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A PROBABILISTIC MODEL OF THE APPARENT RADIANCE OF A  
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# A Probabilistic Model of the Apparent Radiance of a Rough Sea

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<p>A widely used approach to the specification of the background statistics of wind driven sea states is the power spectral method (*). Coupled with a ray trace of the sky radiance, one can create deterministic images of a rough sea in the infrared. In this paper a probabilistic model of a rough sea is presented as an alternative to the power spectral approach. The model takes into account self-shadowing of the sea surface which is important when the sea is viewed at near grazing angles. Given such a model, the apparent radiance of a rough sea is shown to emit a substantially lower amount of radiation when compared to a smooth sea. The sea, being a poor emitter of radiation at near grazing angles, reflects cool sky radiation from higher points in the sky and therefore looks colder than its blackbody temperature.</p>			
<p>(*)I. B. Schwartz and D. Han, Emissivity as a function of surface roughness. A computer model, NRL Memorandum Report No. 5816, 1986.</p>			
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## A PROBABILISTIC MODEL OF THE APPARENT RADIANCE OF A ROUGH SEA

### INTRODUCTION

The infrared signature of ships is most often described in terms of contrast to the background infrared signature of the sea. Since the contrast signature involves the signature of the sea, an important task for signature modeling is to calculate the radiance of the sea surface as a function of viewing angle. This is an interesting problem because of the complication caused by the roughness of the sea surface. Since the infrared wavelength is much shorter than the length scale of disturbances on the water's surface, the rough sea may be thought of as an ensemble of small facets each of which emits and reflects (from the sky, and possibly other facets) infrared radiation. In cases where the length corresponding to the projection of the instantaneous field of view (IFOV) of the sensor onto the ocean surface is greater than the length scale of sea disturbances, reflection and emission properties may be calculated by taking an average over facets weighted, in an appropriate manner, by the probability density function for facet orientation.

It is easy to see that the distribution of facet orientations can have a large effect on the infrared radiation emanating from the sea surface. If the sea were smooth, all of the facets would be horizontally oriented. However when a rough sea surface is viewed near grazing, most of the facets seen by the sensor are tipped away from the horizontal toward the sensor [1,2]. This phenomenon impacts upon both the amount of radiation emitted from the sea and the amount of radiation reflected from the sky. The average tilt angle causes radiation from elevation angles relatively higher in the sky to be reflected than would be the case if the sea were not rough. Since the strength of sky radiation decreases sharply with increasing elevation angle the effect of sea roughness is to decrease the amount of radiation reflected from the sky. At the same time the decreased angle of incidence to the facet increases the emissivity of the facet [3]. This effect increases the amount of radiation emitted from the facet. At angles very near grazing the reflected radiation decrease is larger than the emitted radiation increase. For other angles the situation is reversed.

A major theoretical problem is that some facets are hidden from view by others. This is called the hidden surface problem or shadowing problem. Two approaches to this statistical problem are possible. The first is to make a calculation of the bidirectional reflectance distribution function (BRDF) under the assumption that there is no shadowing and then to introduce some heuristic corrections to take shadowing into account. This approach has been studied by Torrance and Sparrow [4] for the case of rough material surfaces and by Maxwell [5] for the case of rough water. The second approach involves a calculation of the contribution to the total surface area seen by a sensor that is made by facets of a given orientation. This approach permits a simple adjustment to be made for one of the effects of shadowing. It is this second approach which is presented in this paper. It will be shown that both approaches lead to the same result in cases where there is no shadowing. It will be seen that the simple correction allowed by the second approach leads to reasonable results for viewing angles near grazing incidence. For this reason it represents an advance over the corrected BRDF approach.

## STATISTICS OF FACET PROJECTED AREAS

As the first step it is necessary to calculate the wave slope probability density function. An analytic result is available for the case in which the wave amplitudes are sufficiently small such that linear superposition of amplitudes is valid. The essential elements of this calculation have appeared in the literature [6]. For completeness the derivation is presented in appendix A. The central result can be expressed in terms of a probability density for the orientation of a unit vector  $\hat{n}$  normal to the surface of a facet. The simplest description of this density is in terms of a system of spherical polar coordinates with polar axis (the  $z$  axis) in the zenith direction and azimuths measured with respect to the upwind direction (the  $x$  axis). If  $\theta$  and  $\varphi$  are, respectively the polar and azimuthal angles of  $\hat{n}$ , the probability density is:

$$p(\theta, \varphi) = (1/2\pi)(ab)^{-1} \tan(\theta) \sec^2(\theta) e^{-\tan^2(\theta)A(\varphi)} \quad (1)$$

with

$$A(\varphi) = \frac{1}{2} \left[ \frac{\cos^2(\varphi)}{a^2} + \frac{\sin^2(\varphi)}{b^2} \right]$$

where  $a^2$  is the variance of the derivative of the waveslope in the upwind direction and  $b^2$  is the variance in the crosswind direction. This expression may be thought of as the angular probability density for finding the facet at a given place on the surface to be oriented such that its normal has the angular coordinates  $\theta, \varphi$ . We assume there is no correlation between upwind and crosswind directions.

