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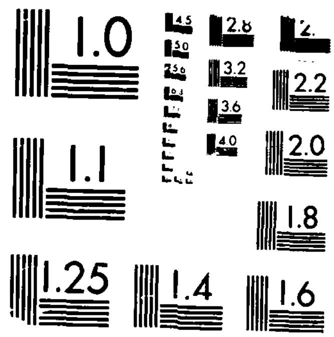

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by

Janine V. England

March 1988

Thesis Advisor: Robert D. Strum

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An overview and investigation of the more popular digital filter design techniques are presented, with the intent of providing the filter design engineer a complete and concise source of information. Advantages and disadvantages of the various techniques are discussed, and extensive design examples used to illustrate their application to specific design problems. Both IIR (Butterworth, Chebyshev and elliptic), and FIR (Fourier coefficient design, windows and frequency sampling) design methods are featured, as well as, the Optimum FIR Filter Design Program of Parks and McClellan, and the Minimum p - Error IIR Filter Design Method of Deczky.

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Digital Filter
Design Techniques

by

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requirements for the degree of

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ABSTRACT

An overview and investigation of the more popular digital filter design techniques are presented, with the intent of providing the filter design engineer a complete and concise source of information. Advantages and disadvantages of the various techniques are discussed, and extensive design examples used to illustrate their application to specific design problems. Both IIR (Butterworth, Chebyshev and elliptic), and FIR (Fourier coefficient design, windows and frequency sampling) design methods are featured, as well as the Optimum FIR Filter Design Program of Parks and McClellan, and the Minimum p - Error IIR Filter Design Method of Deczky.

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I. INTRODUCTION

Digital filter design today is a rapidly growing field with myriad applications in the areas of signal processing, image processing, filtering, prediction, and estimation. Extensive research has resulted in a plethora of available information, which can be overwhelming to the engineer presented with a specific design problem. As the title suggests, the purpose of this thesis is to present an overview of the more popular design techniques in an attempt to consolidate the information available in the literature, and provide an easily readable reference manual. Detailed examples are given to illustrate the application of selected techniques to specific filter design problems.

Due to the wealth of information available, a summary and investigation of every filter design technique is not possible. As stated the more popular methods are outlined in detail, however, references are included as sources of further information.

The survey is threefold, consisting first of recursive IIR design methods, followed by nonrecursive FIR design, and finally computer-aided design (CAD). A summary of the chapter contents follows to enable the reader to immediately locate the section applicable to his/her particular design problem.

Chapter II, Recursive Filter Design, presents Butterworth, Chebyshev and elliptic filter design from both a traditional approach, wherein analog prototype filters based on design specifications are converted to digital versions using the bilinear transformation, and a direct design approach that eliminates the need to determine an analog prototype.

Chapter III, Nonrecursive Filter Design, includes an overview of Fourier coefficient design, windowing, and a derivation and illustration of the method of frequency sampling. Among the design examples presented is the design of a bandpass differentiator and a bandpass integrator.

In Chapter IV, Computer-Aided Design, the Optimum FIR Filter Design Method of Parks and McClellan (that employs the Remez Exchange Algorithm) is investigated and applied to a bandpass filter design problem. Also included is a short discussion of a method to enhance the application of this program to the design of high-order filters.

To complete the chapter, the Minimum p-Error Design Method for IIR Filter Design, that utilizes the Fletcher-Powell algorithm, is presented. Furthermore, a discussion of the advantages and disadvantages of these two iterative techniques is included.

II. RECURSIVE FILTER DESIGN

A. INTRODUCTION

The recursive realization of digital filters is advantageous, in that the desired frequency response can be obtained using a lower order filter than if a nonrecursive realization were used (assuming linear phase is not required). This is because the filter frequency response is influenced by both the poles and zeros of the filter transfer function, whereas the frequency response of a nonrecursive realization is determined only by the filter's zeros.

Traditionally, the design of recursive digital filters involves the determination of an analog prototype using one of several methods: Butterworth, Chebyshev or elliptic, to name a few, and converting these to a digital version using impulse-invariant design or bilinear transformation methods, the latter being the more popular.

This chapter first presents a review of the traditional design techniques using examples involving Butterworth, Chebyshev and elliptic filters; then a Direct Design method is presented, whereby the requirement for determining an analog prototype is eliminated, thus reducing the algebraic complexity of recursive filter design.

B. TRADITIONAL DESIGN TECHNIQUES

Recursive filter design techniques, as stated in the introduction, involve determining an analog prototype filter from given filter specifications, and then converting the prototype to a digital version using a bilinear transformation. Since this is a well known design procedure [1], details of the derivation will not be considered.

However, as a quick review, three examples involving the design of Butterworth, Chebyshev and elliptic filters will be given.

It should be noted that the tables of analog prototype transfer functions found in this section (Tables 2.2, 2.3 and 2.4), all have a critical frequency of unity ($w_C = 1$). Thus, to obtain a filter transfer function with a different critical frequency based on these prototypes requires the use of an appropriate analog frequency transformation (Table 2.1).

For example, suppose a lowpass analog filter with a critical frequency of w_C is desired. The following relationship applies:

$$H_{LP}(jw_C) = H_{LP_P}(j1) \tag{2.1}$$

where H_{LP_P} is the prototype filter with a critical frequency of $w_C = 1$.

To convert the prototype transfer function to the desired transfer function, $H_{LP}(s)$, the s in the lowpass prototype filter is replaced by s/w_C as:

$$H_{LP}(s) = H_{LP_P}(s)\Big|_{s=\frac{s}{w_C}} \tag{2.2}$$

Table 2.1 (Reference 1) lists the appropriate frequency transformations for highpass, bandpass and bandstop filters. A summary of the design procedure follows.

1. Bilinear Transformation Design

From given analog filter specifications, determine the approppriate s-domain design technique (i.e., Butterworth, Chebyshev or elliptic).

Design the analog prototype filter in accordance with the technique selected above. This involves the translation of the filter specifications to those of a lowpass prototype. The lowpass prototype is then designed according to the translated specifications.

Transform the lowpass prototype to the desired lowpass, highpass, bandpass, or bandstop filter according to Table 2.1.

Transform the analog filter transfer function to a digital version using the bilinear transformation.

$$H(z) = H(s)\Big|_{s=\frac{z-1}{z+1}} \tag{2.3}$$

Example 2.1 Butterworth Filter

Design a digital filter that is flat in the passband from 0 to the 3 dB cutoff frequency, f_C, of 2 kHz. For frequencies greater than 4 kHz, the attenuation should be at least 10 dB The sampling rate is 20 kHz.

Step 1: A Butterworth design is called for because a flat passband is desired.

Step 2: Convert the critical analog design frequencies to digital.

Sampling frequency: $f_s = 20$ kHz $\Longrightarrow$ $w_s = 40 \times 10^3\pi$ rad/s

Cutoff frequency: $f_C = 2$ kHz $\Longrightarrow$ $w_C = 4 \times 10^3\pi$ rad/s

Stopband frequency: $f_a = 4$ kHz $\Longrightarrow$ $w_a = 8 \times 10^3\pi$ rad/s

$$\theta'_C = w_C T = w_C/f_s = \frac{(4 \times 10^3)\pi}{20 \times 10^3} = 0.2\pi \text{ rad}$$

$$\theta'_a = w_a T = w_a/f_s = \frac{(8 \times 10^3)\pi}{20 \times 10^3} = 0.4\pi \text{ rad}$$

Step 3: Prewarp the analog frequencies to yield the desired digital frequencies.

$$w'_C = \tan\left(\theta'_C/2\right) = \tan(0.1\pi) = 0.325 \text{ rad}/s$$

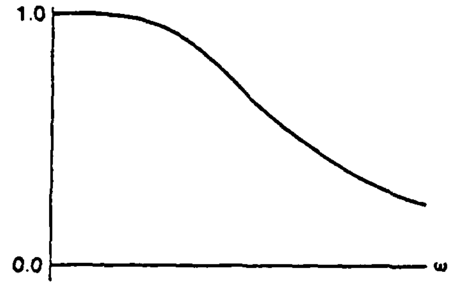
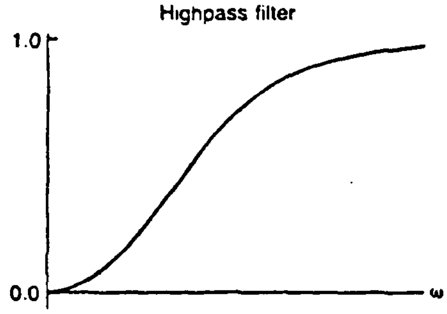
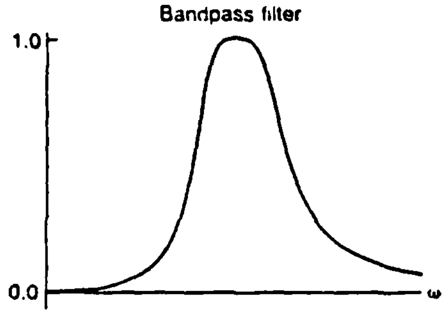
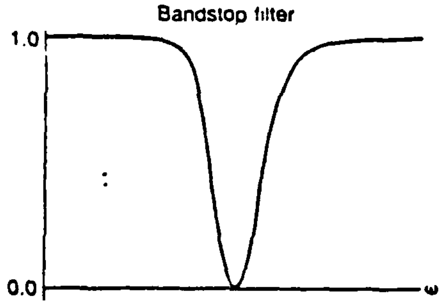
$$w'_a = \tan\left(\theta'_a/2\right) = \tan(0.2\pi) = 0.726 \text{ rad}/s$$

Note: Prewarping is done to ensure the analog filter will ultimately yield a digital filter with the correct critical frequencies. This is best visualized in the following graph (Figure 2.1) of the relationship between the analog filter frequencies and the desired digital frequencies, i.e., $w'_n = \tan(\theta_n/2)$.

TABLE 2.1

ANALOG FILTER FREQUENCY TRANSFORMATIONS

(after Ref. [1])

Filter Type	ω-form to find ω_{LP_p}	s-form (to find $H(s)$ from $H_{LP_p}(s)$ replace s in prototype with)
Lowpass filter 	$\omega_{LP_p} = \frac{\omega_{LP}}{\omega_c}$	$\frac{s}{\omega_c}$
Highpass filter 	$\omega_{LP_p} = \frac{\omega_c}{\omega_{HP}}$	$\frac{\omega_c}{s}$
Bandpass filter 	$\omega_{LP_p} = \frac{\omega_{BP}^2 - \omega_0^2}{B\omega_{BP}}$	$\frac{s^2 + \omega_0^2}{Bs}$
Bandstop filter 	$\omega_{LP_p} = \frac{B\omega_{BS}}{-\omega_{BS}^2 + \omega_0^2}$	$\frac{Bs}{s^2 + \omega_0^2}$

Normalizing to the lowpass prototype, according to Table 2.1:

$$w'_c = 0.325 \longrightarrow w_{c_p} = 1 \text{ rad}$$

$$w'_a = 0.726 \longrightarrow w_{a_p} = 0.726/0.325 = 2.234 \text{ rad}$$

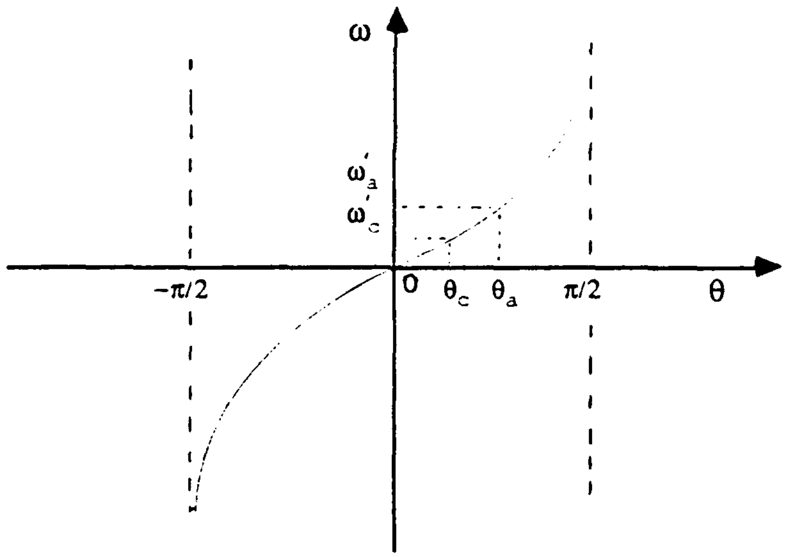


Figure 2.1. Relationship Between Analog and Digital Filter Frequencies

Note: Normalizing is done to transform the prewarped critical frequencies of the analog filter to their relative equivalents in a lowpass prototype filter with a critical frequency of $w_{c_p} = 1$. This is done to enable the designer to take advantage of previously compiled tables or frequency response curves corresponding to the transfer functions of these lowpass prototypes. Thus, a prototype filter can be selected whose frequency response characteristics have the same shape as the filter that is being designed. The transfer function for the selected prototype is then converted to a transfer function that exhibits the desired frequency response characteristics using an appropriate substitution (Step 5). Figure 2.2, depicts the process of normalizing.

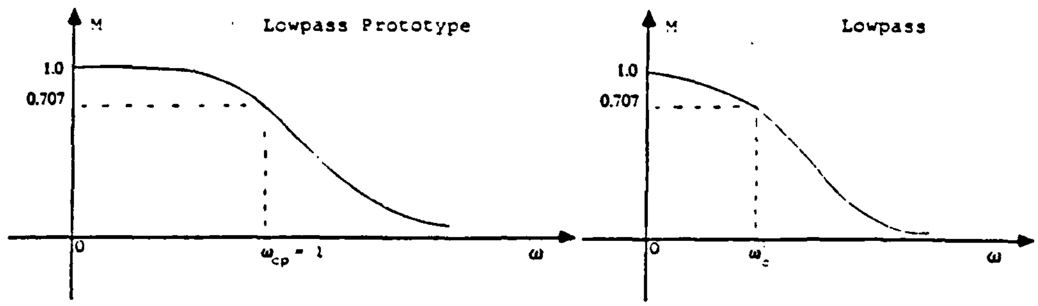


Figure 2.2. Normalization

<u>Step 4</u>: Find the order N of the lowpass Butterworth filter prototype. The order N to provide a gain of M_{dB} at an angular frequency w_a is

$$\begin{aligned} N &= \frac{\log_{10}(10^{(-M_{\text{dB}}/10)} - 1)}{2\log_{10} w_a} \quad ; M_{\text{dB}} = -10\text{dB} \\ &= \frac{\log_{10}(10^1 - 1)}{2\log_{10}\ 2.234} \qquad (2.4) \\ &= 1.37 \text{ (Since } N \text{ has to be an integer choose, } N = 2) \end{aligned}$$

<u>Step 5</u>: From Table 2.2 determine the transfer function for the lowpass filter.

$$\begin{aligned} H_{LP}(s) &= H_{LP_P}(s)\Big|_{s=\frac{s}{w_C'}} = H_{LP_P}(s)\Big|_{s=\frac{s}{0.325}} \\ &= \frac{1}{s^2 + \sqrt{2}s + 1}\Big|_{s=\frac{s}{0.325}} \qquad (2.5) \\ &= \frac{0.106}{s^2 + 0.460s + 0.106} \end{aligned}$$

<u>Step 6</u>: Determine the transfer function for the digital version of the filter using the bilinear transformation.

$$\begin{aligned} H_{LP}(z) &= H_{LP}(s)\Big|_{s=\frac{z-1}{z+1}} \qquad (2.6) \\ &= \frac{z^2 + 2z + 1}{14.84z^2 - 17.0z + 6.14} = \frac{0.068(z+1)^2}{z^2 - 1.14z + 0.413} \end{aligned}$$

TABLE 2.2

BUTTERWORTH PROTOTYPE COEFFICIENTS

(Table after to Ref. 4)

N	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
1	1.0000							
2	1.4141	1.0000						
3	2.0000	2.0000	1.0000					
4	2.6131	3.4142	2.6131	1.0000				
5	3.2361	5.2361	5.2361	3.2361	1.0000			
6	3.8637	7.4641	9.1416	7.4641	3.8637	1.0000		
7	4.4940	10.0978	14.5918	14.5918	10.0978	4.4940	1.0000	
8	5.1258	13.1371	21.8462	25.6884	21.8462	13.1371	5.1258	1.0000

$$H_{LP_P}(s) = \frac{1}{1 + a_1 s + a_2 s^2 + \ldots + a_N s^N}$$

Step 7: Obtain a plot of the digital filter frequency response to see if the design specifications have been met.

Figure 2.3 shows that the design does indeed comply with the given filter specifications, in that the cutoff frequency is $0.628 = 0.2\pi$ rad and the stopband frequency is $1.257 = 0.4\pi$ rad with gains of -3 dB and -14 dB, respectively.

Example 2.2 Chebyshev Filter

Design a digital bandpass filter with the following specifications:

- 1 dB ripple in the frequency band 600 to 900 Hz,
- sampling frequency of 3 kHz,
- maximum gain of -40 dB for $0 < f < 200$ Hz.

Step 1: Convert the critical analog design frequencies, w_i, to the corresponding digital frequencies, θ_i.

