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EXACT LOWER CONFIDENCE LIMITS FOR
THE RELIABILITY OF k OF n SYSTEMS
FOR ZERO FAILURES AND NONCONSERVATIVENESS
OF THE MAXIMUS METHOD

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ABSTRACT

The Buehler (1957) optimal $1-\alpha$ lower confidence limit on the reliability of k of n systems of independent components is derived for the case of zero failures and equal sample sizes. The limiting form of the lower confidence limit is obtained for $n-1$ of n systems as n goes to infinity. This result is used to show the nonconservativeness of the Maximus method given by Spencer and Easterling (1986).

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Andrew P. Soms^{*}

1. The Lower Confidence Limit

We consider a k of n system of independent components, i.e., a system where the components function independently and the system works if and only if at least k of the subsystems do. Let p_i be the probability that the i^{th} subsystem functions. Thus

$$h(\tilde{p}) = P\left(\sum_{i=1}^n Z_i \geq k\right), \quad (1)$$

where $\tilde{p} = (p_1, \dots, p_n)$, $h(\tilde{p})$ is the system reliability corresponding to $\tilde{p}$ and $Z_i = 1$ if the i^{th} subsystem functions, 0 otherwise. The cases $k = 1$ and $k = n$ correspond to parallel and series systems, respectively. A practical matter of great importance is to place a good $1-\alpha$ lower confidence limit on $h(\tilde{p})$ when binomial data is available on the subsystems, i.e., when the observed values $y_1, y_2, \dots, y_n$ of $Y_1, Y_2, \dots, Y_n$ are available, where the Y_i 's are independent binomial random variables with parameters m_i and p_i . In other words, Y_i is the number of successes in m_i trials of the i^{th} subsystem. Buehler (1957) gave a general solution to this problem which is generally computationally difficult. We will specialize this to the case when $\tilde{m} = (m_1, m_2, \dots, m_n) = (m, m, \dots, m)$ and $\tilde{y} = (y_1, y_2, \dots, y_n) = \tilde{m} = (m, m, \dots, m)$. In this case the $1-\alpha$ lower confidence limit $a_{\tilde{m}}$ on $h(\tilde{p})$ is

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$$a_{\tilde{m}} = \inf h(\tilde{p}) \mid \prod_{i=1}^n p_i^m > \alpha \quad (2)$$

More generally, $a_{\tilde{y}}$ is given by

$$a_{\tilde{y}} = \inf h(\tilde{p}) \mid P(\tilde{z} | g(\tilde{z}) > g(\tilde{y})) > \alpha ,$$

where $g(\tilde{y})$ is a reasonable point estimator of $h(\tilde{p})$. The only property of g that we will assume in the sequel is that $g(\tilde{m})$ is the unique maximum of $g(\tilde{y})$. This insures that $a_{\tilde{m}} > a_{\tilde{y}}$, $\tilde{y} \neq \tilde{m}$, and gives the reasonable result that $a_{\tilde{m}}$ is the largest lower confidence limit. Since $h(\tilde{p})$ is a nondecreasing function in each p_i , without loss of generality (2) may be rewritten as

$$a_{\tilde{m}} = \inf h(\tilde{p}) \mid \left(\prod_{i=1}^n p_i \right)^m = \alpha \quad (3)$$

One of the results of Pledger and Proschan (1971, p. 92) states that, subject to $\prod_{i=1}^n p_i = \alpha^{1/m}$, which is the condition in (3), $h(\tilde{p})$ is minimized by choosing the p_i to be equal, i.e., $p_i = \alpha^{1/(nm)}$ (they prove that $h(\tilde{p})$ is a Schur-convex function of $R_i = -\ln p_i$, which implies the above). So the $1-\alpha$ lower confidence limit $a_{\tilde{m}}$ is given by

$$a_{\tilde{m}} = \sum_{i=k}^n \binom{n}{i} \left(\alpha^{1/(nm)} \right)^i \left(1 - \alpha^{1/(nm)} \right)^{n-i} , \quad (4)$$

since for all the p_i equal (1) implies that $\sum_{i=1}^n Z_i$ is a binomial random variable with parameters n and $\alpha^{1/(nm)}$. For $k=1$ and $k=n$, (4) is already known (see, e.g., Spencer and Easterling (1986)). We summarize the above results in Theorem 1.1.

