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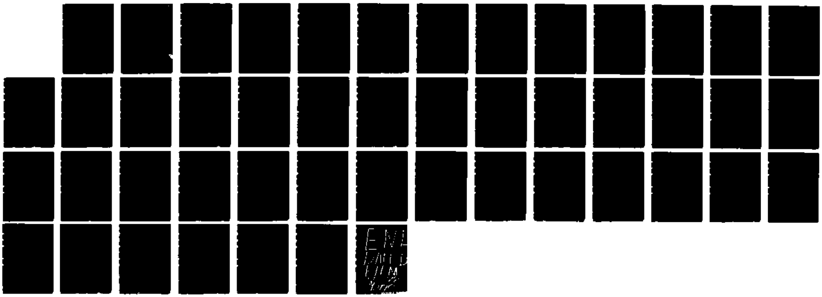

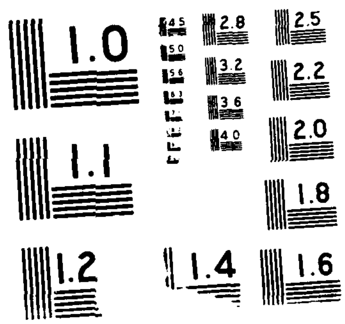
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$$L(t) = \sum_{i_1,i_2,\ldots,i_d=-\infty}^{\infty} \sum_{j=1}^{d} \{a_{i_j}(t,x)^2 \frac{\partial}{\partial x_{i_j}^2} + b_{i_j}(t,x) \frac{\partial}{\partial x_{i_j}}\}.$$

It is proven that the weak solution of a Langevin's equation:

$$dX(t) = dW(t) + L^*(t)X(t)dt,$$

exists uniquely on a generalized functional space on $\mathcal{S}'(\mathbb{Z}^d)$ which is

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19. Abstract

appropriate for the central limit theorem of lattice valued diffusions.

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CENTER FOR STOCHASTIC PROCESSES

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Weak Solution of the Langevin Equation on a

Generalized Functional Space

by

Itaru Mitoma

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Weak Solution of the Langevin Equation on a Generalized Functional Space

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Abstract. Let $\mathcal{S}'(\mathbb{Z}^d)$ be the Schwartz space of tempered distributions on the d-dimensional lattice $\mathbb{Z}^d$ and $L^*(t)$ the adjoint operator of $L(t)$ which has a formal expression:

$$L(t) = \sum_{i_1, i_2, \ldots, i_d = -\infty}^{\infty} \sum_{j=1}^{d} \{a_{i_j}(t,x)^2 \frac{\partial^2}{\partial x_{i_j}^2} + b_{i_j}(t,x) \frac{\partial}{\partial x_{i_j}}\}.$$

It is proven that the weak solution of a Langevin's equation:

$$dX(t) = dW(t) + L^*(t)X(t)dt,$$

exists uniquely on a generalized functional space on $\mathcal{S}'(\mathbb{Z}^d)$ which is appropriate for the central limit theorem of lattice valued diffusions.

Key words and phrases: Weak solution, Langevin's equation, Fréchet derivative, generalized functional space, central limit theorem, lattice valued diffusion.

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§1. Introduction

Recently Deuschel [4] has obtained a fluctuation result for a system of lattice valued diffusion processes. The result is similar to that for mean-field interacting diffusion particles [2], [3], [8], [9], [15], [22].

However the identification problem of limit measures he treated leads us to discuss the uniqueness for weak solutions of the Langevin equation:

$$dX(t) = dW(t) + L^{*}(t)X(t)dt,$$

where $W(t)$ is a generalized functional space valued Brownian motion and $L^{*}(t)$ is the adjoint operator of $L(t)$ which has a formal expression:

$$L(t) = \sum_{i=-\infty}^{\infty} a_i(t,x)^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i=-\infty}^{\infty} b_i(t,x) \frac{\partial}{\partial x_i} .$$

The aim of this paper is to find a suitable space $\mathcal{D}_{E'}$ of smooth functionals on the dual nuclear space E' and to solve the Langevin equation on the dual space $\mathcal{D}'_{E'}$, which is appropriate for the central limit theorem of empirical distributions of the system of lattice valued diffusion processes. This application is another approach to his problem [4].

We will proceed to explain the setting: A stochastic process $X_F(t)$ defined on a complete probability space $(\Omega,\mathcal{F},P)$ indexed by elements in $\mathcal{D}_{E'}$ is called a $\mathcal{L}(\mathcal{D}_{E'})$-process if $X_F(t)$ is a real stochastic process for any fixed $F \in \mathcal{D}_{E'}$ and $X_{\alpha F+\beta G}(t) = \alpha X_F(t) + \beta X_G(t)$ almost surely for each real numbers α,β and elements of $F,G \in \mathcal{D}_{E'}$ and further $E[X_F(t)^2]$ is continuous on $\mathcal{D}_{E'}$ [10]. $X_F(t)$ is called continuous if $\lim_{t\to s} E[(X_F(t)-X_F(s))^2] = 0$ for each $F \in \mathcal{D}_{E'}$. Let $W_F(t)$ be a Wiener $\mathcal{L}(\mathcal{D}_{E'})$-process such that for any fixed $F \in \mathcal{D}_{E'}$, $W_F(t)$ is a real continuous Gaussian additive process with mean 0.

We will prove that a continuous $\mathcal{L}(\mathcal{D}_{E'})$-process solution $X_F(t)$ for the following equation uniquely exists in the case where E' is the space $\mathcal{S}'(\mathbb{Z}^d)$ of tempered distributions on the d-dimensional lattice, (Theorem):

$$(1.1) \qquad dX_F(t) = dW_F(t) + X_{L(t)F}(t)dt.$$

Roughly speaking, if L(t) generates the strongly continuous Kolmogorov evolution operator U(t,s) from $\mathcal{D}_{E'}$ into itself, the unique solution for (1.1) can be given as follows:

$$X_F(t) = X_{U(t,0)F}(0) + W_F(t) + \int_0^t W_{L(s)U(t,s)F}(s)ds.$$

We will now begin by giving the precise definitions of the operator L(t) and the space $\mathcal{D}_{E'}$. Let E be a nuclear Fréchet space whose topology is defined by an increasing sequence of Hilbertian sem-norms $\|\cdot\|_1 \le \|\cdot\|_2 \le \ldots \le \|\cdot\|_p \ldots$. As usual let E' be the dual space, E_p the completion of E by the p-th semi-norm $\|\cdot\|_p$ and E'_p the dual space of E_p. Then we have

$$E = \bigcap_{p=0}^{\infty} \quad \text{and} \quad E' = \bigcup_{p=0}^{\infty} E'_p.$$

Let K be a separable Hilbert space with norm $\|\cdot\|_K$ and F a mapping from E' into K. Then F is said to be E'_p-Fréchet differentiable if for every $x \in E'$, we have a bounded linear operator $\mathcal{D}_pF(x)$ from E'_p into K such that

$$\lim_{t\to 0} \frac{F(x+th) - F(x)}{t} = \mathcal{D}_pF(x)(h) \quad \text{in K}.$$

Suppose that F is E'_p-Fréchet differentiable for every integer $p \ge 0$. Then taking $E' = \bigcup_{p=0}^{\infty} E'_p$ and the strong topology of E' is equivalent to the inductive limit topology of E'_p; p=0,1,2,..., into account, we have a continuous linear operator DF(x) from E' equipped with the strong topology into K such that for any integer $p \geq 0$, $DF(x)(h) = \mathcal{D}_p F(x)(h)$ for $h \in E_p'$. Hence, if F is n-times E_p'-Fréchet differentiable for every integer $p \geq 0$, we have a continuous n-linear operator $D^n F(x)$ from $\underbrace{E' \times E' \times \ldots \times E'}_{n\text{-times}}$ into K such that the restriction of $D^n F(x)$ on $\underbrace{E_p' \times E_p' \times \ldots \times E_p'}_{n\text{-time}}$ = the n-th E_p'-Fréchet derivative $\mathcal{D}_p^n F(x)(\xi_1, \xi_2, \ldots, \xi_n)$, $\xi_i \in E'$. Then if F is infinitely many times E_p'-Fréchet differentiable for every integer $p \geq 0$, the Hilbert-Schmidt norm

$$\|D^n F(x)\|_{H.S.}^{(p)} = \Big(\sum_{i_1, i_2, \ldots, i_n = 1}^{\infty} \|D^n F(x)(h_{i_1}^{(p)}, h_{i_2}^{(p)}, \ldots, h_{i_n}^{(p)})\|_K^2 \Big)^{1/2}$$

is finite for each integer $n \geq 1$ and $p \geq 0$, where $(h_j^{(p)})$ is a C.O.N.S., (complete orthonormal system), in E_p' [13].

From now on, we will often use the conventional notation such that $\|D^0 F(x)\|_{H.S.}^{(p)} = \|F(x)\|_K$.

Let $\beta(t)$ be the standard E'-Wiener process such that for any $\xi \in E$, $\langle \beta(t), \xi \rangle$ is a 1-dimensional Brownian motion, with variance $E[\langle \beta(t), \xi \rangle^2] = t\|\xi\|_0^2$, where $\langle x, \xi \rangle$, $(x \in E', \xi \in E)$, denotes the canonical bilinear form on $E' \times E$.

Without loss of generality, we assume $\beta(t)$ is an E_1'-valued Wiener process throughout this paper, [16], [17].

Definition of L(t). Let A(t,x) and B(t,·) be continuous mappings from E' into itself such that the following conditions are satisfied.

(H1) There exists a natural number p_0 such that A(t,x) maps E_1' into E_{p_0}', B(t,·) maps E' into E_{p_0}' and for each $T > 0$,

$$\sup_{\substack{x \in E' \\ 0 \leq t \leq T}} \|A(t,x)\|_2^2 < \infty \quad \text{and} \quad \sup_{\substack{x \in E' \\ 0 \leq t \leq T}} \|B(t,x)\|_{-p_0} < \infty,$$

where $\|\cdot\|_{-p}$ denotes the dual norm of E_p' and $\| A(t,x) \|_2^2 = \sum_{j=1}^{\infty} \|A(t,x)h_j^{(0)}\|_{-p_0}^2$.

(H2) $A(t,x)$ and $B(t,x)$ are infinitely many times E_p'-Fréchet differentiable for every integer $p \geq 0$ such that for any $T > 0$,

$$\sup_{\substack{x \in E' \\ 0 \leq t \leq T}} \|D^n(t,x)\|_{H.S.}^{(p)} < \infty \quad \text{and} \quad \sup_{\substack{x \in E' \\ 0 \leq t \leq T}} \|D^n B(t,x)\|_{H.S.}^{(p)} < \infty,$$

where $\|D^n A(t,x)\|_{H.S.}^{(p)} = (\sum_{i_1, i_2, \ldots, i_n = 1}^{\infty} \| D^n A(t,x)(h_{i_1}^{(p)}, h_{i_2}^{(p)}, \ldots, h_{i_n}^{(p)} \|_2^2)^{1/2}$ and

$$\|D^n B(t,x)\|_{H.S.}^{(p)} = (\sum_{i_1, i_2, \ldots, i_n = 1}^{\infty} \|D^n B(t,x)(h_{i_1}^{(p)}, h_{1_2}^{(p)}, \ldots, h_{i_n}^{(p)})\|_{-p_0}^2)^{1/2}.$$

(H3) For any integer $n \geq 0$ and any $T > 0$, there exist $\lambda(n,P,T) > 0$ and $\lambda_1(n,p,T) > 0$ such that

$$\sup_{\substack{x \in E' \\ 0 \leq k \leq n}} \max\{\|D^k A(t,x) - D^k A(s,x)\|_{H.S.}^{(p)}, \|D^k B(t,x) - D^k B(s,x)\|_{H.S.}^{p}\}$$

$$\leq \lambda_1(n,p,T) \ |t-s|^{\lambda(n,p,T)}, \quad 0 \leq s,t \leq T.$$

Then for any two times E_p'-Fréchet differentiable real valued function F on E' for every $p \geq 0$, we put

$$(L(t)F)(x) = \frac{1}{2} \operatorname{trace}_{E_0} D^2 F(x) \circ [A(t,x) \times A(t,x)] + DF(x)(B(t,x)),$$

where $D^2F(x) \circ [A(t,x) \times A(t,x)](\xi_1,\xi_2) = D^2F(x)(A(t,x)\xi_1, A(t,x)\xi_2)$ for any $\xi_1, \xi_2 \in E'$.