Sampling frequency: $f_s = 3$ kHz

Lower ripple passband frequency: $f_\ell = 600 \text{ Hz} \Longrightarrow w_\ell = 1200\pi \text{ rad} / s$

Upper ripple passband frequency: $f_u = 900 \text{ Hz} \Longrightarrow w_u = 1800\pi \text{ rad} / s$

Stopband frequency: $f_a = 200 \text{ Hz} \Longrightarrow w_a = 400\pi \text{ rad} /s$

$$\theta'_\ell = w_\ell T = w_\ell / f_s = \frac{1200\pi}{3000} = 0.4\pi = 1.26 \text{ rad}$$
$$\theta'_u = w_u T = w_u / f_s = \frac{1800\pi}{3000} = 0.6\pi = 1.88 \text{ rad}$$
$$\theta'_a = w_a T = w_a / f_s = \frac{400\pi}{3000} = 0.133\pi = 1.54 \text{ rad}$$

Ripple band center frequency : $\theta'_0 = \sqrt{\theta'_\ell \theta'_u} = 0.49\pi = 0.418$ rad

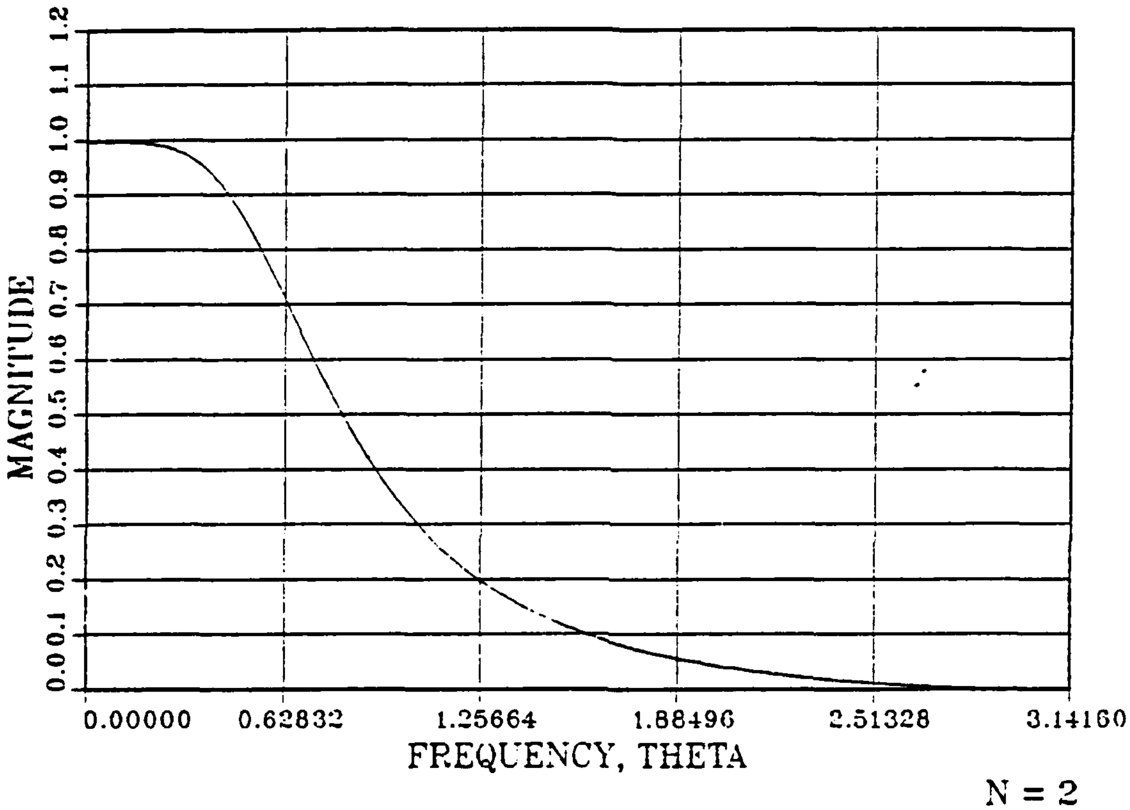


Figure 2.3. Frequency Response for Butterworth Lowpass Filter

Step 2: Prewarp the analog frequencies so that the desired digital frequency characteristics will be achieved.

$$w'_\ell = \tan(\theta'_\ell/2) = \tan(1.26/2) = 0.729 \text{ rad}/s$$

$$w'_u = \tan(\theta'_u/2) = \tan(1.88/2) = 1.369 \text{ rad}/s$$

$$w'_a = \tan(\theta'_a/2) = \tan(0.418/2) = 0.212 \text{ rad}/s$$

$$w'_0 = \sqrt{w'_\ell w'_u} = \sqrt{(0.729)(1.369)} = 1.00 \text{ rad}/s$$

Ripple band is: $B' = w'_u - w'_\ell = 1.369 - 0.729 = 0.64$ rad/s

From Table 2.1 convert the bandpass filter frequencies to the corresponding lowpass prototype filter frequencies.

$$\begin{aligned} w_{LP_P} &= \frac{w'^2_{BP} - w'^2_0}{B' w'_{BP}} \\ &= \frac{w'^2_{BP} - (1.00)^2}{(0.64) w'_{BP}} \end{aligned} \tag{2.7}$$

	w'_{BP}	w_{LP_P}
w'_ℓ	0.729	−1
w'_u	1.369	1
w'_a	0.212	−7.04
w'_0	1.000	0

Step 3: Determine the order of the Chebyshev prototype filter that meets the design requirements.

For a lowpass Chebyshev filter the magnitude - squared characteristic is given by:

$$\left|H_{LP_P}(jw)\right|^2 = \frac{1}{1 + \epsilon^2 C_N^2(w)} \tag{2.8}$$

where $C_N(w)$ is an N^{th}- order Chebyshev polynomial and ϵ indicates the degree of ripple.

In this example the −40 dB gain translates to $M = 10^{-2}$ for the desired filter frequency response in the stopband, i.e., we want $|H(jw)| < 10^{-2}$ or

$\left|H(jw)\right|^2 < 10^{-4}$. The design specification for 1 dB ripple in the ripple band corresponds to a value for ϵ of 0.2589.

Making an initial guess for the filter order of $N = 2$ yields:

$$\begin{aligned}\left|H_2(jw)\right|^2 &= \frac{1}{1 + 0.2589\,(2w_a^2 - 1)^2}\bigg|_{w_a=7.04} \\ &= \frac{1}{1 + 0.2589\,(99.12 - 1)^2} = 4.01 \times 10^{-4}\end{aligned} \tag{2.9}$$

which is not less than 10^{-4}.

Proceeding further with $N = 3$ yields:

$$\begin{aligned}\left|H_3(jw)\right|^3 &= \frac{1}{1 + 0.2589\,(4w_a^3 - 3w_a)^2}\bigg|_{w_a=7.04} \\ &= \frac{1}{1 + 0.2589\,(1395.65 - 21.12)^2} = 2.044 \times 10^{-6}\end{aligned} \tag{2.10}$$

which is less than 10^{-4}. Therefore, the design needs can be met using a third-order Chebyshev lowpass prototype filter.

Step 4: Obtain the transfer function of the third order lowpass prototype filter with 1 dB ripple. Looking at Table 2.3, yields the following prototype transfer function:

$$H_{LP_P}(s) = \frac{0.491}{s^3 + 0.988s^2 + 1.238s + 0.491} \tag{2.11}$$

Since this is an odd order filter, the constant 0.491 in the numerator was selected to make $\left|H(j0)\right| = 1$. Recall that for Chebyshev filter design, the constant K in the numerator of the lowpass prototype filter is selected on the following basis:

$$\begin{aligned}&\text{for } N \text{ odd;} \quad \left|H(j0)\right| = 1 \\ &\text{for } N \text{ even;} \quad \left|H(j0)\right| = \frac{1}{[1 + \epsilon^2]^{1/2}}\end{aligned} \tag{2.12}$$

TABLE 2.3

CHEBYSHEV - PROTOTYPE DENOMINATOR POLYNOMIALS

(Table due to Reference 4)

0.5-dB Ripple Chebvyshev Filter ($\epsilon = 0.3493$)

n	
1	$s + 2.863$
2	$s^2 + 1.425s + 1.516$
3	$s^3 + 1.253s^2 + 1.535s + 0.716 = (s + 0.626)(s^2 + 0.626s + 1.142)$
4	$s^4 + 1.197s^3 + 1.717s^2 + 1.025s + 0.379 = (s^2 + 0.351s + 1.064)(s^2 + 0.845s + 0.356)$
5	$s^5 + 1.17251s^4 + 1.9374s^3 + 1.3096s^2 + 0.7525s + 0.1789$ $(s + 0.3623)[(s + 0.1120)^2 + 1.0116^2][(s + 0.2931)^2 + 0.6252^2]$
6	$s^6 + 1.1592s^5 + 2.1718s^4 + 1.5898s^3 + 1.1719s^2 + 0.4324s + 0.0948$ $[(s + 0.0777)^2 + 1.0085^2][(s + 0.2121)^2 + 0.7382^2][(s + 0.2898)^2 + 0.2702^2]$
7	$s^7 + 1.1512s^6 + 2.4126s^5 + 1.8694s^4 + 1.6479s^3 + 0.7556s^2 + 0.2821s + 0.0447$ $(s + 0.2562)[(s + 0.0570)^2 + 1.0064^2][(s + 0.1597)^2 + 0.8001^2][(s + 0.2308)^2 +$ $0.4479^2]$
8	$s^8 + 1.1461s^7 + 2.6567s^6 + 2.1492s^5 + 2.1840s^4 + 1.1486s^3 + 0.5736s^2 +$ $0.1525s + 0.0237$ $[(s + 0.0436)^2 + 1.0050^2][(s + 0.1242)^2 + 0.8520^2][(s + 0.1859)^2 + 0.5693^2]$ $[(s + 0.2193)^2 + 0.1999^2]$
9	$s^9 + 1.1426s^8 + 2.9027s^7 + 2.4293s^6 + 2.7815s^5 + 1.6114s^4 + 0.9836s^3 +$ $0.3408s^2 + 0.0941s + 0.0112$ $(s + 0.1984)[(s + 0.0345)^2 + 1.0040^2][(s + 0.0992)^2 + 0.8829^2][(s + 0.1520)^2 +$ $0.6553^2][(s + 0.1864)^2 + 0.3487^2]$
10	$s^{10} + 1.1401s^9 + 3.1499s^8 + 2.7097s^7 + 3.4409s^6 + 2.1442s^5 + 1.5274s^4 +$ $0.6270s^3 + 0.2373s^2 + 0.0493s + 0.0059$ $[(s + 0.0279)^2 + 1.0033^2][(s + 0.0810)^2 + 0.9051^2][(s + 0.1261)^2 + 0.7183^2]$ $[(s + 0.1589)^2 + 0.4612^2][(s + 0.1761)^2 + 0.1589^2]$

TABLE 2.3 (Continued)

CHEBYSHEV - PROTOTYPE DENOMINATOR POLYNOMIALS

1.0-dB Ripple Chebyshev Filter ($\epsilon = 0.5089$)

n	Polynomial	Factored form
1	$s + 1.308$	
2	$s^2 + 0.804s + 0.637$	
3	$s^3 + 0.738s^2 + 1.022s + 0.327$	$= (s + 0.402)(s^2 + 0.369s + 0.886)$
4	$s^4 + 0.716s^3 + 1.256s^2 + 0.517s + 0.206$	$= (s^2 + 0.210s + 0.928)(s^2 + 0.506s + 0.221)$
5	$s^5 + 0.7065s^4 + 1.4995s^3 + 0.6935s^2 + 0.4593s + 0.0817$	$(s + 0.2183)[(s + 0.0675)^2 + 0.9735^2][(s + 0.1766)^2 + 0.6016^2]$
6	$s^6 + 0.7012s^5 + 1.7459s^4 + 0.8670s^3 + 0.7715s^2 + 0.2103s + 0.0514$	$[(s + 0.0470)^2 + 0.9817^2][(s + 0.1283)^2 + 0.7187^2][(s + 0.1753)^2 + 0.2630^2]$
7	$s^7 + 0.6979s^6 + 1.9935s^5 + 1.0392s^4 + 1.1444s^3 + 0.3825s^2 + 0.16661s + 0.0204$	$(s + 0.1553)[(s + 0.0346)^2 + 0.9867^2][(s + 0.0968)^2 + 0.7912^2][(s + 0.1399)^2 + 0.4391^2]$
8	$s^8 + 0.6961s^7 + 2.2423s^6 + 1.2117s^5 + 1.5796s^4 + 0.5982s^3 + 0.3587s^2 + 0.0729s + 0.0129$	$[(s + 0.0265)^2 + 0.9898^2][(s + 0.0754)^2 + 0.8391^2][(s + 0.1129)^2 + 0.5607^2][(s + 0.1332)^2 + 0.1969^2]$
9	$s^9 + 0.6947s^8 + 2.4913s^7 + 1.3837s^6 + 2.0767s^5 + 0.8569s^4 + 0.6445s^3 + 0.16844s^2 + 0.0544s + 0.0051$	$(s + 0.1206)[(s + 0.0209)^2 + 0.9919^2][(s + 0.0603)^2 + 0.8723^2][(s + 0.0924)^2 + 0.6474^2][(s + 0.1134)^2 + 0.3445^2]$
10	$s^{10} + 0.6937s^9 + 2.7406s^8 + 1.5557s^7 + 2.6363s^6 + 1.1585s^5 + 1.0389s^4 + 0.3178s^3 + 0.1440s^2 + 0.0233s + 0.0032$	$[(s + 0.0170)^2 + 0.9935^2][(s + 0.0767)^2 + 0.7113^2][(s + 0.0493)^2 + 0.8962^2][(s + 0.0967)^2 + 0.4567^2][(s + 0.1072)^2 + 0.1574^2]$

2-dB Ripple Chebyshev Filter ($\epsilon = 0.7648$)

n	Polynomial	Factored form
1	$s + 1.965$	
2	$s^2 + 1.098s + 1.103$	
3	$s^3 + 0.988s^2 + 1.238s + 0.491$	$= (s + 0.494)(s^2 + 0.490s + 0.994)$
4	$s^4 + 0.953s^3 + 1.454s^2 + 0.743s + 0.276$	$= (s^2 + 0.279s + 0.987)(s^2 + 0.674s + 0.279)$
5	$s^5 + 0.9368s^4 + 1.6888s^3 + 0.9744s^2 + 0.5805s + 0.1228$	$(s + 0.2895)[(s + 0.0895)^2 + 0.9901^2][(s + 0.2342)^2 + 0.6119^2]$
6	$s^6 + 0.9282s^5 + 1.9308s^4 + 1.2021s^3 + 0.9393s^2 + 0.3071s + 0.0689$	$[(s + 0.0622)^2 + 0.9934^2][(s + 0.1699)^2 + 0.7272^2][(s + 0.2321)^2 + 0.2662^2]$
7	$s^7 + 0.9231s^6 + 2.1761s^5 + 1.4288s^4 + 1.3575s^3 + 0.5486s^2 + 0.2137s + 0.0307$	$(s + 0.2054)[(s + 0.0457)^2 + 0.9953^2][(s + 0.1281)^2 + 0.7982^2][(s + 0.1851)^2 + 0.4429^2]$
8	$s^8 + 0.9198s^7 + 2.4230s^6 + 1.6552s^5 + 1.8369s^4 + 0.8468s^3 + 0.4478s^2 + 0.1073s + 0.0172$	$[(s + 0.0350)^2 + 0.9965^2][(s + 0.0997)^2 + 0.8448^2][(s + 0.1492)^2 + 0.5644^2][(s + 0.1759)^2 + 0.1982^2]$
9	$s^9 + 0.9175s^8 + 2.6709s^7 + 1.8815s^6 + 2.3781s^5 + 1.2016s^4 + 0.7863s^3 + 0.2442s^2 + 0.0706s + 0.0077$	$(s + 0.1593)[(s + 0.0277)^2 + 0.9972^2][(s + 0.0797)^2 + 0.8769^2][(s + 0.1221)^2 + 0.6509^2][(s + 0.1497)^2 + 0.3463^2]$
10	$s^{10} + 0.9159s^9 + 2.9195s^8 + 2.1079s^7 + 2.9815s^6 + 1.6830s^5 + 1.2445s^4 + 0.4554s^3 + 0.1825s^2 + 0.0345s + 0.0043$	$[(s + 0.0224)^2 + 0.9978^2][(s + 0.1013)^2 + 0.7143^2][(s + 0.0651)^2 + 0.9001^2][(s + 0.1277)^2 + 0.4586^2][(s + 0.1415)^2 + 0.1580^2]$

Step 5: Convert the lowpass filter prototype to a bandpass filter prototype.

From Table 2.1:

$$
\begin{aligned}
H_{BP}(s) &= H_{LP_P}(s)\Big|_{s=\frac{s^2+w_0^2}{Bs}} \\
&= \frac{0.491}{\left(\frac{s^2+1}{0.64s}\right)^3 + 0.988\left(\frac{s^2+1}{0.64s}\right)^2 + 1.238\left(\frac{s^2+1}{0.64s}\right) + 0.491} \\
&= \frac{0.129s^3}{s^6 + 0.632s^5 + 3.507s^4 + 1.394s^3 + 3.507s^2 + 0.6325 + 1}
\end{aligned}
\tag{2.13}
$$

Step 6: Determine the digital version of the transfer function using the bilinear transformation.

$$
\begin{aligned}
H_{BP}(z) &= H_{BP}(s)\Big|_{s=\frac{z-1}{z+1}} \\
&= \frac{0.129\left(\frac{z-1}{z+1}\right)^3}{\left(\frac{z-1}{z+1}\right)^6 + 0.632\left(\frac{z-1}{z+1}\right)^5 + 3.507\left(\frac{z-1}{z+1}\right)^4 + 1.394\left(\frac{z-1}{z+1}\right)^3 + 3.507\left(\frac{z-1}{z+1}\right)^2 + 1.238\left(\frac{z-1}{z+1}\right) + 0.491} \\
&= \frac{0.011\left(z^6 - 3z^4 + 3z^2 - 1\right)}{z^6 + 2.153z^4 + 1.786z^2 + 0.545}
\end{aligned}
\tag{2.14}
$$

Step 7: Confirm that the filter design meets specifications by obtaining a frequency response plot. Figure 2.4 indicates the specifications for a passband of 1.26 to 1.88 rad with 1 dB ripple have been met.

Example 2.3 Elliptic Filter

A lowpass elliptic filter with the following specifications is desired:

- passband ripple of 0.5 dB,
- passband ripple-edge frequency of 2 kHz,
- stopband gain should be at most −20 dB for frequencies greater than 4 kHz, and
- sampling frequency is 20 kHz.

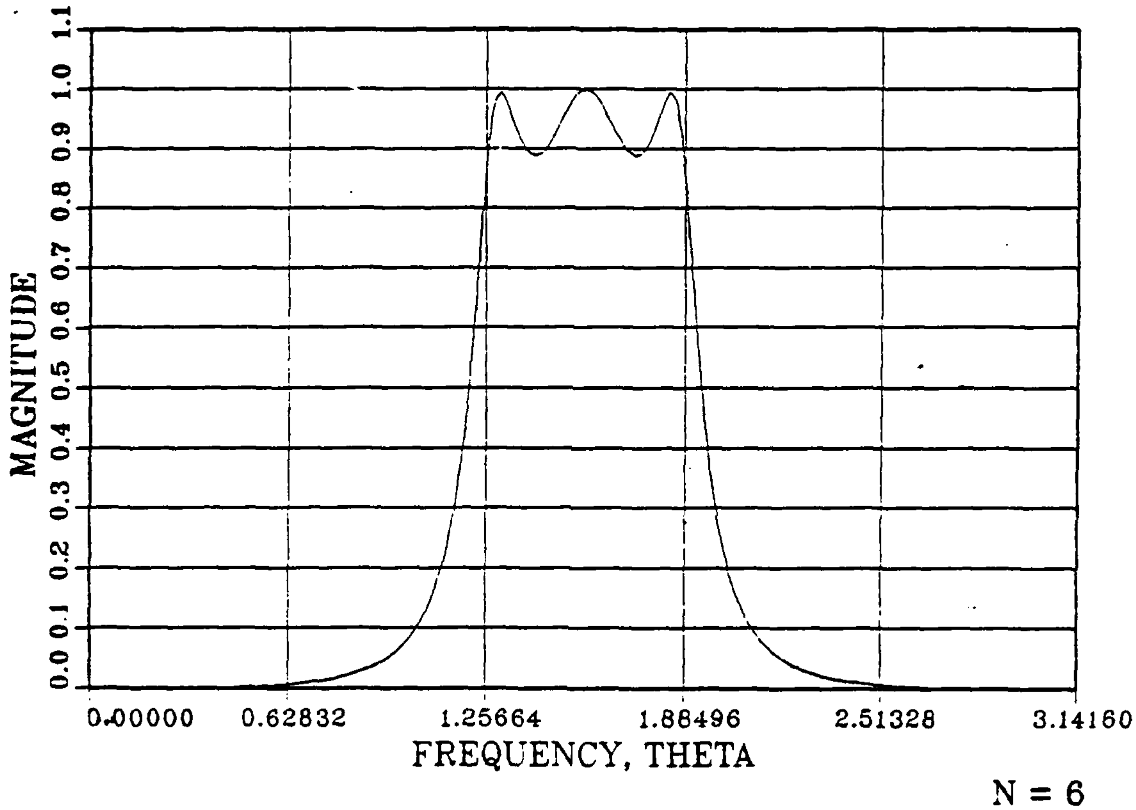


Figure 2.4. Frequency Response for Chebyshev Bandpass Filter

Step 1: Convert the critical analog design frequencies to digital frequencies.