The basis of the approach developed in this paper is the calculation of a fraction,  $f(\theta, \varphi)$ , of the total projected area seen by a sensor. This is the fraction contributed by facets falling in an infinitesimal angular range centered at  $\theta, \varphi$ . Once this fraction is determined it can be used to weight the reflection and emission contributions of the facets to the net observed radiance. To make progress it is necessary to define two types of projected areas. The first type is horizontally projected area. This is the horizontal (parallel to the mean sea level) area obtained by projecting a facet or group of facets onto the horizontal plane. The second is line-of-sight (LOS) projected area. LOS projected area is the area obtained by projecting a facet or group of facets onto a plane normal to the LOS to the sensor. The probability density function  $p(\theta, \varphi)$  is related to the horizontally projected area in the following way: the horizontally projected area,  $\Delta A_h$ , contributed by facets with orientations in the infinitesimal angular range  $\Delta\theta\Delta\varphi$  centered at  $\theta, \varphi$  is given by :

$$\Delta A_h = A_h p(\theta, \varphi) \Delta\theta \Delta\varphi \quad (2)$$

where  $A_h$  is the total horizontally projected area in the region under consideration. Note that the integral over  $\theta$  and  $\varphi$  of the RHS of Eq. (2) correctly yields  $A_h$  due to the fact that  $p(\theta, \varphi)$  is a (properly normalized) probability density. The actual (unprojected) surface area,  $\Delta A$ , of the same group of facets is  $\Delta A_h / \cos(\theta)$ . That  $\Delta A$  is greater than  $\Delta A_h$  is a reflection of the well known fact that the total surface area of a rough surface is greater than the nominal area which it covers. The LOS projected area of this same group of facets,  $\Delta A_{LOS}$ , is given by  $\Delta A \cos(\Omega)$ , where  $\Omega$  is the angle between the normal to the facet and the LOS from the facet to the sensor. The expression for  $\Delta A_{LOS}$  in terms of the probability density function is thus:

$$\Delta A_{LOS} = A_h p(\theta, \varphi) \frac{\cos(\Omega)}{\cos(\theta)} \Delta\theta \Delta\varphi. \quad (3)$$

The total LOS projected area can be obtained by integrating the RHS of Eq. (3) over  $\theta$  and  $\varphi$ .

Caution must be exercised in the use of this result. While  $\theta$  is constrained by  $p$  to be less than  $\pi/2$ , there is no similar constraint on  $\Omega$ . Consequently the RHS of Equation (3) can be less than zero for some orientations of the LOS to the sensor. Eq. (3) associates negative projected areas with facets that are oriented such that their normal vector points away from the sensor. When the RHS of Eq. (3) is integrated, the negative projected areas contributed by facets that point away from the sensor partially cancel the positive projected areas of other facets.

The preceding discussion of projected area makes it possible to give an expression for the fractional projected area  $f(\theta, \varphi)$ . The quantity  $f(\theta, \varphi) \Delta\theta\Delta\varphi$  is defined to be the fraction of the LOS projected area seen by the sensor that is contributed by facets with orientations in the range  $\Delta\theta\Delta\varphi$  centered at  $\theta, \varphi$ . In the absence of shadowing effects  $f$  is given by:

$$f(\theta, \varphi) = \frac{p(\theta, \varphi) \frac{\cos(\Omega)}{\cos(\theta)}}{\int p(\theta, \varphi) \frac{\cos(\Omega)}{\cos(\theta)} d\theta d\varphi} \quad (4)$$