Definition of Space $\mathcal{D}_{E'}$. We define the space $\mathcal{D}_{E'}$ as a collection of all real valued functionals F on E' such that F is infinitely many times E_p'-Fréchet differentiable for every integer $p \geq 0$ and further the space $\mathcal{D}_{E'}$ is a complete separable metric space metrized by the following semi-norms:

$$\|F\|_{p,q,n} = \sum_{k=0}^{n} \|F\|_{p,k}^{(q)}, \quad F \in \mathcal{D}_{E'},$$

where $p \geq p_0$, $q \geq 0$ and $n \geq 0$ are integers and

$$\|F\|_{p,n}^{(q)} = \sup_{x \in E_p'} e^{-\|x\|_{-p}} \|D^n F(x)\|_{H.S.}^{(q)} .$$

Before proceeding to the discussions of the equation (1.1), we will give some remarks on Wiener $\mathcal{L}(\mathcal{D}_{E'})$-process. Taking the continuities of $W_F(t)$ and $E[W_F(t)^2]$ with respect to the parameters t and F into account, we have that $\sup_{0 \leq t \leq T} E[W_F(t)^2] < \infty$ and $\sup_{0 \leq t \leq T} E[W_F(t)^2]$ is lower semi-continuous on $\mathcal{D}_{E'}$. Since $\mathcal{D}_{E'}$ is a complete metric space, by the Banach-Steinhaus theorem we have some positive integers p_1, q_1 and m_1 such that

$$\sup_{0 \leq t \leq T} E[W_F(t)^2] \leq C_1(T) \|F\|^2_{p_1,q_1,m_1}. \tag{1.2}$$

Here and in the sequel, we denote positive constants by C_i or, if necessary, by $C_i(\tau_1, \tau_2, \ldots)$, $i=1,2,\ldots$, in the case where they depend on the parameters $\tau_1, \tau_2, \ldots$.

Now given a functional $V_t(F)$ such that it is positive definite quadratic form on $\mathcal{D}_{E'} \times \mathcal{D}_{E'}$, increasing and continuous in t and $\sup_{0 \leq t \leq T} V_t(F) \leq C_2(T) \|F\|^2_{p,q,n}$ for some natural numbers p, q and n, we can construct a $\mathcal{D}_{E'}$-indexed Gaussian mean-zero continuous process $W_F(t)$ with independent increments and variance $V_t(F)$ by the Kolmogorov theorem, since $V_{t \wedge s}(F)$ is positive definite quadratic form with respect to (t,F), $t \in [0,\infty)$, $F \in \mathcal{D}_{E'}$. Here $t \wedge s = \min\{t,s\}$.

§2. Existence and Uniqueness for solutions of the Langevin equation

Let $\eta_{s,t}(x)$ be a solution of the following stochastic differential equation:

$$\eta_{s,t}(x) = x + \int_s^t A(t,\eta_{s,r}(x))d\beta(r) + \int_s^t B(r,\eta_{s,r}(x))dr,$$

where $\beta(t)$ is the standard E'-Wiener process. By the assumptions (H1) and (H2), if $p \geq p_0$ and $x \in E_p'$, then the solution of the above equation is uniquely obtained by the usual method of successive approximations in E_p'.

We will assume the following condition:

(H4) $(L(t)F)(x)$ and $(U(t,s)F)(x) = E[F(\eta_{s,t}(x))] \in \mathcal{D}_{E'}$ if $F \in \mathcal{D}_{E'}$.

Let $W_F(t)$, $F \in \mathcal{D}_{E'}$, be the Wiener $\mathcal{L}(\mathcal{D}_{E'})$-process and $L(t)$ a diffusion operator defined before. Then we will prove

Proposition 1. Under the assumptions (H1)-(H4) the continuous $\mathcal{L}(\mathcal{D}_{E'})$-process solution of (1.1) such that for some $0<\alpha<1$, $E[|X_F(0)|^{2+\alpha}] < \infty$ is uniquely given as follows:

$$X_F(t) = X_{U(t,0)F}(0) + W_F(t) + \int_0^t W_{L(s)U(t,s)F}(s)ds.$$

Proof. Under the assumptions (H1)-(H4), $L(t)$ is a continuous linear operator from $\mathcal{D}_{E'}$ into itself and we can get the following lemma which will be proved later.

Lemma 1. Suppose that the conditions (H1)-(H4) hold. Then $L(t)$ generates the Kolmogorov evolution operator $U(t,s)$ from $\mathcal{D}_{E'}$ into itself such that

(1) $U(t,s)$ is a continuous linear operator from $\mathcal{D}_{E'}$ into itslef,

(2) for any $F \in \mathcal{D}_{E'}$, $U(t,s)F$ is continuous from $\{(t,s);\ 0\leq s\leq t\}$ into $\mathcal{D}_{E'}$,

(3) $U(t,t) = U(s,s) =$ identity operator,

(4) $\frac{d}{dt}U(t,s)F = U(t,s)L(t)F$, $0 \leq s \leq t$ on $\mathcal{D}_{E'}$,

(5) $\frac{d}{ds}U(t,s)F = -L(s)U(t,s)F$, $0\leq s\leq t$, $t>0$ on $\mathcal{D}_{E'}$.

Further for any integers $p\geq p_0$, $q\geq 0$, $n\geq 0$, $j\geq 1$ and any $T>0$ and $F \in \mathcal{D}_{E'}$, we have

$$(2.1)\quad \|U(t',s')F - U(t,s)F\|^{2j}_{p,q,n} \leq C_3(T,F,p,q,n)\{|t-t'|^j + |s-s'|^j\},$$
$$0 \leq s,t,s',t' \leq T.$$

First we will guarantee the well-definedness of the integral in Proposition 1 by showing that for any fixed $F \in \mathscr{D}_{E'}$, $W_{L(s)U(t,s)F}(s)$ is continuous in (t,s). Since $W_F(t)$ is a Gaussian additive process with mean 0 and variance $V_t(F)$, we get for any integer $n \geq 1$,

$$(2.2)\quad E[|W_F(t_1) - W_F(t_2)|^{2n}] \leq C_4(T)(V_{t_1}(F)-V_{t_2}(F))^n,\ 0 \leq t_1,t_2 \leq T.$$

We choose an integer $k \geq 4$ such that $2k\lambda(m_1,q_1,T) > 2$, where m_1 and q_1 are the numbers which appeared in (1.2) and $\lambda(m_1,q_1,T)$ is the number in (H3). For $0 \leq s,t,s',t' \leq T$, the inequalities (2.1) and (2.2) yield, together with (H3),

$$(2.3)\quad E\{|W_{L(s)(t,s)F}(s') - W_{L(s)U(t,s)F}(s)|^{2k}]$$
$$\leq C_5(T)(V_{s'}(L(s)U(t,s)F) - V_s(L(s)U(t,s)F))^k$$

and

$$(2.4)\quad E[|W_{L(s')U(T',s')F}(s')-W_{L(s)U(t,s)F}(s')|^{2k}]$$
$$\leq C_6(T)\|L(s')U(t',s')F - L(s)U(t,s)F\|^{2k}_{p_1,q_1,m_1}$$
$$\leq C_7(T)\{\|U(t',s')F - U(t,s)F\|^{2k}_{p_1,q_1,m_1+1}+ \|U(t',s')F - U(t,s)F\|^{2k}_{p_1,q_1,m_1+2}$$
$$+ |s'-s|^{2k\lambda(m_1,q_1,T)}\}$$

$$\leq C_8(T)\{|t-t'|^k + |s-s'|^k + |s'-s|^{2k\lambda(m_1,q_1,T)}\}.$$

The inequalities (2.3) and (2.4) are sufficient for the condition of the Kolmogorov-Totoki criterion [24] for continuity in (t,s). Further the continuity of $W_{L(s)U(t,s)L(t)F}(s)$ in (t,s) can be proved similarly.

Now we will proceed to the proof of the existence of solutions for (1.1). Taking the relation $U(t,s)F = F + \int_s^t U(\tau,s)L(\tau)F d\tau$, the continuity of $W_{L(s)U(\tau,s)L(\tau)F}(s)$ in τ, the linearity of $W_\cdot(s)$ and the L^2-continuity of $W_\cdot(s)$, into account, we have

$$\begin{aligned} W_{L(s)U(t,s)F}(s) &= W_{L(s)F}(s) + W_{L(s)\int_s^t U(\tau,s)L(\tau)F d\tau}(s) \\ &= W_{L(s)F}(s) + \int_s^t W_{L(s)U(\tau,s)L(\tau)F}(s) d\tau, \end{aligned}$$

so that by making use of the continuity of $W_{L(s)U(\tau,s)L(\tau)F}(s)$ in (τ,s) again, we get

$$\begin{aligned} &\int_0^t W_{L(s)U(t,s)F}(s) ds \\ &= \int_0^t W_{L(s)F}(s) ds + \int_0^t \left(\int_s^t W_{L(s)U(\tau,s)L(\tau)F}(s) d\tau\right) ds \\ &= \int_0^t W_{L(s)F}(s) ds + \int_0^t \left(\int_0^\tau W_{L(s)U(\tau,s)L(\tau)F}(s) ds\right) d\tau \\ &= \int_0^t \left(W_{L(\tau)F}(\tau) + \int_0^t W_{L(s)U(\tau,s)L(\tau)F}(s) ds\right) d\tau \\ &= \int_0^t \left(X_{L(\tau)F}(\tau) - X_{U(\tau,0)L(\tau)F}(0)\right) d\tau. \end{aligned} \tag{2.5}$$

Combining the L^2-continuity of $X_F(0)$ in the definition of $\mathcal{L}(\mathcal{D}_{E,})$-process and the Jensen inequality such that $E[|X_F(0)|^{2+\alpha}] \le E[|X_F(0)|^2]^\alpha$, we get that $E[|X_F(0)|^{2+\alpha}]$ is continuous in $\mathcal{D}_{E,}$. Hence there exist some positive integers $p_2 \ge p_0, q_2$ and m_2 such that

$$E[|X_F(0)|^{2+\alpha}] \le C_9 \|F\|_{p_2,q_2,m_2}^{2+\alpha}. \tag{2.6}$$

Therefore the Kolmogorov criterion for continuity, together with the inequalities (2.1) in Lemma 1 and (2.6), gives the continuity of $X_{U(\tau,0)L(\tau)F}(0)$ in τ. Thus we get

(2.7) $$\int_0^t X_{U(\tau,0)L(\tau)F^{(0)}}d\tau = X_{U(t,0)F^{(0)}} - X_F(0).$$

The equalities (2.5) and (2.7) mean that $X_F(t)$ is a solution of the Langevin equation (1.1).

Following H. Komatsu [11], we will prove the uniqueness of L^2-continuous solutions for the equation (1.1). Let $Y_1(t,F)$ and $Y_2(t,F)$ be the two continuous $\mathcal{L}(\mathfrak{D}_{E'})$-process solutions for the equation (1.1). First we remark by the Baire category theorem that for each $T > 0$, we have some natural number $p_3 \geq p_0, q_3$ and m_3 such that

(2.8) $$\max_{i=1,2} \sup_{0\leq t\leq T} E[Y_i(t,F)^2] \leq C_{10}(T)\|F\|_{p_3,q_3,m_3}.$$

Define $v(t,F) = Y_1(t,F) - Y_2(t,F)$. Then for any $a > 0$, we will prove $\frac{d}{dt}$ $E[v(t,U(a,t)F)^2] = 0$ for $t \in (0,a]$. The inequality (2.8) and the strong continuity of $U(t,s)$, ((2) in Lemma 1), yield

$$E[|\frac{v(s,U(a,s)F)^2 - v(t,U(a,t)F)^2}{s-t}|]$$

$$\leq C_{11}(T,F)\ E[(\frac{v(s,U(a,s)F)-v(t,U(a,t)F)}{s-t})^2]^{1/2},\quad s,t \in (0,a] \subset [0,T].$$

The inequality (2.8) and the strong continuities of $L(t)$ and $U(t,s)$ imply that

(2.9) $$\lim_{s\to t} E[|\frac{v(s,U(a,t)F) - v(t,U(a,t)F)}{s-t} - v(t,L(t)U(a,t)F)|^2] = 0.$$

By the strong continuity of $U(t,s)$, we get similarly

(2.10) $$\lim_{s\to t} E[|\frac{v(s,[U(a,s) - U(a,t)]F) - v(t,[U(a,s) - U(a,t)]F)}{s-t}$$

$$- v(t,L(t)[U(a,s) - U(a,t)]F)|^2] = 0.$$

Since $L(t)$ generates the Kolmogorov evolution operator $U(t,s)$, we have

$$\lim_{s\to t} E[|v(t,L(t)U(a,s)F) - v(t,L(t)U(a,t)F)|^2] = 0$$

$$\lim_{s\to t} E[|v(t,L(t)U(a,t)F) + v(t,\frac{U(a,s) - U(a,t)}{s - t}F)|^2] = 0,$$

so that we get

$$\lim_{s\to t} E[|v(t,L(t)U(a,s)F) + \frac{v(t,U(a,s)F) - v(t,U(a,t)F)}{s - t}|^2] = 0. \tag{2.11}$$

Summing up the inequalities (2.9), (2.10) and (2.11), we get the desired equality claimed before. Hence $E[v(t,U(a,t)F)^2]$ = constant. Then letting $t \to 0$, we have the constant = 0. Taking the equalities $E[v(t,U(a,t)F)^2] = E[(v(t,F) + v(t,[U(a,t) - U(a,a)]F))^2]$ and $\lim_{t\to a} E[v(t,[U(a,t) - U(a,a)]F)^2] = 0$, into account, we have $E[v(a,F)^2] = 0$ for any $a > 0$, which implies $v(a,F) = 0$ almost surely. Thus the proof is completed. □

§3. Proof of *Lemma 1*.