Sampling frequency: $f_s = 20$ kHz

Passband ripple-edge frequency: $f_1 = 2$ kHz $\Longrightarrow w_1 = 4 \times 10^3 \pi$ rad $/s$

Stopband ripple-edge frequency: $f_2 = 4$ kHz $\Longrightarrow w_2 = 8 \times 10^3 \pi$ rad $/s$

$$\theta_1' = w_1 T = w_1 / f_s = \frac{\left(4 \times 10^3\right) \pi}{20 \times 10^3} = 0.2\pi = 0.628 \text{ rad}$$
$$\theta_2' = w_2 T = w_2 / f_s = \frac{\left(8 \times 10^3\right) \pi}{20 \times 10^3} = 0.6\pi = 1.88 \text{ rad}$$

Step 2: Prewarp the analog frequencies in order to determine the appropriate lowpass prototype filter.

$$w_1' = \tan\left(\theta_1'/2\right) = 0.325 \text{ rad}/s$$

$$w_2' = \tan\left(\theta_2'/2\right) = 1.376 \text{ rad}/s$$

Step 3: Determine the order of the elliptic prototype filter that meets the design specifications. Find the ratio, R, of the stopband frequency, w_2', and the passband frequency, w_1'.

$$R = w_2'/w_1' = \frac{1.376}{0.325} = 4.234 \tag{2.15}$$

From Table 2.4b determine the prototype filter order and the resulting transfer function. Looking at Table 2.4, it can be seen that a filter of order $N = 2$ has a value for R of 2.76261, which is more than sufficient to meet the design specifications.

Step 4: Obtain the transfer function of the second-order lowpass prototype filter. Again, looking at Table 2.4, the prototype transfer function is determined to be:

$$H_{LP_P}(s) = \frac{0.1s^2 + 0.5338}{s^2 + 0.8094s + 0.5667} \tag{2.16}$$

TABLE 2.4a

ELLIPTIC FILTERS

Generalized Transfer Functions

(due to reference 5)

$\underline{N=2}$

$$H_2(s) = H_0 \frac{s^2 + A_{01}}{s^2 + B_{11_s} + B_{01}} = \frac{H_0 s^2 + H_0 A_{01}}{s^2 + B_{11} s + B_{01}}$$

$\underline{N=3}$

$$H_3(s) = \frac{H_0}{s + s_0} \cdot \frac{s^2 + A_{01}}{s^2 + B_{11} s + B_{01}}$$

$$H_3(s) = \frac{H_0 s^2 + H_0 A_{01}}{s^3 + (B_{11} + s_0) s^2 + (B_{01} + B_{11} s_0) s + B_{01} s_0}$$

$\underline{N=4}$

$$H_4(s) = H_0 \cdot \frac{(s^2 + A_{01})}{s^2 + B_{11} s + B_{01}} \cdot \frac{(s^2 + A_{02})}{s^2 + B_{12} s + B_{02}}$$

$$H_4(s) = \frac{H_0 s^4 + H_0 (A_{01} + A_{02}) s^2 + H_0 A_{01} A_{02}}{s^4 + (B_{11} + B_{12}) s^3 + (B_{02} + B_{11} B_{12} + B_{01}) s^2 + (B_{11} B_{02} + B_{01} B_{12}) s + B_{01} B_{02}}$$

$\underline{N=5}$

$$H_s(s) = \frac{H_0 s^4 + H_0 (A_{01} + A_{02}) s^2 H_0 A_{01} A_{02}}{\begin{gathered} s^5 + (B_{11} + B_{12} + s_0) s^4 + [B_{02} + B_{11} B_{12} + B_{01} + B_{11} s_0 + B_{12} s_0] s^3 \\ + [B_{11} B_{02} + B_{01} B_{12} + B_{02} s_0 + B_{11} B_{12} s_0 + B_{01} s_0] s^2 \\ + [B_{01} B_{02} + B_{11} B_{02} s_0 + B_{01} B_{12} s_0] s + B_{01} B_{02} s_0 \end{gathered}}$$

TABLE 2.4b

ELLIPTIC FILTERS

(due to reference 5)

$$H_N(s) = \frac{H_0}{D_N(s)} \prod_{i=1}^{r} \frac{s^2 + A_{0i}}{s^2 + B_{1i}s + B_{0i}} \qquad D_N(s) = \begin{cases} s + s_0, & N \text{ odd} \\ 1, & N \text{ even} \end{cases} \qquad r = \left\lfloor \frac{N}{2} \right\rfloor$$

(a) Passband ripple: 0.5 dB; stopband gains. −20, −30, −40, −50, −60, −70 dB

Passband ripple = 0.5 dB; stopband gain = −20 dB

N	i	A_{0i}	B_{0i}	B_{1i}	H_0/s_0	Ω_r
2	1	5.33789	0.566660	0.809390	0.100220E+000	2.76261
3	1	1.75640	0.808321	0.359160	0.306214E+000	1.42189
					0.667292	
4	1	4.38105	0.611195	0.931959	0.100219E+000	1.13188
	2	1.21841	0.927132	0.136543		
5	1	1.65076	0.827787	0.412816	0.303895E+000	1.04465
	2	1.07211	0.973640	0.049395	0.667292	
6	1	4.36790	0.611899	0.933855	0.100218E+000	1.01553
	2	1.19243	0.934830	0.156221		
	3	1.02486	0.990620	0.017576		
7	1	1.64918	0.828092	0.413652	0.303861E+000	1.00545
	2	1.06401	0.976479	0.056384	0.667292	
	3	1.00870	0.996681	0.006219		
8	1	4.36811	0.611846	0.933864	0.100192E+000	1.00192
	2	1.19207	0.934928	0.156548		
	3	1.02213	0.991634	0.020051		
	4	1.00306	0.998827	0.002197		
9	1	1.64927	0.828047	0.413695	0.303786E+000	1.00068
	2	1.06390	0.976512	0.056505	0.667292	
	3	1.00775	0.997041	0.007093		
	4	1.00108	0.999586	0.000775		

Passband ripple = 0.5 dB; stopband gain = −30 dB

N	i	A_{0i}	B_{0i}	B_{1i}	H_0/s_0	Ω_r
2	1	9.51248	0.318702	0.639007	0.316294E−001	4.80880
3	1	2.46997	0.597384	0.382044	0.121878E+000	1.92322
					0.511761	
4	1	6.46603	0.398996	0.822201	0.316297E−001	1.32446
	2	1.47114	0.798764	0.191032		
5	1	2.14490	0.648724	0.480774	0.118807E+000	1.12912
	2	1.18132	0.907216	0.088080	0.511761	
6	1	6.38228	0.402050	0.828822	0.316296E−001	1.05394
	2	1.38680	0.826821	0.237025		
	3	1.07474	0.958727	0.039181		
7	1	2.13439	0.650591	0.484325	0.118701E+000	1.02299
	2	1.15171	0.920785	0.108419	0.511761	
	3	1.03168	0.981925	0.017159		
8	1	6.37941	0.402154	0.829052	0.316289E−001	1.00989
	2	1.38394	0.827819	0.238663		
	3	1.06301	0.964898	0.048043		
	4	1.01359	0.992137	0.007464		
9	1	2.13409	0.650636	0.484454	0.118683E+000	1.00427
	2	1.15070	0.921256	0.109144	0.511761	
	3	1.02680	0.984652	0.021005		
	4	1.00586	0.996589	0.003238		

TABLE 2.4c

ELLIPTIC FILTERS

(due to reference 5)

Passband ripple 1 dB; stopband gains − 20, − 30, − 40, − 50, − 60, − 70 dB

		Passband ripple = 1 dB; stopband gain = − 20 dB				
N	i	A_{0i}	B_{0i}	B_{1i}	H_0/s_0	Ω_s
2	1	4.42342	0.497233	0.676727	0.100185E + 000	2.32474
3	1	1.58565	0.790229	0.282927	0.281080E + 000 0.565168	1.30797
4	1	3.81475	0.536633	0.768217	0.100185E + 000	1.09029
	2	1.15956	0.926578	0.099029		
5	1	1.51852	0.808049	0.318242	0.279829E + 000 0.565168	1.02826
	2	1.04886	0.975703	0.032771		
6	1	3.80873	0.537071	0.769217	0.100184E + 000	1.00902
	2	1.14367	0.933194	0.110761		
	3	1.01550	0.992101	0.010654		
7	1	1.51782	0.808242	0.318627	0.279814E + 000 0.565168	1.00290
	2	1.04421	0.977941	0.036573		
	3	1.00497	0.997446	0.003444		
8	1	3.80866	0.537076	0.769228	0.100185E + 000	1.00093
	2	1.14350	0.933266	0.110888		
	3	1.01405	0.992834	0.011881		
	4	1.00160	0.999176	0.001111		
9	1	1.51812	0.808099	0.318714	0.279611E + 000 0.565168	1.00030
	2	1.04421	0.977936	0.036644		
	3	1.00451	0.997680	0.003845		
	4	1.00052	0.999734	0.000359		

		Passband ripple = 1 dB, stopband gain = − 30 dB				
N	i	A_{0i}	B_{0i}	B_{1i}	H_0/s_0	Ω_s
2	1	7.88158	0.279699	0.538632	0.316284E − 001	4.00423
3	1	2.20293	0.586596	0.311981	0.113223E + 000 0.430700	1.73254
4	1	5.72672	0.348[illegible]11	0.682880	0.316286E − 001	1.25040
	2	1.37628	0.803477	0.148347		
5	1	1.96934	0.633529	0.383990	0.111180E + 000 0.430700	1.09554
	2	1.13918	0.914405	0.064612		
6	1	5.67743	0.350184	0.687075	0.316278E-001	1.03799
	2	1.31626	0.828467	0.179733		
	3	1.05468	0.964084	0.027111		
7	1	1.96322	0.634884	0.386055	0.111123E + 000 0.430700	1.01536
	2	1.11870	0.925872	0.077672		
	3	1.02200	0.985162	0.011201		
8	1	5.67614	0.350237	0.687189	0.316265E − 001	1.00625
	2	1.31463	0.829171	0.180623		
	3	1.04690	0.969003	0.032478		
	4	1.00893	0.993908	0.004598		
9	1	1.96311	0.634904	0.386117	0.111109E + 000 0.430700	1.00255
	2	1.11814	0.926187	0.078043		
	3	1.01891	0.987211	0.013399		
	4	1.00364	0.997505	0.001883		

TABLE 2.4d

ELLIPTIC FILTERS

(due to reference 5)

Passband ripple. 2 dB. stopband gains: −20, −30, −40, −50, −60, −70 dB

Passband ripple = 2 dB: stopband gain = −20 dB

N	i	A_{0i}	B_{0i}	B_{1i}	H_0/s_0	Ω_r
2	1	3.60961	0.454891	0.537326	0.100103E+000	1.94332
3	1	1.42939	0.793180	0.204089	0.254443E+000	1.20808
					0.458898	
4	1	3.25882	0.486218	0.597266	0.100102E+000	1.05569
	2	1.10765	0.935564	0.063585		
5	1	1.39116	0.807316	0.223995	0.253878E+000	1.01567
	2	1.02976	0.981070	0.018680	0.458898	
6	1	3.25657	0.486437	0.597679	0.100102E+000	1.00447
	2	1.09913	0.940286	0.069417		
	3	1.00845	0.994532	0.005396		
7	1	1.39096	0.807384	0.224146	0.253839E+000	1.00128
	2	1.02749	0.982483	0.020362	0.458898	
	3	1.00242	0.998428	0.001551		
8	1	3.25736	0.486314	0.597662	0.100040E+000	1.00037
	2	1.09913	0.940275	0.069488		
	3	1.00782	0.994938	0.005883		
	4	1.00069	0.999548	0.000446		
9	1	1.39157	0.807065	0.224311	0.253438E+000	1.00011
	2	1.02755	0.982437	0.020419	0.458898	
	3	1.00224	0.998540	0.001696		
	4	1.00020	0.999870	0.000129		

Passband ripple = 2 dB: stopband gain = −30 dB

N	i	A_{0i}	B_{0i}	B_{1i}	H_0/s_0	Ω_r
2	1	6.42917	0.255975	0.433953	0.316259E−001	3.29235
3		1.95290	0.593773	0.238474	0.104183E+000	1.55690
					0.345928	
4	1	4.99348	0.312154	0.537291	0.316258E−001	1.18280
	2	1.28752	0.820026	0.105666		
5	1	1.79533	0.634366	0.285641	0.102947E+000	1.06594
	2	1.10083	0.927270	0.042702	0.345928	
6	1	4.96765	0.313430	0.539562	0.316259E−001	1.02460
	2	1.24776	0.840500	0.124605		
	3	1.03723	0.971673	0.016632		
7	1	1.79218	0.635240	0.286656	0.102920E+000	1.00929
	2	1.08781	0.935985	0.050009	0.345928	
	3	1.01401	0.989124	0.006386		
8	1	4.96715	0.313454	0.539608	0.316253E−001	1.00353
	2	1.24694	0.840933	0.125007		
	3	1.03254	0.975137	0.019424		
	4	1.00531	0.995847	0.002438		
9	1	1.79226	0.635210	0.286689	0.102890E+000	1.00134
	2	1.08756	0.936150	0.050173	0.345928	
	3	1.01227	0.998460	0.007452		
	4	1.00202	0.998417	0.000930		

Step 5: Find the analog transfer function of the desired filter based on the lowpass prototype.

This is done by finding a scaling factor, α, to convert the lowpass prototype's cutoff frequency, w_{1_P}, to the desired cutoff frequency, w_1.

For the prototype:

$$R = 2.76261 \quad ; \quad w_{1_P} = 1/\sqrt{R} = 0.6016$$

For the desired filter:

$$w_1 = 0.325$$

Therefore, the scaling factor, α, is:

$$\alpha = w_{1_P}/w_1 = \frac{0.6016}{0.325} = 1.85 \tag{2.17}$$

Thus, the actual transfer function, $H_{LP}(s)$, can be found as follows:

$$\begin{aligned} H_{LP}(s) = H_{LP_P}(s)\Big|_{s=\alpha s} &= H_{LP_P}(s)\Big|_{s=1.85s} \\ &= \frac{0.1s^2 + 0.5338}{s^2 + 0.8094s + 0.5667}\Bigg|_{s=1.85s} \\ &= \frac{0.1(1.85s)^2 + 0.5338}{(1.85s)^2 + 0.8094(1.85s) + 0.5667} \\ &= \frac{0.3423s^2 + 0.5338}{3.423s^2 + 1.497s + 0.5667} \end{aligned} \tag{2.19}$$

Step 6: Determine the digital version of the transfer function using bilinear transformation.

$$\begin{aligned} H_{LP}(z) &= H_{LP}(s)\Big|_{s=\frac{z-1}{z+1}} \\ &= \frac{0.3423\,(z-1/z+1)^2 + 0.5338}{3.423\,(z-1/z+1)^2 + 1.497\,(z-1/z+1) + 0.5667} \\ &= \frac{0.8761z^2 + 0.383z + 0.8761}{5.487z^2 - 5.7126z + 2.493} \\ &= \frac{0.1597z^2 + 0.0698z + 0.1597}{z^2 - 1.0411z + 0.4543} \end{aligned} \tag{2.20}$$

Step 7: Verify that the design meets the specifications from a frequency response plot (Figure 2.5).

2. Summary

This concludes the review of the use of the bilinear transformation to design recursive filters based on Butterworth, Chebyshev or elliptic analog prototypes. As can be seen, this method is very involved in terms of algebraic manipulation. The following sections will introduce the direct design technique. It will be shown that the direct design procedure reduces considerably the amount of algebraic calculations required, and eliminates the somewhat confusing procedure of prewarping.

C. DIRECT DESIGN TECHNIQUE

The direct design technique is based on the bilinear transformation in the following manner:

The tables of coefficients for analog lowpass prototype filters, i.e., Butterworth - Table 2.2, Chebyshev - Table 2.3, and elliptic -Table 2.4, were converted to z-domain versions through the use of the bilinear transformation, and can be shown to have a critical frequency of $\pi/2$. The reason, as stated in Section A, is that the analog prototype filters comprising these tables all have a critical frequency of $w_c = 1$. For the bilinear transformation, the following substitution for s is used in the analog prototype transfer function to convert it to a digital prototype transfer function:

$$s = \frac{z-1}{z+1} \quad \text{or} \quad z = \frac{1+s}{1-s} \tag{2.21}$$

An analog critical frequency of $s = jw_c = j1$ implies:

$$z = \frac{1+jw_c}{1-jw_c} = \frac{1+j}{1-j} \tag{2.22}$$

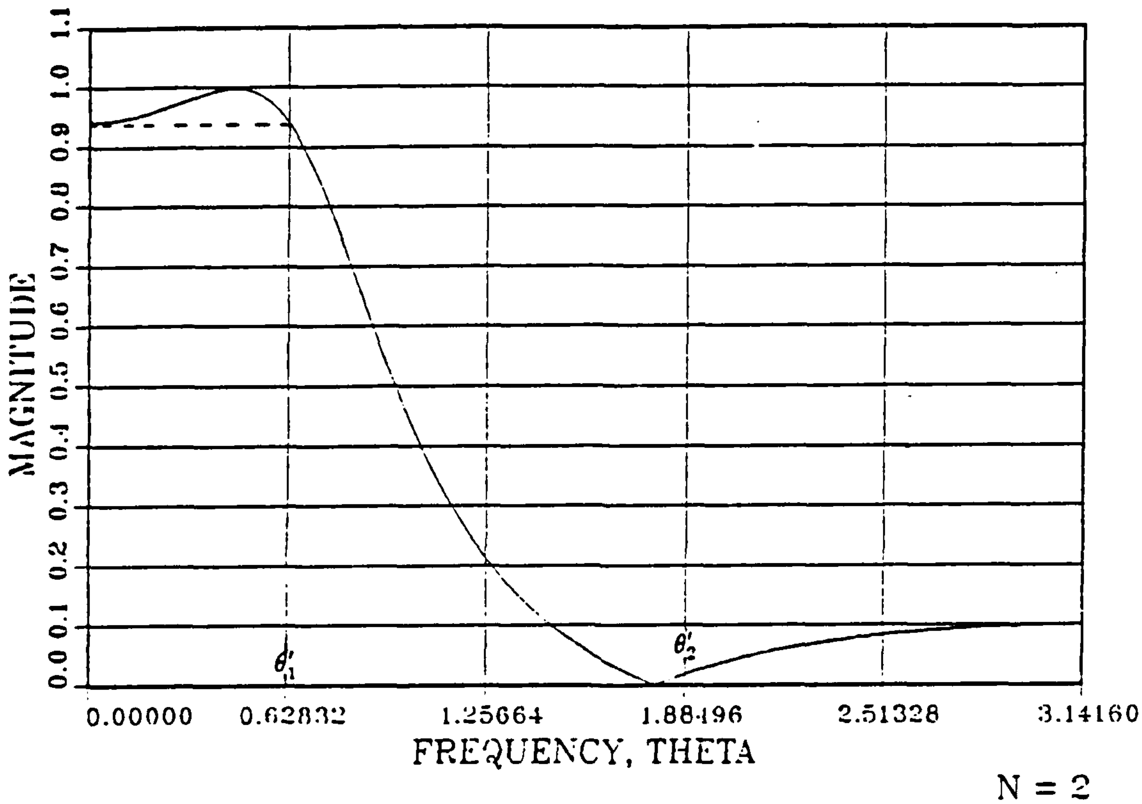


Figure 2.5. Frequency Response for Elliptic Lowpass Filter

since $z = e^{j\theta}$, this gives:

$$e^{j\theta_C} = \frac{1+j}{1-j} = 1e^{j\pi/2} \tag{2.23}$$

Thus, it can be seen that an analog critical frequency of $w_C = 1$ yields a digital critical frequency of $\theta_C = \pi/2$. Tables of prototype filters were thus created, specifically, Butterworth - Table 2.7, Chebyshev - Table 2.8 and elliptic - Table 2.9.

The use of these tables to design a digital filter based on analog specifications involves the following generalized steps. To further illustrate the technique, a detailed description of its derivation and use, including examples for the various filter types, will be presented.

