of the log-fractional stable processes has been studied in [1], [2], [10], [11], [12]. In this section, we study the domain of attraction of the  $H$ -self-similar, symmetric  $\alpha$ -stable process with stationary increments  $\theta_{\alpha, H}(a, b; \cdot)$  introduced in Section 4. We start with  $\theta_{\alpha, H}(1, 1; \cdot)$ .

**Theorem 6.1.** Let  $1 < \alpha < 2$ ,  $1-1/\alpha < H < 1$ ,  $\tilde{M}$  be the complex SaS motion in (4.1),  $\{Y_n\}_{n=-\infty}^{\infty}$  be the harmonizable SaS sequence

$$Y_n = \int_{-\pi}^{\pi} e^{in\lambda} d\tilde{M}(\lambda)$$

and

$$(6.1) \quad Z_n = \sum_{k=-\infty}^{\infty} c_{n-k} Y_k$$

with appropriate coefficients  $\{c_n\}$ . Let  $\gamma = H+1/\alpha-1 \in (0,1)$ . Define  $\{c_n\}$  by  $c_0 = 0$ ,  $c_n = |n|^{\gamma-1}$  for  $n \neq 0$ . Then  $\{Z_n\}$  is well-defined by (6.1) and as  $n \rightarrow \infty$ ,

$$\frac{1}{n^H} \sum_{m=1}^{[nt]} Z_m \xrightarrow{d} C_{\gamma} \theta_{\alpha,H}(1,1;t), \quad t > 0,$$

where  $C_{\gamma} = 4\Gamma(\gamma) \cos(\gamma\pi/2)$ , and  $\xrightarrow{d}$  means the convergence of all finite dimensional distributions of the indicated process.

**Proof. (Step 1)** We show that  $\{Z_n\}$  is well-defined. Since

$$\sum_{|k| \leq K} c_{n-k} Y_k = \int_{-\pi}^{\pi} e^{in\lambda} \left( \sum_{m=n-K}^{n+K} c_m e^{-im\lambda} \right) d\tilde{M}(\lambda),$$

it suffices to show that the Fourier series  $\sum_{m=-\infty}^{\infty} c_m e^{-im\lambda}$  (i.e. the sequence of its partial sums) converges in  $L_{\alpha}(-\pi, \pi)$ . Since  $\gamma < 1$ ,  $c_m = |m|^{\gamma-1} \downarrow 0$  as  $|m| \uparrow \infty$  and thus  $\sum_{m=-\infty}^{\infty} c_m e^{-im\lambda}$  is convergent everywhere except at  $\lambda \equiv 0 \pmod{2\pi}$  (see [3], pp. 87-88). Also since  $\alpha > 1$ , a necessary and sufficient condition for  $\sum_{m=-\infty}^{\infty} c_m e^{-im\lambda} \in L_{\alpha}(-\pi, \pi)$  is  $\sum_{m=-\infty}^{\infty} c_m^{\alpha} |m|^{\alpha-2} < \infty$  (see [3], p. 207), which is satisfied since  $\alpha(\gamma-1)+\alpha-2 = \alpha(H-1)-1 < -1$ . In fact the argument in [3], p. 208, shows that in this case the sequence of partial sums converges in  $L_{\alpha}(-\pi, \pi)$ . For completeness, we show this fact below.

For simplicity, we consider the convergence of

$$f_m(x) := \sum_{k=1}^m c_k \cos kx \quad \text{to} \quad f(x) := \sum_{k=1}^{\infty} c_k \cos kx$$

in  $L_\alpha(0, \pi)$  as  $m \rightarrow \infty$ . By the same argument as in [3], p. 208, for  $n > m$ , if  $\pi/(n-m+1) \leq x < \pi/(n-m)$ ,

$$\begin{aligned} |f_m(x) - f(x)| &\leq \sum_{k=m+1}^n c_k + \left| \sum_{k=n+1}^{\infty} c_k \cos kx \right| \\ &\leq \sum_{k=m+1}^n c_k + \frac{\pi}{x} c_n \leq \sum_{k=m+1}^n c_k + \text{const.} (n-m) c_n \leq 2 \sum_{k=m+1}^n c_k. \end{aligned}$$