That is, the expression of Eq. (3) is divided by the whole projected area seen by the sensor. The limits of the integration here and throughout the paper are understood to be 0 to  $2\pi$  for the azimuthal angles and 0 to  $\pi$  for the polar angles. This expression must be modified to account properly for shadowing effects. The form of this expression and the cautionary observation discussed above together suggest a heuristic modification to Eq. (4) that can account for some of the shadowing effects. The class of hidden facets of interest are the ones that are turned away from the sensor. These facets are always hidden from view. According to the formulas of Eq. (3) and (4) these facets make negative contributions toward projected areas. The correct contribution for these facets is zero. This observation suggests the following modified form of Eq. (4):

$$\hat{f}(\theta, \varphi) = \frac{p(\theta, \varphi) \frac{\cos(\Omega)}{\cos(\theta)} H(\pi/2 - \Omega)}{\int p(\theta, \varphi) \frac{\cos(\Omega)}{\cos(\theta)} H(\pi/2 - \Omega) d\theta d\varphi} \quad (5)$$

Here  $\hat{f}$  is the corrected form of  $f$  and  $H$  is the step function which is equal to unity if its argument is positive and zero otherwise.

An alternative method for arriving at Eq. (5) involves consideration of a shadowing factor. This is the probability that a facet of given orientation is not hidden from view of the sensor. This problem has been analyzed in the context of RADAR detection [7]. The result, ignoring correlation of waveheight and slope, is that the shadowing factor is proportional to  $H(\pi/2 - \Omega)$ . The proportionality factor is independent of facet orientation. The step from Eq. (4) to Eq. (5) can be viewed as the replacement of LOS projected areas with the product of LOS projected area and the appropriate shadowing factor.

The central idea introduced in this paper is to use the expression of Eq. (5) to weight facet contributions to the net radiance. For purposes of comparison with the results of the BRDF approach it is useful to note that the integral in the denominator of Eq. (5) can be explicitly carried out in the nonshadowing case. In this case the function  $H$  is identically equal to unity. The spherical addition theorem may be used to calculate  $\cos(\Omega)$ :

$$\cos(\Omega) = \cos(\theta) \cos(\hat{\theta}) + \sin(\theta) \sin(\hat{\theta}) \cos(\varphi - \hat{\varphi}) \quad (6)$$

where  $\hat{\theta}$  and  $\hat{\varphi}$  are the polar and azimuthal angles of a unit vector,  $\hat{r}$ , which is parallel to the LOS to

the sensor. (See Fig. 1 for specification of the coordinate system.) The integral in the denominator of Eq. (5) thus involves two terms. The first term evaluates immediately to  $\cos(\hat{\theta})$  because the probability density is normalized. The second term involves the integral of the product of  $\cos(\varphi - \hat{\varphi})$  and  $p(\theta, \varphi)$ . The probability functions considered in this paper all have inversion symmetry. That is  $p(\theta, \varphi) = p(\theta, \pi - \varphi)$ . With this condition it is easy to see that the second integral is zero. However it can be shown that this integral vanishes under more general conditions. The integral in the denominator of Eq. (5) thus evaluates to  $\cos \hat{\theta}$  in the nonshadowing case. Thus in this case:

$$\hat{f}(\theta, \varphi) = f(\theta, \varphi) = p(\theta, \varphi) \frac{\cos(\Omega)}{\cos(\theta) \cos(\hat{\theta})}. \quad (7)$$

## EMISSIVITY AND REFLECTIVITY

The facet emissivity,  $\epsilon$ , and reflectivity,  $\rho$ , are both functions of the angle  $\Omega$ . The Fresnel formula for unpolarized radiation may be used to calculate  $\rho(\Omega)$  if the complex optical constant for water is specified. The appropriate value is obtained by computing an average weighted by the response function of the sensor. Kirchhoff's law in the form  $\epsilon = 1 - \rho$  may be used to obtain the angularly dependent emissivity. The intrinsic radiance,  $R_i$ , is given by  $R_i = \epsilon(\Omega)R_B$  where  $R_B$  is the radiance of a blackbody at the sea temperature. The reflected radiance,  $R_r$ , is given by  $R_r = \rho(\Omega)R_s(\theta')$ , where  $R_s$  is the radiance downwelling from the sky (sky radiance). The sky radiance depends on the angle,  $\theta'$ , between the unit vector  $\hat{r}'$  and the zenith direction. The unit vector  $\hat{r}'$  is parallel to the ray generated by the reflection of the LOS from the sensor by the facet. It has the angular coordinates  $\theta'$  and  $\varphi'$ . A vector expression can be given for  $\hat{r}'$  in terms of  $\hat{n}$  and  $\hat{r}$ :

$$\hat{r}' = 2 \cos(\Omega)\hat{n} - \hat{r}. \quad (8)$$

An expression for  $\theta'$  in terms of  $\Omega$  and  $\theta$  can be obtained from this equation by taking the inner product of both sides of the equation with the unit vector pointing in the zenith direction:

$$\cos(\theta') = 2 \cos(\Omega) \cos(\theta) - \cos(\hat{\theta}). \quad (9)$$

## ROUGH SEA RADIANCE

The apparent radiance of the rough sea,  $R_a$ , is the expected value of the sum of  $R_i$  and  $R_r$ . This expected value is given by:

$$\begin{aligned} R_a(\hat{\theta}) &= \int d\theta \int d\varphi \hat{f} (R_i + R_r) \\ &= \int d\theta \int d\varphi \hat{f}(\theta, \varphi) \{ [1 - \rho(\Omega)]R_B + \rho(\Omega)R_s(\theta') \}. \end{aligned} \quad (10)$$

Here the form of  $\hat{f}$  is given by Eqs. (5) and (1),  $\Omega$  is given by Eq. (6) and  $\theta'$  is given by Eq. (9). The Fresnel formula is to be used for the functional form of  $\rho(\Omega)$  and the functional form of  $R_s$  must be externally specified. An atmospheric modeling code such as Lowtran VI can be used to create a tabulated form of this function. Since  $\hat{f}$  is a normalized weighting function, it can be verified that Eq. (10) satisfies an important consistency relation. Specifically, in the case where  $R_s$  is a constant equal to  $R_B$ , one finds that  $R_a = R_B$ . This result is required by the second law of thermodynamics. It forms a useful check on the consistency of the general approach.

As mentioned in the introduction it can be shown that the results obtained in this paper agree with the results obtained using the BRDF approach in the case where there is no shadowing. In this case  $f^2$  is given by Eq. (7). If this form is substituted into Eq. (10) with the  $p(\theta, \varphi)$  given by Eq. (1) with  $a^2 = b^2 = \sigma^2/2$  (isotropic case) the weighting terms in Eq. (10) become:

$$\frac{\tan(\theta) \cos(\Omega) \exp(-\tan^2(\theta)/\sigma^2)}{\cos^3(\theta) \cos(\hat{\theta})} \quad (11)$$

Apart from notational differences, this weighting factor is the same as the one in [8].

## RESULTS

Equation (10) is the central result of this paper. To see the typical magnitude of the effects caused by a rough sea, a functional form for  $R_s$  must be chosen. As an example we have exercised the Lowtran program with summer midlatitude parameters and have obtained the long wave IR (8-12 micron) sky radiance form shown in Fig. 2. For convenience, the value of the blackbody radiance of the sea,  $R_B$ , which appears in Eq. (10) was taken to be equal to the value of  $R_s$  at the horizon. This corresponds to selecting temperatures of the sea and of the air just above the sea to be equal. These Lowtran results, together with Eq. (10) (with  $a^2 = b^2 = \sigma^2/2$ ) were used to calculate  $Z(\theta') = 1 - R_s(\theta')/R_B$  for a set of values of  $\sigma$ . For this calculation the optical constants appropriate to the long wave IR were used in the Fresnel reflectivity formula. Since there is an empirical relationship between  $\sigma$  and the wind speed given by [9]:

$$\sigma^2 = .003 + .00512 w \quad (12)$$

where  $w$  is the wind speed in m/sec, this set of results is effectively parameterized by the wind speed. Figure 3 presents the calculated function  $Z(\theta')$  for selected values of  $\sigma$ . It can be observed that for values of  $\theta'$  near  $75^\circ$  ( $15^\circ$  away from grazing) modest values of the wind speed produce substantial effects. The sea can appear to emit up to 10% less radiation than a calm sea would appear to emit. Effects of this magnitude can have a substantial impact on ship detectability.

## DISCUSSION

In this paper we have presented a probabilistic model of the apparent radiance of a rough sea surface. It is observed that when compared to a smooth sea surface, a rough sea can emit a substantially lower amount of radiation in the infrared. The sea, being a poor emitter of radiation at near grazing angles, reflects cool sky radiation from higher points in the sky and therefore looks colder than its blackbody temperature. Other authors have modelled this phenomenon as well but without the aid of a probability model of the sea slopes [1,2]. In some extreme cases, other authors rely on one emissivity value of the sea surface for all angles [10].

