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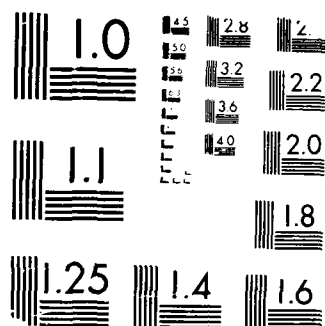
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PARTIAL-DIFFERENTIAL EQUATION
WITH VISCOELASTIC DAMPING**

by

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**Lefschetz Center for Dynamical Systems
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**FEEDBACK CONTROL OF A HYPERBOLIC
PARTIAL-DIFFERENTIAL EQUATION
WITH VISCOELASTIC DAMPING**

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ABSTRACT

In this paper we consider an approximation scheme for an optimal control problem described by a hyperbolic partial-functional differential equation used to model the elastic motion of a viscoelastic body of Boltzmann type. The method is based on combined finite element/averaging approximations. We present theoretical and numerical results for a problem with quadratic cost functional.

Introduction

We discuss the development of numerical schemes for the modeling and control of longitudinal vibrations in a rod with Boltzmann - type viscoelastic damping. It has been thought for some time that a high fidelity model with an accurate description of the damping mechanism is crucial for applications involving control and stabilization of large flexible structures. The model which we describe below may be considered as a prototype for the investigation of such applications. In particular, we consider Boltzmann-type viscoelastic damping in our model (other types of damping which have been studied include viscous, Kelvin-Voigt, structural, hysteretic, etc.).

Our ideas apply and extend to various structures, but for simplicity of exposition we consider a model for the longitudinal motion of a uniform bar with fixed ends. This leads to the following equation (see [5]):

$$(1.1) \quad \rho \frac{\partial^2}{\partial t^2} u(t, x) = \frac{\partial}{\partial x} \left\{ \alpha \frac{\partial}{\partial x} u(t, x) + \int_{-r}^0 g(s) \frac{\partial}{\partial x} u(t+s, x) ds \right\} + f(t, x)$$

where ρ is the density, and f is the (applied) body force. Here, $u(t, x)$ is the longitudinal displacement at position x along the rod at time t . The constant α is a stiffness parameter and the function $g(s)$ (described in more detail below) may be considered as the damping parameter. Note that without the integral term this would be a purely elastic model (the wave equation with no damping). The integral term (referred to as the memory or history term) arises from the fact that in the underlying constitutive equation for the Boltzmann model, the stress is assumed to be a function of the strain and the strain history.

We make the following reasonable physical assumptions on the function

$g: (-\infty, 0] \rightarrow \mathbb{R}$ and constant α :

- (1) $g \in H^1(-r, 0)$, $\alpha > 0$.
- (2) There exists a continuous function $g_0 : (-\infty, 0] \rightarrow (-\infty, 0]$ and constants $\mu > 0$, $\epsilon_0 > 0$ such that
 - (a) $g(s) = g_0(s) = 0$, $s \leq -r$
 - (b) $g(s) \leq g_0(s) < 0$, $-r < s \leq 0$
 - (c) $\epsilon \triangleq \alpha + \int_{-r}^0 g(s) ds \geq \epsilon_0$
 - (d) $\frac{d}{ds} g(s) \leq \mu g_0(s)$ a.e. on $[-r, 0]$.

The existence of ϵ_0 follows from general properties of elastic moduli.

Condition (d) is a "decaying memory" assumption. For a further discussion of the physical basis for these assumptions, see [15], [16], and [24]. The fixed end boundary conditions are given by

$$(1.2) \quad u(t, 0) = 0 = u(t, 1).$$

We consider initial data of the form

$$(1.3) \quad \begin{aligned} u(0, x) &= d(x), \quad \frac{\partial}{\partial t} u(0, x) = v(x), \quad 0 < x < 1 \\ \frac{\partial}{\partial x} u(s, x) &= h(s, x), \quad -r \leq s < 0, \quad 0 < x < 1. \end{aligned}$$

This represents initial displacement, velocity and past history.

We are interested in the problem of constructing a sequence of finite dimensional "approximate" models which can be used for control design. Our ideas are developed in the context of standard results from linear semigroup theory, such as the Hille-Yosida theorem and the Trotter-Kato approximation theorem. In addition, we appeal to many results concerning the abstract linear quadratic cost optimal control problem ([12], [13], [7]). Again we note that our ideas can be easily modified to include other boundary conditions, as well as easily extended to include structures such as Euler-Bernoulli beams with Boltzmann damping.

In section 2, we develop a state space formulation of the class of PFDE's which we will consider. A well-posedness result is given in this context. In section 3, we develop an approximation scheme, and convergence results are given. In section 4, we present some examples, including an application to a quadratic cost optimal control problem.

The notation in this paper is standard. The symbols $\langle \cdot, \cdot \rangle_X$ and $\|\cdot\|_X$ stand for the inner product and norm, respectively, on the Hilbert space X . Often the underlying space X is not specified but will be understood from the context. The symbol $\triangleq$ indicates that the expression on the left is defined by the expression on the right. Also, $\mathfrak{L}(X, Y)$ denotes the space of bounded linear operators from X to Y .

2. A State Space Model

We now proceed with the formulation of the system (1.1)-(1.3) as an equivalent abstract Cauchy problem in an appropriate Hilbert space. In several recent and relevant investigations ([10], [11], [22], [23]), the underlying PFDE is treated as an abstract FDE. In a manner analogous to ordinary FDE's, a solution semigroup is defined, its infinitesimal generator is characterized, and a Cauchy problem is formulated on a "product" space. An important problem in this approach, however, is the prescription of appropriate initial data so that the abstract FDE is well-posed. Kunish and Schappacher have shown ([18]) that a "natural" choice for a product space generally leads to a Cauchy problem which is not well-posed. The correct choice of state space often involves the use of suitable interpolation spaces (see [9], [10]).

We proceed in a different manner by first defining the state space (a "product" space) Z and state operator A . Well-posedness follows when it is shown that A is the infinitesimal generator of a C_0 -semigroup on Z . To proceed, it will be convenient to introduce the space $L_2^0 = L_2^0(0,1)$ defined by

$$L_2^0 = L_2^0(0,1) = \left\{ \varphi \in L_2(0,1) : \int_0^1 \varphi(x)dx = 0 \right\}.$$

To explain the use of the space L_2^0 , we remark that (as will be described below) we choose to formulate the second order (in time) differential equation (1.1) as a first order system using strain $u_x(t,x)$ and velocity $u_t(t,x)$ as states rather than displacement $u(t,x)$ and velocity. (This is done because we use strain and velocity feedback in the optimal control problem to be considered below in Section 3). The space L_2^0 corresponds to that part of the state representing the strain. Heuristically, then, one can think of the zero integral mean condition

$\left[\int_0^1 u_x(t,x) dx = 0 \right]$ as representing the boundary conditions given by (1.2).

Next, let $X = L_2(0,1)$ with norm $\|f\|_X^2 = \int_0^1 |f|^2 dx$,

$Y = L_2^0(0,1)$ with norm $\|f\|_Y^2 = \int_0^1 |f|^2 dx$,

and $W = L_2(-r,0;Y)$ with norm $\|w\|_W^2 = \int_{-r}^0 \|w(s)\|_Y^2 ds$.

Let $G(s)$ and K denote the 2×2 matrices defined by

$$G(s) = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\rho} g(s) \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\alpha}{\rho} \end{bmatrix},$$

respectively. Define the operator $A_0 : \mathcal{D}(A_0) \subset X \times Y \rightarrow X \times Y$ with domain

$$\mathcal{D}(A_0) = H_0^1(0,1) \times [H^1(0,1) \cap L_2^0(0,1)]$$

by

$$A_0 = \begin{bmatrix} 0 & \frac{d}{dx} \\ \frac{d}{dx} & 0 \end{bmatrix}.$$

To obtain a well-posed model, we let $z(t)$ denote the "state"

$$z(t) = \begin{bmatrix} \frac{\partial}{\partial t} u(t,x) \\ \frac{\partial}{\partial x} u(t,x) \\ \frac{\partial}{\partial x} u(t+s,x) \end{bmatrix},$$

which represents the velocity, the strain, and the past history of the strain.

Define the state space Z by

$$Z = X \times Y \times W$$

$$\text{with norm } \left\| \begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} \right\|_Z^2 = \|\varphi\|_X^2 + \|\psi\|_Y^2 + \|w\|_W^2.$$

We shall also consider the equivalent norm on Z given by

$$\left\| \begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} \right\|_g^2 = \rho \|\varphi\|_X^2 + \epsilon \|\psi\|_Y^2 - \int_{-r}^0 g(s) \|w(s)\|_Y^2 ds.$$

Define the operator A on the domain

$$D(A) = \left\{ \begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} \in Z : \begin{array}{l} \varphi \in H_0^1(0,1), \quad w \in H^1(-r,0;Y) \\ \left[\frac{\alpha}{\rho} \psi + \frac{1}{\rho} \int_{-r}^0 g(s) w(s) ds \right] \in H^1(0,1), \quad w(0) = \psi \end{array} \right\}$$

by

$$A \begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} = \begin{bmatrix} \frac{d}{dx} \left(\frac{\alpha}{\rho} \psi + \frac{1}{\rho} \int_{-r}^0 g(s) w(s) ds \right) \\ \frac{d}{dx} \varphi \\ \frac{dw}{ds} \end{bmatrix}$$

$$= \begin{bmatrix} A_0 \left\{ K \begin{bmatrix} \varphi \\ \psi \end{bmatrix} + \int_{-r}^0 G(s) \begin{bmatrix} 0 \\ w(s) \end{bmatrix} ds \right\} \\ \frac{dw}{ds} \end{bmatrix}$$

Let $F(t)$ be given by

$$F(t) = \begin{bmatrix} f(t,x) \\ 0 \\ 0 \end{bmatrix} \in Z.$$

The system (1.1)-(1.3) can now be formulated as the abstract Cauchy problem

$$(2.1) \quad \dot{z}(t) = A z(t) + F(t)$$

$$(2.2) \quad z(0) = z_0 \triangleq (v(x), d'(x), h(s,x))^T.$$

We note that this state space formulation is analogous to the usual state space formulation for ordinary FDE's (see [2] and [6] for a discussion of the "reduced" structure). The well-posedness of (2.1)-(2.2) is demonstrated by showing that the operator A is the infinitesimal generator of a C_0 -semigroup on Z , and it is

sufficient to consider the norm $\|\cdot\|_g$ on Z . The operator A is not dissipative, and it will be useful to introduce a dissipative operator A_1 via a similarity transformation. To this end, define the operator $L \in \mathcal{L}(Z, Z)$ by

$$(2.3) \quad L \begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} = \begin{bmatrix} \varphi \\ \psi \\ \psi - w \end{bmatrix}.$$

Note that $L = L^{-1}$ and L is bijective. Next define the operator A_1 on the domain

$$\begin{aligned} \mathcal{D}(A_1) &= \{z \in Z : Lz \in \mathcal{D}(A)\} \\ &= \left\{ \begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} \in Z : \begin{aligned} &\varphi \in H_0^1(0,1), \quad w \in H^1(-r,0;Y) \\ &\left[\frac{\varepsilon}{\rho} \psi - \frac{1}{\rho} \int_{-r}^0 g(s) w(s) ds \right] \in H^1(0,1) \\ &w(0) = 0 \end{aligned} \right\} \end{aligned}$$

by $A_1 \triangleq L^{-1}AL$. In particular, it follows that

$$A_1 \begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} = \begin{bmatrix} \left[\frac{\varepsilon}{\rho} \psi - \frac{1}{\rho} \int_{-r}^0 g(s) w(s) ds \right]' \\ \varphi' \\ \psi' + \frac{dw}{ds} \end{bmatrix}.$$

Well-posedness is provided by the following result.