Hence, with  $B_n = \sum_{k=m+1}^n c_k$ ,

$$\begin{aligned} \int_0^\pi |f_m(x) - f(x)|^\alpha dx &= \sum_{n=m+1}^{\infty} \int_{\pi/(n-m+1)}^{\pi/(n-m)} |f_m(x) - f(x)|^\alpha dx \\ &\leq \text{const.} \sum_{n=m+1}^{\infty} \int_{\pi/(n-m+1)}^{\pi/(n-m)} B_n^\alpha dx \leq \text{const.} \sum_{n=m+1}^{\infty} B_n^\alpha (n-m)^{-2} \\ &\leq \text{const.} \sum_{\ell=1}^{\infty} B_{\ell+m}^\alpha \ell^{-2}, \end{aligned}$$

by a similar argument as in Theorem 2 in Appendix 22 of [3],

$$\leq o_m(1) + \text{const.} \sum_{\ell=1}^{\infty} c_{\ell+m}^\alpha \ell^{\alpha-2} \leq \text{const.} \sum_{\ell=1}^{\infty} c_{\ell+m}^\alpha \ell^{\alpha-2} \quad \text{for large } m.$$

We will show  $\lim_{m \rightarrow \infty} \sum_{\ell=1}^{\infty} c_{\ell+m}^\alpha \ell^{\alpha-2} = 0$  under our assumptions. Let

$$A := \sum_{\ell=1}^{\infty} c_\ell^\alpha \ell^{\alpha-2} < \infty.$$

For any  $\epsilon > 0$ , there exists  $L = L(\epsilon)$  such that  $\sum_{\ell=L+1}^{\infty} c_\ell^\alpha \ell^{\alpha-2} < \epsilon$ . For any  $m > LA^{1/(1-\gamma)\alpha}/\epsilon$ ,

$$\sum_{\ell=L+1}^{\infty} c_{\ell+m}^\alpha \ell^{\alpha-2} \leq \sum_{\ell=L+1}^{\infty} c_\ell^\alpha \ell^{\alpha-2} < \epsilon.$$

Note that

$$\frac{\ell}{\ell+m} \leq \frac{L}{m} < \frac{\epsilon}{A^{1/(1-\gamma)\alpha}} \quad \text{for } \ell \leq L.$$

Hence

$$c_{\ell+m} = \left[ \frac{1}{\ell+m} \right]^{1-\gamma} < \frac{\epsilon^{1-\gamma}}{A^{1/\alpha}} c_\ell.$$

Therefore any  $m > LA^{1/(1-\gamma)\alpha/\epsilon}$ ,

$$\sum_{\ell=1}^L c_{\ell+m}^{\alpha} \ell^{\alpha-2} < \frac{\epsilon^{(1-\gamma)\alpha}}{A} \sum_{\ell=1}^L c_{\ell}^{\alpha} \ell^{\alpha-2} \leq \epsilon^{(1-\gamma)\alpha}.$$

Hence

$$\limsup_{m \rightarrow \infty} \sum_{\ell=1}^{\infty} c_{\ell+m}^{\alpha} \ell^{\alpha-2} < \epsilon + \epsilon^{(1-\gamma)\alpha},$$

and so

$$\lim_{m \rightarrow \infty} \sum_{\ell=1}^{\infty} c_{\ell+m}^{\alpha} \ell^{\alpha-2} = 0,$$

