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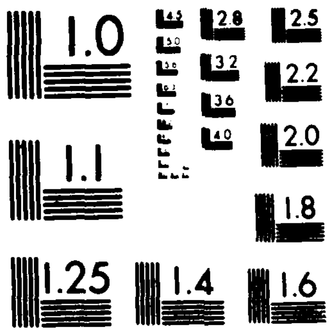

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CMS Technical Summary Report #88-1

ON A CLASS OF FUNCTIONALS INVARIANT UNDER A $\mathbb{Z}^n$ ACTION

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ABSTRACT

Consider a system of ordinary differential equations of the form

$$(*) \qquad \ddot{q} + V_q(t,q) = f(t)$$

where f and V are periodic in t, V is periodic in the components of $q = (q_1,\ldots,q_n)$, and the mean value of f vanishes. By showing that a corresponding functional is invariant under a natural Z^n action, a simple variational argument yields at least $n + 1$ distinct periodic solutions of (*). More general versions of (*) are also treated as is a class of Neumann problems for semilinear elliptic partial differential equations.

AMS (MOS) Subject Classifications: 34C25, 35J60, 58E05, 58F05, 58F22

Key Words: Z^n action periodic solution, critical point, minimax argument, Ljusternik-Schnirelmann category, Neumann problem

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ON A CLASS OF FUNCTIONALS INVARIANT UNDER A $\mathbb{Z}^n$ ACTION

Paul H. Rabinowitz*

Introduction

Consider the system of ordinary differential equations

$$\ddot{q} + V_q(t,q) = f(t) \tag{0.1}$$

where $q = (q_1,\dots,q_n) \in \mathbb{R}^n$ and V satisfies

(V_1) $V \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and is T periodic in t and T_i periodic in q_i, $1 \leq i \leq n$.

Suppose further that f satisfies

(f_1) $f \in C(\mathbb{R},\mathbb{R}^n)$ and is T periodic in t

and

(f_2) $[f] \equiv \frac{1}{T}\int_0^T f(t)dt = 0$.

Note that if $q(t)$ is a solution of (0.1), so is $q(t) + (k_1T_1,\dots,k_nT_n)$ for all $k = (k_1,\dots,k_n) \in \mathbb{Z}^n$. It was shown by Mawhin and Willem [1], Serrin and Pucci [2-3], Li [4], and Franks [5] that if $n = 1$ and V is independent of t, (0.1) possesses at least two T periodic solutions which do not differ by a multiple of T_1. In [5], the proof relies on a generalized version of the Poincaré-Birkhoff Theorem while [1-4] use variational arguments. Part of the difficulty in treating (0.1) in [1-4] is caused by the fact that the corresponding functional:

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$$(0.2) \qquad I(q) = \int_0^T \left[\frac{1}{2}|\dot{q}|^2 - V(t,q) + f \cdot q\right]dt$$

defined on the natural Sobolev space associated with (0.2) does not satisfy the Palais-Smale condition, a compactness criterion very useful for variational problems and henceforth denoted by (PS).

The purpose of this note is to show that in fact if appropriately interpreted, the variational problem does satisfy the (PS) condition. The simple observation that makes this statement precise together with standard techniques leads to a generalization of the above results:

Theorem 0.3: Under the above hypotheses on V and f, (0.1) possesses at least $n + 1$ "distinct" solutions.

What is meant by distinct will be explained in §1. Theorem 0.3 will be obtained from a more general result involving a Lagrangian of the form

$$\mathcal{L}(q,\dot{q}) = \frac{1}{2} L(t,q)\dot{q} \cdot \dot{q} - V(t,q) + f(t) \cdot q$$

where $L(t,q)$ is a symmetric positive definite matrix possessing the same periodicity properties as does V. Actually as will be shown in §1, when L is independent of t and q and $f \equiv 0$, this more general result can be obtained from a theorem of Conley and Zehnder [6, Theorem 3]. They were mainly interested in the much more difficult case of indefinite L.