Following [19], [20], we will treat the generation problem via stochastic method.

For any F in $\mathcal{D}_{E'}$, we recall the definition of $U(t,s)$:

$$(U(t,s)F)(x) = E[F(\eta_{s,t}(x))].$$

To examine that $U(t,s)$ is the evolution operator stated in Lemma 1, we will check some regularities and integrabilities for $\eta_{s,t}(x)$. It is obvious that if $p \geq p_0$ and $x \in E'_p$, $\eta_{s,t}(x) \in E'_p$, so that for $h \in E'_{p_4}$, $\eta_{s,t}(x+h) \in E'_{p_5}$, where $p_5 = p \vee p_4$. Here $a \vee b = \max\{a,b\}$. Following Kunita (p. 219 of [12]), we will show that $\xi_{s,t}(\tau) := \frac{1}{\tau}\{\eta_{s,t}(x+\tau h) - \eta_{s,t}(x)\}$ has a continuous extension at $\tau = 0$ for any s,t a.s. in E'_{p_5}. This can be proved by appealing the following Kolmogorov-Totoki criterion for continuity [24].

Lemma 2. For any $T>0$ and any integer $j \geq 1$, we have

$$E[\|\xi_{s,t}(\tau) - \xi_{s',t'}(\tau')\|_{-p_5}^{2j}] \leq C_{12}(T,h)\{|s-s'|^j + |t-t'|^j + |\tau-\tau'|^j\},$$

$$0 \leq s,s',t,t',\tau,\tau' \leq T.$$

Proof. First we will show that following Burkholder's inequality. Let $A(r)$ be a well measurable random linear operator from E_1' to E_{p_0}' such that $E[\int_s^t \|A(r)\|_2^2 dr] < +\infty$. Then we have

Lemma 3. For any integer $j \geq 1$,

$$E[\|\int_s^t A(r)d\beta(r)\|_{p_0}^{2j}] \leq C_{13}(j)E[(\int_s^t \|A(r)\|_2^2 dr)^j].$$

Proof. Let $(\cdot,\cdot)_{p_0}$ be the inner product in E_{p_0}' such that $(x,x)_{p_0} = \|x\|_{-p_0}^2$. Setting $\theta(x) = (x,x)_{p_0}^j$ and $y(t) = \int_s^t A(r)d\beta(r)$ and applying the Itô formula, (Kuo [14]), for $\theta(y(t))$, we get

$$\text{(3.1)} \qquad E[\|y(t)\|_{-p_0}^{2j}] = \frac{1}{2}E[\int_s^t \mathrm{trace}_{E_0} D^2\theta(y(r))\circ[A(r)\times A(r)]dr]$$

$$= \frac{1}{2}E[\int_s^t \sum_{i=1}^{\infty} \{2^2 j(j-1)(y(r),A(r)h_i^{(0)})_{-p_0}^2 \|y(r)\|_{-p_0}^{2(j-2)}$$

$$+ 2j\|A(r)h_i^{(0)}\|_{-p_0}^2 \|y(r)\|_{-p_0}^{2(j-1)}\}dr]$$

$$\leq (j+2j(j-1))E[\int_s^t \|A(r)\|_2^2 \|y(r)\|_{-p_0}^{2(j-1)} dr].$$

By Hölder's inequality and the martingale inequality, the right hand side of (3.1) is dominated by

$$(j+2j(j-1))E[\sup_{s\leq r\leq t} \|y(r)\|_{-p_0}^{2j}]^{j-1/j} E[(\int_s^t \|A(r)\|_2^2 dr)^j]^{1/j}$$

$$\leq (2j^2-j)(2j/(2j-1))^{2(j-1)}E[\|y(t)\|_{-p_0}^{2j}]^{j-1/j}\ E[(\int_s^t \|A(r)\|_2^2 dr)^j]^{1/j}.$$

which completes the proof of Lemma 3. □

Now for the convenience of notations we will write $dt = d\beta_0(t)$, $d\beta(t) = d\beta_1(t)$, $A_0(t,x) = B(t,x)$, $A_1(t,x) = A(t,x)$, $|||\cdot|||_0 = \|\cdot\|_{-p_0}$ and $|||\cdot|||_1 = \|\cdot\|_2$. Without loss of generality, we may assume $0 \leq s < s' < t < t' \leq T$. Then $\xi_{s,t}(\tau) - \xi_{s',t}(\tau')$ is a sum of the following terms:

$$\sum_k \int_s^{s'} (\int_0^1 DA_k(r,\zeta_{s,r}(\tau,y))(\xi_{s,r}(\tau))dy)d\beta_k(r), \tag{3.2}$$

where $\zeta_{s,r}(\tau,y) = \eta_{s,r}(x) + y(\eta_{s,r}(x+\tau h) - \eta_{s,r}(x))$.

$$\sum_k \int_{s'}^t (\int_0^1 \{DA_k(r,\zeta_{s,r}(\tau,y))(\xi_{s,r}(\tau)) \tag{3.3}$$

$$-DA_k(r,\zeta_{s',r}(\tau',y))(\xi_{s',r}(\tau'))\}dy)d\beta_k(r).$$

By Lemma 3 and the assumption (H2), the expectation of the $2j^{th}$ power of the $\|\cdot\|_{-p_5}$-norm of (3.2) is dominated by

$$C_{14} \sum_k E[(\int_s^{s'} |||\int_0^1 DA_k(r,\zeta_{s,r}(\tau,y))(\xi_{s,r}(\tau))dy\ |||_k^2 dr)^j]$$

$$\leq C_{15} \sum_k |s'-s|^{j-1} E[\int_s^{s'} \|\xi_{s,r}(\tau)\|_{-p_5}^{2j} dr].$$

Again using Lemma 3, assumption (H2) and the Gronwall lemma, we have

$$E[\|\eta_{s,t}(x)-\eta_{s,t}(y)\|_{-p_5}^{2j}] \leq C_{16}\|x-y\|_{-p_5}^{2j},\ x,y \in E'_{p_5}, \tag{3.4}$$

which implies

$$E[\int_s^{s'} \|\xi_{s,r}(\tau)\|_{-p_5}^{2j} dr] \leq C_{16}\|h\|_{-p_5}^{2j} |s'-s|. \tag{3.5}$$

Since the integrand in (3.3)

$$= \int_0^1 DA_k(r,\zeta_{s,r}(\tau,y))(\xi_{s,r}(\tau) - \xi_{s',r}(\tau'))dy$$

$$+ \int_0^1 (\int_0^1 D^2 A_k(r,\gamma_{s,s',r}(\tau,\tau',y_1))(\zeta_{s,r}(\tau,y) - \zeta_{s',r}(\tau',y))dy_1)(\xi_{s',r}(\tau'))dy,$$

where $\gamma_{s,s',r}(\tau,\tau',y_1) = \zeta_{s',r}(\tau',y) + y_1(\zeta_{s,r}(\tau,y) - \zeta_{s',r}(\tau',y))$, the $|||\cdot|||_k$-norm of the integrand is dominated by

$$\text{(3.6)} \qquad C_{17}\ \{\|\xi_{s,r}(\tau) - \xi_{s',r}(\tau')\|_{-p_5} + (\|\eta_{s,r}(x) - \eta_{s',r}(x)\|_{-p_5}$$

$$+ \|\eta_{s,r}(x+\tau h) - \eta_{s',r}(x+\tau' h)\|_{-p_5})\|\xi_{s',r}(\tau')\|_{-p_5}\}.$$

By Lemma 3 and (3.6), the expectation of the 2j-th power of $\|\cdot\|_{-p_5}$-norm of (3.3) is dominated by

$$\text{(3.7)} \quad C_{18}\{\int_s^t, E[\|\xi_{s,r}(\tau) - \xi_{s',r}(\tau)\|_{-p_5}^{2j}]dr$$

$$+ \int_{s}^{t}, E[\|\eta_{s,r}(x) - \eta_{s',r}(x)\|_{-p_5}^{4j}]^{1/2} E[\|\xi_{s',r}(\tau)\|_{-p_5}^{4j}]^{1/2}dr$$

$$+ \int_{s}^{t}, E[\|\eta_{s,r}(x+\tau h) - \eta_{s',r}(x+\tau' h)\|_{-p_5}^{4j}]^{1/2} E[\|\xi_{s',r}(\tau')\|_{-p_5}^{4j}]^{1/2}dr\}.$$

From the assumptions (H1) and (H2), we get

$$|||A_k(r,\eta_{s,t}(x)) - A_k(r,\eta_{s',t'}(x'))|||_k \le C_{19}\|\eta_{s,t}(x) - \eta_{s',t'}(x')\|_{-p_5}$$

and taking the expectations of the 2n-th power of both sides of $\|\cdot\|_{-p_5}$-norm of the following inequality;

$$\|\eta_{s,t}(x) - \eta_{s',t'}(x')\|_{-p_5} \le \|\sum_k \int_s^{s'} A_k(r,\eta_{s,r}(x))d\beta_k(r)\|_{-p_5}$$

$$+ \|\sum_k \int_t^{t'} A_k(r,\eta_{s',r}(x'))d\beta_k(r)\|_{-p_5} + \|\sum_k \int_{s'}^{t} \{A_k(r,\eta_{s,r}(x))$$

$$- A_k(r,\eta_{s',r}(x'))\}d\beta_k(r)\|_{-p_5},$$

we have, by Lemma 3, similarly

$$E[\|\eta_{s,t}(x) - \eta_{s',t'}(x')\|_{-p_5}^{2n}]$$

$$\leq C_{20}(T)\{|t-t'|^n + |s-s'|^n + \int_s^t E[\|\eta_{s,r}(x)-\eta_{s',r}(x')\|_{-p_5}^{2n}]dr\}.$$

Noticing that $\eta_{s,r}(x) = \eta_{s',r}(\eta_{s,s'}(x))$ and $\eta_{s',r}(\cdot)$ is independent of $\eta_{s,s'}(\cdot)$ and using (3.4), we get

$$E[\|\eta_{s,r}(x) - \eta_{s',r}(x')\|_{-p_5}^{2n}] = \int_{E'_{p_5}} E[\|\eta_{s',r}(y)-\eta_{s',r}(x')\|_{-p_5}^{2n}]P(\eta_{s,s'}(x)\in dy)$$

$$\leq \int_{E'_{p_5}} C_{21}\|y-x'\|_{-p_5}^{2n} P(\eta_{s,s'}(x) \in dy)$$

$$= C_{21}E[\|\eta_{s,s'}(x) - x'\|_{-p_5}^{2n}]$$

$$\leq C_{22}\{\|x-x'\|_{-p_5}^{2n} + |s'-s|^n\},$$

where $P(\cdot)$ denotes the fundamental probability measure associated with $\beta(t)$. Hence we obtain

$$(3.8)\quad E[\|\eta_{s,t}(x) - \eta_{s',t'}(x')\|_{-p_5}^{2n} \leq C_{23}(T)\{|t-t'|^n+|s-s'|^n+\|x-x'\|_{-p_5}^{2n}\}.$$

Combining (3.2), (3.3), (3.4), (3.5), (3.7) and (3.8), we have

$$E[\|\xi_{s,t}(\tau) - \xi_{s',t'}(\tau')\|_{-p_5}^{2j}]$$

$$\leq C_{24}(T)\|h\|_{-p_5}^{2j}\{|t-t'|^j + |s-s'|^j + |\tau-\tau'|^{2j}\|h\|_{-p_5}^{2j}\}.$$

This completes the proof of Lemma 2. □

Let τ tend to 0, we have for each $x \in E_p'$,

$$(3.9) \qquad D\eta_{s,t}(x)(h) = h + \sum_k \int_s^t DA_k(r,\eta_{s,r}(x))(D\eta_{s,r}(x)(h))d\beta_k(r).$$

For the higher order differentiations, the formula similar to (3.9) can be proved inductively, together with the following lemma.