concluding

$$\lim_{m \rightarrow \infty} \int_0^{\pi} |f_m(x) - f(x)|^{\alpha} dx = 0.$$

(Step 2) We have for each  $t > 0$ ,

$$\begin{aligned} Z_n(t) &:= \frac{1}{n^H} \sum_{m=1}^{[nt]} Z_m \\ &= \frac{1}{n^H} \int_{-\pi}^{\pi} \left( \sum_{m=1}^{[nt]} e^{im\lambda} \right) \left( \sum_{k=-\infty}^{\infty} c_k e^{-ik\lambda} \right) d\tilde{M}(\lambda) \\ &= \frac{1}{n^{\gamma-1/\alpha+1}} \int_{-n\pi}^{n\pi} \left( \sum_{m=1}^{[nt]} e^{imu/n} \right) \left( \sum_{k=-\infty}^{\infty} c_k e^{-iku/n} \right) d\tilde{M}\left(\frac{u}{n}\right) \\ &\stackrel{d}{=} \int_{-n\pi}^{n\pi} \left( \frac{1}{n} \sum_{m=1}^{[nt]} e^{imu/n} \right) \left( \frac{1}{n^{\gamma}} \sum_{k=-\infty}^{\infty} c_k e^{-iku/n} \right) d\tilde{M}(u) \\ (6.2) \quad &= \int_{-n\pi}^{n\pi} K_n(t, u) L_n(u) d\tilde{M}(u) =: \tilde{Z}_n(t), \end{aligned}$$

where

$$\begin{aligned} K_n(t, u) &= \frac{1}{n} \sum_{m=1}^{[nt]} e^{imu/n}, \quad |u| \leq n\pi, \\ &= \frac{1}{n} e^{iu/n} \frac{e^{iu[nt]/n} - 1}{e^{iu/n} - 1}, \quad 0 < |u| \leq n\pi, \\ &\xrightarrow{n \rightarrow \infty} \frac{e^{itu} - 1}{iu}, \quad u \neq 0, \end{aligned}$$

and

$$L_n(u) = \frac{1}{n^\gamma} \sum_{k=-\infty}^{\infty} c_k e^{-iku/n}, \quad 0 < |u| \leq n\pi.$$

We now follow the approach of [14]. Let  $f(\lambda)$  be a symmetric, twice differentiable function such that  $f(\lambda) = 1$ ,  $|\lambda| \leq \pi/4$ ;  $f(\lambda) = 0$ ,  $|\lambda| \geq \pi/2$ ; and  $f(\lambda)$  is monotone decreasing for  $\lambda > 0$ . Then by [14], p. 138,

$$(6.3) \quad c'_k := \frac{1}{2\pi} C_\gamma \int_{-\pi}^{\pi} e^{ik\lambda} |\lambda|^{-\gamma} f(\lambda) d\lambda = |k|^{\gamma-1} + O(|k|^{-2})$$

and

$$(6.4) \quad \begin{aligned} \sum_{k=-\infty}^{\infty} c'_k e^{-ikx} &= C_\gamma |x|^{-\gamma} f(x) \quad \text{for } |x| \leq \pi, \\ &= C_\gamma |x|^{-\gamma} \quad \text{for } |x| \leq \pi/4, \end{aligned}$$

except for  $x = 0$ . Now put

$$(6.5) \quad \tilde{Z}'_n(t) = \int_{-n\pi}^{n\pi} K_n(t, u) L'_n(u) d\tilde{M}(u),$$

where

$$(6.6) \quad \begin{aligned} L'_n(u) &= \frac{1}{n^\gamma} \sum_{k=-\infty}^{\infty} c'_k e^{-iku/n}, \quad 0 < |u| \leq n\pi, \\ &= C_\gamma |u|^{-\gamma} f\left(\frac{u}{n}\right) \end{aligned}$$

by (6.4) and thus,

$$(6.7) \quad L'_n(u) = C_\gamma |u|^{-\gamma}, \quad 0 < |u| \leq n\pi/4,$$

$$(6.8) \quad L'_n(u) \leq C_\gamma |u|^{-\gamma}, \quad 0 < |u| \leq n\pi.$$

(Step 3) We show that for each  $t > 0$  as  $n \rightarrow \infty$ ,

$$(6.9) \quad \tilde{Z}_n(t) - \tilde{Z}'_n(t) \rightarrow 0$$

in probability, or equivalently, in view of (6.2) and (6.5), that as  $n \rightarrow \infty$

$$\int_{-n\pi}^{n\pi} |K_n(t, u)|^\alpha |L_n(u) - L'_n(u)|^\alpha du \rightarrow 0.$$