Theorem 2.1. *The operator A_1 generates a C_0 -semigroup.*

Proof. The proof proceeds as follows. We first verify that A_1 is dissipative, and then show that $\overline{R(\lambda I - A_1)} = Z$ for some $\lambda > 0$. We then show that A_1 is a closed operator. It follows that $(\lambda I - A_1)^{-1}$ exists and is a closed, bounded and densely defined operator. Thus, $D(\lambda I - A_1)^{-1} = Z$, and hence $R(\lambda I - A_1) = Z$. Therefore, $D(A_1)$ will be dense in Z (see Theorem 1.4.6 in [19]), and the Lumer-Phillips Theorem will imply that A_1 generates a C_0 -semigroup. In order to complete the details, let $\begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} \in D(A_1)$. Then

$$\begin{aligned} 2 \left\langle A_1 \begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix}, \begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} \right\rangle_g &= \\ 2 \int_0^1 \left[\epsilon \psi - \int_{-r}^0 g(s) w(s) ds \right]' \varphi dx + 2 \int_0^1 \epsilon \varphi' \psi & \\ - 2 \int_{-r}^0 g(s) \int_0^1 \left[\varphi' + \frac{dw}{ds} \right] w(s) dx ds & \\ = - 2 \int_{-r}^0 g(s) \int_0^1 \frac{dw}{ds} w(s) dx ds & \\ = - \int_{-r}^0 \frac{d}{ds} \left[g(s) \int_0^1 w^2(s) dx \right] ds + \int_{-r}^0 \left[\frac{d}{ds} g(s) \right] \int_0^1 w^2(s) dx ds & \\ \leq \int_{-r}^0 \frac{d}{ds} g(s) \int_0^1 w^2(s) ds & \\ \leq \mu \int_{-r}^0 g_0(s) \int_0^1 w^2(s) dx ds \leq 0. & \end{aligned}$$

Hence, A_1 is dissipative in Z . To show that $\overline{R(\lambda I - A_1)} = Z$, it is sufficient to show that $\overline{(\lambda I - A_1)D} = Z$, where D is given by

$$D = \left\{ \begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} \in \mathcal{D}(A_1) : \varphi \in H^1(0,1), w \in H^1(-r,0;H^1(0,1)) \right\} \subset \mathcal{D}(A_1).$$

Suppose $\begin{bmatrix} u \\ v \\ z \end{bmatrix}$ is orthogonal to $(\lambda I - A)D$. Then

$$\begin{aligned} 0 &= \left\langle u, \lambda \varphi - \frac{\epsilon}{\rho} \psi' + \frac{1}{\rho} \int_{-r}^0 g(s) w'(s) ds \right\rangle_X + \\ &\quad \left\langle v, \lambda \psi - \varphi' \right\rangle_Y + \\ &\quad \left\langle z, \lambda w - \varphi' - \frac{dw}{ds} \right\rangle_W. \end{aligned}$$

for all $\begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} \in D$. For each $\varphi \in H^2(0,1) \cap H_0^1(0,1)$, let $\psi = \frac{1}{\lambda} \varphi'$ and $w(s) \equiv \frac{1}{\lambda} \psi'$ (hence $w(s)$ is a "constant" function in W). Therefore $\left[\text{using } \epsilon = \alpha + \int_{-r}^0 g(s) ds \right]$,

$$\begin{aligned} 0 &= \left\langle u, \lambda \varphi - \frac{\alpha}{\rho \lambda} \psi'' - \frac{1}{\rho} \left[\frac{1}{\lambda} \varphi'' \right] \int_{-r}^0 g(s) ds + \frac{1}{\rho} \int_{-r}^0 g(s) \left[\frac{1}{\lambda} \varphi'' \right] ds \right\rangle_X \\ &= \left\langle u, \lambda \varphi - \frac{\alpha}{\rho \lambda} \psi'' \right\rangle_X \text{ for all } \varphi \in H^2(0,1) \cap H_0^1(0,1). \end{aligned}$$

The image of $H^2(0,1) \cap H_0^1(0,1)$ under the operator $\left[\lambda^2 I - \frac{d^2}{dx^2} \right]$ is dense in $L_2(0,1)$ which implies that $u \equiv 0$. If $u \equiv 0$ and $\psi \equiv 0$, then

$$0 = \left\langle z, \lambda w - \frac{dw}{ds} \right\rangle_W \text{ for all } w \in H^1(-r,0;H^1(0,1))$$

and $z \equiv 0$ by similar reasoning. Next, choose $\varphi \equiv 0$ and $w \equiv 0$. It follows that

$$0 = \langle v, \lambda \psi \rangle_Y \quad \text{for all } \psi \in H^1(0,1) \cap Y.$$

This implies that $v \equiv \text{constant}$, and hence $v \equiv 0$ since $v \in Y$. Thus, we conclude that $\overline{R(\lambda I - A_1)} = Z$. It remains to be shown that A_1 is closed. Assume that the sequence

$$\left\{ \begin{bmatrix} \varphi_n \\ \psi_n \\ w_n \end{bmatrix} \right\}_{n=1}^{\infty} = \{z_n\}_{n=1}^{\infty}$$

satisfies

$$(2.4) \quad z_n \in \mathcal{D}(A_1),$$

$$(2.5) \quad z_n \rightarrow z = \begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} \text{ in } Z,$$

and

$$(2.6) \quad A_1 z_n \rightarrow y = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \text{ in } Z.$$

We must show that $z \in \mathcal{D}(A_1)$ and $A_1 z = y$. We deduce from (2.5) and (2.6) that $\varphi_n \rightarrow \varphi$ and

$$(2.7) \quad \varphi_n' \rightarrow \beta.$$

Since the differential operator B , defined by $\mathcal{D}(B) = H_0^1(0,1)$ and $Bf = f'$, is closed on $L_2(0,1)$, we conclude that

$$(2.8) \quad \varphi \in H_0^1(0,1)$$

and

$$(2.9) \quad \varphi' = \beta .$$

By a similar argument, we see that

$$(2.10) \quad \left[\frac{\epsilon}{\rho} \psi - \frac{1}{\rho} \int_{-r}^0 g(s) w(s) ds \right] \in H^1(0,1)$$

and

$$(2.10) \quad \left[\frac{\epsilon}{\rho} \psi - \frac{1}{\rho} \int_{-r}^0 g(s) w(s) ds \right]' = \alpha .$$

Next, (2.6) and (2.7) imply that

$$\frac{dw_p}{ds} \rightarrow \gamma - \beta = \gamma - \varphi' \quad \text{in } W .$$

Also, (2.5) implies that $w_n(s) \rightarrow w(s)$ in W . Again, using an argument similar to that used above, we conclude that

$$(2.12) \quad w \in H^1(-r,0;Y) ,$$

$$(2.13) \quad w(0) = 0 ,$$

and

$$(2.14) \quad \gamma = \varphi' + \frac{dw}{ds} .$$

Combining (2.8), (2.10), (2.12) and (2.13), it follows that $z \in \mathcal{D}(A_1)$. Also, (2.9), (2.11) and (2.14) imply that $A_1 z = y$, and the result follows. ■

Observe that since $A = L^{-1}A_1L$, it follows from Theorem 2.1 that A generates a C_0 -semigroup. We state this in the following.

Corollary 2.1. *The operator A generates a C_0 -semigroup.*

We conclude from Corollary 2.1 that the system (2.1)-(2.2) is well posed.

3. An Approximation Scheme

We are interested in constructing an approximation scheme for an optimal control problem associated with the viscoelastic model. For $\delta > 0$ and a partition of $[0,1]$ given by $0 < \bar{x}_i < 1$, $i = 1, 2, \dots, P$, the operator $S_i^\delta : L_2(0,1) \rightarrow \mathbb{R}$ is defined by

$$S_i^\delta(f) = \frac{1}{2\delta} \int_{\bar{x}_i - \delta}^{\bar{x}_i + \delta} f(x) dx .$$

The problem is to minimize the cost functional

$$J^\delta(u) = \int_0^\infty \left\{ \sum_{i=1}^P \left| S_i^\delta \frac{\partial y}{\partial t}(t, x) \right|^2 + \sum_{i=1}^P \left| S_i^\delta \frac{\partial y}{\partial x}(t, x) \right|^2 + R |u(t)|^2 \right\} dt$$

subject to dynamics governed by

$$(3.1) \quad \frac{\partial^2}{\partial t^2} y(t, x) = \frac{\partial}{\partial x} \left[\alpha \frac{\partial}{\partial x} y(t, x) + \int_{-r}^0 g(s) \frac{\partial}{\partial x} y(t+s, x) ds \right] + b(x) u(t),$$

where $b(x) \in L_2(0,1)$ is a given function. Initial data and boundary conditions are as in (1.2)-(1.3). Loosely speaking, the cost functional corresponds to observations of the average velocity and strain in a neighborhood of each $\bar{x}_i$. We point out that we consider observations of average velocity and strain rather than point observations of velocity and strain because this allows us to formulate an LQR problem with bounded (rather than unbounded) operators in the cost functional [i.e. point evaluation is not a bounded linear functional on $L_2(0,1)$.]

Based on the results of the previous section, we can treat this problem equivalently as an abstract regulator problem on the state space Z :

$$(3.2) \quad \text{minimize } J^{\delta}(u) = \int_0^{\infty} \left\{ |C^{\delta} z(t)|^2 + R |u(t)|^2 \right\} dt$$

subject to dynamics governed by

$$(3.3) \quad \begin{aligned} \dot{z}(t) &= A z(t) + B u(t) \\ z(0) &= z_0 . \end{aligned}$$

Here Z and A are as defined in the previous section. The operator $B : \mathbb{R} \rightarrow Z$ is defined by

$$B u = \begin{bmatrix} b(x) \\ 0 \\ 0 \end{bmatrix} u(t) .$$

The operator $C^{\delta} : Z \rightarrow \mathbb{R}^{2p}$ is defined by

$$C^{\delta} \begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} = \begin{bmatrix} S_1^{\delta}(\varphi) \\ \cdot \\ \cdot \\ S_p^{\delta}(\varphi) \\ S_1^{\delta}(\psi) \\ \cdot \\ \cdot \\ S_p^{\delta}(\psi) \end{bmatrix} .$$

It can be shown that (see [12]) there exists a unique optimal control for (3.2)-(3.3) which is given in feedback form by

$$u(t) = -Kz(t) .$$

Further, $K = R^{-1}B^*\pi$, where π is the unique non-negative self-adjoint solution of the algebraic Riccati equation

$$(ARE) \quad A^*\pi + \pi A - \pi BR^{-1}B^*\pi + (C^\delta)^*C^\delta = 0 .$$

Since K is a bounded linear functional on Z , we can write

$$\begin{aligned} u(t) &= -Kz(t) \\ &= -\langle k_1, \varphi \rangle_X - \langle k_2, \psi \rangle_Y - \langle k_3, \omega \rangle_W \end{aligned}$$

where $k_1 \in X$, $k_2 \in Y$, and $k_3 \in W$. We are interested in an approximation scheme which, in addition to accurately simulating the dynamics of (3.1), also gives a reasonable approximation of the functions k_1 , k_2 , and k_3 .