Lemma 4. Suppose that a natural number $q \geq p_0$ and any $T > 0$. Then for $0 \leq s,t,s',t' \leq T$, a natural number j and $x,x',h_i \in E_q'$, $i=1,2,\ldots,n$, we have

$$(3.10) \quad E[\|E^n\eta_{s,t}(x)(h_1,h_2,\ldots,h_n)\|_{-q}^{2j}] \leq C_{25}(T)\|h_1\|_{-q}^{2j}\|h_2\|_{-q}^{2j}\cdots\|h_n\|_{-q}^{2j}.$$

$$(3.11) \qquad E[\|D^n\eta_{s,t}(x)(h_1,h_2,\ldots,h_n) - D^n\eta_{s',t'}(x')(h_1,h_2,\ldots,h_n)\|_{-q}^{2j}]$$
$$\leq C_{26}(T)\{|t-t'|^j + |s-s'|^j + \|x-x'\|_{-q}^{2j}\} \|h_1\|_{-q}^{2j}\|h_2\|_{-q}^{2j}\cdots\|h_n\|_{-q}^{2j} .$$

Proof. First we will show (3.10) for the case n=1. By the assumptions (H1) and (H2), we get

$$|||DA_k(r,\eta_{s,r}(x))(D\eta_{s,r}(x)(h))|||_k \leq C_{27}\|D\eta_{s,r}(x)(h)\|_{-q},$$

so that taking the expectations of 2j-th powers of $\|\cdot\|_{-q}$ norms of both sides of (3.9) and using Lemma 3, we get

$$E[\|D\eta_{s,t}(x)(h)\|_{-q}^{2j}] \leq C_{28}(T)\{\|h\|_{-q}^{2j} + \int_s^t E[\|D\eta_{s,r}(x)(h)\|_{-q}^{2j}]dr\}$$

and the Gronwall lemma gives (3.10) for the case where n=1. For $n \geq 2$, we will prove the inequality by the Mathematical induction. For $h_1,h_2,\ldots,h_n \in E_q'$,

$$(D^n\eta_{s,t}(x))(h_1,h_2,\ldots,h_n) = \sum_k \int_s^t D^n(A_k(r,\eta_{s,r}(x)))(h_1,h_2,\ldots,h_n)d\beta_k(r).$$

Since

$$(3.12) \qquad D^n(A_k(r,\eta_{s,r}(x)))(h_1,h_2,\ldots,h_n)$$

$$= DA_k(r,\eta_{s,r}(x))(D^n\eta_{s,r}(x)(h_1,h_2,\dots,h_n))$$

+ finite sum of terms of the type

$$(D^m A_k(r,\eta_{s,r}(x)))(D^{n_1}\eta_{s,r}(x)(h_{j_1^{(1)}},\ h_{j_2^{(1)}},\dots,h_{j_{n_1}^{(1)}}),$$

$$D^{n_2}\eta_{s,r}(x)(h_{j_1^{(2)}},h_{j_2^{(2)}},\dots,h_{j_{n_2}^{(2)}}),\dots,D^{n_m}\eta_{s,r}(x)(h_{j_1^{(m)}},\ h_{j_2^{(m)}},\dots,h_{j_{n_m}^{(m)}})),$$

where $2 \leq m \leq n$, $n_1 + n_2 + \dots + n_m = n$ and $0 \leq n_i \leq n-1$, so that using the assumption of the mathematical induction, we get (3.10) by the same argument as before.

Before proceeding to the proof of (3.11), we notice that for $h \in E_q'$, $\|D\eta_{s,t}(x)(h)-D\eta_{s',t'}(x')(h)\|_{-q}$ is dominated by

$$\sum_k \|\int_s^{s'} D(A_k(r,\eta_{s,r}(x)))(h)d\beta_k(r)\|_{-q} \tag{3.13}$$

$$+ \sum_k \|\int_t^{t'} D(A_k(r,\eta_{s',r}(x')))(h)d\beta_k(r)\|_{-q}$$

$$+ \sum_k \|\int_{s'}^{t} \{D(A_k(r,\eta_{s,r}(x)))(h) - D(A_k(r,\eta_{s',r}(x')))(h)\}d\beta_k(r)\|_{-q}.$$

Now, by the assumptions (H1) and (H2), we have

$$|||D(A_k(r,\eta_{s,r}(x)))(h)-D(A_k(r,\eta_{s',r}(x')))(h))|||_k \tag{3.14}$$

$$\leq |||\{DA_k(r,\eta_{s,r}(x))-DA_k(r,\eta_{s',r}(x'))\}(D\eta_{s,r}(x)(h))|||_k$$

$$+ |||DA_k(r,\eta_{s',r}(x'))(D\eta_{s,r}(x)(h)-D\eta_{s',r}(x')(h))|||_k$$

$$\leq C_{29}(T)\{\|\eta_{s,r}(x)-\eta_{s',r}(x')\|_{-q}\|D\eta_{s,r}(x)(h)\|_{-q}$$

$$+ \|D\eta_{s,r}(x)(h)-D\eta_{s',r}(x')(h)\|_{-q}\}.$$

Then the same manner as before, together with (3.8), (3.13) and (3.14), leads us to

$$E[\|D\eta_{s,t}(x)(h)-D\eta_{s',t'}(x')(h)\|_{-q}^{2j}]$$
$$\leq C_{30}(T)\{(|t-t'|^{j} + |s-s'|^{j}+\|x-x'\|_{-q}^{2j})\|h\|_{-q}^{2j}$$
$$+ \int_{s'}^{t} E[\|D\eta_{s,t}(x)(h)-D\eta_{s',r}(x')(h)\|_{-q}^{2}]dr\},$$

which gives (3.11) by the Gronwall lemma for the case n=1. By (3.12) and the estimation of $\|D^{n}\eta_{s,t}(x)(h_1,h_2,\dots,h_n)-D^{n}\eta_{s',t'}(x)(h_1,h_2,\dots,h_n)\|_{-q}$ similar to that in (3.13), the mathematical induction and the Gronwall lemma yield the proof of (3.11) for $n \geq 2$.

Now we will proceed to the proof of the generation problem of L(t). By the assumptions (H1) and (H2), (3.8) and (3.10) of Lemma 4, we may exchange the order of the differentiation and the integration. Then by the Itô formula [14], we have the point wise Kolmogorov forward and backward equations like in the finite dimensional case (Theorem 1 (page 73) of [7]):

$$\frac{d}{dt}(U(t,s)F)(x) = (U(t,s)L(t)F)(x)$$

$$\frac{d}{ds}(U(t,s)F)(x) = -(L(s)U(t,s)F)(x).$$

Let $p \geq 0$, $q \geq 0$ and $n \geq 0$ be integers and $x \in E_p'$. Since $D^{n}(F(\eta_{s,t}(x)))(h_{i_1}^{(q)},h_{i_2}^{(q)},\dots,h_{i_n}^{(q)})$ is a finite sum of terms of the type

$$I = D^{m}F(\eta_{s,t}(x))(D^{n_1}\eta_{s,t}(x)(h_{j_1^{(1)}}^{(q)},h_{j_2^{(1)}}^{(q)},\dots,h_{j_{n_1}^{(1)}}^{(q)}),\ D^{n_2}\eta_{s,t}(x)$$

$$(h_{j_1^{(2)}}^{(q)},h_{j_2^{(2)}}^{(q)},\dots,h_{j_{n_2}^{(2)}}^{(q)}),\dots,D^{n_m}\eta_{s,t}(x)(h_{j_1^{(m)}}^{(q)},h_{j_2^{(m)}}^{(q)},\dots,h_{j_{n_m}^{(m)}}^{(q)})),$$

$$n_1 + n_2 + \ldots + n_m = n,$$

so that noticing the nuclearity of E and (3.10), we have an integer $q' > \max\{p, p_0, q\}$ such that

$$\sum_{j=1}^{\infty} \|h_j^{(q)}\|_{-q'}^2 < +\infty \tag{3.15}$$

and

$$\begin{aligned}(3.16)\quad E[|I|^2] \le \|F\|^2_{q',q',n} E[e^{2\|\eta_{s,t}(x)\|_{-q'}} \|D^{n_1}\eta_{s,t}(x)(h^{(q)}_{j_1^{(1)}}, h^{(q)}_{j_2^{(1)}}, \\ \ldots, h^{(q)}_{j_{n_1}^{(1)}})\|^2_{-q'} \|D^{n_2}\eta_{s,t}(x)(h^{(q)}_{j_1^{(2)}}, h^{(q)}_{j_2^{(2)}}, \ldots, h^{(q)}_{j_{n_2}^{(2)}})\|^2_{-q'} \ldots \\ \ldots \|D^{n_m}\eta_{s,t}(x)(h^{(q)}_{j_1^{(m)}}, h^{(q)}_{j_2^{(m)}}, h^{(q)}_{j_2^{(m)}}, \ldots, h^{(q)}_{j_{n_m}^{(m)}})\|^2_{-q'}] \\ \le C_{31} \|F\|^2_{q',q',n} \|h^{(q)}_{i_1}\|^2_{-q'} \|h^{(q)}_{i_2}\|^2_{-q'} \ldots \|h^{(q)}_{i_n}\|^2_{-q'} E[e^{4\|\eta_{s,t}(x)\|_{-q'}}]^{\frac{1}{2}}.\end{aligned}$$

Here we will prove

Lemma 5. For any $\alpha > 0$ and $T > 0$, there exists a constant $C_{32} = C_{32}(\alpha, T)$ such that

$$\sup_{0 \le s,t \le T} E[e^{\alpha\|\eta_{s,t}(x)\|_{-q'}}] \le C_{32} e^{\alpha\|x\|_{-q'}}.$$

Proof. By (H1), $\|\eta_{s,t}(x)\|_{-q'} \le \|x\|_{-q'} + C_{33} + \|\int_s^t A(r, \eta_{s,r}(x)) d\beta(r)\|_{-q'}$. Following [8], it is enough to prove $E[\exp(\|\int_s^t \alpha A(r, \eta_{s,r}(x)) d\beta(r)\|_{-q'})] \le C_{34}$. Setting $y_{s,t}(x) = \int_s^t \alpha A(r, \eta_{s,r}(x)) d\beta(r)$, by the Itô formula and the assumption (H1), we get for any integer $m \ge 2$,

$$E[\|y_{s,t}(x)\|^m_{-q'}] \le E[(1 + \|y_{s,t}(x)\|^2_{-q'})^{m/2}] \tag{3.17}$$

$$\leq E[1 + \int_s^t \tfrac{1}{2}\{2\tfrac{m}{2}(1+\|y_{s,r}(x)\|_{-q'}^2)^{\frac{m}{2}-1}\alpha^2 \|A(r,\eta_{s,r}(x))\|_2^2$$

$$+ 4\tfrac{m}{2}(\tfrac{m}{2}-1)(1+\|y_{s,r}(x)\|_{-q'}^2)^{\frac{m}{2}-2}\ \alpha^2(\sum_{i=1}^{\infty}(y_{s,r}(x),A(r,\eta_{s,r}(x))h_i^{(0)})^2)\}dr]$$

$$\leq 1 + 2(\tfrac{m}{2})^2\alpha^2 C_{35}\int_s^t E[(1+\|y_{s,r}(s)\|_{-q'}^2)^{\frac{m}{2}-1}]dr,$$

where $C_{35} = \sup_{\substack{x\in E' \\ 0\leq t\leq T}} \|A(t,x)\|_2^2$. If we use (3.17) recursively, the rest is similar to that in [8], which completes the proof. □

Therefore (3.15), (3.16) and Lemma 5 yield

$$\|U(t,s)F\|_{p,q,n} \leq C_{36}(T)\|F\|_{q',q',n} \quad , \ t,s \in [0,T],$$

which implies that $U(t,s)$ is a continuous linear operator from $\mathcal{D}_{E'}$ into itself.

By the same reason as in [20], if we prove the strong continuity of $U(t,s)F$ in (t,s), the pointwise Kolmogorov forward and backward equations imply that $L(t)$ generates the evolution operator $U(t,s)$. Since $\|U(t,s)F-U(t',s')F\|_{p,q,n}^{2j}$ is dominated by a finite sum of terms of the type

$$\sup_{x\in E_p'} e^{-2j\|x\|_{-p}} \sum_{\substack{j_1^{(1)},j_2^{(1)},\dots,j_{m_1}^{(1)} \\ \vdots \\ j_1^{(m)},j_2^{(m)},\dots,j_{n_m}^{(m)}}} E[|D^m F(\eta_{s,t}(x))(D^{n_1}\eta_{s,t}(x)$$

$$(h_{j_1^{(1)}}^{(q)},h_{j_2^{(1)}}^{(q)},\dots,h_{j_{n_1}^{(1)}}^{(q)}),\ D^{n_2}\eta_{s,t}(x)(h_{j_1^{(2)}}^{(q)},h_{j_2^{(2)}}^{(q)},\dots,h_{j_{n_2}^{(2)}}^{(q)}),\dots,$$

$$D^{n_m}\eta_{s,t}(x)(h_{j_1^{(m)}}^{(q)},\ h_{j_2^{(m)}}^{(q)},\dots,h_{j_{n_m}^{(m)}}^{(q)})) - D^m F(\eta_{s',t'}(x))(D^{n_1}\eta_{s',t'}(x)$$

$$(h^{(q)}_{j^{(1)}_1}, h^{(q)}_{j^{(1)}_2}, \ldots, h^{(q)}_{j^{(1)}_{n_1}}),\ D^{n_2}\eta_{s',t'}(x)(h^{(q)}_{j^{(2)}_1}, h^{(q)}_{j^{(2)}_2}, \ldots, h^{(q)}_{j^{(2)}_{n_2}}),$$

$$\ldots, D^{n_m}\eta_{s',t'}(x)(h^{(q)}_{j^{(m)}_1}, h^{(q)}_{j^{(m)}_2}, \ldots, h^{(q)}_{j^{(m)}_{n_m}}))|^{2j}],$$

so that by (3.8), Lemmas 4 and 5 and the nuclearity of E, we have an integer $q' > \max\{p, p_0, q\}$ such that $\sum_{j=1}^{\infty} \|h^{(q)}_j\|^2_{-q'} < \infty$ and we get

$$\|U(t,s)F - U(t',s')F\|^{2j}_{p,q,n} \le C_{37}\|F\|^{2j}_{q',q',n+1}\{|t-t'|^j + |s-s'|^j\}$$

which completes the proof of Lemma 1. □

§4. Generation of the Kolmogorov Evolution Operator

In this Section, we will discuss the assumption (H4). Let K be a separable Hilbert space. We call a K-valued functional $G(x) = g(\langle x,\xi_1\rangle, \langle x,\xi_2\rangle, \ldots, \langle x,\xi_n\rangle)$, $\xi_1, \xi_2, \ldots, \xi_n \in E$, the smooth functional if $g(x)\colon \mathbb{R}^n \to K$ is a C^∞-function, where $\mathbb{R}^n$ is the n-dimensional Euclidean space. Further we call G(x) a bounded smooth functional if g(x) itself and all the derivatives of g(x) are bounded. The coefficients A(t,x) and B(t,x) are said to be approximated by sequences of bounded smooth functionals
$A_m(t,x) = a_m(t, \langle x,\xi_1\rangle, \langle x,\xi_2\rangle, \ldots, \langle x,\xi_{k_m}\rangle)$ and
$B_m(t,x) = b_m(t, \langle x,\xi_1\rangle, \langle x,\xi_2\rangle, \ldots, \langle x,\xi_{k_m}\rangle)$ on E' if for any integers, $p \ge p_0$, $q \ge 0$ and $n \ge 0$, the following conditions are satisfied:

(4.1) $A_m(t,x)$ and $B_m(t,x)$ satisfy the conditions (H_1), (H_2) and (H_3).

(4.2) For any $T > 0$,

$$\lim_{m\to\infty} \sup_{\substack{x\in E' \\ 0\le t\le T}} \|A(t,x)-A_m(t,x)\|_2^2 = 0,$$

$$\lim_{m\to\infty} \sup_{\substack{x\in E' \\ 0\le t\le T}} \|B(t,x)-B_m(t,x)\|_{-p_0} = 0,$$

$$\lim_{m\to\infty} \sup_{\substack{x\in E_p' \\ 0\le t\le T}} \|D^kA(t,x)-D^kA_m(t,x)\|_{H.S.}^{(q)} = 0, \ k=1,2,\dots,n,$$

$$\lim_{m\to\infty} \sup_{\substack{x\in E_p' \\ 0\le t\le T}} \|D^kB(t,x)-D^kB_m(t,x)\|_{H.S.}^{(q)} = 0, \ k=1,2,\dots,n.$$

A real valued smooth functional $\Phi(x) = \phi(\langle x,\xi_1\rangle,\langle x,\xi_2\rangle,\dots,\langle x,\xi_n\rangle)$ is said to be a weighted Schwartz functional if $\phi(x) = h(x)\varphi(x)$, $x \in \mathbb{R}^n$, where $\varphi(x)$ is an element of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing C^∞-functions on $\mathbb{R}^n$, $h(x) = 1/g(x)$, $g(x) = \prod_{i=1}^n g_0(x_i)$, $g_0(x_i) = \exp(-\sqrt{\int_{\mathbb{R}}|y|\rho(x_i-y)dy})$ and $\rho(x)$ is the Friedrichs mollifier whose support is contained in $[-1,1]$. Then $F \in \mathfrak{D}_{E'}$ is said to be approximated by a sequence of weighted Schwartz functionals $F_m(x) = f_m(\langle x,\xi_1\rangle,\langle x,\xi_2\rangle,\dots,\langle x,\xi_{k_m}\rangle)$ if for any integer $p \ge 0$, $q \ge 0$ and $n \ge 0$,

$$\lim_{m\to\infty} \|F - F_m\|_{p,q,n} = 0.$$

First we will prove

Proposition 2. Suppose that the coefficients $A(t,x)$ and $B(t,x)$ are approximated by sequences of bounded smooth functionals and also $F \in \mathfrak{D}_{E'}$ is approximated by a sequence of weighted Schwartz functionals. Then $U(t,s)F(x) = E[F(\eta_{s,t}(x))]$ is approximated by a sequence of weighted Schwartz functionals.

Proof. We will use the convenient notations such that $A_0(t,x) = B(t,x)$ and $A_1(t,x) = A(t,x)$. For any integers $p \geq 0$, $q \geq 0$ and $n \geq 0$, we choose an integer $q' > \max\{p, p_0, q\}$ such that

$$(4.3) \qquad \sum_{i=0}^{\infty} \|h_i^{(q)}\|_{-q}^2 < +\infty,$$

since E is a nuclear Fréchet space. Then by the assumptions, for any $\delta > 0$ and $A_k(t,x)$, k=0,1, there exist bounded smooth functionals $\tilde{A}_k(t,x) = \tilde{a}_k(t, \langle x,\zeta_1\rangle, \langle x,\zeta_2\rangle, \ldots, \langle x,\zeta_{m_k}\rangle)$, k=0,1 such that

$$(4.4) \qquad \sum_{\ell=0}^{n} \sup_{\substack{x\in E' \\ 0\leq t\leq T}} \|D^{\ell}A_k(t,x) - D^{\ell}\tilde{A}_k(t,x)\|_{H.S.}^{(q')} < \delta.$$

For sufficiently large N, we put

$$z_{s,t}^N(x) = x + \int_s^t \tilde{A}_k(t_1, x + \int_s^{t_1} \tilde{A}_k(t_2, \ldots,$$
$$x + \int_s^{t_{N-1}} \tilde{A}_k(t_N, x)d\beta_k(t_N)) \ldots)d\beta_k(t_1).$$

Setting

$\hat{z}_{s,t}^{(n)}(x) = x + \int_s^t \tilde{A}_k(t_1, x + \int_s^{t_1} \tilde{A}_k(t_2, \ldots, x + \int_s^{t_{n-1}} \tilde{A}_k(t_n, \eta_{s,t}(x))d\beta_k(t_n)) \ldots)d\beta_k(t_1)$, $n=1,2,\ldots,N$, where $t_0=t$, by Lemma 3, we have for any $x \in E_p'$, $0 \leq s,t \leq T$ and any integer $j \geq 1$,

$$(4.5) \qquad E[\|\eta_{s,t}(x) - z_{s,t}^N(x)\|_{-q'}^{2j}]$$
$$\leq 2^{2j-1}E[\|\eta_{s,t}(x) - \hat{z}_{s,t}^{(1)}(x)\|_{-q}^{2j}]$$
$$+ \sum_{k=2}^{N} (2^{2j-1})^k E[\|\hat{z}_{s,t}^{(k-1)}(x) - \hat{z}_{s,t}^{(k)}(x)\|_{-q'}^{2j}]$$
$$+ (2^{2j-1})^N E[\|\hat{z}_{s,t}^{(N)}(x) - z_{s,t}^N(x)\|_{-q'}^{2j}]$$

$$\leq C_{38}\{2^{2j-1}\delta^{2j}T+\sum_{k=1}^{N-1}(2^{2j-1})^{k+1}M^{2jk}T^{k+1}\delta^{2j}/(k+1)!$$

$$+\ 2(2^{2j-1})^{N+1}M^{2jN}T^{N}/N!$$

$$\leq C_{38}\{\delta^{2j}\exp(2^{2j-1}(MV1)^{2j}T)+2(2^{2j-1})^{N+1}M^{2jN}T^{N}/N!\},$$

where $M = \sup_{\substack{x\in E_p' \\ 0\leq t\leq T}} ⦀\tilde{A}_k(t,x)⦀_k$, $⦀\cdot⦀_k$ is the convenient notation used before and $MV1 = \max\{M,1\}$. Hence for any $\epsilon > 0$, if we take sufficiently small δ and large N, we have

$$\sup_{x\in E_p'} E[\|\eta_{s,t}(x)-z^N_{s,t}(x)\|^{2j}_{-q'}] < \epsilon. \tag{4.6}$$

Next we will verify by the mathematical induction that for any integer $m \geq 1$ and any $\epsilon > 0$, there exists an integer $N(m,\epsilon)$ such that if $N \geq N(m,\epsilon)$,

$$\begin{aligned} E[\|D^m\eta_{s,t}(x)(h^{(q)}_{i_1},h^{(q)}_{i_2},\dots,h^{(q)}_{i_m}) \\ - D^m z^N_{s,t}(x)(h^{(q)}_{i_1},h^{(q)}_{i_2},\dots,h^{(q)}_{i_m})\|^{2j}_{-q'}] < \epsilon. \end{aligned} \tag{4.7}$$

Setting

$$y^{n,N}_{s,t}(x)(h^{(q)}_{i_1})$$

$$= h^{(q)}_{i_1} + \int_s^t D\tilde{A}_k(t_1,z^N_{s,t_1}(x))(h^{(q)}_{i_1}+\int_s^{t_1} D\tilde{A}_k(t_2,z^N_{s,t_2}(x))(h^{(q)}_{i_1}$$

$$+\dots+\int_s^{t_{n-1}} D\tilde{A}_k(t_n,\eta_{s,t_n}(x))(D\eta_{s,t_n}(x)(h^{(q)}_{i_1}))d\beta_k(t_n))\dots)d\beta_k(t_1),$$

$$z^{n,N}_{s,t}(x)(h^{(q)}_{i_1})$$

$$= h^{(q)}_{i_1} + \int_s^t D\tilde{A}_k(t_1,z^N_{s,t_1}(x))(h^{(q)}_{i_1}+\int_s^{t_1} D\tilde{A}_k(t_2,z^N_{s,t_2}(x))(h^{(q)}_{i_1}$$

$$+\ldots+\int_s^{t_{n-1}} D\tilde{A}_k(t_n, z^N_{s,t_n}(x))(D\eta_{s,t_n}(x)(h^{(q)}_{i_1}))d\beta_k(t_n))\ldots)d\beta_k(t_1)$$

and $\delta' = C_{38}\{\delta^{2j}\exp(2^{2j-1}(MV1)^{2j}T)+2(2^{2j-1})^{N+1}M^{2jN}T^N/N!\}$, then we have

$$E[\|D\eta_{s,t}(x)(h^{(q)}_{i_1})-Dz^N_{s,t}(x)(h^{(q)}_{i_1})\|^{2j}_{-q'}]$$

$$\leq 2^{2j-1}E[\|D\eta_{s,t}(x)(h^{(q)}_{i_1})-y^{1,N}_{s,t}(x)(h^{(q)}_{i_1})\|^{2j}_{-q'}]$$

$$+ (2^{2j-1})^2E[\|y^{1,N}_{s,t}(x)(h^{(q)}_{i_1})-z^{1,N}_{s,t}(x)(h^{(q)}_{i_1})\|^{2j}_{-q'}]$$

$$+ \sum_{k=1}^{N-1} \{(2^{2j-1})^{k+2}E[\|z^{k,N}_{s,t}(x)(h^{(q)}_{i_1})-y^{k+1,N}_{s,t}(x)(h^{(q)}_{i_1})\|^{2j}_{-q'}]$$

$$+ (2^{2j-1})^{k+3}E[\|y^{k+1,N}_{s,t}(x)(h^{(q)}_{i_1})-z^{k+1,N}_{s,t}(x)(h^{(q)}_{i_1})\|^{2j}_{-q'}]\}$$

$$+ (2^{2j-1})^{N+2}E[\|z^{N,N}_{s,t}(x)(h^{(q)}_{i_1})-Dz^N_{s,t}(x)(h^{(q)}_{i_1})\|^{2j}_{-q'}]$$

$$\leq C_{39}\{\delta^{2j}2^{2j-1}T+(\delta')^{2j}M^{2j}(2^{2j-1})^2T$$

$$+ \sum_{k=1}^{N-1} \{(2^{2j-1})^{k+2}\delta^{2j}M^{2jk}T^k/k!$$

$$+ (2^{2j-1})^{k+3}(\delta')^{2j}M^{2j(k+1)}T^{k+1}/(k+1)!\} + (2^{2j-1})^{N+2}M^{2jN}T^N/N!\}$$

$$\leq C_{39}\{2^{4j-1}e^{2^{2j-1}(MV1)^{2j}T}(\delta^{2j}+(\delta')^{2j}) + (2^{2j-1})^{N+2}M^{2jN}T^N/N!\},$$

which gives (4.7) for m=1. We assume (4.7) holds for integers $1 \leq m \leq \ell$, $\ell \geq 1$. Since $D^{\ell+1}(A_k(r,\eta_{s,r}(x)))(h^{(q)}_{i_1},h^{(q)}_{i_2},\ldots,h^{(q)}_{i_{\ell+1}}) =$
$DA_k(r,\eta_{s,r}(x))(D^{\ell+1}\eta_{s,r}(x)(h^{(q)}_{i_1}, h^{(q)}_{i_2},\ldots,h^{(q)}_{i_{\ell+1}}))$ + finite sum of terms of the type

$$D^u A_k(r,\eta_{s,r}(x))(D^{n_1}\eta_{s,r}(x)(h^{(q)}_{j^{(1)}_1},h^{(q)}_{j^{(1)}_2},\ldots,h^{(q)}_{j^{(1)}_{n_1}}),$$

$$D^{n_2}\eta_{s,r}(x)(h^{(q)}_{j^{(2)}_1},h^{(q)}_{j^{(2)}_2},\ldots,h^{(q)}_{j^{(2)}_{n_2}}),\ldots,D^{n_u}\eta_{s,r}(x)$$

$$(h^{(q)}_{j^{(u)}_1},h^{(q)}_{j^{(u)}_2},\ldots,h^{(q)}_{j^{(u)}_{n_u}})),$$

where $2 \leq u \leq \ell+1$, $n_1+n_2+\ldots+n_u = \ell+1$, $\{h^{(q)}_{j^{(i)}_{n_i}},\ i=1,2,\ldots,u\} = \{h^{(q)}_{i_j},$ $j=1,2,\ldots,\ell+1\}$ and

$$D^{\ell+1}\eta_{s,t}(x)(h^{(q)}_{i_1},h^{(q)}_{i_2},\ldots,h^{(q)}_{i_{\ell+1}}) = \sum_k \int_s^t D^{\ell+1}(A_k(r,\eta_{s,r}(x)))(h^{(q)}_{i_1},$$

$$h^{(q)}_{i_2},\ldots,h^{(q)}_{i_{\ell+1}})d\beta_k(r),$$

so (4.7) for $m \geq 2$ can be proved similarly.