For  $u \neq 0$  we have

$$|L_n(u) - L'_n(u)| = \frac{1}{n^\gamma} \left| \sum_{k=-\infty}^{\infty} (c_k - c'_k) e^{-iku/n} \right| \leq \frac{\text{const.}}{n^\gamma},$$

since by (6.3),  $c_k - c'_k = O(|k|^{-2})$ . Also for  $0 < |u| \leq n\pi$ ,

$$(6.10) \quad |K_n(t, u)| = \frac{1}{n} \left| \frac{\sin\left(\frac{[nt]u}{2n}\right)}{\sin\left(\frac{u}{2n}\right)} \right| = \frac{2}{|v|} \left| \sin\left(\frac{[nt]u}{2n}\right) \right| \left| \frac{\frac{u}{2n}}{\sin\left(\frac{u}{2n}\right)} \right| \leq \frac{\pi}{|u|}.$$

Since  $\gamma > 0$  and  $\alpha > 1$ , it follows that with  $0 < a < n\pi$ ,

$$(6.11) \quad \int_{a < |u| < n\pi} |K_n(t, u)|^\alpha |L_n(u) - L'_n(u)|^\alpha du \leq \frac{\text{const.}}{n^{\alpha\gamma}} \int_{a < |u| < n\pi} \frac{1}{|u|^\alpha} du$$

$$\leq \frac{\text{const.}}{n^{\alpha\gamma}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus it remains to show that

$$\int_{|u| < a} |K_n(t, u)|^\alpha |L_n(u) - L'_n(u)|^\alpha du \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

However, by the argument in [14] (see Equations (2.10), (2.16), and the next one on p. 139 in [14]), we see that

$$\int_{|u| < n\pi} |K_n(t, u)|^2 |L_n(u) - L'_n(u)|^2 |u|^{q-1} du \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

for any  $q \in (0, 1)$ . Hence

$$\int_{|u| < a} |K_n(t, u)|^2 |L_n(u) - L'_n(u)|^2 |u|^{q-1} du \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which implies (6.11) by Hölder's inequality.

(Step 4) We finally show that for each  $t > 0$ ,

$$(6.12) \quad \tilde{Z}'_n(t) \rightarrow C_{\gamma, \theta, H}(1, 1; t)$$

in probability, or equivalently that

$$\begin{aligned}
 (6.13) \quad & \left| \tilde{Z}'_n(t) - C_\gamma \theta_{\alpha, H}(1, 1; t) \right|_\alpha^\alpha = \int_{-n\pi}^{n\pi} \left| K_n(t, u) L'_n(u) - C_\gamma \frac{e^{itu} - 1}{iu} |u|^{-\gamma} \right|^\alpha du \\
 & + \int_{|u| > n\pi} \left| C_\gamma \frac{e^{itu} - 1}{iu} |u|^{-\gamma} \right|^\alpha du \\
 & \rightarrow 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned}$$

The integral  $\int_{|u| > n\pi}$  tends to 0 as  $n \rightarrow \infty$  because the integrand in Lebesgue integrable over  $\mathbb{R}$ . For the integral over  $|u| < n\pi$  we have from (6.6),

$$\begin{aligned}
 g_n(t, u) &:= K_n(t, u) L'_n(u) - C_\gamma \frac{e^{itu} - 1}{iu} |u|^{-\gamma} \\
 &= C_\gamma |u|^{-\gamma} \left\{ K_n(t, u) f\left(\frac{u}{n}\right) - \frac{e^{itu} - 1}{in} \right\}.
 \end{aligned}$$