1. Direct Design

Step 1: Determine the appropriate desired filter type based on the design specifications, i.e., Butterworth, Chebyshev or elliptic.

Step 2: Convert the analog filter frequency specifications to digital equivalents.

Step 3: Use the digital-digital frequency transformations of Table 2.5 to normalize the desired filter's design frequencies, so that the appropriate lowpass prototype filter may be selected.

Step 4: From the normalized design frequencies obtained in Step 3, determine the lowpass prototype that meets or exceeds these requirements.
For Butterworth and Chebyshev filters - this is done using the design curves of Figures 2.8 and Figures 2.10 - 2.12.
For elliptic filters - Table 2.9 is used.

Step 5: Obtain the actual filter transfer function from the lowpass prototype, based on the digital-digital frequency transformations of Table 2.5.

TABLE 2.5

DIGITAL FILTER FREQUENCY TRANSFORMATIONS

(Due to Reference 2)

Type	Transformation (Replace z in LP digital prototype with)	Design Constants
1. Lowpass	$\frac{z-\alpha}{1-\alpha z}$	$\alpha = \frac{\sin(\theta_C/2-\theta'_C/2)}{\sin(\theta_C/2+\theta'_C/2)}$
2. Highpass	$-\frac{z-\alpha}{1-\alpha z}$	$\alpha = \frac{\cos(\theta_C/2-\theta'_C/2)}{\cos(\theta_C/2+\theta'_C/2)}$
3. Bandpass	$-\frac{z^2-\frac{2\alpha k}{k+1}z+\frac{k-1}{k+1}}{1-\frac{2\alpha k}{k+1}z+\frac{k-1}{k+1}z^2}$	$\alpha = \frac{\cos(\theta'_u/2+\theta'_\ell/2)}{\cos(\theta'_u/2-\theta'_\ell/2)}$ $k = \tan\frac{\theta_C}{2}\cot(\theta'_u/2-\theta'_\ell/2)$
4. Bandstop	$\frac{z^2-\frac{2\alpha}{1+k}z+\frac{1-k}{1+k}}{1-\frac{2\alpha}{1+k}z+\frac{1-k}{1+k}z^2}$	$\alpha = \frac{\cos(\theta'_u/2+\theta'_\ell/2)}{\cos(\theta'_u/2-\theta'_\ell/2)}$ $k = \tan\frac{\theta_C}{2}\tan(\theta'_u/2-\theta'_\ell/2)$

As can be seen from the previous design summary, the crux of this method is the use of the digital-digital frequency transformations of Table 2.5, which merit explanation [2].

These transformations enable the user to convert the lowpass digital prototype, $H_{LP_P}(z)$ (be it Butterworth, Chebyshev or elliptic) to the actual lowpass (LP), highpass (HP), bandpass (BP), or bandstop (BS) filter transfer function.

The transformations provide a means of transferring the stability, inherent in the prototype filter, to the actual filter; that is, the poles of the actual filter will lie inside the unit circle, as they do for the prototype filter. For this reason the frequency response of the lowpass prototype filter at a specific frequency of θ_a must be the same as the desired value of the frequency response of the actual filter at its corresponding frequency of θ'_a.

$$H_{LP_P}\left(e^{j\theta_a}\right) = H_{Type}\left(e^{j\theta'_a}\right) \tag{2.24}$$

where,

$$\theta_a = \text{ prototype frequency}$$

$$\theta'_a = \text{ desired filter frequency}$$

$$Type = LP,\ HP,\ BP, \text{ or } BS$$

For example, when transforming a lowpass prototype filter to a desired lowpass filter, one wishes to maintain the integrity of the magnitude characteristic of the prototype filter, while expanding or compressing the frequency scale so that the critical frequency is changed from the prototype value of $\theta_C = \pi/2$ to the critical frequency of the actual filter θ'_C, (see Figure 2.6).

According to Table 2.5, this involves the following substitution for z in the lowpass prototype transfer function.

$$z = \frac{z - \alpha}{1 - \alpha z} \tag{2.25}$$

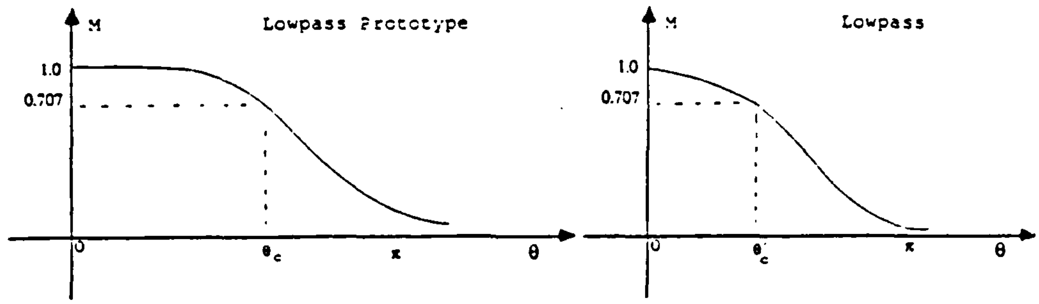


Figure 2.6. Lowpass Prototype - Lowpass Transformation

For the critical frequency of the prototype, θ_C, to map to the critical frequency of the actual filter θ'_C, the following must be true:

Substituting, $e^{j\theta_C} = z$ on the left in Equation (2.25), and $e^{j\theta'_C} = z$ on the right gives,

$$e^{j\theta_C} = \frac{e^{j\theta'_C} - \alpha}{1 - \alpha e^{j\theta'_C}} \tag{2.26}$$

and solving for the design constant, α, yields,

$$\alpha = \frac{\sin\left(\theta_C/2 - \theta'_C/2\right)}{\sin\left(\theta_C/2 + \theta'_C/2\right)}.$$

Similar arguments apply to HP, BP, and BS filters.

Thus, the transformations of Table 2.5 accomplish the appropriate mapping from the lowpass prototype digital filter to the actual filter, and ensure that stabilty is maintained.

2. Butterworth Filters - A Direct Design Approach

The design curves shown in Figures 2.8a - d, describe lowpass prototype filters with a critical frequency of $\theta_c = \pi/2$, and of order $N = 1$ through $N = 8$. Higher order filters can be realized by cascading the appropriate lower order filters characterized by these curves.

The curves were constructed by using the s-domain version of the Butterworth filter coefficients for $N = 1$ through $N = 8$, and then determining the z-domain version of these transfer functions through the use of the bilinear transformation and the conversion tables obtained from Reference 3, (see Table 2.6). These tables establish the necessary relationships between the coefficients of an analog filter and its digital counterpart.

a. Generalized Analog Transfer Function

$$H(s) = \frac{A_0 + A_1 s + A_2 s^2 + \ldots + A_k s^k}{B_0 + B_1 s + B_2 s^2 + \ldots + B_k s^k} \tag{2.27}$$

b. Generalized Digital Transfer Function

$$H(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_k z^{-k}}{1 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_k z^{-k}} \tag{2.28}$$

For example, in the case of $N = 1$ the bilinear transformation digital filter coefficients, in terms of the analog filter coefficients are:

A	$B_0 + B_1$
a_0	$(A_0 + A_1)/A$
a_1	$(A_0 - A_1)/A$
b_1	$(B_0 - B_1)/A$

Note: C is the critical frequency, $w_c = 1$.

For a first-order Butterworth prototype filter described by:

$$H(s) = \frac{1}{s + 1} \tag{2.29}$$

TABLE 2.6

BILINEAR TRANSFORMATION DIGITAL FILTER COEFFICIENTS IN TERMS OF ANALOG FILTER COEFFICIENTS

(due to reference 3)

A	B_0+B_1C
a_0	$(A_0+A_1C)/A$
a_1	$(A_0-A_1C)/A$
b_1	$(B_0-B_1C)/A$

(1st order)

A	$B_0+B_1C+B_2C^2$
a_0	$(A_0+A_1C+A_2C^2)/A$
a_1	$(2A_0-2A_2C^2)/A$
a_2	$(A_0-A_1C+A_2C^2)/A$
b_1	$(2B_0-2B_2C^2)/A$
b_2	$(B_0-B_1C+B_2C^2)/A$

(2nd order).

A	$B_0+B_1C+B_2C^2+B_3C^3$
a_0	$(A_0+A_1C+A_2C^2+A_3C^3)/A$
a_1	$(3A_0+A_1C-A_2C^2-3A_3C^3)/A$
a_2	$(3A_0-A_1C-A_2C^2+3A_3C^3)/A$
a_3	$(A_0-A_1C+A_2C^2-A_3C^3)/A$
b_1	$(3B_0+B_1C-B_2C^2-3B_3C^3)/A$
b_2	$(3B_0-B_1C-B_2C^2+3B_3C^3)/A$
b_3	$(B_0-B_1C+B_2C^2-B_3C^3)/A$

(3rd order).

A	$B_0+B_1C+B_2C^2+B_3C^3+B_4C^4$
a_0	$(A_0+A_1C+A_2C^2+A_3C^3+A_4C^4)/A$
a_1	$(4A_0+2A_1C-2A_3C^3-4A_4C^4)/A$
a_2	$(6A_0-2A_2C^2+6A_4C^4)/A$
a_3	$(4A_0-2A_1C+2A_3C^3-4A_4C^4)/A$
a_4	$(A_0-A_1C+A_2C^2-A_3C^3+A_4C^4)/A$
b_1	$(4B_0+2B_1C-2B_3C^3-4B_4C^4)/A$
b_2	$(6B_0-2B_2C^2+6B_4C^4)/A$
b_3	$(4B_0-2B_1C+2B_3C^3-4B_4C^4)/A$
b_4	$(B_0-B_1C+B_2C^2-B_3C^3+B_4C^4)/A$

(4th order)

A	$B_0+B_1C+B_2C^2+B_3C^3+B_4C^4+B_5C^5$
a_0	$(A_0+A_1C+A_2C^2+A_3C^3+A_4C^4+A_5C^5)/A$
a_1	$(5A_0+3A_1C+A_2C^2-A_3C^3-3A_4C^4-5A_5C^5)/A$
a_2	$(10A_0+2A_1C-2A_2C^2-2A_3C^3+2A_4C^4+10A_5C^5)/A$
a_3	$(10A_0-2A_1C-2A_2C^2+2A_3C^3+2A_4C^4-10A_5C^5)/A$
a_4	$(5A_0-3A_1C+A_2C^2+A_3C^3-3A_4C^4+5A_5C^5)/A$
a_5	$(A_0-A_1C+A_2C^2-A_3C^3+A_4C^4-A_5C^5)/A$
b_1	$(5B_0+3B_1C+B_2C^2-B_3C^3-3B_4C^4-5B_5C^5)/A$
b_2	$(10B_0+2B_1C-2B_2C^2-2B_3C^3+2B_4C^4+10B_5C^5)/A$
b_3	$(10B_0-2B_1C-2B_2C^2+2B_3C^3+2B_4C^4-10B_5C^5)/A$
b_4	$(5B_0-3B_1C+B_2C^2+B_3C^3-3B_4C^4+5B_5C^5)/A$
b_5	$(B_0-B_1C+B_2C^2-B_3C^3+B_4C^4-B_5C^5)/A$

(5th order)

the values for the digital filter coefficients are:

$$\begin{array}{cc} A & 1+1=2 \\ a_0 & (1+0)/2=0.5 \\ a_1 & (1-0)/2=0.5 \\ b_1 & (1-1)/2=0 \end{array}$$

The transfer function $H(z)$ is therefore:

$$H_1(z) = \frac{0.5+0.5z^{-1}}{1} \Longrightarrow H_1(z) = \frac{0.5z+0.5}{z} \qquad (2.30)$$

A plot of this transfer function is illustrated by the curve ($N=1$) in Figure 2.7.

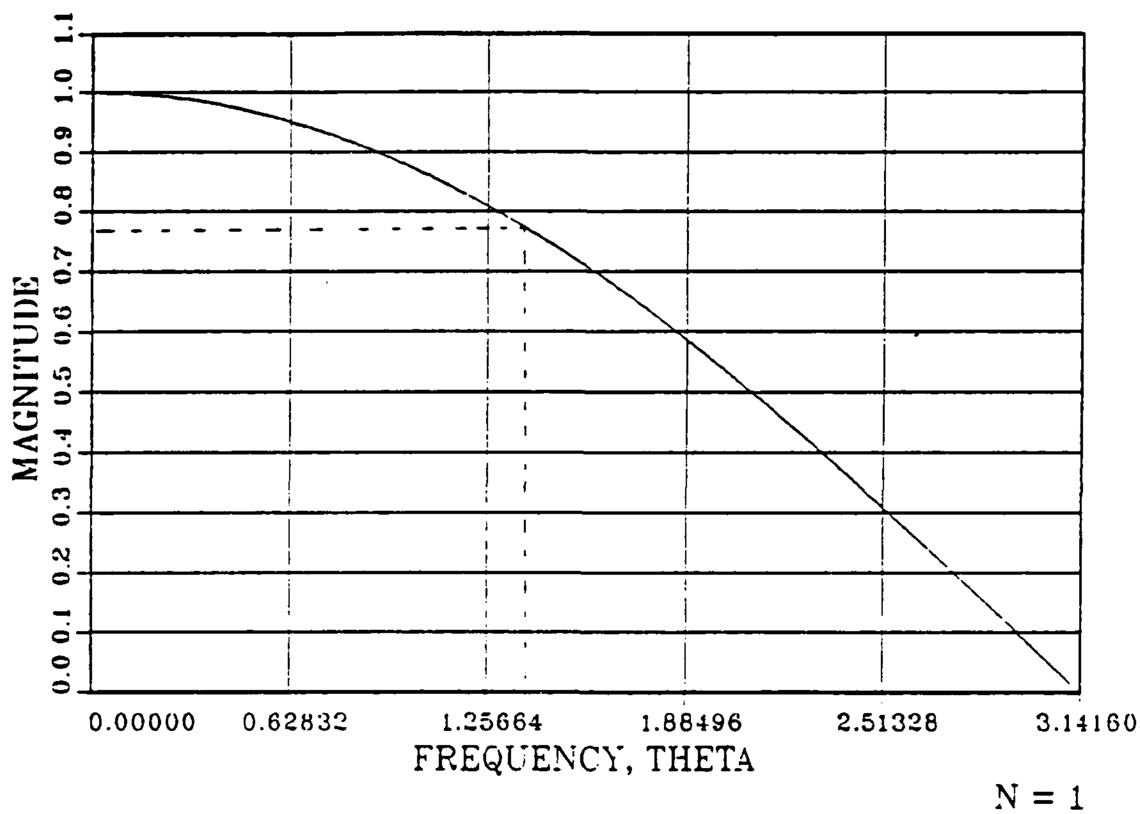


Figure 2.7. Frequency Response for Butterworth Lowpass Filter

Table 2.7 is a summary of the analog and digital versions of lowpass prototype Butterworth filter transfer functions for filter orders up to $N = 8$.

To illustrate the direct design approach the design problem of the previous section will be used (Example 2.1), where the problem statement was:

Design a digital filter for a 20 kHz sampling rate that is flat in the passband of 0 to the 3 dB cutoff frequency of 2 kHz, and has a gain of not more than -10 dB for frequencies greater than 4 kHz.

<u>Step 1</u>: A Butterworth design is called for because a flat passband is desired.

<u>Step 2</u>: Convert the critical analog design frequencies to digital.

$$\begin{array}{llcl}
\text{Sampling frequency:} & f_s = 20 \text{ kHz} & & \\
\text{Cutoff frequency:} & f_C = 2 \text{ kHz} & \Longrightarrow & w_C = 4 \times 10^3 \pi \text{ rad}/s \\
\text{Stopband frequency:} & f_a = 4 \text{ kHz} & \Longrightarrow & w_a = 8 \times 10^3 \pi \text{ rad}/s
\end{array}$$

$$\theta'_C = w_C T = w_C / f_s = \frac{\left(4 \times 10^3\right)\pi}{20 \times 10^3} = 0.2\pi \text{ rad}$$
$$\theta'_a = w_a T = w_a / f_s = \frac{\left(8 \times 10^3\right)\pi}{20 \times 10^3} = 0.4\pi \text{ rad}$$

<u>Step 3</u>: Since the "design chart" is based on a critical frequency of $\theta_C = \pi/2$, digitized design specifications need to be normalized to this frequency, so that the order of the required lowpass prototype filter can be determined. For this process the following frequency transformations apply (from Table 2.5, prime denotes desired filter).

- Prototype filter from desired filter:

$$e^{j\theta_a} = \frac{e^{j\theta'_a} - \alpha}{1 - \alpha e^{j\theta'_a}} \tag{2.31}$$

TABLE 2.7

LOWPASS PROTOTYPE BUTTERWORTH FILTER TRANSFER FUNCTIONS

(due to reference 4)

Filter Order $\underline{N}$	$\underline{H_{LP_P}(s)}$	$\underline{H_{LP_P}(z)}$
1	$\frac{1}{s+1}$	$\frac{0.5(z+1)}{z}$
2	$\frac{1}{s^2+1.4142s+1}$	$\frac{(z+1)^2}{3.41z^2+0.59}$
3	$\frac{1}{s^3+2s^2+2s+1}$	$\frac{(z+1)^3}{6z^3+2z}$
4	$\frac{1}{s^4+2.6131s^3+3.4142s^2+2.6131s+1}$	$\frac{(z+1)^4}{10.6383z^4+5.17z^2}$
5	$\frac{1}{s^5+3.2361s^4+5.2361s^3}$ $+5.2361s^2+3.2361s+1$	$\frac{(z+1)^5}{18.94z^5+11.996z^3+1.0549z}$
6	$\frac{1}{s^6+3.8637s^5+7.4641s^4+9.1416s^3+7.4641s^2}$ $+3.8637s+1$	$\frac{(z+1)^6}{33.797z^6+26.284z^4+3.86z^2+0.0592}$
7	$\frac{1}{s^7+4.4940s^6+10.0978s^5+14.5918s^4}$ $+14.5918s^3+10.0978s^2+4.4940s+1$	$\frac{(z+1)^7}{60.367z^7+55.537z^5+11.633z^3+0.4624z}$
8	$\frac{1}{s^8+5.1258s^7+13.1317s^6+21.8462s^5}$ $+25.6884s^4+21.8462s^3+13.1371s^2$ $+5.1258s+1$	$\frac{(z+1)^8}{107.896z^8+114.438z^6+31.496z^4}$ $+2.162z^2+0.008$