By the assumption for F, for any $\epsilon' > 0$, we have a weighted Schwartz functional $\tilde{F}(x) = f(\langle x,\xi_1\rangle,\langle x,\xi_2\rangle,\ldots,\langle x,\xi_m\rangle)$ such that

$$(4.8) \qquad \sum_{k=0}^{n} \sup_{s\in E'_{q'}} e^{-\|x\|_{-q'}} \|D^k(F(x)-\tilde{F}(x))\|^{(q')}_{H.S.} < \epsilon'.$$

Then to prove Proposition 2, it is enough to show $(U(t,s)F)(x)$ is approximated by weighted Schwartz functionals in $\|\cdot\|^{(q)}_{p,k}$, $0\leq k\leq n$. Since $D^k(F(\eta_{s,t}(x)))(h^{(q)}_{i_1},h^{(q)}_{i_2},\ldots,h^{(q)}_{i_k})$ is a finite sum of terms of the type

$$(4.9) \qquad I_{h^{(q)}_{i_1},h^{(q)}_{i_2},\ldots,h^{(q)}_{i_k}}(\eta_{s,t}(x))$$

$$= D^u F(\eta_{s,t}(x))(D^{n_1}\eta_{s,t}(x)(h^{(q)}_{j_1^{(1)}}, h^{(q)}_{j_2^{(1)}}, \ldots, h^{(q)}_{j_{n_1}^{(1)}}),\ D^{n_2}\eta_{s,t}(x)$$

$$(h^{(q)}_{j_1^{(2)}}, h^{(q)}_{j_2^{(2)}}, \ldots, h^{(q)}_{j_{n_2}^{(2)}}), \ldots, D^{n_u}\eta_{s,t}(x)(h^{(q)}_{j_1^{(u)}}, h^{(q)}_{j_2^{(u)}}, \ldots, h^{(q)}_{j_{n_u}^{(u)}})),$$

where $0 \leq u \leq k$ and $n_1+n_2+\ldots+n_u = k$, so that setting

$$(4.9)' \qquad J_{h^{(q)}_{i_1}, h^{(q)}_{i_2}, \ldots, h^{(q)}_{i_k}}(z^N_{s,t}(x))$$

$$= D^u\tilde{F}(z^N_{s,t}(x))(D^{n_1}z^N_{s,t}(x)(h^{(q)}_{j_1^{(1)}}, h^{(q)}_{j_2^{(1)}}, \ldots, h^{(q)}_{j_{n_1}^{(1)}}),\ D^{n_2}z^N_{s,t}(x)$$

$$(h^{(q)}_{j_1^{(2)}}, h^{(q)}_{j_2^{(2)}}, \ldots, h^{(q)}_{j_{n_2}^{(2)}}), \ldots, D^{n_u}z^N_{s,t}(x)(h^{(q)}_{j_1^{(u)}},\ h^{(q)}_{j_2^{(u)}}, \ldots, h^{(q)}_{j_{n_u}^{(u)}}),$$

then we have that $(\|U(t,s)F-E[\tilde{F}(z^N_{s,t}(\cdot))]\|^{(q)}_{p,k})^2$ is dominated by a finite sum of terms of the type

$$(4.10)\ C_{40} \sup_{x\in E_p'} e^{-2\|x\|_{-p}} \sum_{i_1,i_2,\ldots,i_k=1}^{\infty} E[|I_{h^{(q)}_{i_1}, h^{(q)}_{i_2}, \ldots, h^{(q)}_{i_k}}(\eta_{s,t}(x))$$

$$- J_{h^{(q)}_{i_1}, h^{(q)}_{i_2}, \ldots, h^{(q)}_{i_k}}(z^N_{s,t}(x))|^2]$$

$$\leq C_{41}\{\sup_{x\in E_p'} e^{-2\|x\|_{-p}} \sum_{i_1,i_2,\ldots,i_k=1}^{\infty} E[e^{2\|\eta_{s,t}(x)\|_{-q'}}(\epsilon')^2\|D^{n_1}\eta_{s,t}(x)(h^{(q)}_{j_1^{(1)}},$$

$$h^{q}_{j_2^{(1)}}, \ldots, h^{(q)}_{j_{n_1}^{(1)}}\|^2_{-q'}\|D^{n_2}\eta_{s,t}(x)(h^{(q)}_{j_1^{(2)}}, h^{(q)}_{j_2^{(2)}}, \ldots, h^{(q)}_{j_{n_2}^{(2)}})\|^2_{-q'}$$

$$\ldots \| D^{n_u} \eta_{s,t}(x)(h^{(q)}_{j_1^{(u)}}, h^{(q)}_{j_2^{(u)}}, \ldots, h^{(q)}_{j_{n_u}^{(u)}}) \|^2_{-q},]$$

$$+ \sup_{x \in E_p'} e^{-2\|x\|_{-p}} \sum_{i_1, i_2, \ldots, i_n = 1}^{\infty} E[\, | J_{h^{(q)}_{i_1}, h^{(q)}_{i_2}, \ldots, h^{(q)}_{i_n}}(\eta_{s,t}(x))$$

$$- J_{h^{(q)}_{i_1}, h^{(q)}_{i_2}, \ldots, h^{(q)}_{i_k}}(z^N_{s,t}(x)) |^2]\}$$

By the manner similar to that in the proofs of (3.10) and Lemma 5, we get

Lemma 6. For any integers $q \geq p_0$, $j \geq 1$, $n \geq 1$ and any $T > 0$, we have

$$\text{(4.11)} \qquad E[\| D^n z^N_{s,t}(x)(h_1, h_2, \ldots, h_n) \|^{2j}_{-q}]$$

$$\leq C_{42}(T) \|h_1\|^{2j}_{-q} \|h_2\|^{2j}_{-q} \ldots \|h_n\|^{2j}_{-q}, \quad x, h_i, i = 1, 2, \ldots, n \in E_q',$$

$$0 \leq s, t \leq T.$$

For any $\xi \in E$ and any $\alpha > 0$ and $T > 0$, there exists $C_{43} = C_{43}(\xi, \alpha, T)$ such that

$$\text{(4.12)} \qquad \sup_{0 \leq s,t \leq T} \max\{E[\exp(\alpha\sqrt{|\langle \eta_{s,t}(x), \xi \rangle|})],\ E[\exp(\alpha\sqrt{|\langle z^N_{s,t}(x), \xi \rangle|})]\}$$

$$\leq C_{43} \exp(\alpha\sqrt{|\langle x, \xi \rangle|}).$$

Since $f(x) = h(x)\varphi(x)$ and $|h^{(\ell)}(x)| \leq C_{44} \exp(\sum_{i=1}^{m} \sqrt{|x_i|})$, where $h^{(\ell)}(x) = (\frac{d}{dx})^{\ell} h(x)$, we get by Lemma 6,

$$\text{(4.13)} \qquad \sup_{x \in E_p'} e^{-\|x\|_{-p}} \max\{E[(\| D^u F(z^N_{s,t}(x)) \|^{(q')}_{H.S.})^2]^{1/2}, E[(\| D^{u+1} F(z^N_{s,t}(x)$$

$$+ \tau(\eta_{s,t}(x) - z^N_{s,t}(x))) \|^{(q')}_{H.S.})^2]^{1/2}\} \leq C_{45}(T),\ 0 \leq \tau \leq 1,\ 0 \leq s, t \leq T.$$

Hence noticing (3.10), (4.3), (4.13) and Lemma 5, we have some constants C_{46}

independent of ϵ', $C_{47} = C_{47}(\epsilon')$ and some natural number N_0 such that (4.10) is dominated by

$$(4.14)\quad C_{46}\epsilon' + C_{47} \sum_{i_1, i_2, \ldots, i_k=1}^{N_0} E[\|\eta_{s,t}(x)-z^N_{s,t}(x)\|^2_{-q} \cdot \|D^{n_1}\eta_{s,t}(x)(h^{(q)}_{j_1^{(1)}},$$

$$h^{(q)}_{j_2^{(1)}}, \ldots, h^{(q)}_{j_{n_1}^{(1)}})\|^2_{-q} \cdots \|D^{n_u}\eta_{s,t}(x)(h^{(q)}_{j_1^{(u)}}, h^{(q)}_{j_2^{(u)}}, \ldots, h^{(q)}_{j_{n_u}^{(u)}})\|^2_{-q},$$

$$+ \sum_{r=1}^{u} \|D^{n_1} z^N_{s,t}(x)(h^{(q)}_{j_1^{(1)}}, h^{(q)}_{j_2^{(1)}}, \ldots, h^{(q)}_{j_{n_1}^{(1)}})\|^2_{-q} \cdots \|D^{n_{r-1}} z^N_{s,t}(h^{(q)}_{j_1^{(r-1)}},$$

$$h^{(q)}_{j_2^{(r-1)}}, \ldots, h^{(q)}_{j_{n_{r-1}}^{(r-1)}})\|^2_{-q} \cdot \|D^{n_r}\eta_{s,t}(x)(h^{(q)}_{j_1^{(r)}}, h^{(q)}_{j_2^{(r)}}, \ldots, h^{(q)}_{j_{n_r}^{(r)}})$$

$$- D^{n_r} z^N_{s,t}(x)(h^{(q)}_{j_1^{(r)}}, h^{(q)}_{j_2^{(r)}}, \ldots, h^{(q)}_{j_{n_r}^{(r)}})\|^2_{-q} \cdot \|D^{n_{r+1}}\eta_{s,t}(x)$$

$$(h^{(q)}_{j_1^{(r+1)}}, h^{(q)}_{j_2^{(r+1)}}, \ldots, h^{(q)}_{j_{n_{r+1}}^{(r+1)}})\|^2_{-q} \cdots \|D^{n_u}\eta_{s,t}(x)(h^{(q)}_{j_1^{(u)}}, h^{(q)}_{j_2^{(u)}}, \ldots, h^{(q)}_{j_{n_u}^{(u)}})\|^2_{-q}].$$

Therefore noticing (3.10), (4.6), (4.7), (4.10), (4.11) and (4.14) and further for any $\epsilon > 0$, taking sufficiently small ϵ', δ and large N, we obtain

$$\sup_{x \in E'_p} e^{-\|x\|_{-p}} \|D^k((U(t,s)F)(x)) - D^k(E[\tilde{F}(z^N_{s,t}(x))])\|^{(q)}_{H.S.} < \epsilon.$$

The rest is to prove that $E[\tilde{F}(z^N_{s,t}(x))]$ is a weighted Schwartz functional. Of course $E[\tilde{F}(z^N_{s,t}(x))] = \phi_{s,t}(\langle x,\xi_1\rangle, \langle x,\xi_2\rangle, \ldots, \langle x,\xi_\ell\rangle, \langle x,\zeta_1\rangle, \langle x,\zeta_2\rangle, \ldots, \langle x,\zeta_m\rangle)$ is a smooth functional. Without loss of generality, we may assume that ξ_i, $i=1,2,\ldots,\ell$, ζ_j, $j=1,2,\ldots,m$, are all distinct elements in E. We will prove $g(x)\phi_{s,t}(x)$, $x \in \mathbb{R}^{\ell+m} \in \mathcal{S}(\mathbb{R}^{\ell+m})$. For any integer $n \geq 0$, by the Leibniz formula, it is sufficient to examine the finiteness of

$$\sup_{x\in\mathbb{R}^{\ell+m}} (1+|x|^2)^n |(\frac{d}{dx})^r g(x)(\frac{d}{dx})^k \phi_{s,t}(x)|, \text{ for integers } 0\leq r,k\leq n.$$