Another popular but realistic method of computing the apparent radiance of a rough sea is to create a surface which has the same amplitudes of its Fourier components as a rough sea. The method is called the power spectral approach and has been used either directly or indirectly in simulating images of a rough sea [11,12,13,3,14,15]. (See [15] for a description of the technique.) The power spectral method is computationally intensive since a large number of facets must be considered. Furthermore, a lot of surfaces must be generated to compute any statistical information about the sea surface in question for each wind speed. Analyzing the power spectral model is indeed difficult. Our probabilistic approach, however, immediately generates the statistics of the rough sea surface with a minimum amount of computational effort. The physics is contained in the model at a glance, thus making it easy to analyze and verify in certain trivial cases.

Finally, our slope magnitude model of the sea surface is a two dimensional model based upon on linear superposition of waves. Real sea surfaces, however, have nonlinear waves, as pointed out in [16,17]. In particular, Huang and co-workers [7,8] have derived the probability slope distribution for a nonlinear wave in one dimension. In [16], a small parameter,  $\sigma k$ , is introduced where  $\sigma$  is the standard deviation of the surface elevation and  $k$  is the wave number. Using the assumption that the parameter is small, the probability slope density can be shown to be normal to order  $(\sigma k)^2$ . Therefore, the generalization to a nonlinear sea model should not impact greatly on our results.

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## Appendix A

The starting point is to consider the function  $z = h(x, y)$  defined to be the water height above mean sea level at the position  $(x, y)$  in the horizontal plane. The function  $h$  may be regarded to be a random function; that is,  $h(x, y)$  is to be regarded as a random variable for each value of  $x$  and  $y$ . Corresponding to this type of random variable is a probability density function. Because the surface of the ocean is translationally invariant in a statistical sense, the density does not depend on  $x$  and  $y$ . Accordingly, we denote it by  $p_h$ . The value of  $h(x, y)$  is the sum of contributions from waves of a large range of wavelengths. To the extent that the contributions of the different wavelengths are statistically independent, the central limit theorem applies and  $p_h$  is of normal form. This assumption of independent contributions is an approximation which breaks down for large values of  $h$  due to non-linear hydrodynamic interactions. However, it is a good approximation for our purposes here. The wave slope is the gradient of  $h$ . The statistics of spatial derivatives can be calculated by the use of finite differences, to wit:

$$\frac{\partial h}{\partial x} = \frac{h(x', y) - h(x, y)}{x' - x} \quad (\text{A1})$$

for  $x'$  sufficiently close to  $x$ . (A similar relation holds for  $\partial h / \partial y$ .) It has been established that wavelengths shorter than a certain cutoff contribute very little to the slope statistics. Consequently Eq. (A1) may be considered valid as long as  $|x' - x|$  is less than the cutoff length. If the anisotropy introduced by the wind direction is ignored,  $h(x', y)$  and  $h(x, y)$  are identically normally distributed random variables. They are correlated with a correlation length depending on  $|x' - x|$ . Since the difference of two correlated normally distributed random variables is itself a normally distributed random variable, it follows that  $\partial h / \partial x$  and  $\partial h / \partial y$  are normally distributed. The wave slope,  $s$ , is defined to be magnitude of the gradient of  $h$ ; i.e.,

$$s = \left[ \left( \frac{\partial h}{\partial x} \right)^2 + \left( \frac{\partial h}{\partial y} \right)^2 \right]^{1/2} \quad (\text{A2})$$

Since  $\partial h / \partial x$  and  $\partial h / \partial y$  are normal, independent with zero mean and equal variance, then the random variable  $s$  has a Rayleigh density, given by

$$p_s(s) \propto s e^{-s^2/\sigma^2} \quad (\text{A3})$$

where  $\sigma^2$  is the slope variance. It is more convenient to work with a description in terms of spherical coordinates on a unit sphere. We introduce the angle coordinates,  $\theta$ ,  $\phi$ , to describe the direction of the unit normal,  $n$ , to the facet face. The angle  $\theta$  is the angle between and the normal to the sea surface (the  $z$  direction) while  $\phi$  is the azimuthal angle defined such that it is zero when  $n$  is in the  $z-x$  plane. Using the fact that  $s = \tan \theta$ , it follows from a change of variables that

$$p_n(\theta, \phi) = (1/2\pi) 2\sigma^2 \tan \theta \sec^2 \theta e^{-\tan^2 \theta / \sigma^2} \quad (\text{A4})$$

It is easy to see how the nonisotropic case in which the upwind and crosswind variances are unequal generalizes the slope probability distribution. Assume that the  $x$  and  $y$  directions correspond to the upwind and crosswind directions, respectively, and also suppose that the mean slopes are zero in both  $x$  and  $y$  directions. A normal vector to the surface at  $(x, y)$  is given by

$$N(x, y) = (\partial h / \partial x, \partial h / \partial y, 1) = [s^2 + 1]^{1/2} n(x, y). \quad (\text{A5})$$

Decomposing the vector  $N(x,y)$  into spherical coordinates  $(\theta, \phi)$  on the sphere of radius  $[s^2 + 1]^{1/2}$ , it follows that  $\sin \theta = s/[s^2 + 1]^{1/2}$  and  $\tan \theta = s$ .