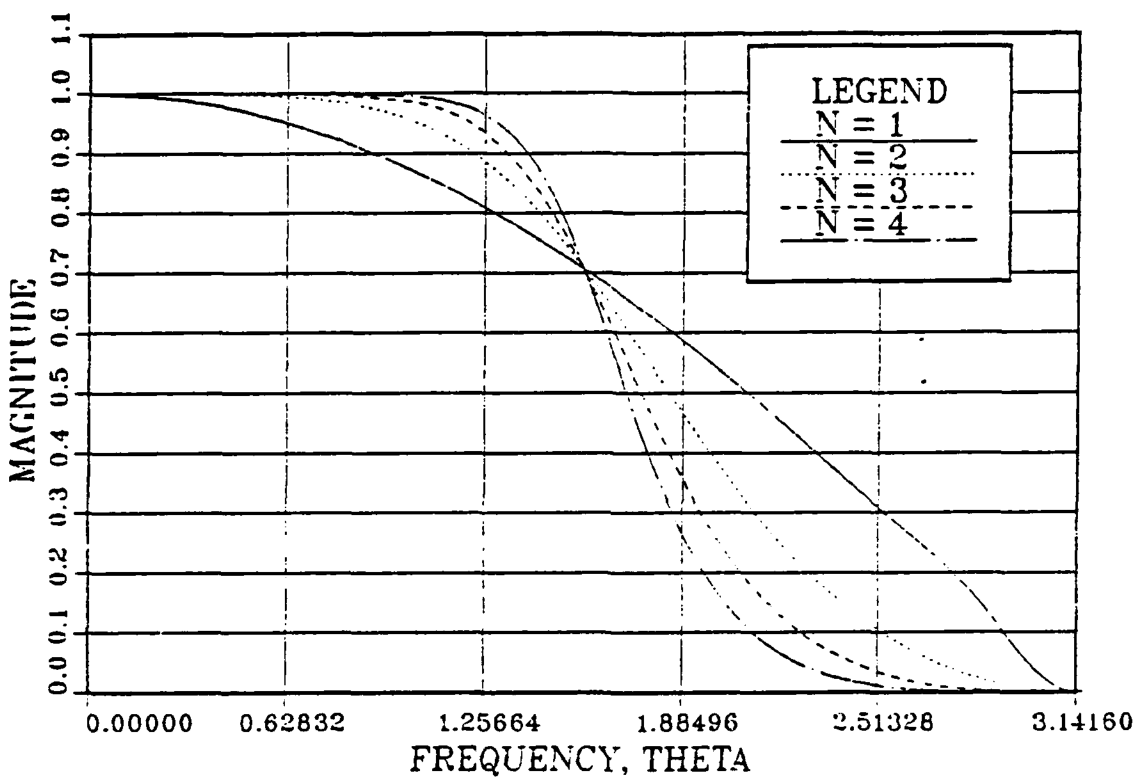


Figure 2.8a. Butterworth Filter Design Curves

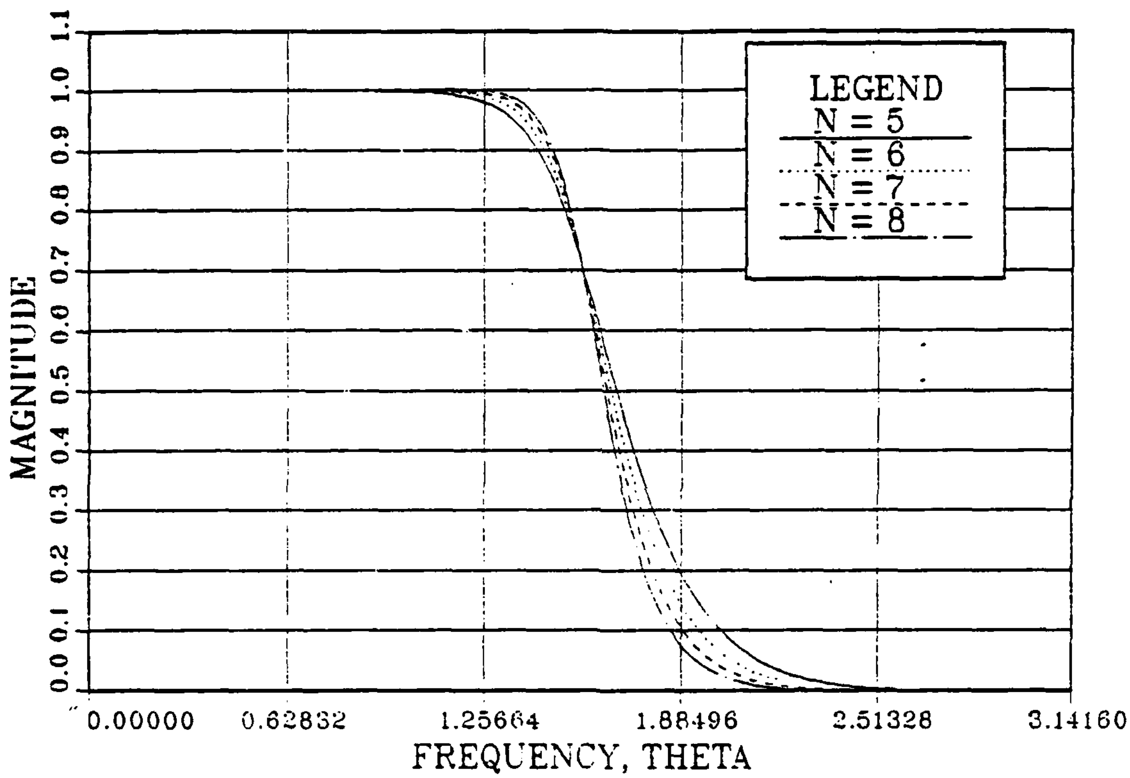


Figure 2.8b. Butterworth Filter Design Curves

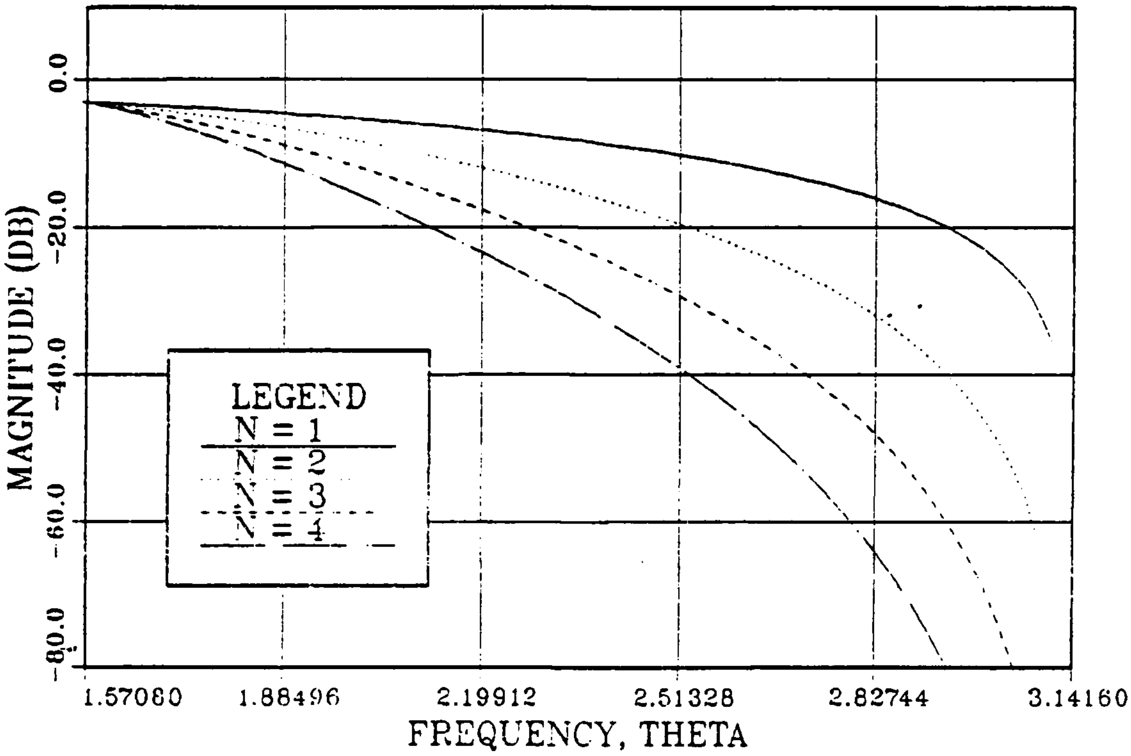


Figure 2.8c. Butterworth Filter Design Curves

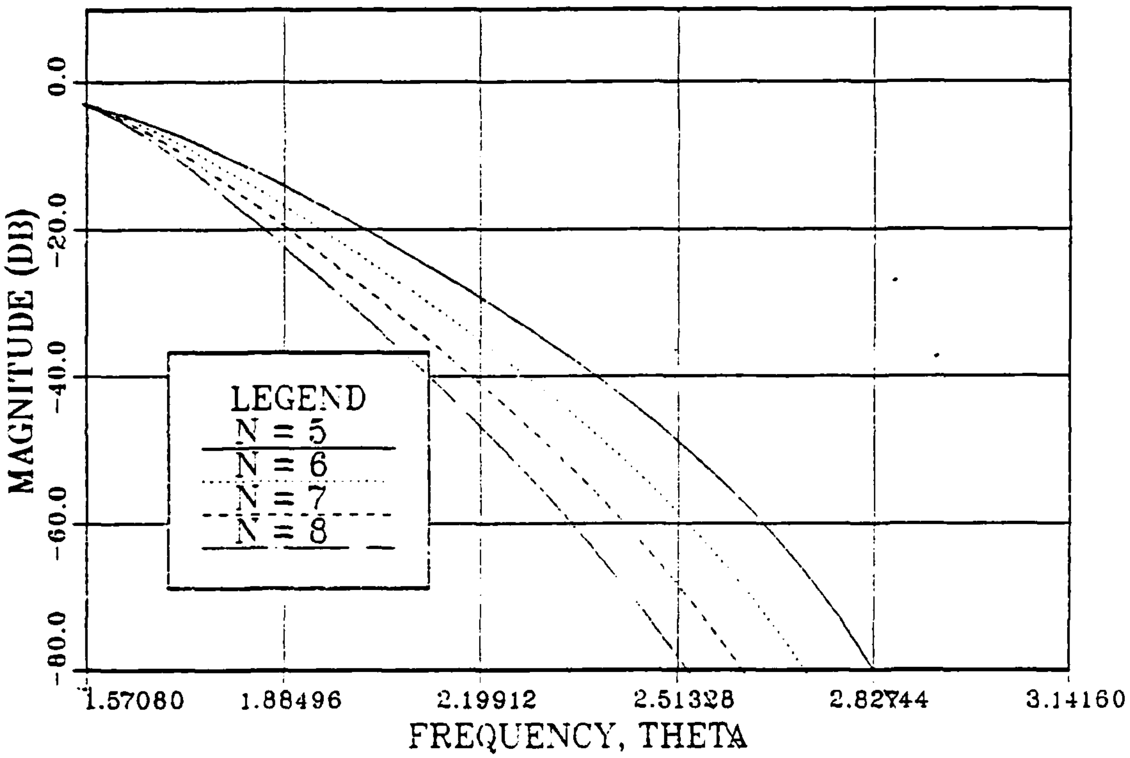


Figure 2.8d. Butterworth Filter Design Curves

- Desired filter from prototype filter:

$$e^{j\theta_a} = \frac{e^{j\theta'_a} + \alpha}{1 + \alpha e^{j\theta_a}}$$

where (2.32)

$$\alpha = \frac{\sin\left(\theta_c/2 - \theta'_c/2\right)}{\sin\left(\theta_c/2 + \theta'_c/2\right)}$$

θ'_c and θ'_a are the critical and stopband frequencies (respectively) used to determine the optimum prototype filter order.

Equation (2.31) is applicable to this problem, and thus:

$$\theta_a = ?$$

$$\theta'_a = 0.4\pi$$

$$\theta_c = 0.5\pi$$

$$\theta'_c = 0.2\pi$$

$$\alpha = \frac{\sin(0.5\pi/2 - 0.2\pi/2)}{\sin(0.5\pi/2 + 0.2\pi/2)}$$
$$= \frac{\sin(0.15\pi)}{\sin(0.35\pi)} = \frac{0.454}{0.891} = 0.510$$

Therefore,

$$e^{j\theta_a} = \frac{e^{j0.4\pi} - 0.510}{1 - 0.519e^{j0.4\pi}}$$
$$= \frac{0.97e^{j1.779}}{0.97e^{-j0.57}} = e^{j2.349} \tag{2.33}$$
$$\Longrightarrow \quad \theta_a = 2.349 \text{ rad } = 0.75\pi \text{ rad}$$

Step 4: Once the normalized stopband frequency is obtained, (in this case 0.75π), the design chart is used to determine the lowest order filter that will give:

$$-3dB \text{ at } \theta_c = 0.5\pi$$

$$\text{less than } -10dB \text{ at } \theta_a = 0.7\pi$$

Turning to Figure 2.8, it can be seen that the curve for $N = 2$ meets these requirements. At this point, it is worth noting that a second-order Butterworth filter was also determined to be required using the traditional approach in the previous section.

In Table 2.7 it can be seen the prototype low-pass filter transfer function for $N = 2$ is:

$$H_{LP_P}(z) = \frac{(z+1)^2}{3.41z^2 + 0.59} = \frac{z^2 + 2z + 1}{3.41z^2 + 0.59} \tag{2.34}$$

<u>**Step 5**</u>: Once the prototype filter is obtained from the design curves, it is necessary to determine the actual transfer function that will meet the design specification, such that θ'_c is 0.2π and θ'_a is 0.4π.

Using the digital filter frequency transformations of Table 2.5, the transfer function is determined as follows:

$$\begin{aligned} H_{LP}(z) &= H_{LP_P}(z)\Big|_{z=\frac{z-\alpha}{1-\alpha z}} \\ &= \frac{z^2 + 2z + 1}{3.41z^2 + 0.59}\Big|_{z=\frac{z-0.510}{1-0.510z}} \\ &= \frac{z^2 + 2z + 1}{14.84z^2 - 17.0z + 6.14} \end{aligned} \tag{2.35}$$

which is the same transfer function obtained in the previous section.

<u>**Step 6**</u>: A computer generated frequency response plot of the filter is used to verify that the design fulfills the design specifications; Figure 2.9 reveals that this is indeed the case.

D. LOWPASS PROTOTYPE TO HIGHPASS, BANDPASS BAND-STOP FILTERS

Once the lowpass prototype filter is obtained, a highpass, bandpass or band-stop filter can be derived using the digital frequency transformations of Table 2.5. In the next section, an example is presented that illustrates this process.

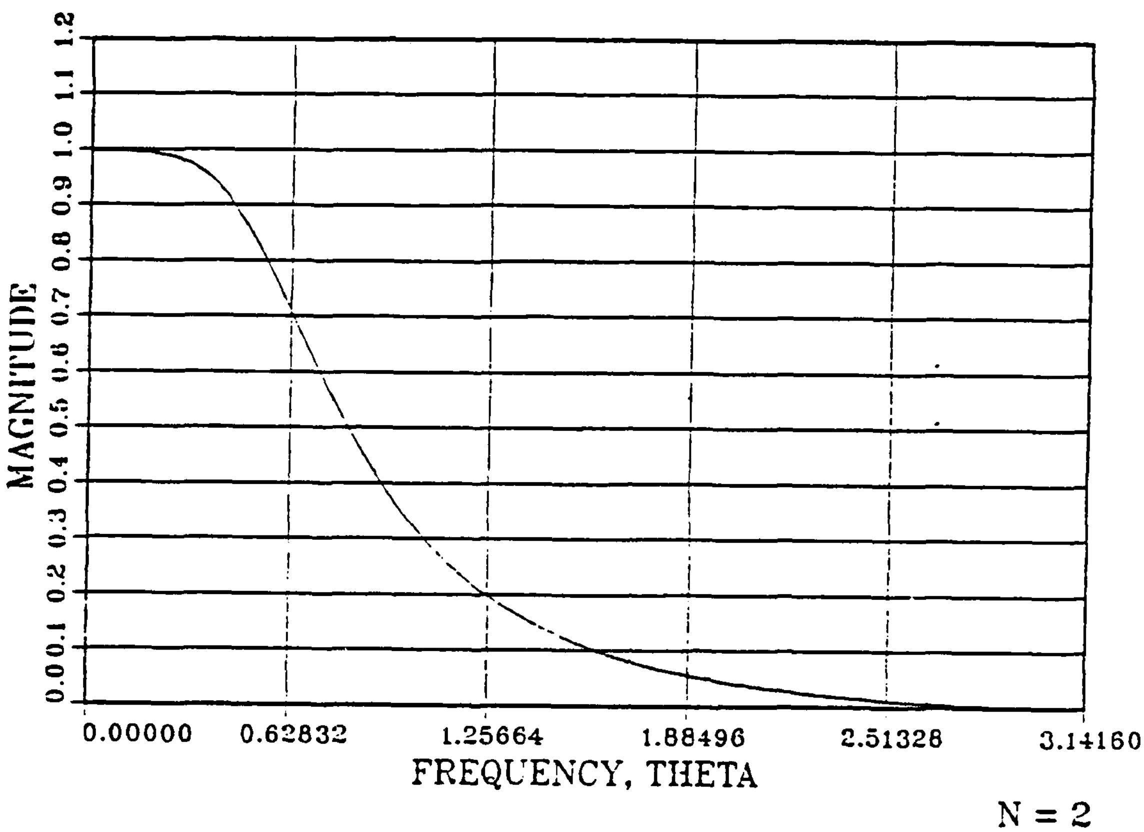


Figure 2.9. Frequency Response for Butterworth Lowpass Filter

1. Chebyshev Filters - A Direct Design Approach

As stated in the introduction to this section, this design approach is of course applicable to Chebyshev filter design problems. Again using the bilinear transformation techniques cited in [2], the digital filter coefficients corresponding to the analog filter coefficients for prototype Chebyshev filters were determined. The frequency responses of the resulting transfer functions listed in Table 2.8 were then plotted to obtain a set of design curves with a cutoff frequency of $\pi/2$ (see Figures 2.10 thru 2.12).

The design problem of the previous section will be used to illustrate application of this new technique (Example 2.2).