By the expression (4.9)' of $D^k(\tilde{F}(z^N_{s,t}(x)))(h^{(q)}_{i_1}, h^{(q)}_{i_2}, \ldots, h^{(q)}_{i_k})$, (4.11) and the fact that $f(x) = h(x)\varphi(x)$, $x \in \mathbb{R}^\ell$ and $|(\frac{d}{dx})^r g(x)| \leq C_{48}\exp(-\sum_{i=1}^{\ell+m} \sqrt{|x_i|})$, it is enough to show the finiteness of

$$(4.15) \quad \sup_Q (1 + \sum_{i=1}^{\ell} \langle x,\xi_i\rangle^2 + \sum_{j=1}^{m} \langle x,\zeta_i\rangle^2)^n \exp(-\sum_{i=1}^{\ell} \sqrt{|\langle x,\xi_i\rangle|} - \sum_{j=1}^{m} \sqrt{|\langle x,\zeta_j\rangle|})$$

$$\times E[(h^{(\mu)}(z^N_{s,t}(x))\varphi^{(\nu)}(z^N_{s,t}(x)))^2]^{1/2},$$

where $Q = \{x; (\langle x,\xi_1\rangle, \langle x,\xi_2\rangle, \ldots, \langle x,\xi_\ell\rangle, \langle x,\zeta_1\rangle, \langle x,\zeta_2\rangle, \ldots, \langle x,\zeta_m\rangle) \in \mathbb{R}^{\ell+m}\}$ and $h^{(\mu)}(x) = (\frac{d}{dx})^\mu h(x)$, $\varphi^{(\nu)}(x) = (\frac{d}{dx})^\nu \varphi(x)$, $x \in \mathbb{R}^\ell$.

Since $|h^{(\mu)}(x)| \leq C_{49}\exp(\sum_{i=1}^{\ell} \sqrt{|x_i|})$, (4.12) of Lemma 6 yields that (4.15) is dominated by

$$(4.16) \quad \sup_Q (1 + \sum_{i=1}^{\ell} \langle x,\xi_i\rangle^2 + \sum_{j=1}^{m} \langle x,\zeta_j\rangle^2)^n \exp(-\sum_{j=1}^{m} \sqrt{|\langle x,\zeta_j\rangle|}) E[(\varphi^{(\nu)}(z^N_{s,t}(x)))^4]^{1/4}$$

$$\leq C_{50} \sup_Q (1+ \sum_{i=1}^{\ell} \langle x,\xi_i\rangle^2 + \sum_{j=1}^{m} \langle x,\zeta_j\rangle^2)^n \exp(-\sum_{j=1}^{m} \sqrt{|\langle x,\zeta_j\rangle|})$$

$$\times E\left[\frac{(1+\sum_{i=1}^{\ell} \langle z^N_{s,t}(x),\xi_i\rangle^2)^{4n}}{(1+\sum_{i=1}^{\ell} \langle z^N_{s,t}(x),\xi_i\rangle^2)^{4n}} |\varphi^{(\nu)}(z^N_{s,t}(x))|^4\right]^{1/4}$$

$$\leq C_{51}\|\varphi\| \sup_Q (1+ \sum_{i=1}^{\ell} \langle x,\xi_i\rangle^2 + \sum_{j=1}^{m} \langle x,\zeta_j\rangle^2)^n \exp(-\sum_{j=1}^{m} \sqrt{|\langle x,\zeta_j\rangle|})$$

$$\times E\left[\frac{1}{(1+\sum_{i=1}^{\ell}\langle z^N_{s,t}(x),\xi_i\rangle^2)^{4n}}\right]^{1/4}.$$

where $\|\varphi\|_n = \sup_{\substack{x\in\mathbb{R}^\ell \\ 0\le k\le n}} (1+|x|^2)^n\,|\varphi^{(k)}(x)|$.

On the other hand, we can verify the following lemma.

Lemma 7. For any $\xi_1,\xi_2,\ldots,\xi_\ell \in E$ and any integer $p \geq 1$, we have

$$E\left[\frac{1}{(1+\sum_{i=1}^{\ell}\langle z^N_{s,t}(x),\xi_i\rangle^2)^p}\right] \le C_{52}(T)\,\frac{1}{(1+\sum_{i=1}^{\ell}\langle x,\xi_i\rangle^2)^p},\quad 0 \le s,t \le T.$$

Proof. Setting $\theta(x) = \dfrac{1}{(1+\sum_{i=1}^{\ell}\langle x,\xi_i\rangle^2)^p}$ and applying the Itô formula for $\theta(z^N_{s,t}(x))$, we get

$$E\left[\frac{1}{(1+\sum_{i=1}^{\ell}\langle z^N_{s,t}(x),\xi_i\rangle^2)^p}\right] = \frac{1}{(1+\sum_{i=1}^{\ell}\langle x,\xi_i\rangle^2)^p} \tag{4.17}$$

$$+\,E\left[\int_s^t \frac{-2p\,(1+\sum_{i=1}^{\ell}\langle z^N_{s,r}(x),\xi_i\rangle^2)^{p-1}(\sum_{i=1}^{\ell}\langle z^N_{s,r}(x),\xi_i\rangle\,\langle\tilde{B}(r,w^{N,2}_{s,r}(x)),\xi_i\rangle)}{(1+\sum_{i=1}^{\ell}\langle z^N_{s,r}(x),\xi_i\rangle^2)^{2p}}\,dr\right]$$

$$+E\left[\int_s^t \frac{1}{2}\times\right.$$

$$\sum_{j=1}^{\infty}\{\frac{-4p(p-1)(1+\sum_{i=1}^{\ell}\langle z^N_{s,r}(x),\xi_i\rangle^2)^{p-2}(\sum_{i=1}^{\ell}\langle z^N_{s,r}(x),\xi_i\rangle\,\langle\tilde{A}(r,w^{N,1}_{s,r}(x))h^{(0)}_j,\xi_i\rangle)^2}{(1+\sum_{i=1}^{\ell}\langle z^N_{s,r}(x),\xi_i\rangle^2)^{4p}}$$

$$+\frac{-2p(1+\sum_{i=1}^{\ell}\langle z_{s,r}^N(x),\xi_i\rangle^2)^{p-1}(\sum_{i=1}^{\ell}\langle\tilde{A}(r,w_{s,r}^{N,1}(x))h_j^{(0)},\xi_i\rangle^2)}{(1+\sum_{i=1}^{\ell}\langle z_{s,r}^N(x),\xi_i\rangle^2)^{4p}}$$

$$\left.+\frac{8p^2(1+\sum_{i=1}^{\ell}\langle z_{s,r}^N(x),\xi_i\rangle^2)^{3p-2}(\sum_{i=1}^{\ell}\langle z_{s,r}^N(x),\xi_i\rangle\langle\tilde{A}(r,w_{s,r}^{N,1}(x))h_j^{(0)},\xi_i\rangle)^2}{(1+\sum_{i=1}^{\ell}\langle z_{s,r}^N(x),\xi_i\rangle^2)^{4p}}\}dr\right],$$

where

$$w_{s,r}^{N,k}(x) = x+\int_s^r\tilde{A}_k(r_1,x+\int_s^{r_1}\tilde{A}_k(r_2,\dots,x+\int_s^{r_{N-2}}\tilde{A}_k(r_{N-1},x)d\beta_k(r_{N-1}))\dots)d\beta_k(r_1).$$

By the boundedness of $\tilde{A}_k(t,x)$, (4.17) is dominated by

$$\frac{1}{(1+\sum_{i=1}^{\ell}\langle z_{s,r}^N(x),\xi_i\rangle^2)^p}+C_{53}\int_s^t E\left[\frac{1}{(1+\sum_{i=1}^{\ell}\langle z_{s,r}^N(x),\xi_i\rangle^2)^p}\right]dr,$$

which yields the proof of the lemma, together with the Gronwall lemma.

Using this lemma, we have that the right hand side of (4.10) is dominated by

$$C_{54}\|\varphi\|_n \sup_Q (1+\sum_{i=1}^{\ell}\langle x,\xi_i\rangle^2+\sum_{j=1}^{m}\langle x,\zeta_j\rangle^2)^n\exp(-\sum_{j=1}^{m}\sqrt{|\langle x,\zeta_j\rangle|})$$

$$\times\frac{1}{(1+\sum_{i=1}^{\ell}\langle x,\xi_i\rangle^2)^n}<\infty,$$

which guarantees that $E[\tilde{F}(z_{s,t}^N(x))]$ is a weighted Schwartz functional. This completes the proof of Proposition 2.

Now the following remark is immediate.

Remark. Under the assumptions of Proposition 2, $(L(t)F)(x)$ is also approximated by a sequence of weighted Schwartz functionals.

Then we will proceed to the discussion for a concrete example $\mathcal{D}_{\mathcal{S}'(\mathbb{Z}^d)}$. We will begin by giving the definition based on the sequential Schwartz space. Let $\mathbb{Z}^d$ be the d-dimensional lattice, $i=(i_1,i_2,\ldots,i_d) \in \mathbb{Z}^d$ and $\mathcal{S} = \mathcal{S}(\mathbb{Z}^d)$ the Schwartz space of rapidly decreasing sequences $\xi = (\xi_i)$, $\xi_i = (\xi_{i_1},\xi_{i_2},\ldots,\xi_{i_d})$, metrized by the countably many semi-norms:

$$\|\xi\|_p^2 = \sum_{i_1,i_2,\ldots,i_d=-\infty}^{\infty} (1+|i|)^{2p}|\xi_i|^2, \quad p=0,1,2,\ldots .$$

The dual space $\mathcal{S}' = \mathcal{S}'(\mathbb{Z}^d)$ of $\mathcal{S}$ is a collection of all slowly increasing sequences $S = (S_i)$ such that for some integer $p \geq 0$,

$$\|S\|_{-p}^2 = \sum_{i_1,i_2,\ldots,i_d=-\infty}^{\infty} (1+|i|)^{-2p}|S_i|^2 < \infty.$$

Let $x \in \mathbb{R}^m$ and $S(\mathbb{R}^m) = \{\phi(x) = h(x)\varphi(x);\ \varphi \in \mathcal{S}(\mathbb{R}^m)\}$. We will define the p-th semi-norm of $S(\mathbb{R}^m)$ by

$$|\phi|_p^S = \sup_{\substack{x\in\mathbb{R}^m \\ 0\leq k\leq m}} (1+|x|^2)^p|(\tfrac{d}{dx})^k(g(x)\phi(x))|.$$

Then $S(\mathbb{R}^m)$ is a nuclear Fréchet space metrized by the countably many semi-norms. $|\cdot|_p^S$, $p=0,1,2,\ldots$,[6]. For finite lattice V in $\mathbb{Z}^d$, $C_0^\infty(\mathcal{S}',V)$ is a collection of all functions $\Phi(S)$ such that there exists a weighted Schwartz function $\phi(x) \in S(\mathbb{R}^{|V|d})$ and $\Phi(S) = \phi(S|_V)$, where $S|_V$ means the restriction of S on V and $|V|$ denotes the number of lattice points in V. We will introduce the nuclear Fréchet topology on this space by the countably many semi-norms

$$\|\Phi\|_p = |\phi|_p^S, \quad p = 0,1,2,\ldots,$$

where $|\cdot|_p^S$ denotes the p-th semi-norm of $S(\mathbb{R}^{|V|d})$. Let $C_0^\infty(\mathcal{S}')$ be a collection of all functionals $\Phi(S)$ such that $\Phi(S) = \phi(S|_V)$ for some finite lattice V in $\mathbb{Z}^d$ and weighted Schwartz function $\phi(x) \in S(\mathbb{R}^{|V|d})$.

Since $C_0^\infty(\mathcal{S}',V) \subset C_0^\infty(\mathcal{S}',U)$ if $V \subset U$, setting $V_n = [-n,n]^d$, we will introduce on $C_0^\infty(\mathcal{S}')$ the strict inductive limit topology of $C_0^\infty(\mathcal{S}',V_n)$.

Since $\mathcal{S}(\mathbb{Z}^d)$ is a nuclear Fréchet space, we use the same notations defined before. For any integers $p \geq 0$, $q \geq 0$ and $n \geq 0$, let $\mathfrak{D}_{p,q,n}$ be the completion of $C_0^\infty(\mathcal{S}')$ by the semi-norm $\|\cdot\|_{p,q,n}$.