In §1, the generalizations of Theorem 0.3 will be carried out. Some of the technical details will be given in §2. The ideas used in §1 and 2 can also be applied to a class of Neumann problems for semilinear elliptic partial differential equations. Consider

$$(0.4) \qquad \begin{aligned} -\Delta u &= p(x,u) + h(x), \quad x \in \Omega, \\ \frac{\partial u}{\partial \nu} &= 0, \quad x \in \partial\Omega. \end{aligned}$$

Here Ω denotes a bounded domain in $\mathbb{R}^n$ with a smooth boundary and outward pointing normal $\nu(x)$ and $\frac{\partial u}{\partial \nu} = \nu(x) \cdot \nabla u$. Suppose $p(x,\xi) = \frac{\partial P(x,\xi)}{\partial \xi}$ and $|\Omega|$ denotes the volume of Ω. Then we have

Theorem 0.5: Suppose P satisfies

(p_1) $P \in C^2(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ and $P(x,\xi + r) = P(x,\xi)$ for all $(x,\xi) \in \bar{\Omega} \times \mathbb{R}$

and h satisfies

(h_1) $h \in C^1(\Omega, \mathbb{R})$,

and

(h_2) $[h] \equiv \frac{1}{|\Omega|} \int_\Omega h(x)dx = 0$.

Then (0.4) possesses at least two classical solutions which do not differ by a multiple of r.

The proof of Theorem 0.5 will be carried out in §3. We thank Ed Fadell and Sufian Husseini for helpful conversations.

§1. A generalized version of Theorem 0.3

Let $E = W_T^{1,2}(\mathbb{R},\mathbb{R}^n)$ denote the Sobolev space of T periodic functions with values in $\mathbb{R}^n$ under the norm

$$\|q\| = \left(\int_0^T |\dot{q}|^2 dt + [q]^2\right)^{1/2} .$$

Suppose L satisfies

(L_1) $L \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^{n^2})$ is a symmetric matrix with $L(t,q)$ being T periodic in t and T_i periodic in q_i, $1 \leq i \leq n$

and

(L_2) there is an $\alpha > 0$ such that $L(t,q)\xi \cdot \xi \geq \alpha|\xi|^2$ for all $t \in \mathbb{R}$ and $q,\xi \in \mathbb{R}^n$.

Suppose V and f satisfy (V_1) and (f_1)-(f_2) respectively. Let $q \in E$. Then

$$I(q) \equiv \int_0^T \left[\frac{1}{2} L(t,q)\dot{q} \cdot \dot{q} - V(t,q) + f \cdot q\right]dt \tag{1.1}$$

is well defined and the argument of [7, Prop. B10] shows that $I \in C^1(E,\mathbb{R})$ and critical points of I are classical solutions of

$$\frac{d}{dt}(L(t,q)\dot{q}) - \frac{1}{2}\frac{\partial L}{\partial q}(t,q)\dot{q} \cdot \dot{q} + V_q(t,q) = f(t) . \tag{1.2}$$

Suppose $Q,q \in E$ and

$$Q - q = (k_1T_1,\ldots,k_nT_n) \tag{1.3}$$

where $k = (k_1,\ldots,k_n) \in \mathbb{Z}^n$. The properties of L, V, and f then imply

$$I(Q) = I(q) . \tag{1.4}$$

A functional J on E satisfies (PS) if any sequence q_m such that $I(q_m)$ is bounded and $I'(q_m) \to 0$ possesses a convergent subsequence. Equation (1.4) shows that I does not satisfy (PS) on E. However, (1.4) implies I possesses a free $\mathbb{Z}^n$ action on E and this will provide the topological basis for an existence result for (1.2). Before stating it some further preliminaries are required.

We introduce an equivalence relation $\sim$ on E via $Q \sim q$ if (1.3) holds for some $k \in \mathbb{Z}^n$. Let $\tilde{E} = E/\sim$ with the corresponding quotient topology. One could now consider I on $\tilde{E}$. However, we find it more convenient technically to work on E itself and proceed as follows. For $k \in \mathbb{Z}^n$, set $\theta_k = (k_1T_1,\dots,k_nT_n)$. For $q \in \mathbb{R}^n$, let $g_k(q) = q + \theta_k$. Then $G \equiv \{g_k | k \in \mathbb{Z}^n\}$ is a group of mappings of E onto E and I is invariant under G, i.e. $I(g(q)) = I(q)$ for all $q \in E$ and $g \in G$. A set $A \subset E$ is called <u>invariant</u> (with respect to G) if $g(A) \subset A$ for all $g \in G$. A is an invariant set if and only if there is a set $\tilde{A} \subset \tilde{E}$ such that $A/\sim = \tilde{A}$.