It then follows from (6.7) that as  $n \rightarrow \infty$ ,

$$g_n(t, u) \rightarrow 0, \quad u \neq 0,$$

and from (6.8) and (6.10) that

$$|g_n(t, u)| \leq C_\gamma |u|^{-\gamma} \min \left\{ 1+t, \frac{2+\pi}{|u|} \right\}, \quad 0 < |u| \leq n\pi,$$

where the function on the right hand side is in  $L_\alpha(\mathbb{R})$  since  $\alpha\gamma = 1 - \alpha(1-H) < 1$  and  $\alpha(\gamma+1) = \alpha H + 1 > 1$ . Hence the dominated convergence theorem implies

$$\int_{-n\pi}^{n\pi} |g_n(t, u)|^\alpha du \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and thus (6.13) is established.

It then follows from (6.12) that as  $n \rightarrow \infty$ ,

$$\sum_{j=1}^J a_j \tilde{Z}'_n(t_j) \xrightarrow{d} C_\gamma \sum_{j=1}^J a_j \theta_{\alpha, H}(1, 1; t_j)$$

in probability, and the result follows from



$$\sum_{j=1}^J a_j Z_n(t_j) \stackrel{d}{=} \sum_{j=1}^J a_j \tilde{Z}_n(t_j) \quad .$$

which is established as (6.2), and from (6.9).  $\square$

The restriction  $1-1/\alpha < H < 1$  in Theorem 6.1 was used only in the estimate (6.11). If the left hand side of (6.11) could be shown to converge to zero for  $0 < H < 1$ , then Theorem 6.1 would hold for all  $0 < H < 1$ .

With some modification of the proof of Theorem 6.1 along the idea of Major [14], we obtain the following generalization.

Theorem 6.2 Let  $a \geq 0$ ,  $b \geq 0$ ,  $a+b > 0$  and define

$$A = \frac{1}{8\Gamma(\gamma)} \left\{ \frac{a+b}{\cos(\gamma\pi/2)} - i \frac{a-b}{\sin(\gamma\pi/2)} \right\},$$

where  $\gamma = H+1/\alpha-1$  as in Theorem 6.1. In the assumption of Theorem 6.1, we replace  $c_n = |n|^{\gamma-1}$ ,  $n \neq 0$ , by

$$c_n = \begin{cases} An^{\gamma-1}, & n > 0, \\ \bar{A}|n|^{\gamma-1}, & n < 0. \end{cases}$$

Then we have

$$\frac{1}{n^H} \sum_{m=1}^{[nt]} Z_m \stackrel{d}{\rightarrow} \theta_{\alpha, H}(a, b; t).$$

Sketch of Proof. We give only the outline of the proof here and omit the details.

With the same function  $f$ , as in the paragraph preceding (6.3), we define, instead of the  $\{c'_k\}$  of (6.3),

$$a'_k := \frac{1}{2\pi} K_\gamma \int_{-\pi}^{\pi} e^{ik\lambda} |\lambda|^{-\gamma} f(\lambda) d\lambda, \quad b'_k := \frac{1}{2\pi} K'_\gamma \int_{-\pi}^{\pi} e^{ik\lambda} |\lambda|^{-\gamma} f(\lambda) \operatorname{sgn} \lambda d\lambda,$$

where

$$K_\gamma = 2(A+\bar{A})\Gamma(\gamma)\cos(\gamma\pi/2), \quad K'_\gamma = 2(A-\bar{A})\Gamma(\gamma)\sin(\gamma\pi/2).$$

Then by the same argument as in [14],

$$a'_k = \frac{1}{2}(A+\bar{A}) |k|^{\gamma-1} + O(k^{-2}), \quad b'_k = \frac{1}{2}(A-\bar{A}) |k|^{\gamma-1} \operatorname{sgn} k + O(k^{-2})$$

implying

$$(6.14) \quad c'_k := a'_k + b'_k = \begin{cases} A|k|^{\gamma-1} + O(k^{-2}), & k > 0, \\ \bar{A}|k|^{\gamma-1} + O(k^{-2}), & k < 0, \end{cases}$$

and also

$$\sum_{k=-\infty}^{\infty} a'_k e^{-ikx} = K_{\gamma} |x|^{-\gamma} f(x), \quad \sum_{k=-\infty}^{\infty} b'_k e^{-ikx} = iK'_{\gamma} |x|^{-\gamma} f(x) \operatorname{sgn} x.$$