We shall be interested in approximation schemes consisting of sequences of finite dimensional spaces $Z^N \subset Z$ and operators $A^N : Z \rightarrow Z^N$, each A^N the infinitesimal generator of a C_0 -semigroup $T^N(t)$. If $P^N : Z \rightarrow Z^N$ is the orthogonal projection defined by Z^N , then for each N we have the following finite dimensional regulator problem:

$$(3.2)^N \quad \text{minimize } J_N^\delta(u) = \int_0^\infty \left\{ |C^\delta z^N(t)|^2 + |Ru(t)|^2 \right\} dt$$

subject to

$$\dot{z}^N(t) = A^N z^N(t) + B^N u(t)$$

(3.3)

$$z^N(0) = P^N z_0 .$$

Here $B^N = P^N B$.

This problem is finite dimensional and can be solved numerically. Naturally, the nature of the approximation scheme will determine how well the solution of this problem "approximates" the solution of the original problem (in particular, whether the approximating feedback gains converge). Gibson [13] (see also [4]) studied this problem and showed that, in addition to satisfying $P^N z \rightarrow z$ for all $z \in Z$, an approximation scheme $\{Z^N, A^N\}$ should have the property that

$$(3.4) \quad T^N(t)z \rightarrow T(t)z , \quad \text{for all } z$$

and

$$(3.5) \quad T^{N*}(t)z \rightarrow T^*(t)z , \quad \text{for all } z ,$$

with uniform convergence on bounded t -intervals. Here, $T(t)$ denotes the semigroup generated by A and $T^*(t)$ ($= T(t)^*$) is the semigroup generated by A^* . We note that an approximation scheme $\{Z^N, A^N\}$ which satisfies only (3.4) is useful for "simulation" purposes; i.e., approximation of the open loop problem. However, for the closed loop problem, (3.5) is important in order to guarantee strong convergence of the approximating feedback gains. For the approximation scheme which we shall develop, convergence results (i.e. (3.4) and (3.5)) are given in the context of following version of the Trotter-Kato semigroup approximation theorem (see [19]).

Theorem 3.1. Let $A \in G(M, \beta)$ be the infinitesimal generator of a C_0 -semigroup $T(t)$ in a Hilbert space Z , and suppose there is a sequence of linear operators A^N each of which generates a C_0 -semigroup on Z . If

$$H1) \quad A^N \in G(M, \beta) \quad \text{for } N = 1, 2, \dots,$$

$$H2) \quad A^N z \rightarrow Az \quad \text{for } z \in D, \quad D \text{ a dense subset of } Z,$$

and $H3)$ there exists λ_0 with $\operatorname{Re} \lambda_0 > \beta$ such that

$$(A - \lambda_0)D \text{ is a dense subset of } z.$$

then $T^N(t)z \rightarrow T(t)z$ for all $z \in Z$, $t \geq 0$, and the convergence is uniform in compact t -intervals.

■

With this preliminary discussion in mind (especially the importance of (3.4) and (3.5)), we proceed to develop an approximation scheme for our problem. Recall that the state space Z is defined by

$$(3.6) \quad Z = X \times Y \times W = L_2(0,1) \times L_2^0(0,1) \times L_2(-r,0; L_2^0(0,1))$$

with norm

$$(3.7) \quad \left\| \begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} \right\|_Z^2 = \|\varphi\|_X^2 + \|\psi\|_Y^2 + \|w\|_W^2 = \int_0^1 (\varphi^2 + \psi^2) + \int_{-r}^0 \int_0^1 w^2(s) dx ds.$$

The state operator A is defined on the domain

$$(3.8) \quad D(A) = \left\{ \begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} \in Z : \begin{array}{l} \varphi \in H_0^1(0,1), \quad w \in H^1(-r,0; Y) \\ \left[\frac{\alpha}{\rho} \psi + \frac{1}{\rho} \int_{-r}^0 g(s) w(s) ds \right] \in H^1(0,1), \quad w(0) = \psi \end{array} \right\}$$

by

$$(3.9) \quad A \begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} = \begin{bmatrix} \frac{d}{dx} \left[\frac{\alpha}{\rho} \psi + \frac{1}{\rho} \int_{-r}^0 g(s) w(s) ds \right] \\ \frac{d}{dx} \varphi \\ \frac{dw}{ds} \end{bmatrix}$$

$$= \begin{bmatrix} A_0 \left\{ K \begin{bmatrix} \varphi \\ \psi \end{bmatrix} + \int_{-r}^0 G(s) \begin{bmatrix} 0 \\ w(s) \end{bmatrix} ds \right\} \\ \frac{dw}{ds} \end{bmatrix}.$$

We will also need to consider the adjoint of A . It is straightforward to verify that A^* is defined on the domain

$$(3.10) \quad D(A^*) = \left\{ \begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} \in Z : \begin{array}{ll} \varphi \in H_0^1(0,1), & \psi \in H^1(0,1) \\ w \in H^1(-r,0;Y) & w(-r) = 0 \end{array} \right\}$$

by

$$(3.11) \quad A^* \begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} = \begin{bmatrix} -\psi' \\ -\frac{\alpha}{\rho} \varphi' + w(0) \\ -\dot{w} - \frac{1}{\rho} g(s) \varphi' \end{bmatrix} = \begin{bmatrix} A_0 \begin{bmatrix} -\frac{\alpha}{\rho} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} + \begin{bmatrix} 0 \\ w(0) \end{bmatrix} \\ -\frac{dw}{ds} - \frac{1}{\rho} g(s) \varphi' \end{bmatrix}$$

Equation (3.9) suggests a two-stage approximation for A . In particular, one must approximate the operator A_0 by discretizing the spatial variable x . Further, one must approximate $\frac{d}{ds}$ by discretizing the delay variable. Roughly speaking, we will use a finite element approximation of A_0 , and the AVE approximation scheme (see [2]) for the delay variable.

To proceed, divide $[0,1]$ into N equal subintervals and let $h_i^N(x)$, $i = 0,1,\dots,N$, denote the standard "hat" functions which are continuous, piecewise linear, and satisfy $h_i^N(j/N) = \delta_{ij}$, for $j = 0,1,\dots,N$. For convenience in the ensuing discussion, we shall use the shorthand notation $h_i = h_i^N(x)$ and $h_i' = \frac{d}{dx}h_i^N(x)$. Next, we define the subspace X^N of X by

$$X^N = \text{span} \{h_i \mid i = 1,2,\dots,N-1\}$$

and the subspace Y^N of Y by

$$Y^N = \text{span} \{h_i' \mid i = 1,2,\dots,N-1\}.$$

Next, define the bilinear form $a(\cdot, \cdot)$ on $X \times Y$ by

$$(3.12) \quad a \left(\begin{bmatrix} \varphi_1 \\ \psi_1 \end{bmatrix}, \begin{bmatrix} \varphi_2 \\ \psi_2 \end{bmatrix} \right) = \int_0^1 (\varphi_1' \psi_2 - \psi_1 \varphi_2').$$

We define the operator A_0^N (the finite element approximation of A_0) as the restriction of $a(\cdot, \cdot)$ to $X^N \times Y^N$:

$$(3.13) \quad \langle A_0^N u, v \rangle_{X \times Y} = a(u, v)$$

for $u, v \in X^N \times Y^N$.

To complete the "first stage" of the approximation, let

$$(3.14) \quad W^N = L_2(-r, 0; Y^N) \subset W$$

$$Z^N = X^N \times Y^N \times W^N,$$

and define A^N on the domain

$$(3.15) \quad D(A^N) = \left\{ \begin{bmatrix} \varphi^N \\ \psi^N \\ w^N \end{bmatrix} \in Z^N \mid w^N \in H^1(-r, 0; Y^N), w^N(0) = \psi^N \right\}$$

by

$$(3.16) \quad A^N \begin{bmatrix} \varphi^N \\ \psi^N \\ w^N \end{bmatrix} = \begin{bmatrix} A_0^N \left\{ K \begin{bmatrix} \varphi^N \\ \psi^N \end{bmatrix} + \int_{-r}^0 G(s) \begin{bmatrix} 0 \\ w^N(s) \end{bmatrix} ds \right\} \\ \frac{d}{ds} w^N \end{bmatrix}.$$

Let $P_Z^N : Z \rightarrow Z^N$ denote the orthogonal projection onto Z^N , and note that

$$(3.17) \quad P_Z^N \begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} = \begin{bmatrix} P_X^N \varphi \\ P_Y^N \psi \\ P_W^N w \end{bmatrix},$$

where $P_X^N : X \rightarrow X^N$, $P_Y^N : Y \rightarrow Y^N$, $P_W^N : W \rightarrow W^N$ are the respective orthogonal projections.

Observe that for each N , A^N is the infinitesimal generator of the C_0 -semigroup on Z^N corresponding to the ordinary delay-differential equation

$$(3.18) \quad \frac{d}{dt} \begin{bmatrix} \varphi^N(t) \\ \psi^N(t) \end{bmatrix} = A_0^N K \begin{bmatrix} \varphi^N(t) \\ \psi^N(t) \end{bmatrix} + \int_{-r}^0 (A_0^N G)(s) \begin{bmatrix} \varphi^N(t+s) \\ \psi^N(t+s) \end{bmatrix} ds.$$

Thus, for the "second stage" of approximation, we can use any of several

schemes in the literature for this type of problem (see [2], [3], [17]). Since we are interested in optimal control, in this paper we will use the AVE scheme (see [2]), which is known to have desirable properties (recall (3.4) and (3.5)) for such problems (see [13]).

Subdivide $[-r, 0]$ into M equal subintervals $[t_j^M, t_{j-1}^M]$, $j = 1, 2, \dots, M$, where $t_j^M = -\frac{jr}{M}$. Let $\chi_j^M(\cdot)$ denote the characteristic function of $[t_j^M, t_{j-1}^M]$, $[t_j^M, t_{j-1}^M]$, and define the spaces $W^{N,M} \subset W^N$ and $Z^{N,M} \subset Z^N$ by

$$(3.19) \quad W^{N,M} = \left\{ w^N \in L_2(-r, 0; Y) \mid w^N = \sum_{j=1}^M v_j^N \chi_j^M(\cdot), v_j^N \in Y^N \right\}$$

and

$$(3.20) \quad Z^{N,M} = X^N \times Y^N \times W^{N,M},$$

respectively. Next define the operators $Q^{N,M} : Z^N \rightarrow Z^{N,M}$ by

$$(3.21) \quad Q^{N,M} \begin{bmatrix} \varphi^N \\ \psi^N \\ w^N \end{bmatrix} = \begin{bmatrix} A_0^N \left\{ K \begin{bmatrix} \varphi^N \\ \psi^N \end{bmatrix} + \frac{r}{M} \sum_{j=1}^M G_j^M \begin{bmatrix} 0 \\ (w^N)_j^M \end{bmatrix} \right\} \\ \sum_{j=1}^M \frac{M}{r} [(w^N)_{j-1}^M - (w^N)_j^M] \chi_j^M(\cdot) \end{bmatrix}$$

where

$$(3.22) \quad (w^N)_0^M = \psi^N, \quad (w^N)_j^M = \frac{M}{r} \int_{t_j^M}^{t_{j-1}^M} w^N(s) ds$$

and

$$(3.23) \quad G_j^M = \frac{M}{r} \int_{t_j^M}^{t_{j-1}^M} G(s) ds, \quad \text{and} \quad g_j^M = \frac{M}{r} \int_{t_j^M}^{t_{j-1}^M} g(s) ds,$$

for $j = 1, 2, \dots, M$. Combining the two stages of approximation, we define the operator $A^{N,M} : Z \rightarrow Z^{N,M}$ by

$$(3.24) \quad A^{N,M} = Q^{N,M} P_Z^N.$$

Remark 3.1. In view of hypothesis (H2) of Theorem 3.1, the approximation scheme should satisfy

$$\|A^{N,M} z - Az\|_Z \rightarrow 0$$

for all z in some dense subset of Z . From the triangle inequality,

$$\|A^{N,M} z - Az\| \leq \|Q^{N,M} P_Z^N - A^N P_Z^N\| + \|A^N P_Z^N z - Az\| = S_1 + S_2.$$

Standard spline estimates imply that $S_2 \rightarrow 0$ as $N \rightarrow \infty$ for each z , but convergence of the AVE scheme implies only that $S_1 \rightarrow 0$ as $M \rightarrow \infty$ for each fixed N and z . In particular, the rate of convergence in M of $Q^{N,M}$ to A^N is bounded by $\|A_0^N\|$. Although A_0 is an unbounded operator, it is straightforward to show that $\|A_0^N\| = O(N)$ (see Lemma 3.2). This estimate provides the key to the choice of the index M as a function of the index N so that the convergence in M dominates the unbounded behavior of $\|A_0^N\|$ (see 3.43).