With the aid of the above notions, a generalized version of Theorem 0.3 can be stated.

<u>Theorem 1.5</u>: Let V satisfy (V_1), f satisfy (f_1)-(f_2) and L satisfy (L_1)-(L_2). Then equation (1.2) at least $n + 1$ distinct solutions $Q_1,\dots,Q_{n+1}$, i.e. for $i \neq j$, $Q_i \neq g(Q_j)$ for all $g \in G$.

These solutions will be determined as critical points of I by means of a minimax argument. To characterize the corresponding critical values of I, let $A \subset E$ be closed and define

$$(1.6) \qquad \Gamma_k = \{A \subset E | A \text{ is invariant and } \operatorname{cat}_{\tilde{E}} \tilde{A} \geq k\} .$$

In (1.6), $\operatorname{cat}_X Y$ denotes the Ljusternik-Schnirelmann category of the closed set Y in the topological space X. (See e.g. [8].)

<u>Lemma 1.7</u>: $\Gamma_k \neq \emptyset$, $1 \leq k \leq n + 1$.

<u>Proof</u>: It suffices to show $\Gamma_{n+1} \neq \emptyset$. Let $A = \{q \in E | q = [q]\}$. Then A is isomorphic to $\mathbb{R}^n$ and $A/\sim$ to T^n. Therefore $\tilde{E}/\sim = T^n \oplus F$ where $F = A^{\perp}$, the orthogonal complement of A. Since F is a linear space, it is contractible to a point in itself. Therefore by standard results on

Ljusternik-Schnirelmann category - see p. 459-460 of [9]

$$\text{(1.8)} \qquad \mathrm{cat}_{\tilde{E}}\,\tilde{E} = \mathrm{cat}_{\tilde{E}}\,T^n \oplus \{0\} = \mathrm{cat}_{T^n} T^n = n + 1 \ .$$

Now $n + 1$ critical values c_k of I can be defined as

$$\text{(1.9)} \qquad c_k = \inf_{B \in \Gamma_k} \max_{q \in B} I(q), \qquad 1 \le k \le n + 1 \ .$$

To prove *Theorem* 1.5, it will be shown that each of the numbers c_k is a critical value of I and there are at least $n + 1$ corresponding critical points. To carry out the proof some further preliminaries are required. A mapping $\Psi : E \to E$ is called <u>equivariant</u> with respect to G if $\Psi \circ g = g \circ \Psi$ for all $g \in G$. For $s \in \mathbb{R}$, let $A_s \equiv \{q \in E | I(q) \le s\}$ and $K_s = \{q \in E | I(q) = s$ and $I'(q) = 0\}$. The following version of a standard "Deformation Theorem" is needed:

<u>Proposition 1.10</u>: For any $c \in \mathbb{R}$ and invariant neighborhood $\mathcal{O}$ of K_c, there is an $\varepsilon > 0$ and $\eta \in C([0,1] \times E, E)$ such that

1° $\eta(0,q) = q$ for all $q \in E$

2° $\eta(t,q)$ is equivariant for each $t \in [0,1]$

3° $\eta(q, A_{c+\varepsilon} \backslash \mathcal{O}) \subset A_{c-\varepsilon}$

4° If $K_c = \emptyset$, $\eta(1, A_{c+\varepsilon}) \subset A_{c-\varepsilon}$.

The proof of Proposition 1.10 will be given in §2.

