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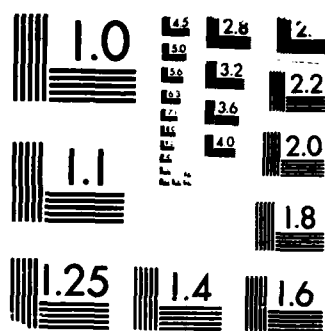
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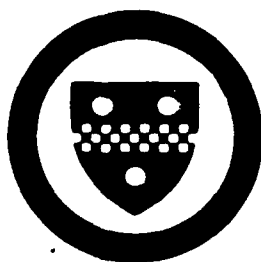
STRONG LAW FOR MIXING SEQUENCE*

Xiru Chen and Yuehua Wu

Center for Multivariate Analysis
University of Pittsburgh

Technical Report No. 87-47

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Xiru Chen and Yuehua Wu

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Center for Multivariate Analysis
Fifth Floor Thackeray Hall
University of Pittsburgh
Pittsburgh, PA 15260

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STRONG LAW FOR MIXING SEQUENCE *

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ABSTRACT

In this note we present some theorems on the strong law for the mixing sequence which is not necessarily stationary, and the mixing coefficient involving only a pair of variables in the sequence.

AMS 1980 Subject Classifications: Primary 60F15.

Key words and phrases: mixing coefficient, stationary sequence, strong law of large numbers.

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1. INTRODUCTION

In this article we present some results concerning the strong law of a mixing sequence $\{X_n, n \geq 1\}$. We do not assume that $\{X_n\}$ is stationary, and we use mixing coefficients involving only a pair of variables X, Y (in that order): The Rosenblatt mixing coefficient

$$\alpha(X, Y) = \sup\{|P(X \in A, Y \in B) - P(X \in A)P(Y \in B)| : A \in B', B \in B'\}$$

and the Ibragimov mixing coefficient

$$\beta(X, Y) = \sup\{|P(Y \in B | X \in A) - P(Y \in B)| : A \in B', B \in B', P(X \in A) > 0\}$$

where B' is the σ -field of all Borel sets in R' .

THEOREM 1. Suppose that $\{X_n, n \geq 1\}$ is a sequence of random variables, and for some $p > 1$ the following conditions are satisfied:

$$1^\circ. \sup_n E|X_n|^p < \infty. \quad (1)$$

$$2^\circ. \text{ There exists } \epsilon > 0 \text{ such that as } |i - j| \rightarrow \infty,$$

$$\alpha(X_i, X_j) \leq \rho(|i - j|) = \begin{cases} O(|i - j|^{-p/(2p-2)-\epsilon}), & 1 < p < 2, \\ O(|i - j|^{-2/p-\epsilon}), & p \geq 2. \end{cases} \quad (2)$$

Then

$$\lim_{n \rightarrow \infty} (S_n - ES_n)/n = 0, \quad \text{a.s.} \quad (3)$$

Here and in the sequel $S_n = \sum_{i=1}^n X_i$.

THEOREM 2. Suppose that $\{X_n, n \geq 1\}$ is a sequence of random variables, and one of the following conditions are satisfied:

$$(I) \quad \sum_{n=1}^{\infty} \text{var}(X_n)/n^2 < \infty, \quad \sup_n E|X_n| < \infty,$$

and

$$\beta(X_i, X_j) \leq \mu(|i - j|), \quad \sum_{n=0}^{\infty} \mu^{1/2}(n) < \infty;$$

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(II) $\sup_n \text{var}(X_n) < \infty$ and there exists $\epsilon > 0$ such that

$$\sum_{i=1}^n \mu^{1/2}(i) = O(n/(\log n)^{1+\epsilon}); \quad (5)$$

(III) (4) holds, $X_1, X_2, \dots$ are identically distributed and $E|X_1| < \infty$ (the existence of variance is not assumed). Then (3) is true.

Remarks:

1. Part (I) of Theorem 2 can be compared with a result of Blum *et al* [1], who assumes that $\{X_n\}$ is a $\star$ -mixing sequence instead of (4). Note that this assumption does not follow from (4). We can easily construct a pairwise independent sequence which is not $\star$ -mixing.