We now state some useful properties of the operator A_0^N and the projections given in (3.17).

Lemma 3.1.

i) If $f \in X$, then

$$(3.25) \quad \|P_X^N f - f\|_X \rightarrow 0 \text{ as } N \rightarrow \infty.$$

ii) If $f \in H^1(0,1) \subset X$, then

$$(3.26) \quad \|P_X^N(f') - f'\|_X \rightarrow 0 \text{ as } N \rightarrow \infty,$$

and

$$(3.27) \quad \|(P_X^N f)' - f'\|_X \rightarrow 0 \text{ as } N \rightarrow \infty.$$

iii) If $f \in Y$, then

$$(3.28) \quad \|P_Y^N f - f\|_Y \rightarrow 0 \text{ as } N \rightarrow \infty.$$

iv)

$$(3.29) \quad P_W^N w(s) = P_Y^N w(s) \text{ in the } L_2 \text{ sense.}$$

v) If $w \in W$, then

$$(3.30) \quad \|P_W^N w - w\|_W \rightarrow 0 \text{ as } N \rightarrow \infty.$$

vi) If $\begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in \mathcal{D}(A_0)$, then

$$(3.31) \quad \left\| A_0^N \begin{bmatrix} P_X^N \varphi \\ P_Y^N \psi \end{bmatrix} - A_0 \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \right\|_{X \times Y} \rightarrow 0 \text{ as } N \rightarrow \infty.$$

vii) If $w \in H^1(-r,0;Y) \subset W$, then $P_W^N w \in H^1(-r,0;Y^N) \subset W^N$, and

$$(3.32) \quad P_W^N \left[\frac{dw}{ds} \right] = \frac{d}{ds} P_W^N w(s) \quad a.e.$$

Proof. For i) and ii), one can use virtually the same argument used by Banks and Kappel in ([3], Theorem 4.1), with the slight modification that our splines satisfy zero boundary conditions. For iii), let $f \in Y = L_2^0(0,1)$ and define $F(x) = \int_0^x f(\tau) d\tau$. Then since P_Y^N is the orthogonal projection of Y onto Y^N , it follows that

$$\begin{aligned} \|P_Y^N f - f\|_Y &= \min_{u \in Y^N} \|u - f\|_Y \leq \left\| \frac{d}{dx} (P_X^N F) - f \right\|_Y \\ &\leq \|(P_X^N F)' - F'\|_Y \rightarrow 0 \end{aligned}$$

by (3.27). To prove iv), we note that it follows from the definition of P_Y^N as an orthogonal projection that

$$0 = \int_0^1 [P_Y^N[w(s)] - [w(s)]] \cdot c^N(x) dx \quad \text{for almost all } s \in (-r,0),$$

for all $c^N(x) \in Y^N$. Hence, it follows that

$$0 = \int_{-r}^0 \int_a^b [P_Y^N[w(s)] - [w(s)]] \cdot b^N(s) dx ds$$

for all $b^N(s) \in W^N$. Now iv) follows since P_W^N is the orthogonal projection of W onto W^N . Next, v) follows from (3.28), (3.29) and the dominated convergence theorem. To see that (3.31) holds, assume that $[\psi] \in \mathcal{D}(A_0)$ and that

$$(3.32) \quad A_0^N \begin{bmatrix} P_X^N \varphi \\ P_Y^N \psi \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N-1} a_i h_i \\ \sum_{i=1}^{N-1} b_i h_i' \end{bmatrix} .$$

It follows from a straightforward calculation using the definition of A_0^N in (3.13) that

$$(3.33) \quad \sum_{i=1}^{N-1} a_i \int_a^b h_i h_j = \int_a^b \psi' h_j \quad j = 1, \dots, N-1$$

and

$$(3.34) \quad \sum_{i=1}^{N-1} b_i \int_a^b h_i' h_j' = \int_a^b (P_X^N \varphi)' h_j' \quad j = 1, 2, \dots, N-1 .$$

It follows that

$$(3.35) \quad \sum_{i=1}^{N-1} a_i h_i = P_X^N(\psi')$$

and

$$(3.36) \quad \sum_{i=1}^{N-1} b_i h_i' = (P_X^N \varphi)' .$$

Therefore, (3.26) and (3.27) imply that

$$\left\| A_0^N \begin{bmatrix} P_X^N \varphi \\ P_Y^N \psi \end{bmatrix} - A_0 \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \right\|_{X \times Y}^2 =$$

$$\|P_X^N(\psi') - \psi'\|_X^2 + \|(P_X^N \varphi)' - \varphi'\|_X^2 \rightarrow 0 .$$

Finally, to verify that vii) is true, let $w \in H^1(-r,0;Y) \subset W$. Then $w(s) = w(-r) + \int_{-r}^s \dot{w}(\tau) d\tau$. Using (3.29) we have, for all $c(s) \in W$,

$$\begin{aligned} 0 &= \left\langle P_W^N w(s) - P_W^N \left[w(-r) + \int_{-r}^s \dot{w}(\tau) d\tau \right], c(s) \right\rangle_W \\ &= \left\langle P_W^N w(s) - \left[P_W^N w(-r) + P_Y^N \int_{-r}^s \dot{w}(\tau) d\tau \right], c(s) \right\rangle_W \end{aligned}$$

Since P_Y^N is bounded, it can be moved inside the integral (see [8], p. 91).

Hence,

$$\begin{aligned} &= \left\langle P_W^N w(s) - \left[P_W^N w(-r) + \int_{-r}^s P_Y^N \dot{w}(\tau) d\tau \right], c(s) \right\rangle_W \\ &= \left\langle P_W^N w(s) - \left[P_W^N w(-r) + \int_{-r}^s P_W^N \dot{w}(\tau) d\tau \right], c(s) \right\rangle_W. \end{aligned}$$

Thus, $P_W^N w(s) = P_W^N w(-r) + \int_{-r}^s P_W^N \dot{w}(\tau) d\tau$, which implies that $P_W^N w(s) \in H^1(-r,0;Y)$ and $\frac{d}{ds} P_W^N w(s) = P_W^N \frac{dw}{ds}$. ■

Thus, $P_W^N w(s) = P_W^N w(-r) + \int_{-r}^s P_W^N \dot{w}(\tau) d\tau$, which implies that $P_W^N w(s) \in H^1(-r,0;Y)$ and $\frac{d}{ds} P_W^N w(s) = P_W^N \frac{dw}{ds}$. ■

The next lemma provides the growth rate for $\|A_0^N\|$.

Lemma 3.2. *There exists a constant K , independent of N and u^N , so that*

$$(3.38) \quad \|A_0^N u^N\|_{X \times Y}^2 \leq K N^2 \|u^N\|_{X \times Y}^2$$

for all $u^N \in X^N \times Y^N$.

Proof. Let $u^N = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, where $u_1 = \sum_{i=1}^{N-1} a_i h_i$ and $u_2 = \sum_{i=1}^{N-1} b_i h_i'$. Let

$A_0^N \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, where $v_1 = \sum_{i=1}^{N-1} \alpha_i h_i$ and $v_2 = \sum_{i=1}^{N-1} \beta_i h_i'$. By a direct

calculation using the definition of A_0^N we have

$$(3.39) \quad \sum_{i=1}^{N-1} \alpha_i \langle h_i, h_j \rangle_X = - \sum_{i=1}^{N-1} b_i \langle h_i', h_j' \rangle_X$$

and

$$(3.40) \quad \sum_{i=1}^{N-1} \beta_i \langle h_i', h_j' \rangle_Y = \sum_{i=1}^{N-1} a_i \langle h_i', h_j' \rangle_Y,$$

for each $j = 1, 2, \dots, N-1$. Hence, using (3.40) twice we have that

$$\begin{aligned} \|v_2\|_Y^2 &= \left\langle \sum_{i=1}^{N-1} \beta_i h_i', \sum_{i=1}^{N-1} \beta_i h_i' \right\rangle_Y \\ &= \left\langle \sum_{i=1}^{N-1} a_i h_i', \sum_{i=1}^{N-1} a_i h_i' \right\rangle_Y \\ (3.41) \quad &= \left\| \sum_{i=1}^{N-1} a_i h_i' \right\|_Y^2 \\ &\leq K \cdot N^2 \cdot \left\| \sum_{i=1}^{N-1} a_i h_i \right\|_Y^2 = KN^2 \|u_1\|_Y^2, \end{aligned}$$

where the last inequality follows by applying the Schmidt inequality ([20]) over each interval. Also, by using (3.39) and the Cauchy-Schwartz inequality, we have that

$$\begin{aligned}
 \|v_1\|_X^2 &= \left\langle \sum_{i=1}^{N-1} \alpha_i h_i, \sum_{i=1}^{N-1} \alpha_i h_i \right\rangle_X \\
 &= - \left\langle \sum_{i=1}^{N-1} b_i h_i', \sum_{i=1}^{N-1} \alpha_i h_i \right\rangle_X \\
 &\leq \|u_2\|_X \cdot \|v_1'\|_X \leq \sqrt{k} \cdot N \|u_1\|_X \cdot \|v_1\|_X,
 \end{aligned}$$

where the last inequality follows by again applying the Schmidt inequality over each interval. Hence,

$$(3.42) \quad \|v_1\|_X^2 \leq KN^2 \|u_1\|_X^2.$$

The result follows from (3.41) and (3.42). ■

We are now in position to prove the necessary convergence results for the approximation scheme defined above in (3.20) and (3.24). As discussed above in Remark 3.1, it will be necessary to impose the following condition on the indices N and M .

Definition. A sequence (N, M_N) satisfies condition C_γ if for all $N = 1, 2, \dots$,

$$(3.43) \quad \left[\int_{-r}^0 \left| \sum_{j=1}^{M_N} g_j^{M_N} \chi_j^{M_N} - g(s) \right|^2 ds \right]^{1/2} < \frac{K}{N^\gamma}, \quad \gamma > 1,$$

where the constant K is independent of N .