<u>Completion of proof of Theorem 1.5</u>: By 1° and 2° of Proposition 1.10, $\eta(1,\cdot)$ is homotopic to the identity map and is equivariant. Hence by Lemma 5.3 of [8], $\eta(1,\cdot) : \Gamma_k \to \Gamma_k$, $1 \le k \le n + 1$. Now the conclusions of Theorem 1.5 follow in a standard way: suppose $c_k = \cdots = c_{k+p} \equiv c$. Then

$$\text{(1.11)} \qquad \mathrm{cat}_{\tilde{E}}\,\tilde{K}_c \ge p + 1$$

for if not, by the proof of Proposition 1.10 and Lemma 5.6 of [8], we can find a uniform neighborhood $\mathcal{O}$ of K_c such that $\mathrm{cat}_{E^*}\mathcal{O}^* < p + 1$. Choose

$B \in \Gamma_{k+p}$ such that

(1.12) $$\max_{B} I \leq c + \varepsilon .$$

By the subadditivity property of category - [8, Lemma 5.3], $\operatorname{cat}_{\tilde{E}} \overline{\tilde{B}\backslash\tilde{\mathcal{O}}} \geq k + p - p = k$ so $\overline{B\backslash\mathcal{O}} \in \Gamma_k$ as is $\eta(1,\overline{B\backslash\mathcal{O}}) \equiv B_1$. But by (1.12) and 3° (or if $p = 0$, 4°) of Proposition 1.10,

(1.13) $$\max_{B_1} I \leq c - \varepsilon ,$$

contrary to the definition of $c = c_k$. Finally, if $p \geq 1$, the definition of category implies $\tilde{K}_c$ contains infinitely many distinct points completing the proof.

Remark 1.14: In [4], Li proved an abstract theorem about critical points of periodic functionals which the above arguments can be used to generalize and put in a more natural form.

Remark 1.15: If e.g. L and V are independent of t and $f = 0$, the hypotheses and therefore conclusions of Theorem 1.5 hold for all $T \in \mathbb{R}$. However, by the periodicity properties of V, it belongs to $C^1(T^n,\mathbb{R})$. Therefore V possesses at least $\operatorname{cat}_{T^n} T^n = n + 1$ distinct critical points which will be equilibrium solutions of (1.2). Thus it is not clear that there need exist any time dependent solutions for this special case without more structure for V. In this regard, see Theorem 5.2 of [4] when $L = id$ and $n = 1$.

Remark 1.16: In [6], Conley and Zehnder proved that if (i) $H(t,p,q) \in C^2(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n,\mathbb{R})$ where H is 1-periodic in t and the components of q, and (ii) there is an $R > 0$ such that for $|p| \geq R$,

$$H(t,p,q) = \frac{1}{2} Mp \cdot p + a \cdot p$$

where $a \in \mathbb{R}^n$ and M is a symmetric nonsingular time independent matrix, then the corresponding Hamiltonian system possesses at least $n + 1$ distinct periodic solutions. This theorem applies to a subclass of the problems treated here including Theorem 0.3. Note however, that the Conley-Zehnder result permits indefinite matrices M. Assuming for the moment its applicability here to Hamiltonians of the form

$$H = \frac{1}{2} Mp \cdot p + V(t,q) \ , \tag{1.17}$$

observe that the corresponding Hamiltonian system is

$$\dot{p} = -V_q, \quad \dot{q} = Mp \ . \tag{1.18}$$

Therefore

$$\dot{p} = \frac{d}{dt} M^{-1}\dot{q} = -V_q \ . \tag{1.19}$$

Choosing $M = L^{-1}$, (1.19) now gives (1.2).

To applicability of the Conley-Zehnder result is not quite immediate here since (1.17) does not satisfy condition (ii). However, using a trick, a modified Hamiltonian can be constructed which satisfies (ii) and whose solutions satisfy (1.2). For simplicity we will just verify this for (0.3) where $M = \text{id}$. For this case by (1.2) or (0.1), if q is a T periodic solution,

$$\|\ddot{q}\|_{L^\infty} \le \max_{t\in\mathbb{R},\xi\in\mathbb{R}^n} |V_q(t,\xi)| \equiv K_1 \le K_1 + \max_{t\in\mathbb{R},\xi\in\mathbb{R}^n} |V(t,\xi)| \equiv K \ . \tag{1.20}$$