2. Parts (I) and (II) of Theorem 2 can also be compared with some results (see Theorem 3.7.2 and Theorem 3.7.4 of Stout [5]) derived from Serfling [4]. The conditions of these results involve correlation coefficients between two variables in the sequence.

3. Part (III) of Theorem 2 extends Theorem 1 of Etemadi [2]. The assumption that $\{X_i\}$ is identically distributed can be somewhat relaxed, for example, it can be replaced by the condition that there exists a random variable Y such that $P(|X_n| \geq x) \leq P(|Y| \geq x)$ for all $n \geq 1$ and $x \geq 0$. We also mention a related result of Blum *et al* [1] Theorem 1. They assume that $\{X_n\}$ is identically distributed, the distribution of X_1 has a moment generating function in the neighborhood of zero and that $\{X_n\}$ is $\star$ -mixing. Under these more stronger conditions they prove that $P(|S_n - ES_n|/n \geq \epsilon)$ tends to zero exponentially.

2. PROOF OF THE THEOREMS

In deducing our results we shall borrow a trick from Etemadi [2].

The following well-known facts concerning $\alpha(X,Y)$ and $\beta(X,Y)$ will be used:

$$|\text{cov}(X,Y)| \leq 10(\alpha(X,Y))^{\delta/(2+\delta)}(E|X|^{2+\delta}E|Y|^{2+\delta})^{1/(2+\delta)}, \quad \delta > 0 \quad (6)$$

$$|\text{cov}(X,Y)| \leq 2(\beta(X,Y)\text{var}(X)\text{var}(Y))^{1/2}. \quad (7)$$

For a proof, see Ibragimov and Linnik [3]. Also it is trivially true that

$$\alpha(XI_C(X), YI_D(Y)) \leq \alpha(X,Y), \quad \beta(XI_C(X), YI_D(Y)) \leq \beta(X,Y) \quad (8)$$

$$\alpha(X-a, Y-b) = \alpha(X,Y), \quad \beta(X-a, Y-b) = \beta(X,Y), \quad (9)$$

where C and D are Borel sets in R^1 and a, b are constants.

Proof of Theorem 1. In view of (9), by defining $X_n^+ = X_n I(X_n > 0)$, $X_n^- = -X_n I(X_n \leq 0)$, $n \geq 1$, we can assume without loss of generality that $X_n \geq 0$, $n \geq 1$. Define

$$Y_n = (X_n - EX_n)I(|X_n - EX_n| < n^{1/p+\epsilon_1}), \quad n \geq 1, \quad (10)$$

$$S_n^* = \sum_{i=1}^n (Y_i - EY_i),$$

where $\epsilon_1 > 0$ is a constant to be chosen later.

From condition (1) we have $\sum_{n=1}^{\infty} P(X_n - EX_n \neq Y_n) < \infty$ and $\lim_{n \rightarrow \infty} EY_n = 0$.

Therefore, (3) is equivalent to

$$\lim_{n \rightarrow \infty} S_n^*/n = 0, \quad \text{a.s.} \quad (11)$$

Now fix $\alpha > 1$ and let $k_n = [\alpha^n]$. For positive integer m sufficiently large, there exists n such that $k_n \leq m < k_{n+1}$, and $n \rightarrow \infty$ as $m \rightarrow \infty$. From (1) we have

$$\sup_n E|Y_n| \equiv C < \infty. \quad (12)$$

Here and in the sequel C is an unimportant constant which is allowed to change. Since $Y_n \geq 0$, it follows that

$$\begin{aligned} S_m^* - S_{k_n}^* &\geq -(m - k_n)C, \quad \text{when } S_m^* < S_{k_n}^*, \\ S_m^* - S_{k_n}^* &\leq S_{k_{n+1}}^* - S_{k_n}^* + (k_{n+1} - m)C, \quad \text{when } S_m^* \geq S_{k_n}^*. \end{aligned}$$

Hence

$$|S_m^*/m - S_{k_n}^*/k_n| \leq \left| \frac{k_{n+1}}{k_n} \frac{S_{k_{n+1}}^*}{k_{n+1}} - \frac{S_{k_n}^*}{k_n} \right| + \frac{k_{n+1} - k_n}{k_n} C. \quad (13)$$

From (13) it follows that if we have shown that

$$\lim_{n \rightarrow \infty} S_{k_n}^*/k_n = 0, \quad \text{a.s.} \quad (14)$$

Then we would have

$$\limsup_{m \rightarrow \infty} |S_m^*/m| \leq (\alpha - 1)C, \quad \text{a.s.}$$

For any $\alpha > 1$, hence (11).