We note that due to the convergence properties of the AVE scheme, a sequence satisfying condition C_γ exists for any $g(s) \in L_2(-r,0)$. In particular, for $g \in C^1(-r,0)$, the condition is satisfied by choosing $M_N = N^\gamma$, $\gamma > 1$.

Given a sequence (N, M_N) , let

$$(3.44) \quad A^N = A^{N, M_N},$$

and denote by $T^N(t)$ the C_0 -semigroup generated by A^N on Z .

Lemma 3.3. (Stability) *Let (N, M_N) be a sequence satisfying condition C_γ . Then there exists constants M, β such that for all $N \geq 1$,*

$$\|T^N(t)\|_Z \leq M e^{\beta t}.$$

Proof. For each M , define the norm $\|\cdot\|_{g^M}$ on Z by

$$\left\| \begin{pmatrix} \varphi \\ \psi \\ w \end{pmatrix} \right\|_{g^M}^2 = \int_0^1 (\rho \varphi^2 + \epsilon \psi^2) - \int_{-r}^0 \left(\sum_{j=1}^M g_j^M \chi_j^M \right) \int_a^b w^2(s) dx ds.$$

These norms are uniformly equivalent to the original unweighted norm on Z .

It is sufficient to show that $L^{-1}A^{N,M}L$ is dissipative in the $\|\cdot\|_{g^M}$ norm.

To show this, let $z = \begin{pmatrix} \varphi \\ \psi \\ w \end{pmatrix} \in Z$. Then

$$\langle L^{-1}A^{N,M}Lz, z \rangle_{g^M} = \langle Q^{N,M}P^N Lz, L^* z \rangle_{g^M}$$

where L^* is the adjoint of L with respect to the g^M norm and is given by

$$L^* \begin{pmatrix} \varphi \\ \psi \\ w \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi - \frac{1}{\epsilon} \int_{-r}^0 \left(\sum_{j=1}^M g_j^M \chi_j^M \right) w \\ -w \end{pmatrix}$$

Therefore,

$$\begin{aligned}
 \langle L^{-1} A^{N,M} L z, z \rangle_{g^M} &= \\
 &= \left\langle Q^{N,M} \begin{bmatrix} \varphi^N \\ \psi^N \\ \sum_{j=1}^M (\psi^N - w^N)_j^M \chi_j^M \end{bmatrix}, \begin{bmatrix} \varphi \\ \psi - \frac{1}{\epsilon} \int_{-r}^0 \left(\sum_{j=1}^M g_j^M \chi_j^M \right) w \\ -w \end{bmatrix} \right\rangle_{g^M} \\
 &= \left\langle \begin{bmatrix} A_0^N \begin{bmatrix} \varphi^N \\ \frac{\alpha}{\rho} \psi^N \end{bmatrix} + A_0^N \begin{bmatrix} 0 \\ \frac{1}{\rho} \frac{r}{M} \sum_{j=1}^M g_j^M (\psi^N - w^N)_j^M \end{bmatrix} \\ \sum_{j=1}^M \frac{M}{r} [(w^N)_{j-1}^M - (w^N)_j^M] \chi_j^M \end{bmatrix}, \begin{bmatrix} \varphi \\ \psi - \frac{1}{\epsilon} \int_{-r}^0 \left(\sum_{j=1}^M g_j^M \chi_j^M \right) w \\ -w \end{bmatrix} \right\rangle_{g^M} \\
 &= \left\langle \begin{bmatrix} A_0^N \begin{bmatrix} \varphi^N \\ \frac{\alpha}{\rho} \psi^N \end{bmatrix} + A_0^N \begin{bmatrix} 0 \\ \frac{1}{\rho} \frac{r}{M} \sum_{j=1}^M g_j^M (\psi^N - w^N)_j^M \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \rho \psi^N \\ \epsilon \psi^N - \int_{-r}^0 \left(\sum_{j=1}^M g_j^M \chi_j^M \right) w^N \end{bmatrix} \right\rangle_{X \times Y} \\
 &+ \int_{-r}^0 \left(\sum_{j=1}^M g_j^M \chi_j^M \right) \int_0^1 [(w^N)_{j-1}^M - (w^N)_j^M] \chi_j^M w^N dx ds \\
 &= \int_0^1 \varphi^N \left[\epsilon \psi^N - \int_{-r}^0 \left(\sum_{j=1}^M g_j^M \chi_j^M \right) w^N ds \right] dx \\
 &- \int_0^1 \rho \varphi^N \left[\frac{\alpha}{\rho} \psi^N + \frac{1}{\rho} \left\{ \psi^N \int_{-r}^0 g(s) ds - \frac{r}{M} \sum_{j=1}^M g_j^M (w^N)_j^M \right\} \right] dx \\
 &+ \sum_{j=1}^M \int_0^1 [(w^N)_{j-1}^M - (w^N)_j^M] g_j^M (w^N)_j^M dx.
 \end{aligned}$$

The first two lines cancel exactly, leaving only the term

$$\sum_{j=1}^M \int_0^1 [(w^N)_{j-1}^M - (w^N)_j^M] g_j^M (w^N)_j^M dx \triangleq E_1.$$

Using Cauchy-Schwarz and the inequality $2ab \leq a^2 + b^2$, we get

$$\begin{aligned} E_1 &\leq \sum_{j=1}^M \frac{1}{2} g_j^M \int_0^1 \left\{ [(w^N)_{j-1}^M]^2 - [(w^N)_j^M]^2 \right\} dx \\ &= \frac{1}{2} g_1^M \int_0^1 [(w^N)_0^M]^2 dx - \frac{1}{2} g_M^M \int_0^1 [(w^N)_M^M]^2 dx + \sum_{j=1}^{M-1} \frac{1}{2} (g_{j+1}^M - g_j^M) \int_0^1 [(w^N)_j^M]^2 dx \\ &\leq \frac{1}{2} g_1^M \int_a^b [(w^N)_0^M]^2 dx = \frac{1}{2} g_1^M \int_0^1 (\psi^N)^2. \end{aligned}$$

The last inequality follows since g is increasing, and hence $g_{j+1}^M \leq g_j^M$.

Finally, because g is bounded, the g_1^M are bounded inifirmly in M . We conclude that

$$\langle L^{-1} A^{N,M} L z, z \rangle_g \leq K \langle z, z \rangle_g$$

uniformly in N and M . ■

Next, define the set $D \subset \mathcal{D}(A)$ by

$$D = \left\{ \begin{pmatrix} \varphi \\ \psi \\ w \end{pmatrix} \in \mathcal{D}(A) : w \in C^1(-r, 0; Y) \right\}.$$

Lemma 3.4. (Consistency) *Let (N, M_N) be a sequence satisfying condition C_γ and such that $M_N \rightarrow \infty$ as $N \rightarrow \infty$. Then $A^{N, M_N} z \rightarrow Az$ as $N \rightarrow \infty$ for all $z \in D$.*

Proof. Let $z = \begin{pmatrix} \varphi \\ \psi \\ w \end{pmatrix} \in D$. Then, setting $M = M_N$ for notational ease, we

have

$$\begin{aligned} \|A^{N,M_N} z - Az\|_z^2 &= \\ &= \left\| A_0^N \left\{ K \begin{pmatrix} \varphi^N \\ \psi^N \end{pmatrix} + \sum_{j=1}^M \frac{r}{M} G_j^M \begin{pmatrix} 0 \\ (w^N)_j^M \end{pmatrix} \right\} - A_0 \left\{ K \begin{pmatrix} \varphi \\ \psi \end{pmatrix} + \int_{-r}^0 G(s) \begin{pmatrix} 0 \\ w \end{pmatrix} ds \right\} \right\|_{X \times Y}^2 \\ &+ \int_{-r}^0 \left\| \sum_{j=1}^M \frac{M}{r} \left[[P_w^N w]_{j-1}^M - [P_w^N w]_j^M \right] \chi_j^M - \frac{dw}{ds} \right\|_Y^2 ds \\ &= S_1 + S_2 . \end{aligned}$$

Estimating the term S_1 , we have

$$\begin{aligned} S_1 &\leq \left\| \int_{-r}^0 A_0^N \sum_{j=1}^M \left[G_j^M \chi_j^M - G(s) \right] \begin{pmatrix} 0 \\ w^{N(s)} \end{pmatrix} ds \right\|_{X \times Y}^2 \\ &+ \left\| A_0^N \left\{ K \begin{pmatrix} \varphi^N \\ \psi^N \end{pmatrix} + \int_{-r}^0 G(s) \begin{pmatrix} 0 \\ w^{N(s)} \end{pmatrix} ds \right\} - A_0 \left\{ K \begin{pmatrix} \varphi \\ \psi \end{pmatrix} + \int_{-r}^0 G(s) \begin{pmatrix} 0 \\ w \end{pmatrix} ds \right\} \right\|_{X \times Y}^2 \\ &= F_1 + F_2 . \end{aligned}$$

It follows from (3.31) of Lemma 3.1 that $F_2 \rightarrow 0$. For the term F_1 , apply first the Cauchy-Schwarz inequality and then Lemma 3.2 to get

$$F_1 \leq \int_{-r}^0 \sum_{j=1}^M |g_j^M \chi_j^M - g(s)|^2 ds \cdot N^2 \cdot \|w^N\|_w^2$$

Since N, M_N satisfy condition C_γ , it follows (see 3.43) that $F_1 \rightarrow 0$.

Finally, considering the term S_2 , we have the estimate

$$\begin{aligned}
 S_2 &\leq \int_{-r}^0 \left\| \sum_{j=1}^M \frac{M}{r} \left[[P_w^N w]_{j-1}^M - [P_w^N w]_j^M \right] \chi_j^M - \frac{d}{ds} (P_w^N w) \right\|_Y^2 ds \\
 &\quad + \int_{-r}^0 \left\| \frac{d}{ds} P_w^N w - \frac{dw}{ds} \right\|_Y^2 ds \\
 &= E_1 + E_2 .
 \end{aligned}$$

It follows from (3.30) and (3.32) of Lemma 3.1 that $E_2 \rightarrow 0$. For the term E_1 , we follow an argument analogous to that used by Banks and Burns in the proof of Corollary 3.1 in [2]. This leads to the estimate

$$E_1 \leq r \cdot \left\{ \sup_{1 \leq j \leq M} \frac{5}{2} \gamma_j^M \right\}^2 + \frac{r}{M} \left\{ \frac{3}{2} \gamma_1^M + \frac{K}{2} \right\}^2$$

where

$$\begin{aligned}
 \gamma_j^M &= \sup \left\{ \left\| \left[\frac{d}{ds} P_w^N w \right](\theta) - \left[\frac{d}{ds} P_w^N w \right](\tau) \right\|_Y : \theta, \tau \in [t_j^M, t_{j-1}^M] \right\} \\
 &\leq \sup \left\{ \left\| \frac{dw}{ds}(\theta) - \frac{dw}{ds}(\tau) \right\|_Y : \theta, \tau \in [t_j^M, t_{j-1}^M] \right\}
 \end{aligned}$$

and

$$K = \sup \left\{ \left\| \frac{dw}{ds}(\theta) \right\|_Y : \theta \in [-r, 0] \right\} .$$

Since $N \rightarrow \infty$ and $M \rightarrow \infty$, it follows from the uniform continuity of $\frac{dw}{ds}$ on $[-r, 0]$ that $E_1 \rightarrow 0$. ■

Lemma 3.5. *For the set D defined above, there exists a real number γ_0 such that $(A - \lambda I)D$ is dense in Z for all $\operatorname{Re} \lambda > \gamma_0$.*

Proof. Since A generates a C_0 -semigroup, there exists γ_0 such that $\operatorname{Re} \lambda > \gamma_0$ implies $\lambda \in \rho(A)$. Fix λ with $\operatorname{Re} \lambda > \gamma_0$. For $\begin{bmatrix} \alpha \\ \beta \\ z \end{bmatrix} \in Z$, the equation $(A-\lambda) \begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ z \end{bmatrix}$ has a unique solution $\begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} \in \mathcal{D}(A)$. Define the dense subset $S \subset Z$ by $S = X \times Y \times C(-r, 0; Y)$. If $\begin{bmatrix} \alpha \\ \beta \\ z \end{bmatrix} \in S$, then $\begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix}$ satisfies

$$(3.45) \quad \left[\frac{\alpha}{\rho} \psi + \frac{1}{\rho} \int_{-r}^0 g(s) w(s) ds \right]' - \lambda \varphi = \alpha$$

$$(3.46) \quad \varphi' - \lambda \psi = \beta$$

$$(3.47) \quad \frac{dw}{ds} - \lambda w = z.$$

Since $w \in H^1(-r, 0; Y)$ and $z \in C(-r, 0; Y)$, it follows from (3.47) that $w \in C^1(-r, 0; Y)$. Hence, $S \subset (A-\lambda)\mathcal{D}$ and the result follows. ■

The following is an immediate consequence of the above results.