Simple estimates then show

$$\|\dot{q}\|_{L^\infty} \le T\|\ddot{q}\|_{L^\infty} \le 2TK \ . \tag{1.21}$$

Let $\varphi \in C^\infty(\mathbb{R},\mathbb{R})$ such that $\varphi(s) = 1$ if $s \le (1 + T)K \equiv K_2$, $\varphi(s) = 0$ if $s \ge 2K_2$, and $-1 < \varphi'(s) < 0$ if $s \in (K_2,2K_2)$. The Hamiltonian

$$\hat{H}(t,p,q) = \frac{1}{2} |p|^2 + \varphi(|p|)V(t,q) \tag{1.22}$$

satisfies (i) and (ii). The corresponding Hamiltonian system is

$$\dot{p} = -V_q\varphi, \quad \dot{q} = p + V\varphi_p \ . \tag{1.23}$$

If (p,q) is a T-periodic solution of (1.23), the second equation implies

(1.24) $$[p] = -[V\varphi_p] \ .$$

By the definition of K_2, $|\varphi_p| < 1$ and therefore

(1.25) $$|[p]| < K.$$

Since

(1.26) $$p(t) = [p] + \frac{1}{T}\int_0^T \left(\int_\tau^T \dot{p}(s)ds\right)d\tau \ ,$$

(1.26), (1.23), and (1.25) imply

(1.27) $$\|p\|_{L^\infty} < K + TK < K_2 \ .$$

Consequently $\varphi(|p|) = 1$, $\varphi_p(|p|) = 0$, and any T-periodic solution of the modified equation satisfies (0.3).

§2. The proof of Proposition 1.10

A mapping $\eta \in C([0,1] \times E, E)$ and $\varepsilon > 0$ must be found satisfying 1°-4° of Proposition 1.10. Much of this construction is standard and therefore we will only indicate the modifications that must be made in the proof the "usual case" that can be found in [7, Appendix A].

The mapping η is determined as the solution of an ordinary differential equation of the form

$$\frac{d\eta}{dt} = \omega(\eta)\Psi(\eta) \tag{2.1}$$

where ω is a cut-off function with $0 \leq \omega \leq 1$ and Ψ is a pseudogradient vector field for I'. In [7], the choice of ε and construction of ω depend on the fact that I satisfies (PS). As was already noted, this is not the case here. Therefore we must show that because of the invariance of I under G, the proof works nevertheless. Heuristically, we will show that $I|_{\tilde{E}}$ satisfies (PS). First the form of I' must be studied. Let D denote the duality map from E to E^*.

<u>Proposition 2.2</u>: Under the hypotheses of Theorem 1.5,

$$I'(q) = D(P_1(q) + P(q))$$

where P is compact and

$$\frac{d}{dt} P_1(q) = L(t,q)\dot{q} - [L(t,q)\dot{q}] .$$

<u>Proof</u>: By (1.1),

$$I'(q)\varphi = \int_0^T [L\dot{q} \cdot \dot{\varphi} + \frac{1}{2} \sum_{i=1}^n L_{q_i}\dot{q} \cdot \dot{q}\varphi_i - V_q \cdot \varphi + f \cdot \varphi]dt . \tag{2.3}$$

Let

$$\begin{cases} I_1(q)\varphi \equiv \int_0^T L\dot{q} \cdot \dot{\varphi}dt \\ I_2(q)\varphi = \frac{1}{2}\int_0^T \sum_{i=1}^n L_{q_i}\dot{q} \cdot \dot{q}\varphi_i dt \\ I_3(q)\varphi = \int_0^T (-V_q + f) \cdot \varphi dt . \end{cases} \tag{2.4}$$

By the argument of Proposition B.10 of [7], $I_3(q) = D\mathcal{P}_3(q)$ where $\mathcal{P}_3$ is compact. Next consider $I_2(q)\varphi$. It has the form

$$(2.5) \qquad I_2(q)\varphi = \int_0^T b \cdot \varphi dt$$

where $b = b(t,q,\dot{q}) \in L^1$. Since $I_2(q) \in E^*$, there is a unique $x = \xi + y \in \mathbb{R}^n \oplus F$ such that

$$(2.6) \qquad I_2(q)\varphi = \int_0^T \dot{y} \cdot \dot{\varphi} dt + \xi \cdot [\varphi] .$$

Comparing (2.5)-(2.6) shows y is a weak solution of

$$(2.7) \qquad -\ddot{y} = b - [b] .$$

Hence $y \in W^{2,1}$ and

$$(2.8) \qquad \|y\|_{W^{2,1}} \le 2\|b\|_{L^1} .$$

Similarly $\xi = T[b]$ and

$$(2.9) \qquad |\xi| \le T\|b\|_{L^1} .$$

Let $\mathcal{P}_2(q) = y(q) + \xi(q)$. The form of b shows that for q belonging to a bounded set in E, $\mathcal{P}_2(q)$ will be bounded in $W^{2,1}$. Hence $\mathcal{P}_2(q)$ and $I_2(q) = D\mathcal{P}_2$ are compact on E.