Now put

$$\tilde{Z}'_n(t) = \int_{-n\pi}^{n\pi} K_n(t,u) L'_n(u) d\tilde{M}(u),$$

with

$$\begin{aligned} L'_n(u) &= \frac{1}{n^{\gamma}} \sum_{k=-\infty}^{\infty} c'_k e^{-iku/n}, \quad 0 < u \leq n\pi, \\ &=: L'_{n,1}(u) + L'_{n,2}(u), \end{aligned}$$

where

$$(6.15) \quad L'_{n,1}(u) = K_{\gamma} |u|^{-\gamma} f\left(\frac{u}{n}\right), \quad L'_{n,2}(u) = iK'_{\gamma} |u|^{-\gamma} f\left(\frac{u}{n}\right) \operatorname{sgn} u.$$

Note that from (6.14),  $c_k - c'_k = O(k^{-2})$ . So the same argument as in Step 2 of the proof of Theorem 6.1 concludes that as  $n \rightarrow \infty$ ,

$$\int_{-n\pi}^{n\pi} |K_n(t,u)|^{\alpha} |L_n(u) - L'_n(u)|^{\alpha} du \rightarrow 0.$$

Step 4 will be handled as follows. First note that

$$K_{\gamma} = \frac{1}{2}(a+b), \quad K'_{\gamma} = -\frac{1}{2}(a-b),$$

so that

$$a = K_\gamma + iK'_\gamma, \quad b = K_\gamma - iK'_\gamma.$$

Hence we have

$$\begin{aligned} \theta_{\alpha, H}(a, b; t) &= \int_{-\infty}^{\infty} \frac{e^{itu} - 1}{iu} \{ (K_\gamma + iK'_\gamma) u_+^{-\gamma} + (K_\gamma - iK'_\gamma) u_-^{-\gamma} \} d\tilde{M}(u) \\ &= \int_{-\infty}^{\infty} \frac{e^{itu} - 1}{iu} \{ K_\gamma |u|^{-\gamma} + iK'_\gamma |u|^{-\gamma} \operatorname{sgn} u \} d\tilde{M}(u), \end{aligned}$$

and

$$\begin{aligned} ||\tilde{Z}'_n(t) - \theta_{\alpha, H}(a, b; t)||_\alpha^\alpha &= \int_{-n\pi}^{n\pi} |K_n(t, u) L'_n(u) - \frac{e^{itu} - 1}{iu} |u|^{-\gamma} (K_\gamma + iK'_\gamma \operatorname{sgn} u)|^\alpha du \\ &\quad + \int_{|u| > n\pi} | \frac{e^{itu} - 1}{iu} |u|^{-\gamma} (K_\gamma + iK'_\gamma \operatorname{sgn} u) |^\alpha du. \end{aligned}$$

The second integral above tends to zero. As to the first integral we have from (6.15) that the integrand without the  $\alpha$  power is equal to

$$\begin{aligned} K_n(t, u) L'_{n,1}(u) - K_\gamma \frac{e^{itu} - 1}{iu} |u|^{-\gamma} + K_n(t, u) L'_{n,2}(u) - iK'_\gamma \frac{e^{itu} - 1}{iu} |u|^{-\gamma} \operatorname{sgn} u \\ = K_\gamma |u|^{-\gamma} \{ K_n(t, u) f(\frac{u}{n}) - \frac{e^{itu} - 1}{iu} \} + iK'_\gamma |u|^{-\gamma} \operatorname{sgn} u \{ K_n(t, u) f(\frac{u}{n}) - \frac{e^{itu}}{iu} \}. \end{aligned}$$

By the same reasoning as in the proof of Theorem 6.1, the right hand side above tends to zero, and the integral itself also converges to zero.  $\square$

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