Theorem 3.2. Let N, M_N be a sequence satisfying condition C_γ , and such that $M_N \rightarrow \infty$ as $N \rightarrow \infty$. If the sequence of operators $A^N : Z \rightarrow Z^{N, M}$ is defined as above (see (3.44)), then the hypotheses of the Trotter-Kato theorem hold; that is, $T^N(t) \rightarrow T(t)$ strongly on Z . ■

Next, we verify a similar result for the adjoint A^* (see (3.10) and (3.11)). Since $\|T^{N, M}(t)\| = \|T^{N, M}(t)^*\|$, we need only verify hypotheses H2) and H3) of the Trotter-Kato theorem. Recall that

$$A^{N,M} = Q^{N,M}P^N = Q^{N,M}\Pi^M P^N = P^N \Pi^M Q^{N,M} \Pi^M P^N$$

where $\Pi^M = \Pi^{N,M} : Z^N \rightarrow Z^{N,M}$ is the orthogonal projection.

It is useful to distinguish between $Q^{N,M}$ (defined on all of Z^N) and its restriction to $Z^{N,M}$. Let $R^{N,M} = Q^{N,M}|_{Z^{N,M}} : Z^{N,M} \rightarrow Z^{N,M}$. Hence,

$$A^{N,M} = P^N \Pi^M R^{N,M} \Pi^M P^N$$

and

$$A^{N,M*} = P^N \Pi^M (R^{N,M})^* \Pi^M P^N,$$

where the adjoint is computed in the unweighted inner product (so that the projections P^N , Π^M are orthogonal). To determine the action of $A^{N,M*}$, we compute matrix representations relative to the following basis. For $i = 1, \dots, N-1$ and $j = 2, \dots, M+1$, let

$$e_i = \begin{bmatrix} h_i \\ 0 \\ 0 \end{bmatrix} \quad e_{(N-1)+i} = \begin{bmatrix} 0 \\ h_i' \\ 0 \end{bmatrix}$$

and

$$e_{j(N-1)+i} = \begin{bmatrix} 0 \\ 0 \\ h_i' \chi_j^M \end{bmatrix}.$$

The matrix representation of $(R^{N,M})^*$ is given by

$$M^{-1}(R^{N,M})^T M,$$

where $M = [\langle e_i, e_j \rangle_Z]$ is the $(M+2)(N-1) \times (M+2)(N-1)$ matrix whose i - j

entry is $\langle e_i, e_j \rangle_Z$, and $R^{N, M}$ is the matrix representation of $R^{N, M}$. Here, the superscript "T" denotes the matrix transpose. $R^{N, M}$ is calculated according to the formula

$$R^{N, M} = M^{-1} [\langle R^{N, M} e_i, e_j \rangle_Z]^T.$$

We have

$$M = \begin{bmatrix} H & 0 & 0 & 0 & . & . & . & 0 \\ 0 & D & 0 & 0 & . & . & . & . \\ 0 & 0 & \frac{r}{M}D & 0 & . & . & . & . \\ 0 & 0 & 0 & \frac{r}{M}D & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & 0 \\ 0 & . & . & . & . & . & 0 & \frac{r}{M}D \end{bmatrix}$$

where

$$H = \left[\int_0^1 h_i h_j \right] \text{ is an } (N-1) \times (N-1) \text{ matrix, and}$$

$$D = \left[\int_0^1 h_i' h_j' \right] \text{ is also an } (N-1) \times (N-1) \text{ matrix.}$$

Also,

$$\left[\langle R^{N,M} c_i, c_j \rangle_z \right] =$$

$$\begin{bmatrix} 0 & D & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ -\frac{\alpha}{\rho} D & 0 & D & 0 & \cdot & \cdot & \cdot & 0 \\ -\frac{r}{\rho M} g_1^M D & 0 & -D & D & 0 & \cdot & \cdot & 0 \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & 0 \\ & & & & & & -D & D \\ -\frac{r}{\rho M} g_M^M D & 0 & 0 & 0 & \cdot & \cdot & \cdot & -D \end{bmatrix}$$

Hence, $R^{N,M}$, the matrix representation of $R^{N,M}$, is

$$\begin{bmatrix} 0 & -\frac{\alpha}{\rho} H^{-1} D & -\frac{r}{\rho M} g_1^M H^{-1} D & \cdot & \cdot & \cdot & \cdot & -\frac{r}{\rho M} g_M^M H^{-1} D \\ I & 0 & 0 & 0 & & & & \\ 0 & \frac{M}{r} I & -\frac{M}{r} I & 0 & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & -\frac{M}{r} I & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & \frac{M}{r} I & -\frac{M}{r} I \end{bmatrix}.$$

Also, the matrix representation of R^{N,M^*} , is

$$\begin{bmatrix} 0 & H^{-1}D & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\alpha}{\rho}I & 0 & I & 0 & & & & \\ -\frac{1}{\rho}g_1^M I & 0 & -\frac{M}{r}I & \frac{M}{r}I & & & & \\ & & & & & & & \\ & & & & & & & 0 \\ & & & & & & -\frac{M}{r}I & \frac{M}{r}I \\ & & & & & & 0 & -\frac{M}{r}I \\ -\frac{1}{\rho}g_M^M I & 0 & & & & & & \end{bmatrix}$$

Next, define the set $D^* \subset \mathcal{D}(A^*)$ by

$$D^* = \left\{ \begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} \in \mathcal{D}(A^*) : w \in C^1(-r, 0; Y) \right\}.$$

If $\begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} \in D^*$, then from the above matrix representations it is clear that

$$(A^{N,M})^* \begin{bmatrix} \varphi \\ \psi \\ w \end{bmatrix} = \begin{bmatrix} A_0^N \begin{bmatrix} -\frac{\alpha}{\rho} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \varphi^N \\ \psi^N \end{bmatrix} + \begin{bmatrix} 0 \\ (w^N)_1^M \end{bmatrix} \\ -\frac{1}{\rho} \varphi^N \sum_{j=1}^M g_j^M \chi_j^M - \sum_{j=1}^M \frac{M}{r} [(w^N)_j^M - (w^N)_{j+1}^M] \chi_j^M \end{bmatrix}$$

where $(w^N)_{M+1}^M$ is defined by $(w^N)_{M+1}^M \triangleq w(-r) = 0$ (compare with (3.11)).

Lemma 3.6. *Let (N, M_N) be a sequence satisfying condition C_γ , and such that $M_N \rightarrow \infty$ as $N \rightarrow \infty$. Then $(A^{N,M})^* \rightarrow A^* z$ as $N \rightarrow \infty$ for all $z \in D^*$.*

Proof. The proof is quite similar to that of Lemma 3.4. Let $\begin{pmatrix} \varphi \\ \psi \\ w \end{pmatrix} \in D^*$, then

$$\begin{aligned} & \left\| (A^{N,M})^* z - A^* z \right\|_Z^2 \leq \\ & \left\| A_0^N \begin{pmatrix} -\frac{\alpha}{\rho} \varphi^N \\ -\psi^N \end{pmatrix} - A_0 \begin{pmatrix} -\frac{\alpha}{\rho} \varphi \\ -\psi \end{pmatrix} \right\|_{X \times Y}^2 \\ & + \left\| [w^N]_1^M - w(0) \right\|_Y^2 \\ & + \int_{-r}^0 \left\| \frac{1}{\rho} \varphi^N, \sum_{j=1}^M g_j^M \chi_j^M - \frac{1}{\rho} \varphi, g(s) \right\|_Y^2 ds \\ & + \int_{-r}^0 \left\| \sum_{j=0}^M \frac{M}{r} [(w^N)_j^M - (w^N)_{j+1}^M] \chi_j^M + \frac{dw}{dw} \right\|_Y^2 ds \\ & = F_1 + F_2 + F_3 + F_4 . \end{aligned}$$

$F_1 \rightarrow 0$ and $F_4 \rightarrow 0$ as before. For the term F_2 ,

$$\begin{aligned} F_2 & \leq \left\| [w^N]_1^M - w^N(0) \right\|_Y^2 + \left\| w^N(0) - w(0) \right\|_Y^2 \\ & = S_1 + S_2 . \end{aligned}$$

Lemma 3.1 implies $S_2 \rightarrow 0$. Also,

$$S_1 \leq \frac{K^2 r^2}{M^2} , \quad \text{where } K = \sup \{ \|\dot{w}(\theta)\| : \theta \in [-r, 0] \}$$

(use (3.22) and Cor. 3.1 in [2]). Hence, $F_2 \rightarrow 0$.

Finally,

$$\begin{aligned} F_3 &\leq \int_{-r}^0 \left\| \frac{1}{\rho} \varphi^{N'} \left[\sum_{j=1}^M g_j^M \chi_j^M - g(s) \right] \right\|_Y^2 ds \\ &\quad + \int_{-r}^0 \left\| \frac{1}{\rho} [\varphi^{N'} - \varphi] g(s) \right\|_Y^2 ds \\ &= E_1 + E_2 . \end{aligned}$$

Clearly $E_2 \rightarrow 0$, and it follows from the Schmidt inequality (see the proof of Lemma 3.2) and condition C_γ that $E_1 \rightarrow 0$. The result follows. ■

Lemma 3.7. *For the set D^* defined above, there exists a real number γ_0 such that $(A^* - \lambda I)D^*$ is dense in Z for all $\operatorname{Re} \lambda > \gamma_0$.*

Proof. Use an argument similar to that used in the proof of Lemma 3.5. ■

Theorem 3.3. *Let N, M_N satisfy condition C_γ . Then $T^{N, M}(t)^* z \rightarrow T(t)^* z$ for all $z \in Z$ uniformly on compact t -intervals.*

Therefore, from Theorem 3.2 and Theorem 3.3, we see that the approximation scheme defined above satisfies (3.4) and (3.5). As discussed above, we conclude that this scheme provides a reasonable approximation for simulation purposes (the open-loop problem) as well as for purposes of constructing feedback gains for the (closed-loop) regulator problem. In the next

section, we give the results of some numerical experiments using this approximation scheme.