The problem statement called for a Chebyshev digital bandpass filter to be designed to meet the following specification:

- 1 dB ripple in the range of 600 to 900 Hz
- Sampling frequency is 3 kHz
- Maximum gain of -40dB for $0 < f < 200$ Hz

__Step 1__: Convert the critical analog design frequencies to digital:

Sampling frequency:	$f_s = 3$ kHz
Lower Passband frequency:	$f_\ell = 600$ Hz
Upper Passband frequency:	$f_u = 900$ Hz
Stopband frequency:	$f_a = 200$Hz

$$\theta'_\ell = w_\ell T = w_\ell / f_s = \frac{2(600)\pi}{3000} = 0.4\pi = 1.26 \text{ rad}$$
$$\theta'_u = w_u T = w_u / f_s = \frac{2(900)\pi}{3000} = 0.6\pi = 1.88 \text{ rad}$$
$$\theta'_a = w_a T = w_a / f_s = \frac{2(200)\pi}{3000} = 0.133\pi = 0.418 \text{ rad}$$

Ripple band center frequency: $\theta'_0 = \sqrt{\theta'_\ell \theta'_u} = \sqrt{2.35} = 1.54$ rad

TABLE 2.8

LOWPASS PROTOTYPE CHEBYSHEV FILTER TRANSFER FUNCTIONS

(due to reference 4)

1/2 - dB ripple ($\epsilon = 0.3493$)

N	$H(s)$	$H(z)$
1	$\frac{2.863}{s+2.863}$	$\frac{2.863z+2.863}{3.863z+1.863}$
2	$\frac{1.43}{s^2+1.425s+1.516}$	$\frac{1.43z^2+2.86z+1.43}{3.941z^2+1.032z+1.091}$
3	$\frac{0.716}{s^3+1.253s^2+1.535s+0.716}$	$\frac{0.716z^3+2.148z^2+2.148z+0.716}{4.504z^3-0.570z^2+2.360z-0.566}$
4	$\frac{0.944}{s^4+1.197s^3+1.717s^2+1.025s+0.379}$	$\frac{0.358z^4+1.432z^3+2.148z^2+1.432z+0.358}{5.318z^4-2.828z^3+4.840z^2-2.140z+0.874}$

1 - dB ripple ($\epsilon = 0.5088$)

N	$H(s)$	$H(z)$
1	$\frac{1.965}{s+1.965}$	$\frac{1.965z+1.965}{2.965z+0.965}$
2	$\frac{0.983}{s^2+1.098s+1.103}$	$\frac{0.983z^2+1.966z+0.983}{3.201z^2+0.206z+1.005}$
3	$\frac{0.491}{s^3+0.988s^2+1.238s+0.491}$	$\frac{0.491z^3+1.473z^2+1.473z+0.491}{3.717z^3-1.277z^2+2.247z-0.759}$
4	$\frac{0.891}{s^4+0.953s^3+1.454s^2+0.743s+0.276}$	$\frac{0.246z^4+0.984z^3+1.476z^2+0.984z+0.246}{4.426z^4-3.316z^3+4.748z^2-2.476z+1.034}$

2 - dB ripple ($\epsilon = 0.7648$)

N	$H(s)$	$H(z)$
1	$\frac{1.308}{s+1.308}$	$\frac{1.308z+1.308}{2.308z+0.308}$
2	$\frac{0.506}{s^2+0.804s+0.637}$	$\frac{0.506z^2+1.012z+0.506}{2.441z^2-0.726z+0.833}$
3	$\frac{0.327}{s^3+0.738s^2+1.022s+0.327}$	$\frac{0.327z^3+0.981z^2+0.981z+0.327}{3.087z^3-1.735z^2+2.221z-0.957}$
4	$\frac{0.164}{s^4+0.716s^3+1.256s^2+0.517s+0.206}$	$\frac{0.164z^4+0.656z^3+0.984z^2+0.656z+0.164}{3.695z^4-3.574z^3+4.724z^2-2.778z+1.229}$

TABLE 2.8 (Continued)

LOWPASS PROTOTYPE CHEBYSHEV FILTER TRANSFER FUNCTIONS

1/? - dB ripple ($\epsilon = 0.3493$)

N	$H(s)$	$H(z)$
5	$\frac{0.179}{s^5+1.173s^4+1.937s^3+1.310s^2+0.753s+0.179}$	$\frac{0.179z^5+0.895z^4+1.79z^3+1.79z^2+0.895z+0.179}{6.351z^5-5.993z^4+9.145z^3-6.115z^2+3.367z-1.029}$
6	$\frac{0.09}{s^6+1.159s^5+2.172s^4+1.600s^3+1.172s^2+0.432s+0.095}$	$\frac{0.090z^6+0.537z^5+1.343z^4+1.79z^3+1.343z^2+0.537z+0.09}{7.620z^6-10.338z^5+16.267z^4-14.104z^3+9.890z^2-4.524z+1.257}$
7	$\frac{0.045}{s^7+1.151s^6+2.413s^5+1.869s^4+1.648s^3+0.756s^2+0.282s+0.045}$	$\frac{0.045z^7+0.313z^6+0.939z^5+1.565z^4+1.565z^3+0.939z^2+0.313z+0.045}{9.164z^7-16.225z^6+27.455z^5-28.832z^4+24.109z^3-14.561z^2+6.134z-1.522}$
8	$\frac{0.022}{s^8+1.146s^7+2.657s^6+2.149s^5+2.184s^4+1.149s^3+0.574s^2+0.153s+0.024}$	$\frac{0.022z^8+0.179z^7+0.627z^6+1.254z^5+1.568z^4+1.254z^3+0.627z^2+0.179z+0.022}{11.035z^8-24.106z^7+44.435z^6-54.247z^5+52.46z^4-36.434z^3+21.264z^2-8.18z+1.8415}$

1-dB ripple ($\epsilon = 0.5088$)

N	$H(s)$	$H(z)$
5	$\frac{0.123}{s^5+0.937s^4+1.689s^3+0.974s^2+0.581s+0.123}$	$\frac{0.123z^5+0.614z^4+1.23z^3+1.23z^2+0.614z+0.123}{5.303z^5-6.169z^4+8.936z^3-6.631z^2+3.725z-1.235}$
6	$\frac{0.061}{s^6+0.928s^5+1.931s^4+1.202s^3+0.939s^2+0.307s+0.069}$	$\frac{0.061z^6+0.368z^5+0.921z^4+1.228z^3+0.921z^2+0.368z+0.061}{6.376z^6-10.054z^5+15.734z^4-14.656z^3+10.593z^2-5.085z+1.502}$
7	$\frac{0.031}{s^7+0.923s^6+2.176s^5+1.429s^4+1.358s^3+0.549s^2+0.214s+0.031}$	$\frac{0.031z^7+0.215z^6+0.645z^5+1.075z^4+1.075z^3+0.645z^2+0.215z+0.031}{7.679z^7-15.286z^6+26.242z^5-29.121z^4+25.126z^3-15.812z^2+6.919z-1.816}$
8	$\frac{0.015}{s^8+0.920s^7+2.423s^6+1.655s^5+1.837s^4+0.847s^3+0.448s^2+0.107s+0.017}$	$\frac{0.015z^8+0.12z^7+0.42z^6+0.84z^5+1.05z^4+0.84z^3+0.42z^2+0.12z+0.015}{9.254z^8-22.255z^7+41.993z^6-53.661z^5+53.517z^4-40.611z^3+23.242z^2-9.271z+2.196}$

2 - dB ripple ($\epsilon = 0.7648$)

N	$H(s)$	$H(z)$
5	$\frac{0.082}{s^5+0.707s^4+1.500s^3+0.694s^2+0.459s+0.082}$	$\frac{0.082z^5+0.409z^4+0.817z^3+0.817z^2+0.409z+0.081}{4.441z^5-6.139z^4+8.763z^3-7.077z^2+4.104z-1.477}$
6	$\frac{0.041}{s^6+0.701s^5+1.746s^4+0.867s^3+0.772s^2+0.210s+0.051}$	$\frac{0.041z^6+0.245z^5+0.612z^4+0.816z^3+0.612z^2+0.245z+0.041}{5.347z^6-9.604z^5+15.210z^4-15.074z^3+11.297z^2-5.677z+1.790}$
7	$\frac{0.020}{s^7+0.698s^6+1.994s^5+1.039s^4+1.44s^3+0.383s^2+0.167s+0.020}$	$\frac{0.020z^7+0.143z^6+0.428z^5+0.714z^4+0.714z^3+0.428z^2+0.143z+0.020}{6.445z^7-14.242z^6+25.034z^5-29.203z^4+26.062z^3-17.085z^2+7.765z-2.165}$
8	$\frac{0.010}{s^8+0.696s^7+2.242s^6+1.212s^5+1.580s^4+0.596s^3+0.359s^2+0.073s+0.013}$	$\frac{0.010z^8+0.082z^7+0.286z^6+0.571z^5+0.714z^4+0.571z^3+0.286z^2+0.082z+0.010}{7.772z^8-20.397z^7+39.593z^6-52.787z^5+54.371z^4-42.699z^3+25.30z^2-10.465z+2.615}$