By Borel-Cantelli lemma, in order to prove (14), we have only to show that

$$\sum_{n=1}^{\infty} \text{var}(S_{k_n}^*)/k_n^2 < \infty. \quad (15)$$

By (6), (8) and (9), we have for any $\delta > 0$:

$$\begin{aligned} \text{Var}(S_{k_n}^*) &= \sum_{i,j=1}^{k_n} \text{cov}(Y_i, Y_j) \\ &\leq C \sum_{i,j=1}^{k_n} (\alpha(X_i, X_j))^{\delta/(2+\delta)} (E|Y_i|^{2+\delta} E|Y_j|^{2+\delta})^{1/(2+\delta)}. \end{aligned} \quad (16)$$

From (1) it follows that

$$E|Y_n|^{2+\delta} \leq Cn^{(2+\delta-p)(1/p+\epsilon_1)}, \quad n = 1, 2, \dots \quad (17)$$

First consider the case $p > 2$. From (2), (16) and (17) we obtain

$$\begin{aligned} \text{var}(S_{k_n}^*) &\leq C \sum_{i,j=1}^{k_n} (\alpha(X_i, X_j))^{\delta/(2+\delta)} (ij)^{(2+\delta-p)(1/p+\epsilon_1)/(2+\delta)} \\ &\leq C \sum_{i,j=1}^{k_n} (\alpha(X_i, X_j))^{\delta/(2+\delta)} i^{2(2+\delta-p)(1/p+\epsilon_1)/(2+\delta)} \\ &\leq C \sum_{i,j=1}^{k_n} i^{-(2/p+\epsilon)\delta/(2+\delta)} \sum_{i=1}^{k_n} i^{2(2+\delta-p)(1/p+\epsilon_1)/(2+\delta)}. \end{aligned} \quad (18)$$

Noticing $2/p < 1$, we can assume that $2/p + \epsilon < 1$. Hence from (18) we have

$$\text{var}(S_{k_n}^*) \leq Ck_n^{-\frac{(2/p+\epsilon)\delta}{(2+\delta)} + \frac{2(2+\delta-p)(1/p+\epsilon_1)}{(2+\delta)} + 2}. \quad (19)$$

This inequality holds for any $\delta > 0$. Now we choose $\epsilon_1 \in (0, \epsilon/2)$, then

$$\lim_{\delta \rightarrow \infty} \left\{ -\frac{(2/p+\epsilon)\delta}{(2+\delta)} + \frac{2(2+\delta-p)(1/p+\epsilon_1)}{(2+\delta)} \right\} = -\epsilon + 2\epsilon_1 \equiv \eta < 0.$$

Therefore, choosing δ sufficiently large, from (19) we obtain

$$\text{var}(S_{k_n}^*) \leq Ck_n^{2-\eta}. \quad \text{Hence (15) is true in view of } \sum_{n=1}^{\infty} k_n^{-\eta} < \infty.$$

Next assume that $p = 2$. Again, choose $\epsilon_1 \in (0, \epsilon/2)$. Choose $\delta > 0$ sufficiently small, such that $(1+\epsilon)\delta/(2+\delta) < 1$. We still have (19), with $p = 2$. Since

$$-(1+\epsilon)\delta/(2+\delta) + 2\delta(1/2+\epsilon_1)/(2+\delta) = -(\epsilon - 2\epsilon_1)\delta/(2+\delta) < 0,$$

(15) holds again.