Finally, consider $I_1(q)\varphi$. As was the case with $I_2(q)$,

$$(2.10) \qquad I_1(q)\varphi = \int_0^T \dot{w} \cdot \dot{\varphi} dt$$

where $w \in F$ and there is no mean value term due to the form of $I_1(q)$. Comparing with (2.4) yields

$$(2.11) \qquad \dot{w} = L\dot{q} - [L\dot{q}] .$$

Now (2.11) together with $[w] = 0$ determines $w = \mathcal{P}_1(q)$. Setting $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$, the result follows.

Lemma 2.12: Let $\hat{K}_c = \{q \in K_c \mid 0 < [q_i] < T_i,\ 1 \le i \le n\}$. Then $\hat{K}_c$ is compact.

<u>Proof</u>: Suppose $(Q_m) \subset \hat{K}_c$. Let $Q_m = \xi_m + Y_m$ where $\xi_m \in R^n$ and $Y_m \in F$. The form of I and (L_2) imply that (Y_m) is bounded and the definition of $\hat{K}_c$ shows that (ξ_m) is bounded. Since (Q_m) is bounded, along a subsequence, Q_m converges weakly in E and strongly in L^∞ to Q and $P(Q_m)$ converges in E. Therefore from Proposition 2.2,

$$\dot{Q}_m = L^{-1}(t,Q_m)[T^{-1}\int_0^T L(\tau,Q_m)\dot{Q}_m d\tau - \frac{d}{dt}P(Q_m)] \ . \tag{2.13}$$

Now equation (2.13) shows that along this subsequence $\dot{Q}_m$ converges in L^2 and Q_m converges in E and $I'(Q) = 0$. Hence $Q \in \hat{K}_c$.

<u>Completion of the proof of Proposition 1.10</u>: In the proof of the version of Proposition 1.10 given in [7], (PS) is used in two places. The first is in showing that K_c is compact and therefore for any neighborhood $\mathcal{O}$ of K_c, there exists a uniform neighborhood $N_\delta \equiv N_\delta(K_c) = \{q \in E \mid \|q - K_c\| < \delta\} \subset \mathcal{O}$. For the current setting, if $\mathcal{O}$ is an invariant neighborhood of K_c, since $\hat{K}_c \subset K_c$ and is compact, there is a $\delta > 0$ such that $\mathcal{O} \supset N_\delta(\hat{K}_c)$. Therefore by the invariance of $\mathcal{O}$,

$$\mathcal{O} \supset \bigcup_{g \in G} g(N_\delta(\hat{K}_c)) = N_\delta(K_c) \ . \tag{2.14}$$

The second place at which (PS) is required in [7] is in showing that there exists constants $b, \hat{\varepsilon} > 0$ such that if $q \in A \equiv A_{c+\hat{\varepsilon}} \backslash (A_{c-\hat{\varepsilon}} \cup N_{\delta/8})$,

$$\|I'(q)\| > b \ . \tag{2.15}$$

Arguing indirectly, (2.15) follows for the subclass of $q \in A$ such that $0 < [q_i] < T_i$, $1 < i < n$ by an argument paralleling the proof of Lemma 2.12 which will be omitted. Since A_s and $N_{\delta/8}$ are invariant sets, (2.15) then holds for all $q \in A$.