Reasoning as before, the slope density,  $p(\partial h/\partial x, \partial h/\partial y)$ , is normal in both arguments with standard deviations  $\sigma_x, \sigma_y$ , and correlation coefficient  $r$ . We make the change variables by letting

$$\partial h/\partial x = \sqrt{s^2 + 1} \sin \theta \cos \phi = \tan \theta \cos \phi \quad (\text{A6})$$

$$\partial h/\partial y = \sqrt{s^2 + 1} \sin \theta \sin \phi = \tan \theta \sin \phi.$$

Since the Jacobian of the transformation is  $\sec^2 \theta \tan \theta$ , the new density in terms of  $(\theta, \phi)$  is given by

$$\hat{p}_n(\theta, \phi) = \frac{\sec^2 \theta \tan \theta}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} \exp \left\{ -\frac{\tan^2 \theta}{2(1-r^2)} E(\phi, r) \right\}, \quad (\text{A7})$$

where

$$E(\phi, r) = \frac{\cos^2 \phi}{\sigma_x^2} + \frac{\sin^2 \phi}{\sigma_y^2} - \frac{2r \cos \phi \sin \phi}{\sigma_x \sigma_y} \quad (\text{A8})$$

and  $n$  is  $N/|N|$ . Notice that if  $\sigma_x = \sigma_y$  and  $r = 0$ , then the probability density is given in terms of  $\theta$  only and reduces to Eq. (A4).