Theorem 1.1. The lower $1-\alpha$ confidence limit $a_{\tilde{m}}$ on $h(\tilde{p})$, the reliability of a k of n system, when all the subsystem test results have equal sample sizes m and zero failures are observed, is given by (4).

For purposes of the next section, we shall let $k = n-1$ and consider the limit of (4) as $n \rightarrow \infty$. When $k = n-1$, (4) becomes

$$a_{\tilde{m}} = n \left(\alpha^{\frac{1}{nm}} \right)^{n-1} \left(1 - \alpha^{\frac{1}{nm}} \right) + \alpha^{\frac{1}{m}} \quad (5)$$

and, as $n \rightarrow \infty$, this tends to

$$b_{\tilde{m}} = \alpha^{1/m} \frac{(-\ln \alpha)}{m} + \alpha^{1/m} = e^{\ln \alpha / m} \left(1 + \frac{-\ln \alpha}{m} \right) \quad (6)$$

(in fact, it is readily verified that the limit for a $n-k$ of n system is $e^{\ln \alpha / m} \sum_{i=0}^k \left(\frac{-\ln \alpha}{m} \right)^i / i!$, but this is of no interest here).

2. Nonconservativeness of the Maximus Method.

We first consider the limiting behavior of the Maximus method under the same assumptions as above, i.e., an $n-1$ of n system as $n \rightarrow \infty$ with zero observed failures. Using the description of Spencer and Easterling (1986), we have $Q_0 = 0$ and $Q_1 = 1/(m/(n-1) + 1)^n$, where we have used the unpooling on the $n(n-1)$ component series systems in parallel to give independent subsystem data with $m/(n-1)$ trials and 0 failures. This yields the effective binomial sample size N_S ,

$$N_S = (1 - Q_1)/Q_1 = \frac{1 - 1/(m/(n-1) + 1)^n}{1/(m/(n-1) + 1)^n} \quad (7)$$

and observed failures $F_S = 0$. As $n \rightarrow \infty$, the limit in (7) is

$$N_S = e^m - 1 \quad (8)$$

and so the limiting Maximus lower confidence limit a_M is

$$a_M = \alpha^{1/(e^m - 1)} = 1 + \ln \alpha / e^m \quad (9)$$

For the purpose of comparison, we take $\alpha = .01$ and $m = 20$. Then the exact lower confidence limit $b_{\tilde{m}}$ is .9772. The corresponding Maximus figure

b_M , obtained from (9) is $b_M = 1 - 9.5 \times 10^{-9}$. The true α , α_t , corresponding to a lower confidence limit of $1 - 9.5 \times 10^{-9}$ is $\alpha_t = .998$ (we emphasize this is α , not $1-\alpha$) and is obtained by solving (6) in reverse, i.e., letting $b_{\tilde{m}} = 1 - 9.5 \times 10^{-9}$ and solving for the corresponding α . So there are points $\tilde{p}$ in the parameter space whose coverage probability comes arbitrarily close to .002 from above when the nominal or desired minimal coverage probability is .99. These are n-dimensional parameter points $\tilde{p}$ such that $\tilde{p} = (p, \dots, p)$ and $p^n < (.998)^{1/m}$ but is arbitrarily close to it.

We also consider a finite case. Let $n=5$, $\alpha = .01$, $m = 20$. Then $a_{\tilde{m}}$ is .9815 and Maximus gives .9994, which corresponds to a true α , α_t , of $\alpha_t = .448$. So there are points $\tilde{p} = (p, p, p, p, p)$ in the parameter space with $p^5 < (.448)^{1/20}$, but arbitrarily close to it, for which the actual coverage is arbitrarily close to .552 from above as contrasted with a nominal or desired minimal coverage probability of .99.

3. Conclusions.

We have shown in this paper that there can be large differences in nominal and actual coverages when the Maximus method is used. This method is used extensively by government agencies. The use of such an apparently nonconservative method should be carefully considered.

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