Finally, consider the case $1 < p < 2$. In this case we have, instead of (18),

$$\text{var}(S_{k_n}^*) \leq C \sum_{i=1}^{k_n} i^{-\left(\frac{p}{2p-2} + \epsilon\right)\delta/(2+\delta)} \sum_{i=1}^{k_n} i^{2(2+\delta-p)(1/p + \epsilon_1)/(2+\delta)}. \quad (20)$$

Write $\delta_0 = 2(p/(2p-2) - 1 + \epsilon)^{-1}$. Since $1 < p < 2$, we have $\delta_0 > 0$. Choose $\epsilon_1 > 0$ sufficiently small, such that

$$0 < \delta < \delta_0 \Rightarrow 2(2+\delta-p)(1/p + \epsilon_1)/(2+\delta) \leq 1 - \eta$$

where $\eta > 0$ does not depend on δ , as long as $0 < \delta < \delta_0$. Because $(p/(2p-2) + \epsilon)\delta/(2+\delta) < 1$ for $0 < \delta < \delta_0$ and $(p/(2p-2) + \epsilon)\delta_0/(2+\delta_0) = 1$, one can find $\delta \in (0, \delta_0)$, such that

$$1 - \eta/2 < (p/(2p-2) + \epsilon)\delta/(2+\delta) < 1.$$

For this δ we have, by (20),

$$\text{var}(S_{k_n}^*) \leq C k_n^{-(1-\eta/2) + 1 + (1-\eta) + 1} \leq C k_n^{-\eta/2}.$$

So we obtain (15) again. Theorem 1 is proved.

Proof of Theorem 2. Part (I): Again we can assume $X_n \geq 0$. Write $Y_n = X_n - EX_n$ and $S_n^* = \sum_{i=1}^n Y_i$. From $\sup E|X_n| < \infty$ we have $\sup E|Y_n| < \infty$. Using the same argument employed in proving Theorem 1, we reduce the proof of (11) to that of (15). From (4), (7) and (9),

$$\begin{aligned} \sum_{n=1}^{\infty} \text{var}(S_{k_n}^*)/k_n^2 &= \sum_{n=1}^{\infty} k_n^{-2} \sum_{i,j=1}^{k_n} \text{cov}(Y_i, Y_j) \\ &\leq C \sum_{n=1}^{\infty} k_n^{-2} \sum_{i,j=1}^{k_n} (\mu(|i-j|) \text{var}(X_i) \text{var}(X_j))^{1/2} \\ &\leq C \sum_{n=1}^{\infty} k_n^{-2} \sum_{i=0}^{k_n} \mu^{1/2}(i) \sum_{i=1}^{k_n} \text{var}(X_i) \\ &\leq C \sum_{n=1}^{\infty} k_n^{-2} \sum_{i=1}^{k_n} \text{var}(X_i) \end{aligned} \quad (21)$$

$$\begin{aligned} &\leq C \sum_{n=1}^{\infty} \text{var}(X_n)/n^2 < \infty. \end{aligned} \quad (22)$$

Part (II) is proved in much the same way as Part (I), only that we replace Ck_n for $\sum_{i=1}^{k_n} \text{var}(X_i)$ and $Ck_n/(\log n)^{1+\epsilon}$ for $\sum_{i=1}^{k_n} \mu^{1/2}(i)$ in (21) to obtain (22). Part (III) is proved by truncating X_n at n and combining the reasoning above and that of Etemadi [2].

3. AN EXAMPLE

Consider the autoregression model

$$X_n = a_1 X_{n-1} + \dots + a_m X_{n-m} + e_n, \quad n = 0, \pm 1, \pm 2, \dots \quad (23)$$

We want to show that under certain conditions it is true that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n X_i / n = 0, \quad \text{a.s.} \quad (24)$$

for any solution of (23). Suppose that the following conditions are satisfied:

1. $\{e_n, n = 0, \pm 1, \dots\}$ is a sequence of independent real random variables, and

$$Ee_n = 0, \quad n = 0, \pm 1, \dots, \quad \sup_{-\infty < n < \infty} E|e_n|^p = C < \infty \quad \text{for some } p > 1. \quad (25)$$

where, as before, C is an unimportant constant which is allowed to change.

2. e_n has a density f_n satisfying the Lipschitz condition over R' :

$$|f_n(x) - f_n(y)| \leq C|x - y|, \quad n = 0, \pm 1, \pm 2, \dots \quad (26)$$

where C does not depend on n .