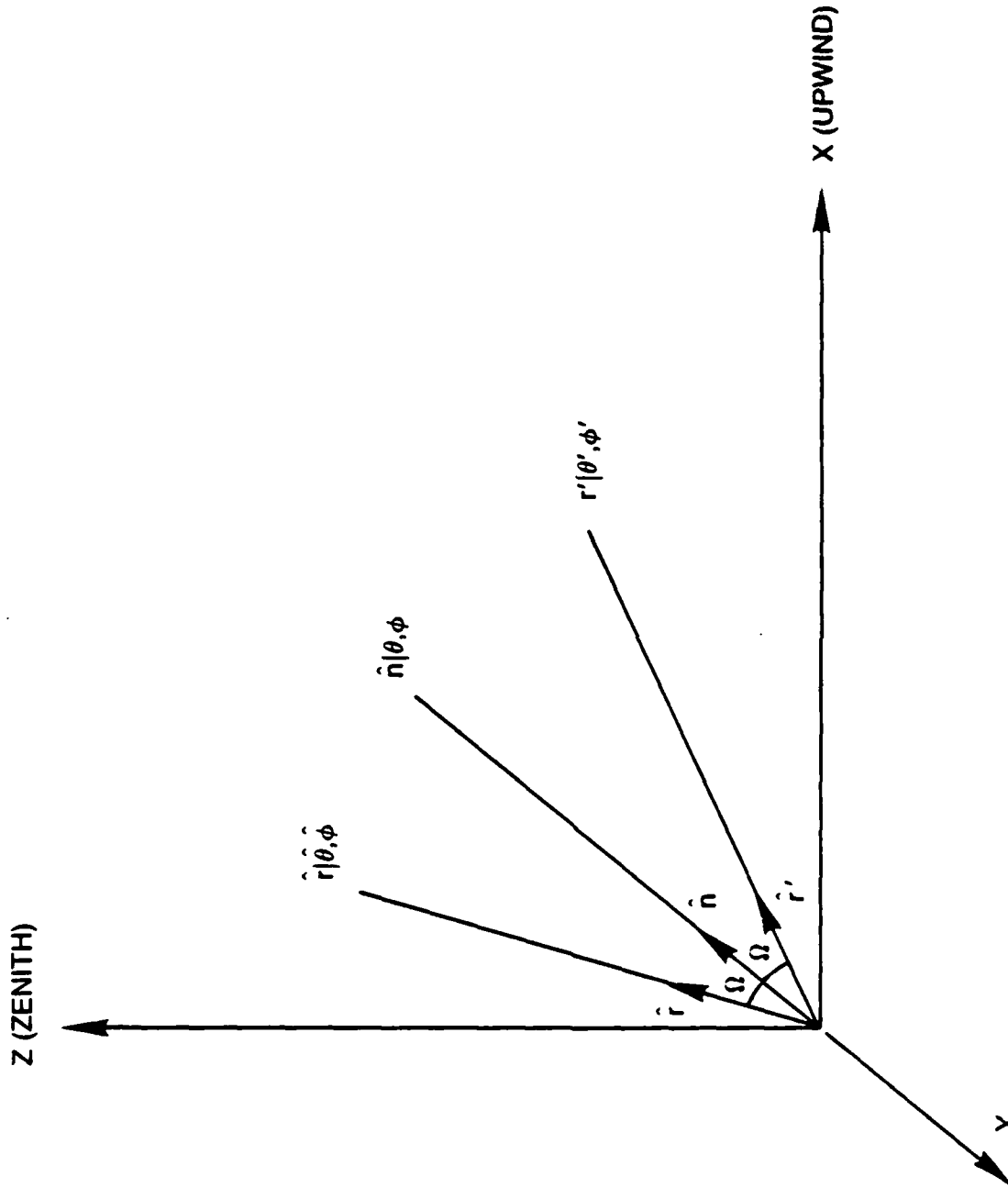


Figure 1

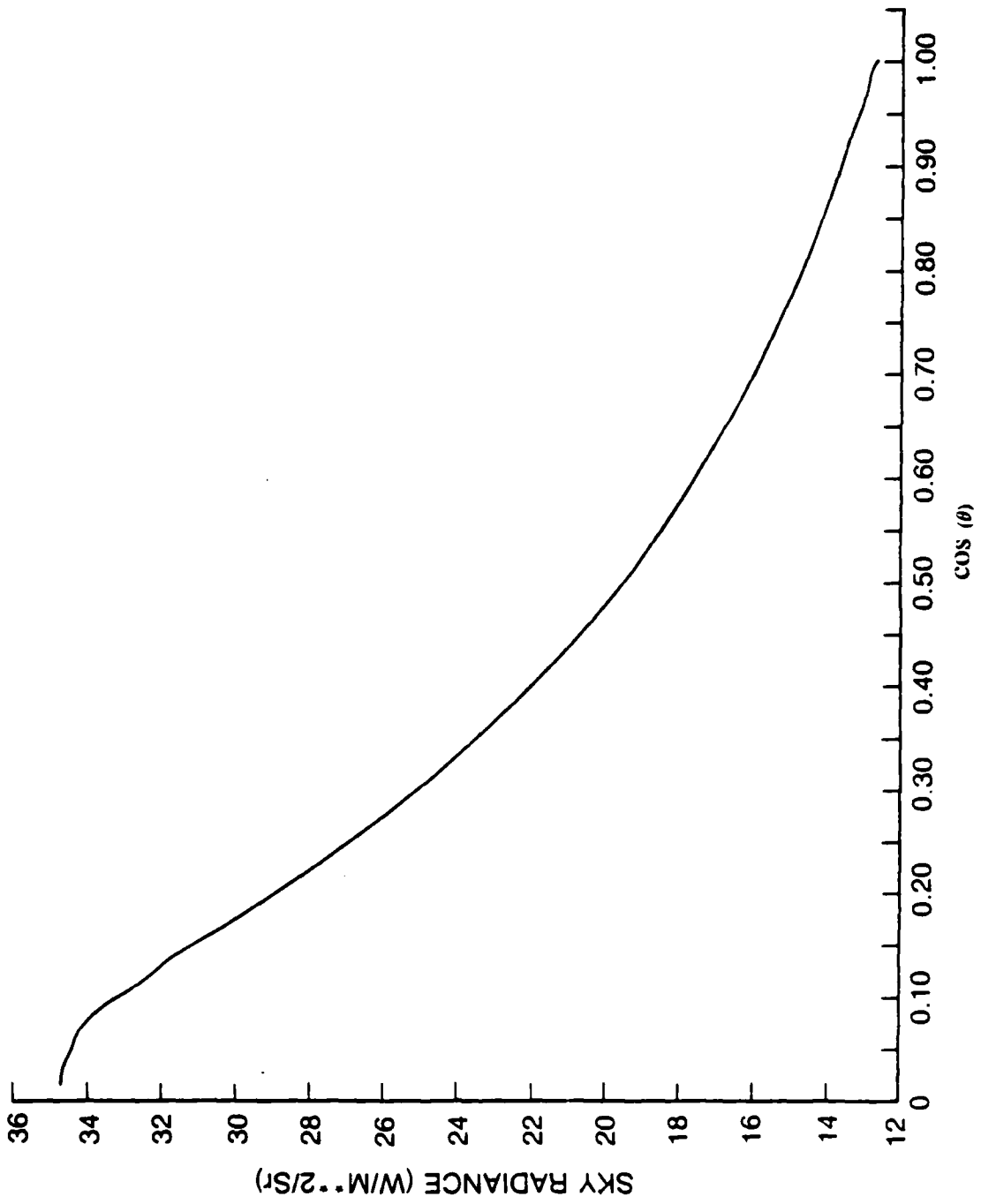


Figure 2