4. Numerical Results

In this section we present some numerical results related to the approximation scheme which we have discussed. Let $g_p(s)$ denote the function

$$g_p(s) = \begin{cases} -e^{5s}(-s)^{-1/2} & , -r \leq s \leq -r/p \\ -e^{5s}[\Gamma(s + \frac{r}{p}) + (\frac{p}{r})^{1/2}] & , -r/p \leq s \leq 0, \end{cases} \quad (4.1)$$

where the constants Γ and p are related by

$$\Gamma = \left[\frac{1 + (e^{-5r(p/r)^{1/2}})/5}{((e^{-5r/p}-1) + (5r/p))/25} \right] \quad (4.2)$$

and $1 \leq p < +\infty$. Condition (4.2) implies that $\int_{-r}^0 g_p(s)ds < 1$ and for the parameters $\rho = \alpha = 1$ used in the numerical runs below, it can be shown that g_p satisfies Conditions (1) and (2) in Section 1 above. Observe that as $p \rightarrow \infty$, $g_p(s) \rightarrow -e^{5s}(-s)^{-1/2}$ and $g_1(s)$ is a linear function. The selection of the form of the function $g_p(s)$ is made to insure that Conditions (1) and (2) hold and to investigate the case where $g_p(s) \approx -(-s)^{-1/2}$ at $s = 0$. This singular case is important in damping studies. In particular, it can be shown that if $g(s) = -e^{5s}(-s)^{-1/2}$, then the (open-loop) eigenvalues of A are asymptotic to a quadratic curve and if $g \in H^1(-r,0)$, then the (open-loop) eigenvalues of A are asymptotic to a vertical line (see [14] for details). Although $g_p(s)$ belongs to $H^1(-r,0)$, for $p = 2^{10}$ the function $g_p(s)$ is "numerically singular" and the

numerical results for this case should be indicative of a truly singular kernel. Recall that for $g(s) \equiv 0$, the undamped system is the wave equation with fundamental frequencies $\omega_k = k\pi$, $k = 1, 2, \dots$.

Example 4.1. For this run we set $p = 2^{10}$ and constructed the approximate operator $A^{N,M}$. The IMSL routine EIGRF was used to compute the eigenvalues of $A^{N,M}$ for various values of M . Since we are interested in the damping properties, we display only those eigenvalues $\lambda^{N,M}(k)$, $k = 1, 2, \dots, N-1$, corresponding to the first $N-1$ fundamental frequencies.

Figure 1 illustrates the behavior of $\lambda^{8,M}(k)$, for $M = 4, 8, 16, 32$ and 64 . The interesting feature here is that for low values of M the damping curve predicts near viscous damping and as M increases the damping curve becomes quadratic (as to be expected for "singular kernels"). This figure supports the remark made earlier that condition C_γ is indeed necessary. We made several other runs for other values of N with precisely the same qualitative results.

Example 4.2. We next consider the optimal control problem $((3.2)^N, (3.3)^N)$. For this example we used the kernel $g(s) = g_p(s)/5$ with $p = 2^{10}$, and selected the observation points at $\bar{x}_i = .25, .32, .50$, and $.67$, $i = 1, 2, 3, 4$. As before, $\alpha = \rho = 1$ and $b(x) = x^2$, $0 < x < 1$. Potters method was employed to compute the optimal feedback gains. Shown in Figure 2 are the open loop poles for $N = 8$ and $M = 4, 8, 12, 16$, and 32 . The solid dots are the closed loop poles for $M = 32$.

Example 4.3. Using the same data as in the previous example, we next examine the convergence of the feedback gains. Recall that the feedback gain for the

infinite dimensional problem (3.2), (3.3) can be characterized by functions $k_1 \in X$, $k_2 \in Y$ and $k_3 \in W$. Similarly, the gain for the approximating problem $(3.2)^N$, $(3.3)^N$ can be characterized by functions $k_1^N \in X^N$, $k_2^N \in Y^N$, and $k_3^{N,M} \in W^{N,M}$. In Figure 3 we observe convergence of the functions $k_1^N(x)$ as N and M increase. We used values of M much larger than those of N in order to satisfy condition C_γ . In Figure 4 and Figure 5, we observe similar convergence for k_2^N and $k_3^{N,M}$.

We conclude with the following remarks. Due to the size of the approximating system (for given N and M , $N \times (M+2)$ is the dimension of $Z^{N,M}$), and the requirements on the discretization parameters N and M (i.e. condition C_γ), the time and storage requirements for computation quickly become unreasonable. Hence these numerical results are preliminary in the sense that we could not "push" the scheme much beyond $N = 10$. We are currently working on a related unconditionally convergent scheme (i.e. no condition like C_γ) with the hope of reducing this computational burden. We are also investigating the applicability of these schemes to fractional-power damping models in beams (see [1], [20]).