3. $a_1, a_2, \dots, a_m$ are real constants, and the equation $1 - a_1 z - \dots - a_m z^m = 0$ has all its root outside the unit circle.

Under the condition 1 and 3, the general real solution of (23) has

the form

$$X_n = \sum_{t=0}^{\infty} b_t e_{n-t} + \sum_{j=0}^J \rho_j^n \sum_{\ell=0}^{m_j-1} n^{\ell} (\xi_{j\ell} \cos n\omega_j + \eta_{j\ell} \sin n\omega_j) \equiv \tilde{X}_n + X_n^* \quad (27)$$

where $b_0 = 1$, $b_2, b_3, \dots$ are real constants such that

$$|b_t| \leq CH^t, \quad t = 0, 1, 2, \dots \quad \text{for some } H \in (0, 1). \quad (28)$$

ρ_j and ω_j , $j = 1, \dots, J$, are real constants, $0 < \rho_j < 1$, $j = 1, \dots, J$, $m_1 + \dots + m_J = m$, and $\xi_{j\ell}, \eta_{j\ell}$, $\ell = 1, \dots, m_j$, $j = 1, \dots, J$, are arbitrary random variables. From (25), (27) and (28) it follows that

$$E\tilde{X}_n = 0, \quad n = 0, 1, 2, \dots, \quad \sup_{-\infty < n < \infty} E|\tilde{X}_n|^p = C < \infty. \quad (29)$$

Let n, N be positive integers, $n < N$. Define

$$Y_{nN} = \sum_{t=0}^{N-n-1} b_t e_{N-t}, \quad Z_{nN} = \sum_{t=N-n}^{\infty} b_t e_{N-t}.$$

Since $b_0 = 1$, from (26) it follows that the density g_{nN} of Y_{nN} obeys Lipschitz's condition with the same constant C as in (26). Also

$$\sup\{E|Y_{nN}|^p : 1 \leq n < N < \infty\} = C < \infty. \quad (30)$$

Now let q_1 be a positive constant, $q_2 = 2q_1$. Define the event

$$D_{nN} = \{|Z_{nN}| \geq (N-n)^{-q_2}\}. \quad (31)$$

(25) entails $\sup_{-\infty < n < \infty} E|e_n| = C < \infty$. Hence

$$P(D_{nN}) \leq C(N-n)^{q_2} \sum_{t=N-n}^{\infty} H^t \leq C(N-n)^{q_2} H^{N-n}. \quad (32)$$

Let G be a Borel set in R' , h be a constant. $G - h$ is defined as the

set $\{g-h: g \in H\}$. Write $\tilde{G} = G \cap \{u: |u| \leq (N-n)^{q_1}\}$, $G^* = G \setminus \tilde{G}$. If $|h| < 1$, we have

$$\begin{aligned}
 & |P(Y_{nN} \in G) - P(Y_{nN} \in G-h)| \\
 & \leq |P(Y_{nN} \in \tilde{G}) - P(Y_{nN} \in \tilde{G}-h)| + P(Y_{nN} \in G^*) + P(Y_{nN} \in G^*-h) \\
 & \leq \int_{\tilde{G}} |g_{nN}(u) - g_{nN}(u-h)| du + P(|Y_{nN}| > (N-n)^{q_1}) + P(|Y_{nN}| > (N-n)^{q_1} - 1) \\
 & \leq C(N-n)^{q_1} h + C(N-n)^{-q_1} + C[(N-n)^{q_1} - 1]^{-1} \\
 & \leq C(N-n)^{q_1} h + C(N-n)^{-q_1}.
 \end{aligned} \tag{33}$$

Now let A and B be two Borel sets in R' . We proceed to estimate

$|P(\tilde{X}_n \in A, \tilde{X}_N \in B) - P(\tilde{X}_n \in A)P(\tilde{X}_N \in B)|$. From (32), (33) and the independence of $e_1, e_2, \dots$, we have

$$\begin{aligned}
 |P(\tilde{X}_n \in B | e_n, e_{n-1}, \dots) - P(Y_{nN} \in B)| &= |P(Y_{nN} \in B - Z_{nN} | Z_{nN}) - P(Y_{nN} \in B)| \\
 &\leq C(N-n)^{-(q_2-q_1)} + C(N-n)^{-q_1} \\
 &\leq C(N-n)^{-q_1},
 \end{aligned} \tag{34}$$