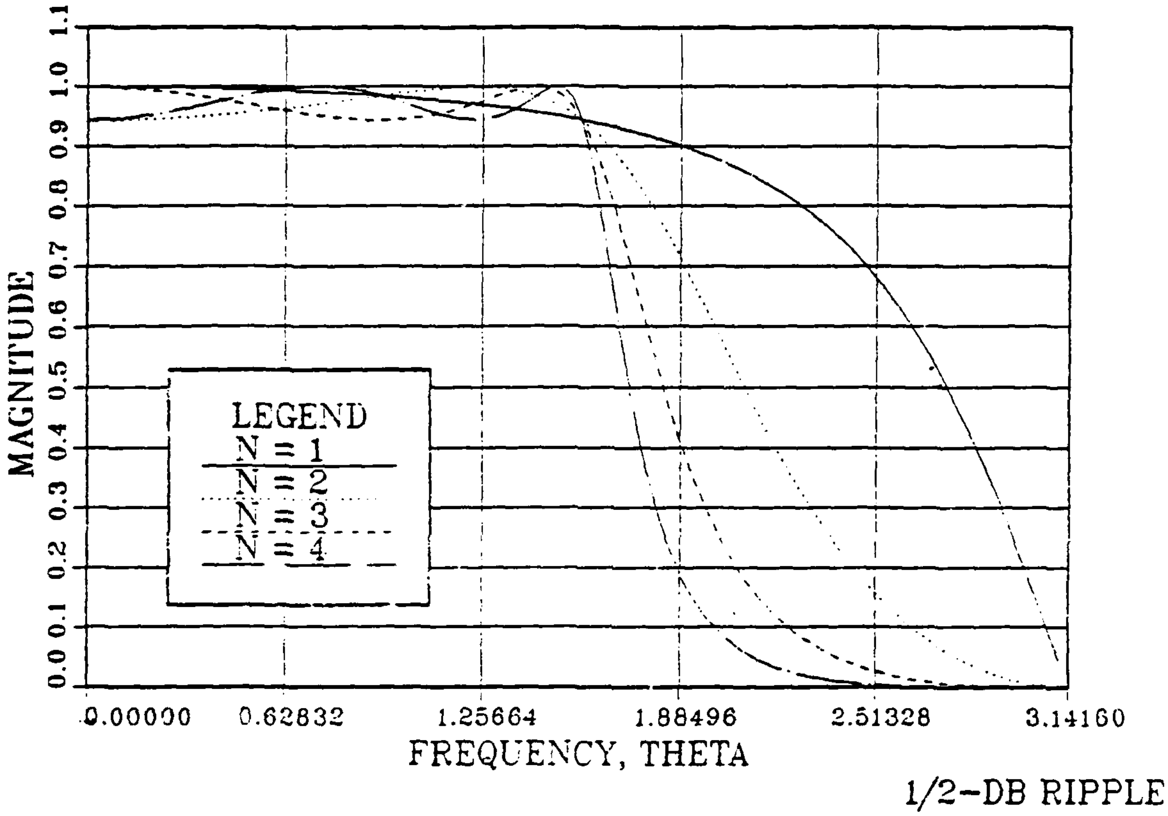


Figure 2.10a. Chebyshev Filter Design Curves

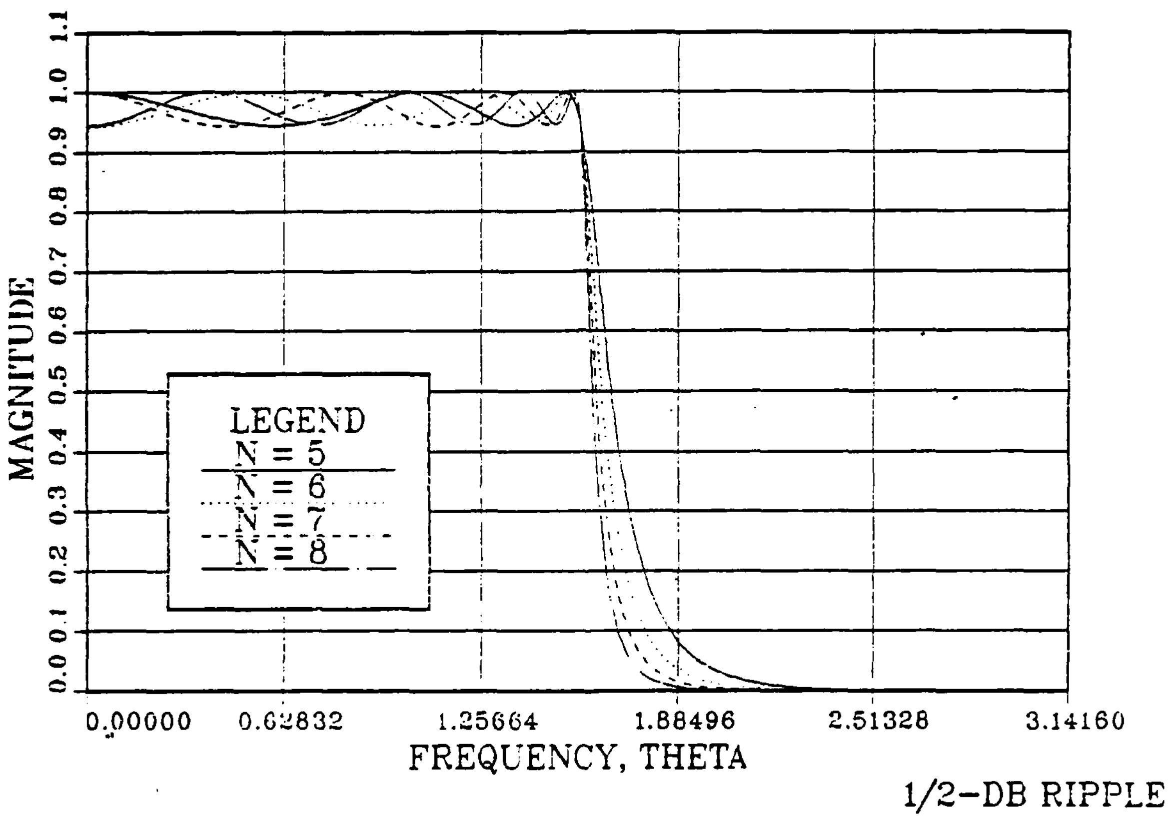


Figure 2.10b. Chebyshev Filter Design Curves

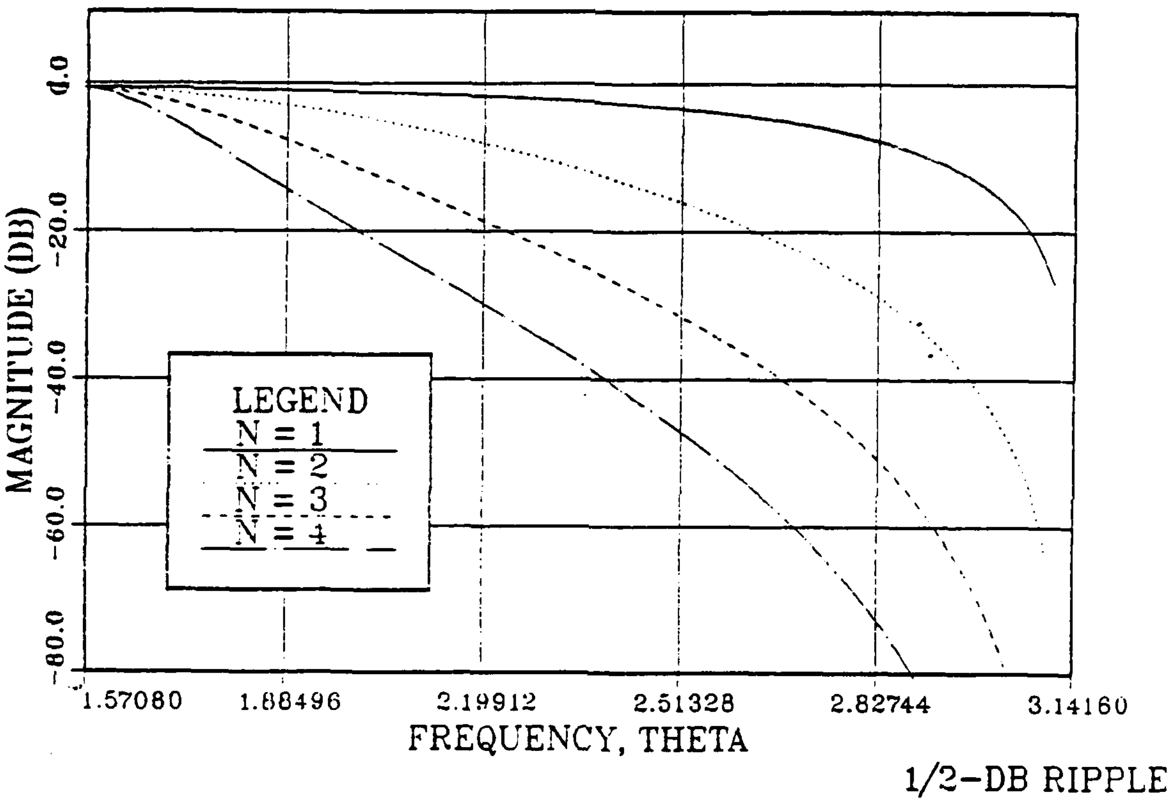


Figure 2.10c. Chebyshev Filter Design Curves

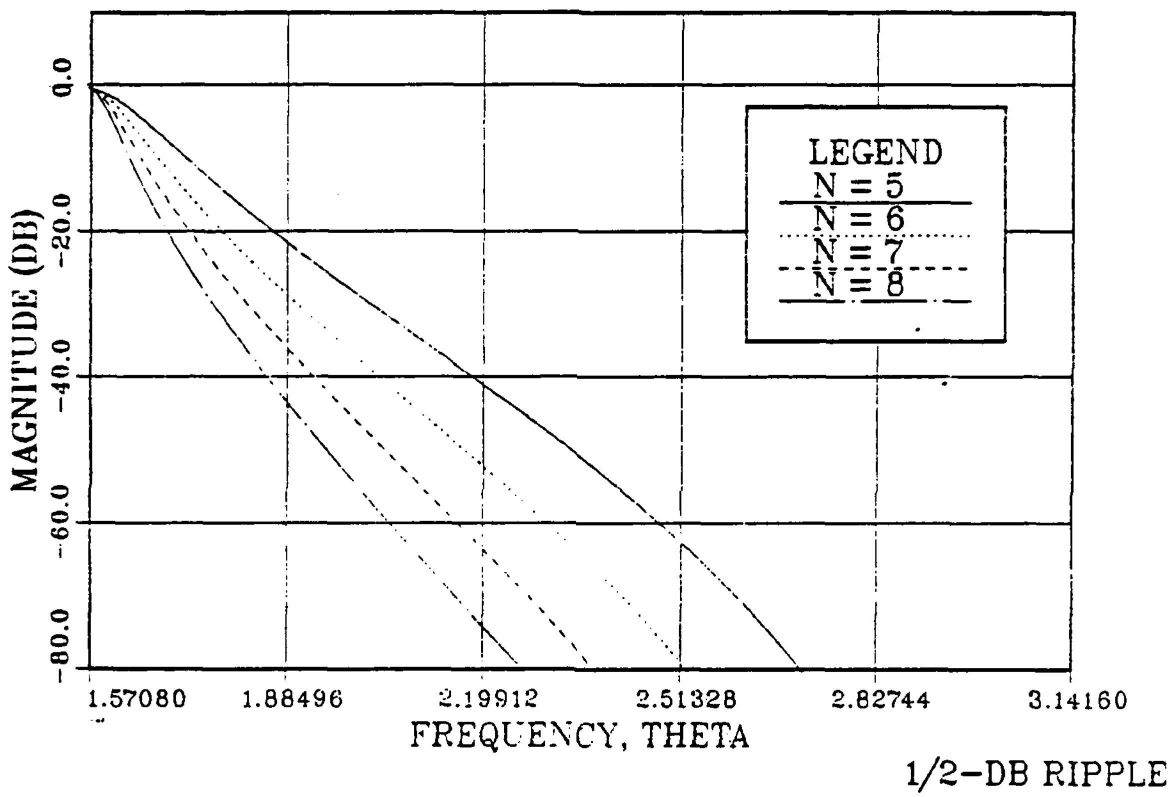


Figure 2.10d. Chebyshev Filter Design Curves

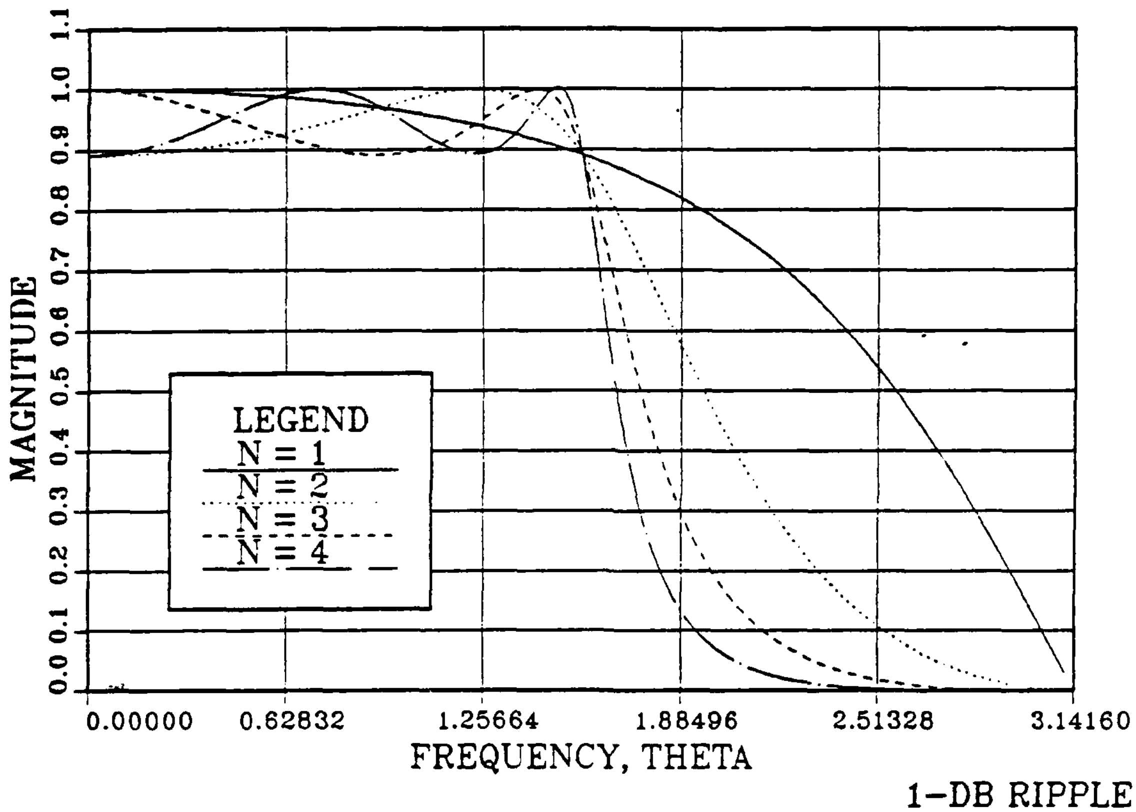


Figure 2.11a. Chebyshev Filter Design Curves

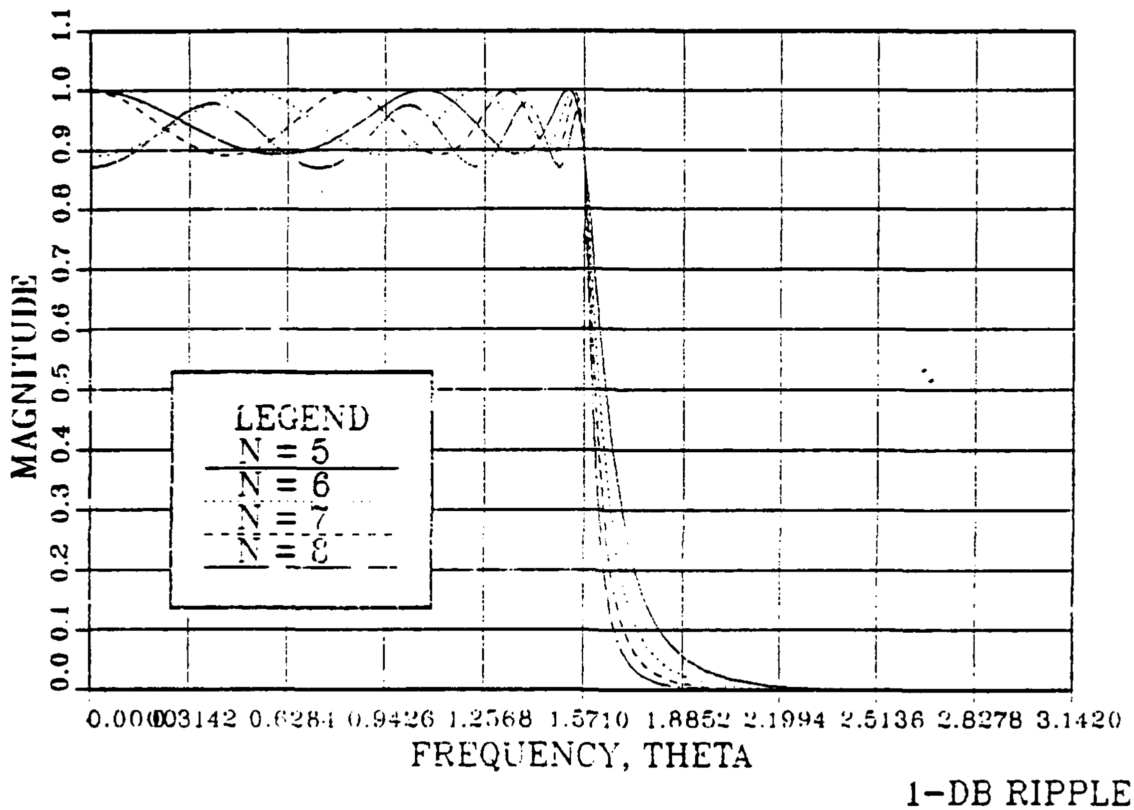


Figure 2.11b. Chebyshev Filter Design Curves

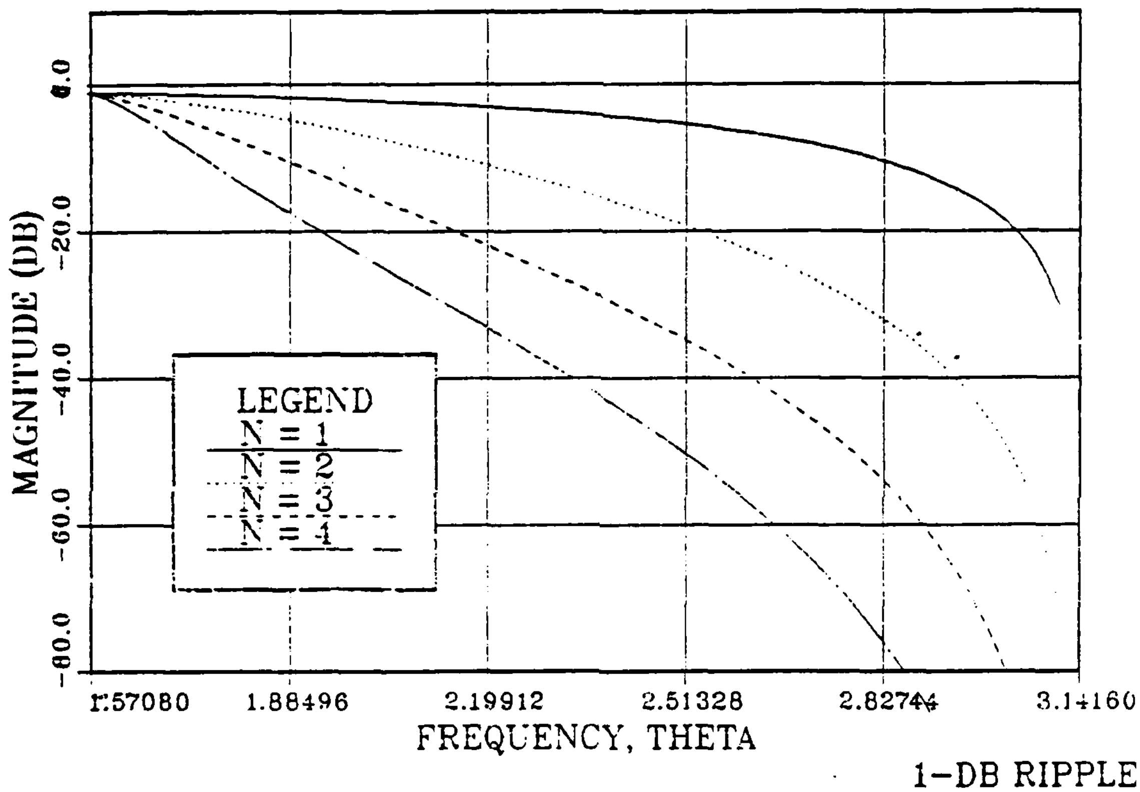


Figure 2.11c. Chebyshev Filter Design Curves

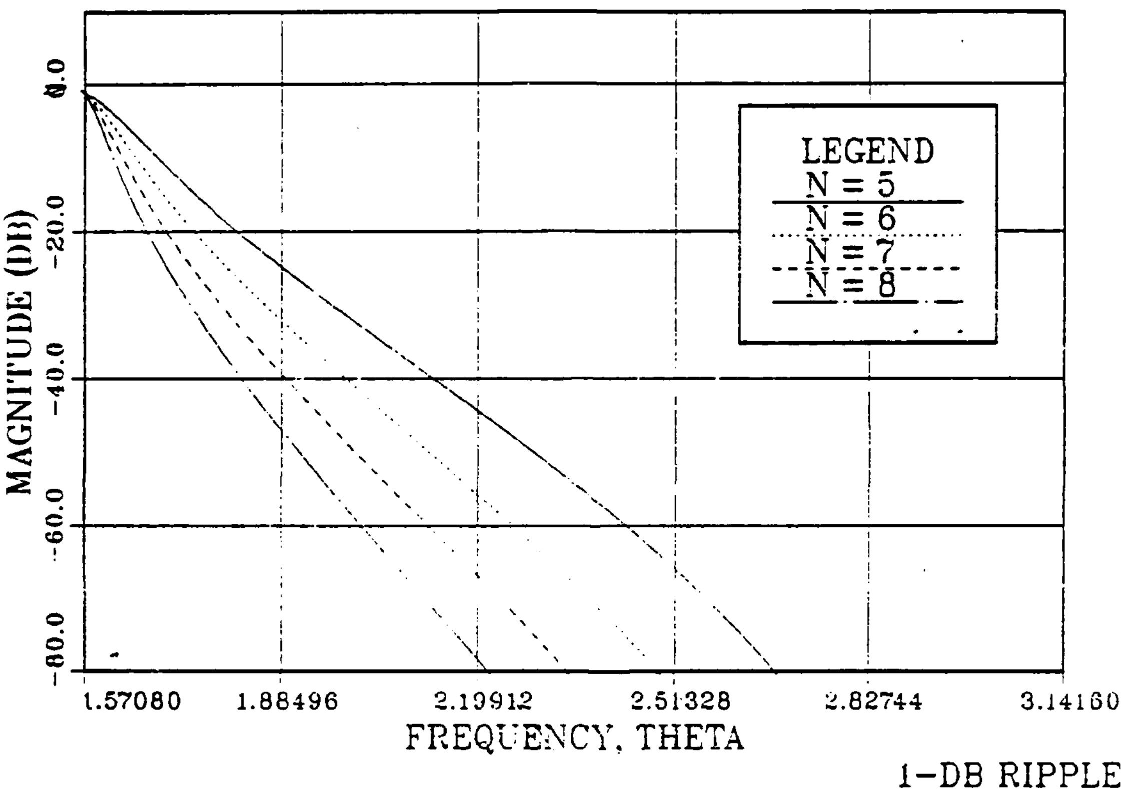


Figure 2.11d. Chebyshev Filter Design Curves

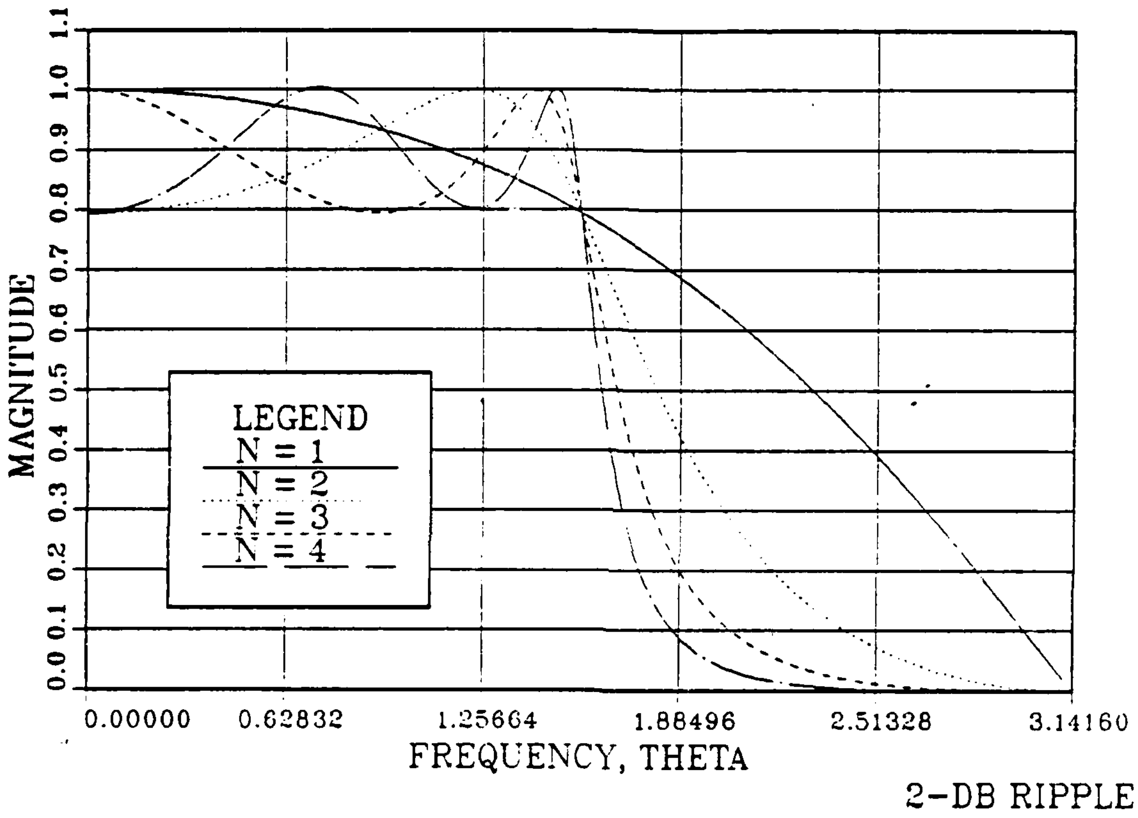


Figure 2.12a. Chebyshev Filter Design Curves

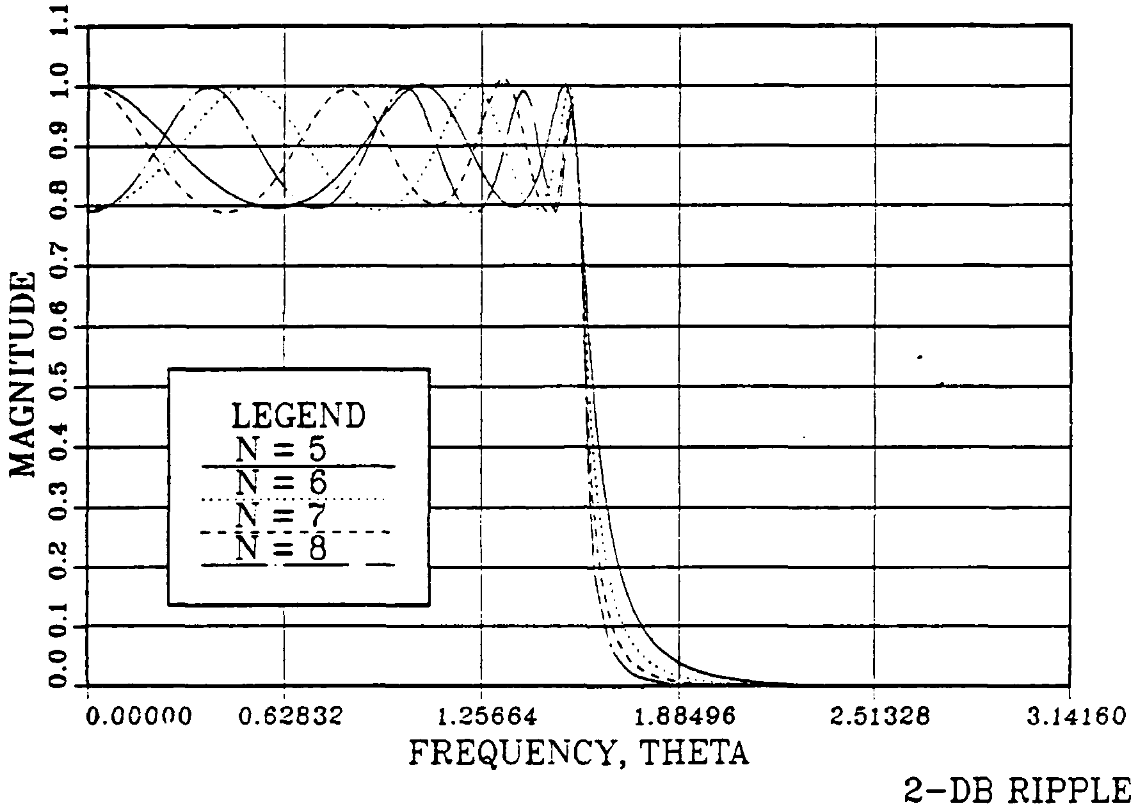


Figure 2.12b. Chebyshev Filter Design Curves

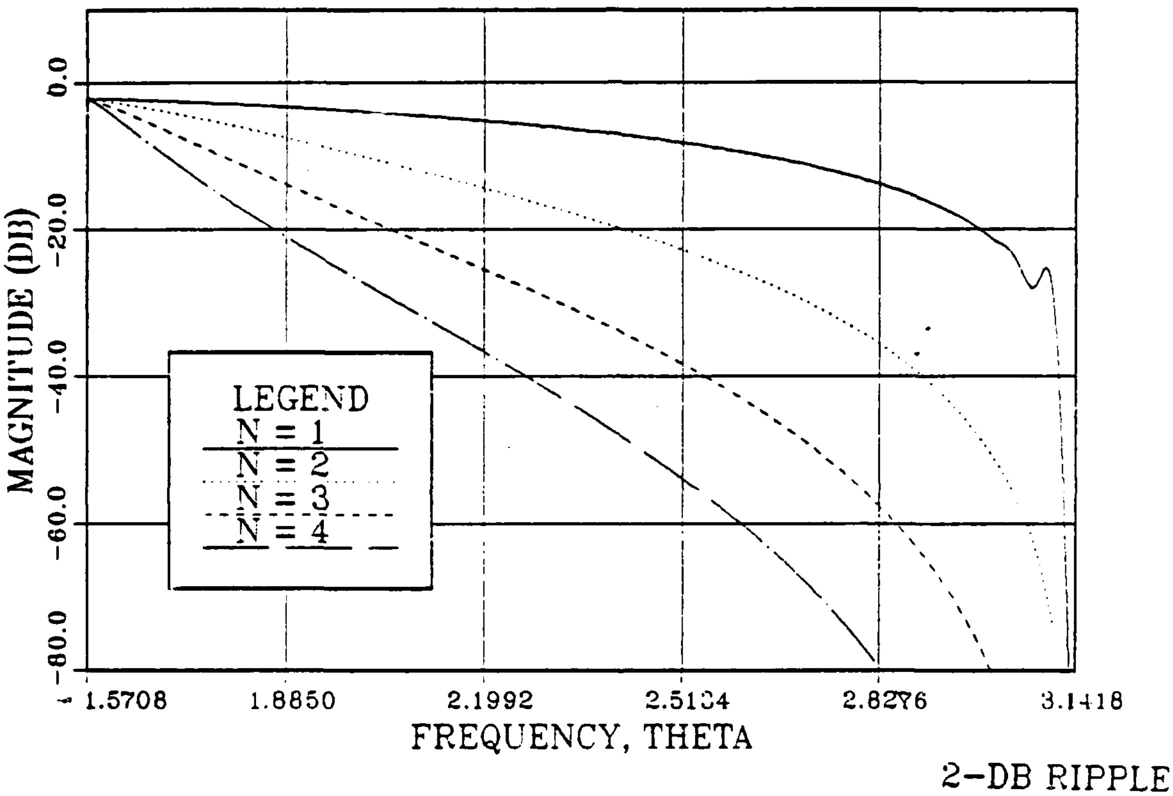


Figure 2.12c. Chebyshev Filter Design Curves

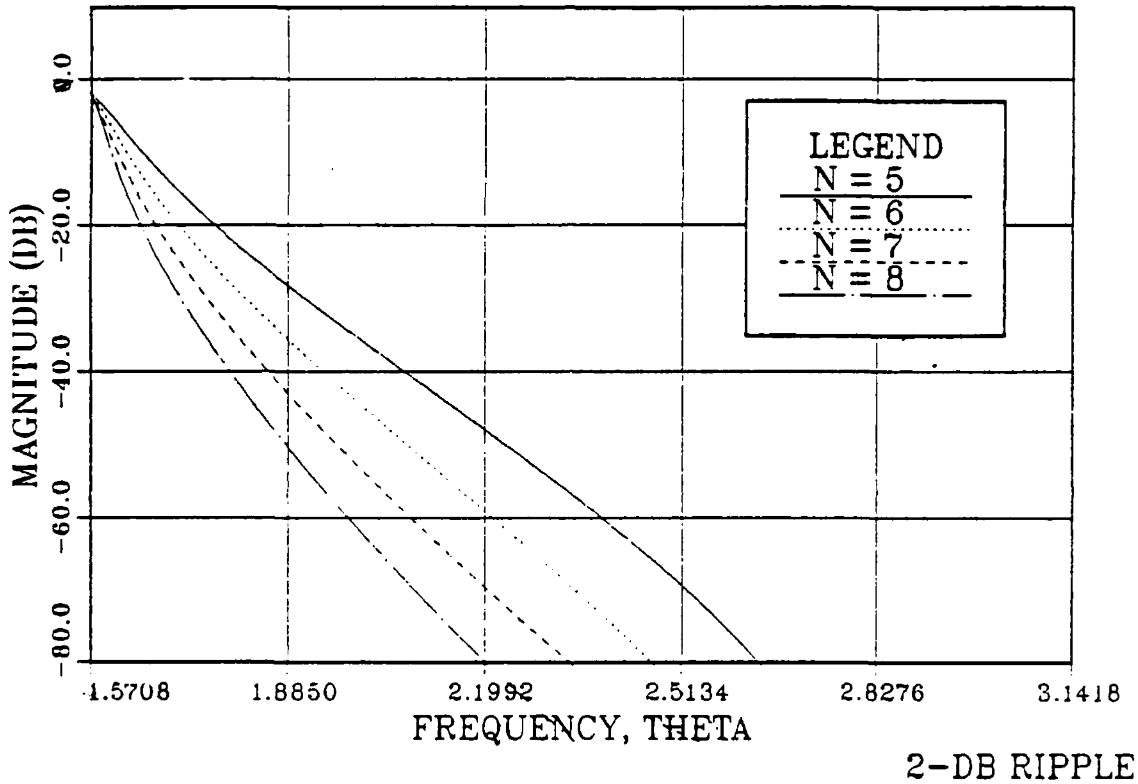


Figure 2.12d. Chebyshev Filter Design Curves

Step 2: Again, the design chart is based on a ripple edge frequency of $\pi/2$, necessitating the normalization of the design specification frequencies. Looking at the Table of Digital Filter Frequency Transformations (Table 2.5), the following frequency transformations apply for a bandpass filter:

Prototype filter from desired filter:

$$e^{j\theta_a} = -\frac{e^{j2\theta'_a} - \frac{2\alpha k}{k+1} e^{j\theta'_a} + \frac{k-1}{k+1}}{1 - \frac{2\alpha k}{k+1} e^{j\theta'_a} + \frac{k-1}{k+1} e^{j2\theta'_a}} \tag{2.36}$$

where:

$$\alpha = \frac{\cos(\theta'_u/2 + \theta'_\ell/2)}{\cos(\theta'_u/2 - \theta'_\ell/2)}$$

$$k = \tan(\theta_c/2)\cot\left(\frac{\theta'_u}{2} - \frac{\theta'_\ell}{2}\right)$$

θ_a is the stopband frequency used to determine the filter order N.

θ_c is the ripple edge frequency on the design chart, in this case $\theta_c = \pi/2$.

For this problem:

$$\theta_a = ?$$

$$\theta'_a = 0.133\pi$$

$$\theta'_u = 0.6\pi$$

$$\theta'_\ell = 0.4\pi$$

$$\theta_c = 0.5\pi$$

$$\alpha = \frac{\cos(0.3\pi + 0.2\pi)}{\cos(0.3\pi - 0.2\pi)} = \frac{\cos(0.5\pi)}{\cos(0.1\pi)}$$

$$k = \tan(\pi/4)\cot(0.1\pi) = 3.07$$

Since $\alpha = 0$

$$\begin{aligned} e^{j\theta_a} &= -\frac{e^{j2\theta'_a} + \frac{k-1}{k+1}}{1 + \frac{k-1}{k+1} e^{j2\theta'_a}} = -\frac{e^{j2\theta'_a} + 0.509}{1 + (0.509) e^{j2\theta'_a}} \\ &= \frac{-e^{j0.836} - 0.509}{1 + 0.509 e^{j0.836}} = \frac{1.394 e^{-j2.58}}{1.394 e^{j0.282}} = e^{-j2.86} \end{aligned} \tag{2.37}$$

$\Longrightarrow \quad \theta_a = -2.86$ rad $= 2.86$ rad (due to symmetry)

<u>**Step 4**</u>: Determine the lowest order Chebyshev lowpass filter prototype from the design chart that gives:

$$-3 \text{ db at } \theta_c = 0.5\pi \text{ rad}$$

$$\text{less than } -40 \text{ db at } \theta_a = 2.86 \text{ rad}$$

Turning to Figure 2.11, the filter order is determined to be $N = 3$. Again, this agrees with the order obtained using the approach in the previous section.

From Table 2.8 the lowpass prototype transfer function for $N = 3$ is:

$$H_{LP_P}(z) = \frac{0.491z^3 + 1.473z^2 + 1.473z + 0.491}{3.717z^3 - 1.277z^2 + 2.247z - 0.759} \tag{2.38}$$

<u>**Step 5**</u>: As with the Butterworth design, the Digital Filter Frequency Transformation Table 2.5, is used to convert the lowpass prototype filter to a bandpass filter.

$$\begin{aligned} H_{BP}(z) &= H_{LP_P}(z)\Big|_{z = -\frac{z^2 - \frac{2\alpha k}{k+1} z + \frac{k-1}{k+1}}{1 - \frac{2\alpha k}{k+1} z + \frac{k-1}{k+1} z^2}} \\ &= \frac{0.491z^3 + 1.473z^2 + 1.473z + 0.491}{3.717z^3 - 1.277z^2 + 2.247z - 0.759} \\ &= \frac{0.012z^6 - 0.036z^4 + 0.036z^2 + 0.012}{z^6 + 2.136z^4 + 1.768z^2 + 0.540} \end{aligned} \tag{2.39}$$

Again, this is nearly identical to the transfer function obtained using the "traditional approach".

Step 6: Verification of design using a computer generated frequency response plot. From Figure 2.13 it can be seen the design specifications of:

$$\theta'_\ell = 0.4\pi = 1.257 \text{ rad}$$

$$\theta'_u = 0.6\pi = 1.885 \text{ rad}$$