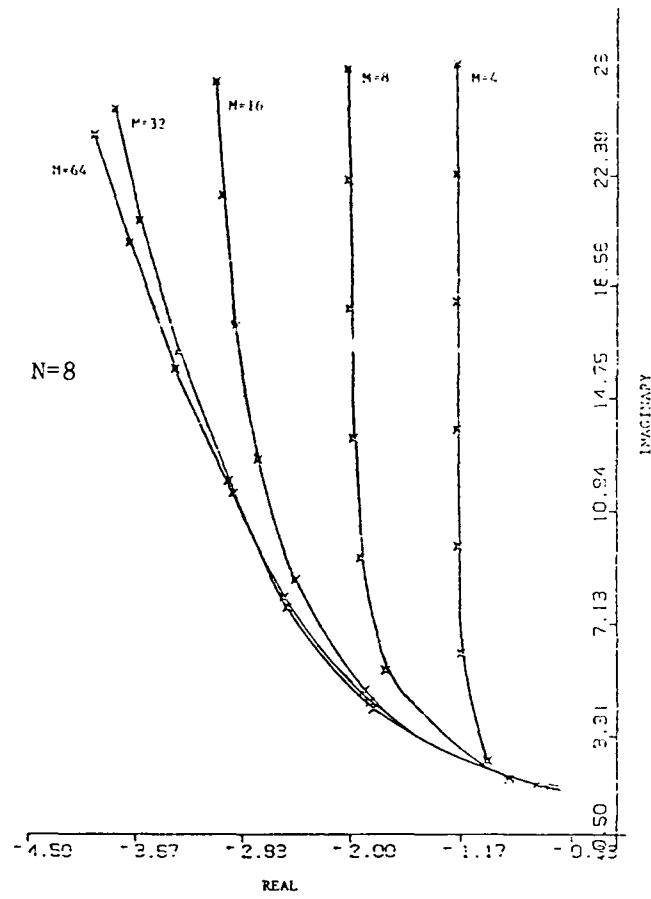


Figure 1

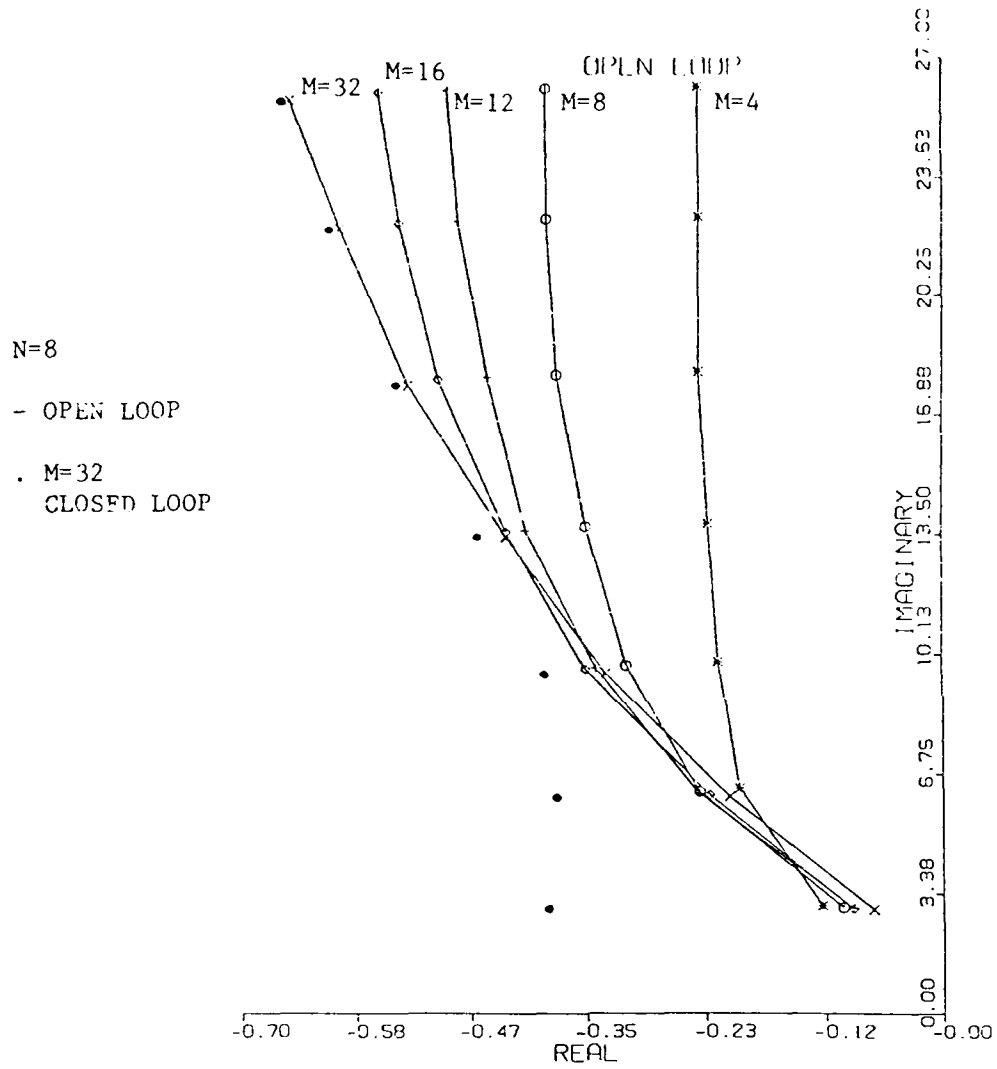


Figure 2

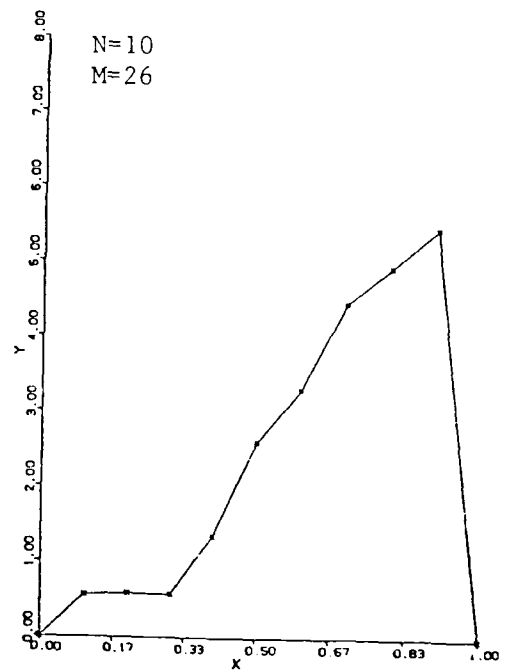
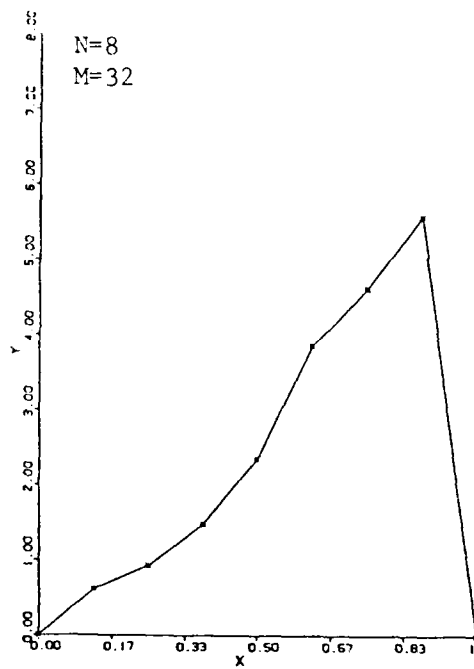
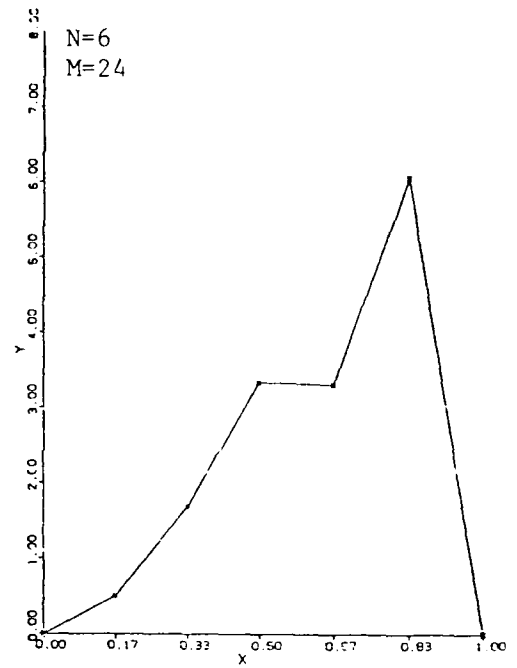
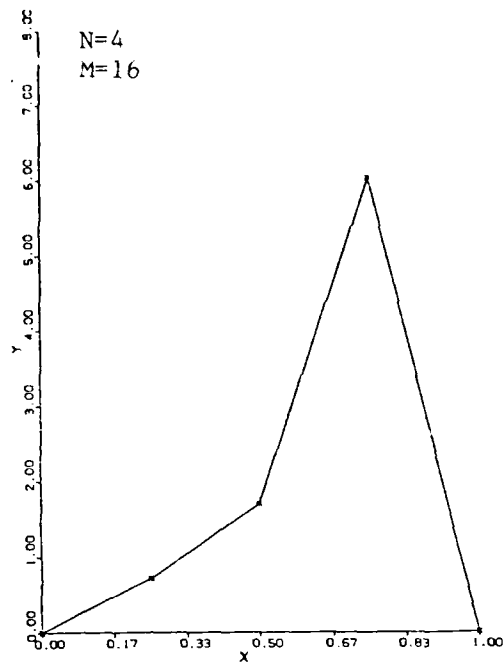


Figure 3

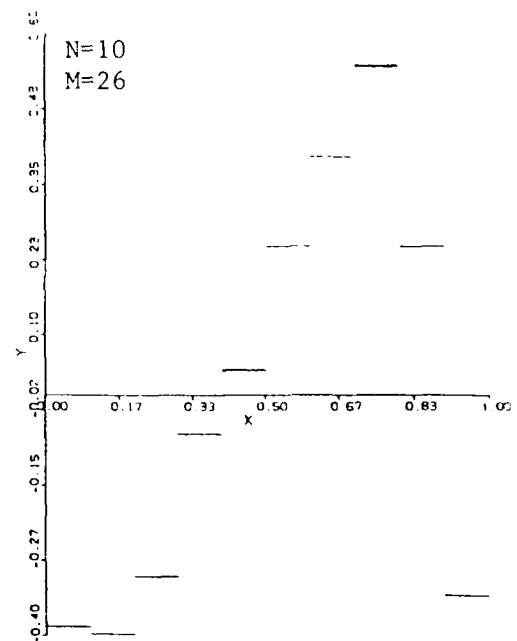
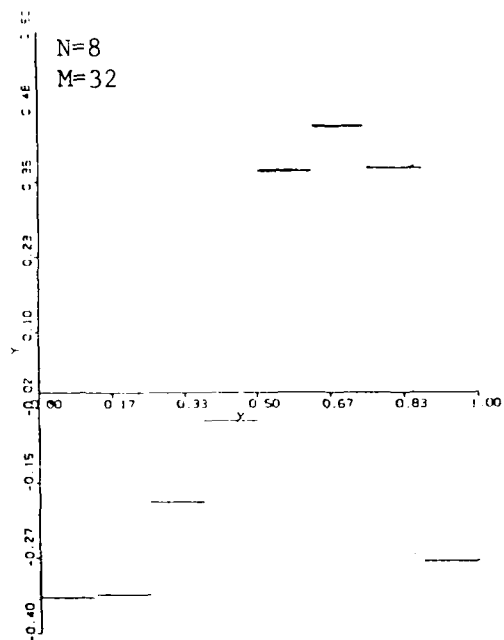
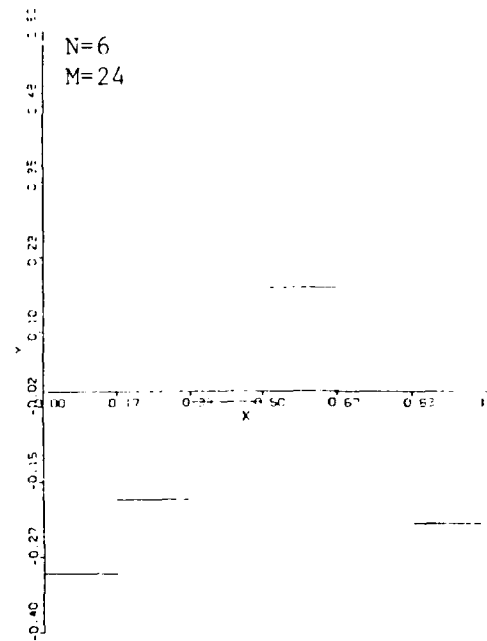
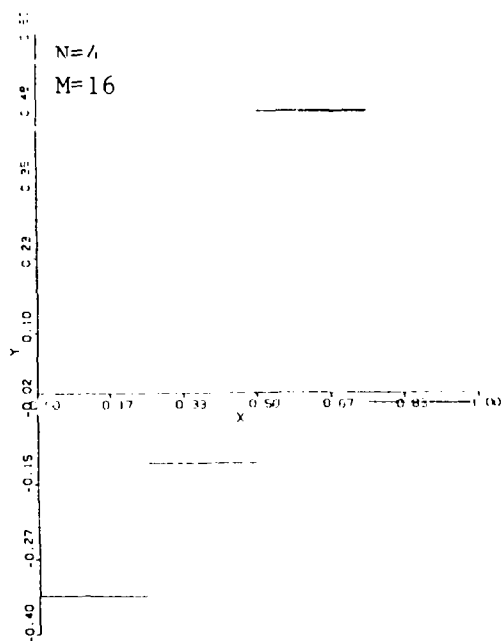


Figure 4

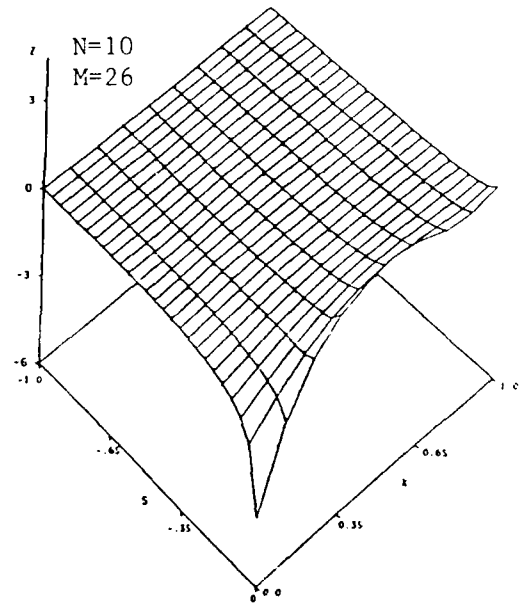
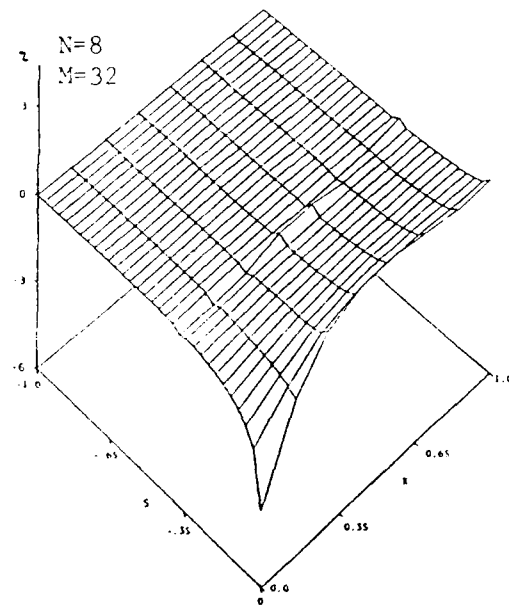
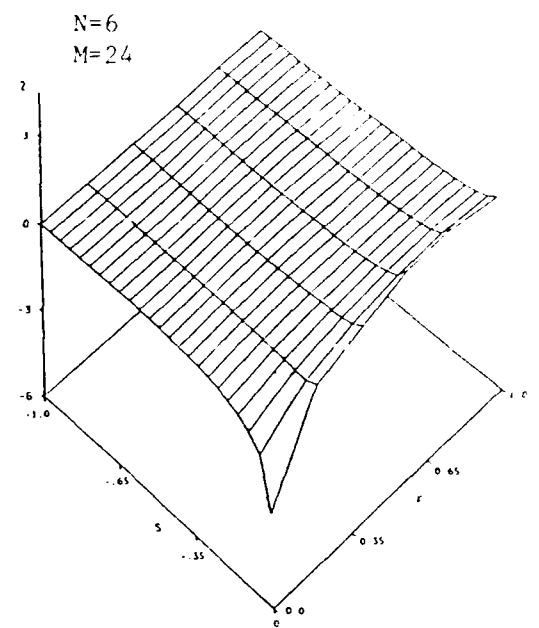
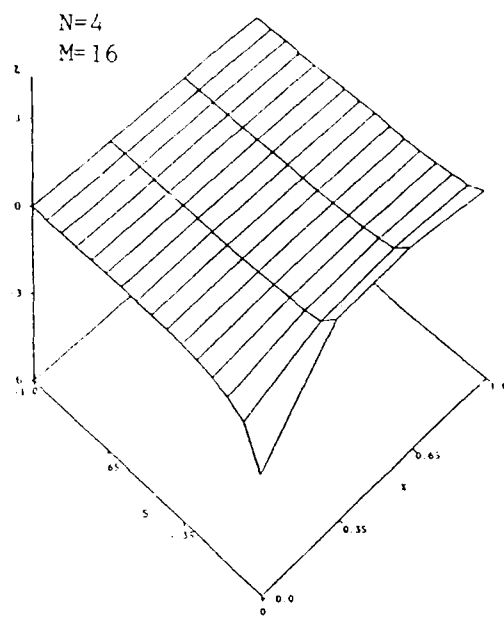


Figure 5

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