when D_{nN} does not occur. But

$$\begin{aligned}
 |P(\tilde{X}_N \in B) - P(Y_{nN} \in B)| &= |P(Y_{nN} \in B - Z_{nN}) - P(Y_{nN} \in B)| \\
 &= |P(D_{nN}^C)P(Y_{nN} \in B - Z_{nN}) + P(D_{nN})P(Y_{nN} \in B - Z_{nN} | D_{nN}) \\
 &\quad - P(Y_{nN} \in B)| \\
 &\leq P(D_{nN}) + |P(Y_{nN} \in B - Z_{nN} | D_{nN}^C) - P(Y_{nN} \in B)| + P(D_{nN}) \\
 &\leq 2P(D_{nN}) + C(N-n)^{-q_1} \leq C(N-n)^{q_2} H^{N-n} + C(N-n)^{-q_1} \\
 &\leq C(N-n)^{-q_1}.
 \end{aligned} \tag{35}$$

From (34) and (35) we get

$$|P(\tilde{X}_N \in B | e_n, e_{n-1}, \dots) - P(\tilde{X}_N \in B)| \leq C(N-n)^{-q_1}$$

when D_{nN} does not occur. If $P(\tilde{X}_n \in B) \geq C(N-n)^{-q_1}$, then from (33) and (35) we obtain

$$P(\tilde{X}_n \in A, \tilde{X}_N \in B) \geq [P(\tilde{X}_N \in B) - C(N-n)^{-q_1}][P(\tilde{X}_n \in A) - C(N-n)^{q_2} H^{N-n}]. \quad (36)$$

Also

$$P(\tilde{X}_n \in A, X_N \in B) \leq [P(\tilde{X}_N \in B) + C(N-n)^{-q_1}][P(\tilde{X}_n \in A) + C(N-n)^{q_2} H^{N-n}]. \quad (37)$$

From (36) and (37) we have

$$\begin{aligned} |P(\tilde{X}_n \in A, \tilde{X}_N \in B) - P(\tilde{X}_n \in A)P(\tilde{X}_N \in B)| &\leq C(N-n)^{-q_1} + C(N-n)^{q_2} H^{N-n} \\ &\quad + C(N-n)^{q_1} H^{N-n} \leq C(N-n)^{-q_1}, \end{aligned} \quad (38)$$

where C does not depend on A, B . (38) is proved when $P(\tilde{X}_n \in B) \geq C(N-n)^{-q_1}$. If $P(\tilde{X}_N \in B) < C(N-n)^{-q_1}$, (38) is trivially true. Therefore we get

$$\alpha(\tilde{X}_n, \tilde{X}_N) \leq C(N-n)^{-q_1}. \quad (39)$$

Now choose $q_1 = p/(2p-2) + 2$. From (39) we see that the condition (2) is satisfied. This, together with (29), gives, by Theorem 1,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \tilde{X}_i / n = 0, \quad \text{a.s.} \quad (40)$$

From the expression of X_n^* , it is readily seen that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n X_i^* / n = 0, \quad \text{a.s.} \quad (41)$$

From (27), (40) and (41), we obtain (24).

The conclusion (40) does not follow from the ergodic theorem of stationary process, since $\{e_n\}$ is not assumed to be identically distributed, so $\{X_n\}$ may not be a strictly stationary process.

REFERENCES

- [1] BLUM, J.R., HANSON, D.L. and KOOPMANS, L.H. (1963). On the strong law of large numbers for a class of stochastic process. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 2, 1-11.
- [2] ETEMADI, N. (1981). An elementary proof of strong law of large numbers. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 55, 119-122.
- [3] IBRAGIMOV, I.A. and LINNIK, J.V. (1965). *Independent and Stationary Connected Variables* (Nauka, Moscow), English translation (Noordhoff, Groningen, 1971).
- [4] SERFLING, R.J. (1970). Convergence properties of S_n under moment restrictions. *Ann. Math. Statist.* 41, 1235-1248.
- [5] STOUT, W.F. (1974). *Almost Sure Convergence*. Academic Press.

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