Definition of Space $\mathfrak{D}_{\mathcal{S}'(\mathbb{Z}^d)}$. We define $\mathfrak{D}_{\mathcal{S}'(\mathbb{Z}^d)} = \bigcap_{p,q,n} \mathfrak{D}_{p,q,n}$, where $p \geq 0$, $q \geq 0$ and $n \geq 0$. We introduce a topology on $\mathfrak{D}_{\mathcal{S}'(\mathbb{Z}^d)}$ by the countably many semi-norms $\|\cdot\|_{p,q,n}$, $p \geq 0$, $q \geq 0$ and $n \geq 0$.

Then $\mathfrak{D}_{\mathcal{S}'(\mathbb{Z}^d)}$ becomes a complete separable metric space [6].

Propositions 1 and 2 yield.

Theorem. Suppose that the coefficients A(t,x) and B(t,x) satisfy the conditions (H1)-(H3) and are approximated by sequences of bounded smooth functionals on $\mathcal{S}'(\mathbb{Z}^d)$. Then L(t) generates the Kolmogorov evolution operator U(t,s) from $\mathfrak{D}_{\mathcal{S}'(\mathbb{Z}^d)}$ into itself. Further under the same assumption of the initial value as in Proposition 1, the continuous $\mathcal{L}(\mathfrak{D}_{\mathcal{S}'(\mathbb{Z}^d)})$-process solution of (1.1) is uniquely given by

$$X_F(t) = X_{U(t,0)F}(0) + W_F(t) + \int_0^t W_{L(s)U(t,s)F}(s)ds.$$

For a real valued functional F(t,S) on $\mathcal{S}'$ such that F(t,S) is infinitely many times $\mathcal{S}'_p$-Fréchet differentiable with respect to S for every integer $p \geq 0$, we set $|F| = \sup_{0 \leq t \leq T} \sup_{S \in \mathcal{S}'} |F(t,S)|$ and $\|F\|_{p,n} = \sup_{0 \leq t \leq T} \sup_{S \in \mathcal{S}'} \|D^n F(t,S)\|_{H.S.}^{(p)}$.

Let $a_i(t,S)$, $b_i(t,S)$, $i \in \mathbb{Z}^d$, be real valued mappings defined on $\mathcal{S}'$ and infinitely many times $\mathcal{S}'_p$-Fréchet differentiable with respect to S for every integer $p \geq 0$. We assume the following conditions:

(AI) We have some natural number p_0 such that

$$\sum_{i_1, i_2, \ldots, i_d=-\infty}^{\infty} (1+|i|)^{-2p_0} \max\{|a_i|^2, |b_i|^2\} < \infty.$$

(AII) For any integers $n \geq 1$ and $p \geq 0$,

$$\sum_{i_1, i_2, \ldots, i_d=-\infty}^{\infty} (1+|i|)^{-2p} \max\{\|a_i\|_{p,n}^2, \|b_i\|_{p,n}^2\} < \infty.$$

(AIII) For any integer $n \geq 0$ and any $T > 0$, there exist $\lambda_2(n,p,T) > 0$ and $\lambda_3(n,p,T) > 0$ such that

$$\sup_{\substack{S \in \mathcal{S}' \\ 0 \leq k \leq n}} \max\{\|D^k a_i(t,S) - D^k a_i(s,S)\|_{H.S.}^{(p)}, \|D^k b_i(t,S) - D^k b_i(s,S)\|_{H.S.}^{(p)}\}$$

$$\leq \lambda_2(n,p,T)|t-s|^{\lambda_3(n,p,T)}.$$

(AIV) $a_i(t,S)$, $b_i(t,S)$, $i \in \mathbb{Z}^d$ are approximated by sequences of real valued bounded smooth functionals $a_i^{(m)}(t,S)$, $b_i^{(m)}(t,S)$, $i \in \mathbb{Z}^d$ such that

$$\lim_{m\to\infty} \sup_{0 \leq t \leq T} \sup_{S \in \mathcal{S}'_p} \|D^n a_i(t,S) - D^n a_i^{(m)}(t,S)\|_{H.S.}^{(q)} = 0,$$

$$\lim_{m\to\infty} \sup_{0 \leq t \leq T} \sup_{S \in \mathcal{S}'_p} \|D^n b_i(t,S) - D^n b_i^{(m)}(t,S)\|_{H.S.}^{(q)} = 0,$$

for any integer $p \geq p_0$, $q \geq 0$ and $n \geq 0$.

Under the assumption (AI), we define a continuous linear operator $A(t,S)$ from $\mathcal{S}'$ into itself by $A(t,S)Y = (a_i(t,S)Y_i)$, $S=(S_i)$, $Y=(Y_i)$, $i \in \mathbb{Z}^d$. Further set $B(t,S) = (b_i(t,S))$, $i \in \mathbb{Z}^d$. Under the conditions (AI)-(AIV), the coefficients $A(t,S)$ and $B(t,S)$ satisfy the assumptions of the theorem. Then for the diffusion operator

$$(L(t)F)(S) = \frac{1}{2} \operatorname{trace}_{\ell^2(\mathbb{Z}^d)} D^2F(S)\circ[A(t,S)\times A(t,S)] + DF(S)(B(t,S)), \quad F \in \mathcal{D}_{\mathcal{S}'(\mathbb{Z}^d)},$$

we get

Corollary. Suppose that $a_i(t,S)$, $b_i(t,S)$, $i \in \mathbb{Z}^d$ satisfy the conditions (AI)-(AIV). Then $L(t)$ generates the Kolmogorov evolution operator from $\mathcal{D}_{\mathcal{S}'(\mathbb{Z}^d)}$ into itself and the same conclusion stated in Theorem holds.

§5. Central limit theorem for a lattice system of interacting diffusions.

First we begin to explain the system that Deuschel considered [4]. Let $b_i(S)$, $i \in \mathbb{Z}^d$, be real valued infinitely many times $\mathcal{S}'_p$-Fréchet differentiable mappings on $\mathcal{S}'$ for every integer $p \geq 0$ such that $b_i(S) = \hat{b}(\theta_i S)$, $\theta_i S = (S_{j+i})$ and $\hat{b}(S)$ is also real valued mapping on $\mathcal{S}'$.

(VI) We have some natural number p_0 such that

$$\sum_{i_1,i_2,\ldots,i_d=-\infty}^{\infty} (1+|i|)^{-2p_0}(\sup_{S\in\mathcal{S}'} |b_i(S)|)^2 < \infty.$$

(V2) For any integers $n \geq 1$ and $p \geq 0$.

$$\sum_{i_1,i_2,\ldots,i_d=-\infty}^{\infty} (1+|i|)^{-2p}(\sup_{S\in\mathcal{S}'} \|D^n b_i(S)\|_{H.S.}^{(p)})^2 < \infty.$$

(V3) There exists a sequence of real valued bounded smooth functionals $b_i^{(m)}(S)$ such that

$$\lim_{m\to\infty} \sup_{S\in\mathcal{S}'_p} \|D^n b_i(S)-D^n b_i^{(m)}(S)\|_{H.S.}^{(q)} = 0$$

for any integers $p \geq p_0$, $q \geq 0$ and $n \geq 0$.

Let $S(t) = (S_i(t),\ i \in \mathbb{Z}^d)$ be an $\mathcal{S}'(\mathbb{Z}^d)$-valued solution of the following equation:

(5.1) $$S_i(t) = \sigma_i + B_i(t) + \int_0^t b_i(S(s))ds,$$

$$b_i(S) = \hat{b}(\theta_i S), \ \theta_i S = (S_{j+i}),$$

where $(B_i(t))$ are independent copies of the d-dimensional standard Brownian motion $B(t)$ and (σ_i) are also independent copies of the d-dimensional random variable σ independent of $B(t)$ and for any $\epsilon > 0$, $E[\exp(\epsilon \|(\sigma_i)\|_{-p_0})] < \infty$. For a finite lattice $V \in \mathbb{Z}^d$, consider

$$T_V(t) = |V|^{-1/2} \sum_{i \in V} \delta_{\theta_i S(t)},$$

where δ_S denotes the Dirac measure at S in $\mathcal{S}'$. Then we will study the limit behavior of $T_V(t)$ after him [4].

Now put

$$\langle U_V(t), \Phi \rangle = \langle T_V(t), \Phi \rangle - E[\langle T_V(t), \Phi \rangle], \ \Phi \in C_0^\infty(\mathcal{S}'),$$

where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form on $C_0(\mathcal{S}')' \times C_0(\mathcal{S}')$. Then it can be proved by [17], [21] that $U_V(t)$ becomes a strongly continuous $C_0^\infty(\mathcal{S}')'$-valued stochastic process. We will prove the tightness for $U_V(t)$, $V \in \mathbb{Z}^d$ following [5], [18], in $C([0,\infty); C_0^\infty(\mathcal{S}')')$ which is the space of continuous mappings from $[0,\infty)$ into $C_0^\infty(\mathcal{S}')'$ equipped with the strong topology. Let $\Phi(S) = \phi(S_{n_1}, S_{n_2}, \dots, S_{n_q})$, $\phi \in S(\mathbb{R}^{dq})$ and L_0 be an operator such that

$$(L_0 F)(S) = \frac{1}{2} \operatorname{trace}_{\ell^2(\mathbb{Z}^d)} D^2 F(S) + DF(S)(b(S)), \ F \in \mathfrak{D}_{\mathcal{S}'(\mathbb{Z}^d)},$$

where $b(S) = (b_i(S))$.

By the conditions (V1) and (V2), the equation (5.1) is solved in $\mathcal{S}'_{p_0}$, so that $S(t) \in \mathcal{S}'_{p_0}$. Then we have

$$E[\langle T_V(t),\Phi\rangle^2] \le C_{55}\|\Phi\|^2_{p_0,0,0}$$

and since $C_0^\infty(\mathscr{S}')$ is dense in $\mathscr{D}_{\mathscr{S}'(\mathbb{Z}^d)}$, $T_V(t)$ is extended to a continuous $\mathscr{L}(\mathscr{D}_{\mathscr{S}'(\mathbb{Z}^d)})$-process. We denote the extension by $T_{\Phi,V}(t)$.

By the Itô formula, we get

$$\langle T_V(t),\Phi\rangle - \langle T_V(0),\Phi\rangle = M_{\Phi,V}(t) + \int_0^t T_{L_0\Phi,V}(s)ds, \tag{5.2}$$

where

$$M_{\Phi,V}(t) = |V|^{-1/2}\sum_{i\in V}\int_0^t \sum_j \frac{\partial}{\partial S_{n_j}}\phi(S_{n_1+i}(s),S_{n_2+i}(s),\dots,S_{n_q+i}(s))dB_{n_j+i}(s).$$

Noticing the independence of $B_i(t)$, $i \in V$ and the fact that $S(t) \in \mathscr{S}'_{p_0}$, we have for $t \in [0,T]$,

$$E[M_{\Phi,V}(t)^4] \le C_{56}\|\Phi\|^4_{p_0,0,1}\ . \tag{5.3}$$

Then $M_{\Phi,V}(t)$ can be extended to a continuous $\mathscr{L}(\mathscr{D}_{\mathscr{S}'(\mathbb{Z}^d)})$-process and has the same regularities that Wiener $\mathscr{L}(\mathscr{D}_{\mathscr{S}'(\mathbb{Z}^d)})$-process has. Conditions (V1)-(V3) guarantee that L_0 belongs to the class dealt in Corollary. We use the same notation $U(t,s)$ that represents an evolution operator generated by L_0. Thus the solution of (5.2) is given as follows:

$$\langle T_V(t),\Phi\rangle = T_{U(t,0)\Phi,V}(0) + M_{\Phi,V}(t) + \int_0^t M_{L_0U(t,s)\Phi,V}(s)ds$$

by the same manner as in the proof of Proposition 1. Hence by (5.3) and the Kolmogorov test for real Wiener process, we get

$$E[\,|\langle U_V(t)-U_V(s),\Phi\rangle|^4] \le C_{57}|t-s|^2$$

and further

$$E[|\langle U_V(t),\Phi\rangle|^2] \le C_{58}\{\|\Phi\|^2_{p_0,0,1} + \sup_{0\le s\le t} \|U(t,s)\Phi\|^2_{p_0,0,3}\}$$

which proves the tightness in $C([0,\infty);C_0^\infty(\mathcal{S}')')$, [5], [18]. By the Skorohod theorem and the usual limiting argument, the limit process $N(t)$ of $U_V(t)$ satisfies the Langevin equation

$$\langle N(t)-N(0),\Phi\rangle = W_\Phi(t) + \int_0^t N_{L_0\Phi}(s)ds, \tag{5.4}$$

where $N_F(t)$, $F \in \mathcal{D}_{\mathcal{S}'(\mathbb{Z}^d)}$, is the extension of $N(t)$ and $W_F(t)$ is a Wiener $\mathcal{L}(\mathcal{D}_{\mathcal{S}'(\mathbb{Z}^d)})$-process [8].

The uniqueness for solutions of the equation (5.4) discussed in Corollary implies the identification of the distribution of the limit process, ([19], [20]), which implies that $U_V(t)$ converges to a Gaussian field in $C([0,\infty);C_0^\infty(\mathcal{S}')')$.

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