Given (2.14)-(2.15), the argument of Theorem A.4 of [7] yields 1°, 3°-4° of Proposition 1.10. It remains only to show that by an appropriate construction of $\eta(t,\cdot)$, 2° also holds. Since I is invariant under G, i.e. $I(g_k) = I(q + \theta_k) = I(q)$ for all $g_k \in G$,

$$I'(q + \theta_k) = I'(q) = I'(g_k q) \tag{2.16}$$

for all $g_k \in G$. The invariance of I' under G allows us to slightly modify the construction of Lemma A.2 of [7] to obtain a pseudogradiant vector field Ψ for I' which is also invariant under G. Also the definition of ω in [7] shows that if I is invariant under G, so is ω. Thus the right-hand side of (2.1) is invariant under G. We claim $\eta(t,\cdot)$ satisfies 2°, i.e. $\eta(t,gq) = g\eta(t,q)$ for all $t \in [0,1]$, $g \in G$, $q \in E$. Indeed set $w = g\eta(t,q)$. Then $w = \eta(t,q) + \theta_k$ for some $k \in \mathbb{Z}^n$ and

$$\begin{cases} \dfrac{dw}{dt} = \dfrac{d\eta}{dt} = \omega(\eta)\Psi(\eta) = \omega(g\eta)\Psi(g\eta) = \omega(w)\Psi(w) \\ w(0,q) = gq \; . \end{cases} \tag{2.17}$$

Therefore $w(t,q) = \eta(t,gq) = g\eta(t,q)$ so 2° holds. The proof is complete.

§3. The Neumann Problem

In this section, the Neumann problem

$$(3.1)\qquad \begin{cases} -\Delta u = p(x,u) + h(x), & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases}$$

will be studied and Theorem 0.5 will be proved. The ideas used here are so close to those of §1-2 that we will be sketchy.

Let $E = W^{1,2}(\Omega)$. For $u \in E$, define

$$(3.2)\qquad I(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - P(x,u) - h(x)u\right]dx$$

where P, h, Ω satisfy the hypotheses of Theorem 0.5. The proof of Proposition B.10 of [7] shows $I \in C^1(E,\mathbb{R})$. By e.g. [10, Chapter 2] critical points of I are classical solutions of (3.1). Note that $I(u + kr) = I(u)$ for all $u \in E$ and $k \in \mathbb{Z}$ via (p_1) and (h_2). Therefore I does not satisfy (PS) on E. For $u,v \in E$, we say u is equivalent to v, $u \sim v$, if $u - v = kr$ for some $k \in \mathbb{Z}$. Let $\tilde{E} = E/\sim$. As in §1, $A \subset E$ is an invariant set if and only if there is an $\tilde{A} \subset \tilde{E}$ such that $\tilde{A} = A/\sim$. Again E can be identified with $\mathbb{R} \oplus F$ where F is the orthogonal complement of span $\{1\}$ in E and $\tilde{E}$ can be identified with $S^1 \oplus F$.

The argument of Lemma 1.7 shows that

$$\Gamma_k = \{A \subset E \mid \operatorname{cat}_{\tilde{E}} \tilde{A} \geq k\} \neq \emptyset$$

for $k = 1,2$. Noting that $(\int_\Omega |\nabla u|^2 dx)^{1/2}$ is a norm on F and that

$$(3.3)\qquad I'(u) = D(u - [u]) + P(u)$$

where D again denotes the duality map and P is compact (see Proposition B.10 of [7]). Therefore there is an analogue of Proposition 1.10 here which implies that

$$(3.4)\qquad c_k = \inf_{A\in\Gamma_k} \max_{u\, A} I(u), \quad k = 1,2$$

is a critical value of I, $k = 1,2$ with a multiplicity result if $c_1 = c_2$. This completes the proof of Theorem 0.5.

Remark 3.5: If in (0.6), u is an n vector, $p(x,u) = \nabla_u P(x,u)$, and P is periodic in the components of u, a version of Theorem 0.5 analogous to Theorem 0.3 obtains.

Remark 3.6: An alternate way to prove Theorem 0.5 is to obtain a first solution as a minimum and a second using a variant of the Mountain Pass Theorem as in [1]-[4] for (0.1) with $n = 1$. Such an approach, however, does not extend to cover the vector case mentioned in Remark 3.5.

Remark 3.7: Theorem 0.5 can also be extended by replacing $-\Delta$ by a more general second order divergence structure uniformly elliptic operator with appropriate changes in the boundary conditions. A similar result also obtains if Ω is replaced by a rectangular domain and the Neumann boundary conditions by periodic ones.

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