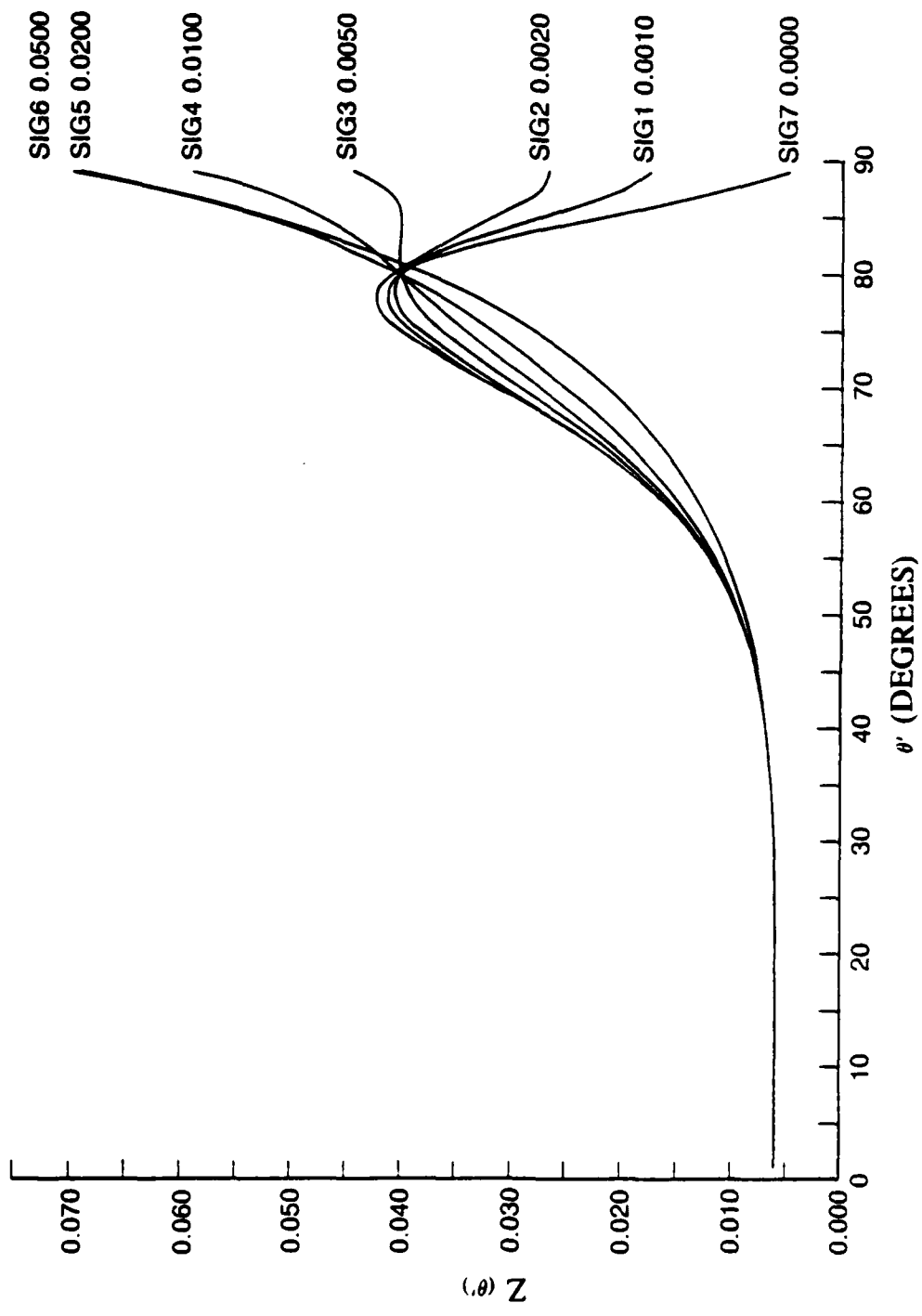


Figure 3

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