$$\theta'_a = 0.133\pi = 0.418 \text{ rad}$$

are met.

2. Elliptic Filters - A Direct Design Approach

As was shown in Example 2.3, the design of elliptic filters is algebraically intensive, an extremely undesirable feature, especially for high-order filters. Elliptic filters characteristically exhibit a sharper cutoff, that is, a narrower transition width for a given filter order than their Butterworth or Chebyshev counterparts. For this reason they are very popular, making the pursuit of a more efficacious design approach a worthwhile endeavor.

Just such an approach was found by taking the analog prototype filters of Table 2.4 and normalizing them to have a passband ripple edge (critical) frequency of one. They were then transformed to digital versions using the bilinear transformation, resulting in digital prototype filters with a critical frequency of $\pi/2$ (Table 2.9).

Thus, the amount of calculation involved in the design of a digital elliptic filter is cut in half because the requirement for finding an analog prototype is eliminated. The designer need only convert the filter design specifications to digital frequencies, find the digital lowpass prototype filter that meets these requirements (from Table 2.9), and then find the actual filter transfer function through the use of the digital frequency transformations of Table 2.5.

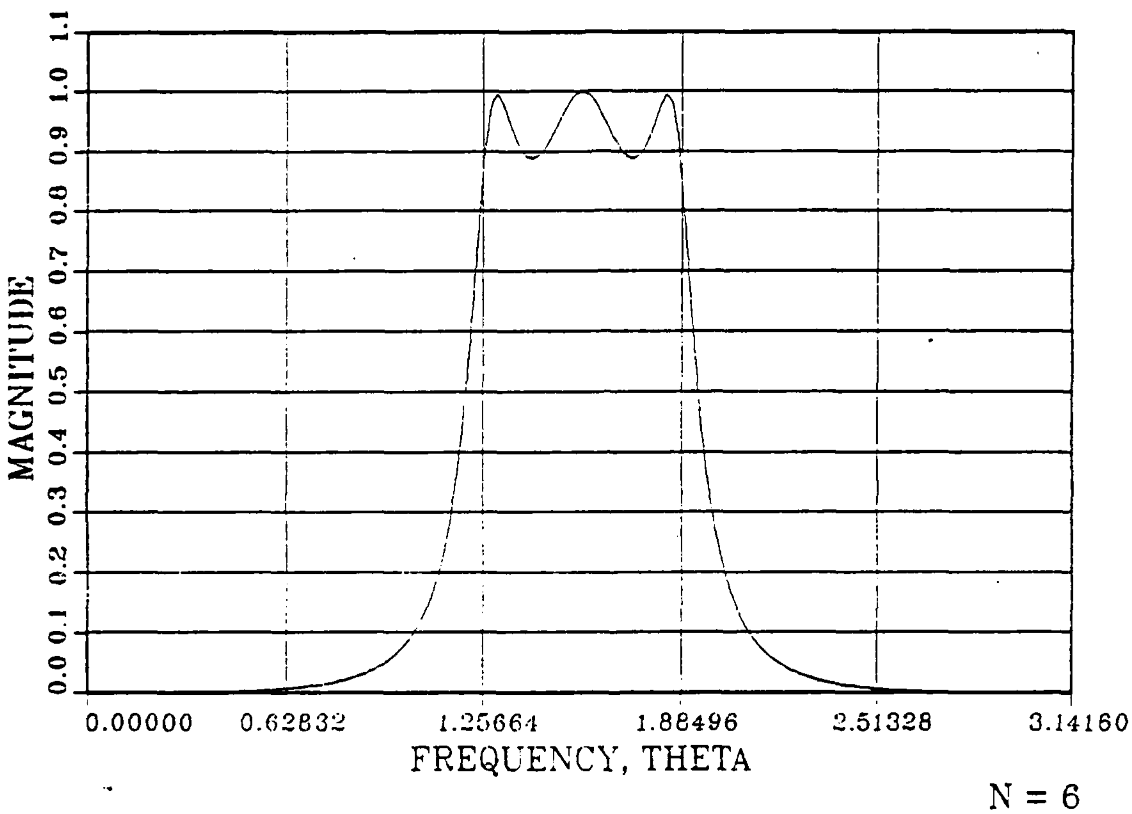


Figure 2.13a. Frequency Response for Chebyshev Bandpass Filter

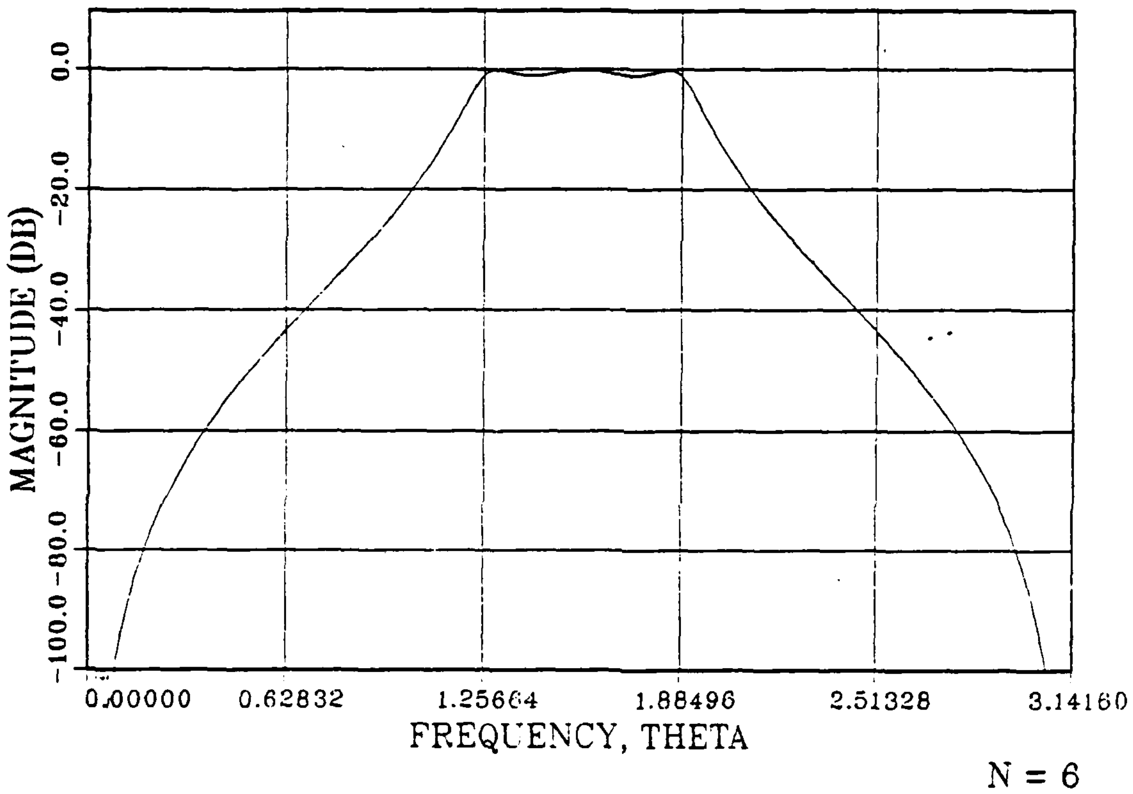


Figure 2.13b. ***Frequency Response for***

Chebyshev Bandpass Filter (dB)

Two examples will be presented that illustrate this process; however, a more detailed explanation of how the transfer functions of Table 2.9 were arrived at is in order.

For purposes of explanation, a second-order filter with 0.5 dB passband ripple, and a stopband gain of −20 dB will used.

Turning to Table 2.4, the lowpass prototype transfer function is of the form:

$$H_{LP_P}(s) = \frac{H_0 s^2 + H_0 A_{01}}{s^2 + B_{11} s + B_{01}}$$

where,

$$H_0 = 0.100220$$

$$A_{01} = 5.33789$$

$$B_{01} = 0.566660$$

$$B_{11} = 0.809390$$

therefore,

$$H_{LP_P}(s) = \frac{0.1s^2 + 0.5338}{s^2 + 0.8094s + 0.5667} \tag{2.40}$$

This prototype has a value for R of 2.76261. The parameter R is the ratio of the stopband frequency, w_{2_P}, to the passband cutoff frequency, w_{1_P}(i.e., $R = w_{2_P} / w_{1_P}$). Thus, it describes the sharpness of the transition region. A large value of R is indicative of a broad transition region, while, conversely, a small value corresponds to a narrow transition region. As expected, the higher the filter order, the narrower the transition region, and the smaller the value for R. Figure 2.14 (due to [5]), illustrates the magnitude squared frequency response of the normalized lowpass elliptic filters of Table 2.4.

For this particular example, w_{1_P} and w_{2_P} are subject to the constraint that their ratio R be 2.76261. Furthermore, it should also be noted, the filters

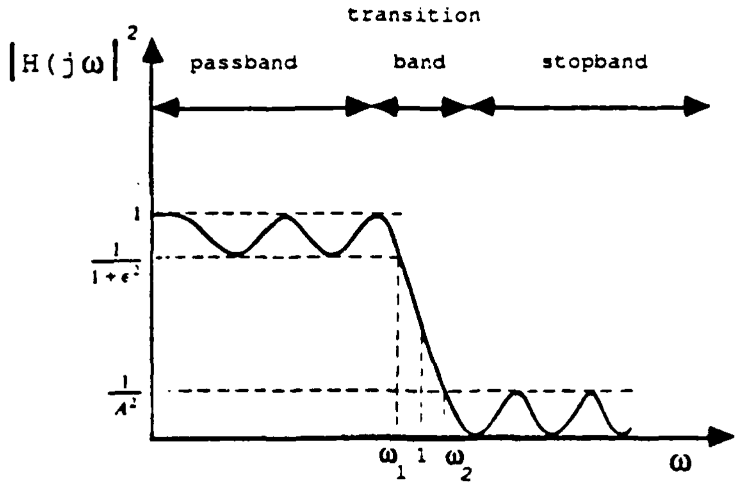


Figure 2.14. Magnitude Squared Frequency Response of a Normalized Lowpass Elliptic Filter

comprising this table have been normalized, so that the geometric mean of w_{1_P} and w_{2_P} is one.

$$(w_{1_P} w_{2_P}) = 1 \tag{2.41}$$

Combining these two constraints, the following relationships between w_{1_P} and w_{2_P} apply:

$$w_{1_P} = 1/\sqrt{R} = 0.6016 \text{ rad}/s \tag{2.42}$$

$$w_{2_P} = \sqrt{R} = 1.6620 \text{ rad}/s \tag{2.43}$$

The goal of the direct design procedure is to find a digital parallel to the analog prototype filters, and tabulate a table of digital lowpass prototypes.

Step 1 of this process is to normalize the transfer functions of Table 2.4 so that they all have a value for w_{1_P} of one. This is depicted Figure 2.15.

For this example:

$$H_{LP_P}(s) = \frac{0.1s^2 + 0.5338}{s^2 + 0.8094s + 0.5667} \tag{2.44}$$

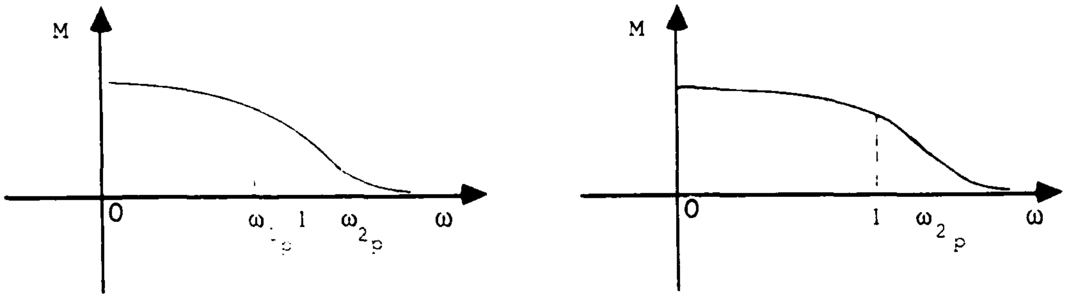


Figure 2.15. Normalization of Elliptic Filter

where

$$w_{1_P} = 1/\sqrt{R} = 1/\sqrt{2.76261} = 0.6061$$

To normalize $H_{LP_P}(s)$ to $w_{1_P} = 1$, let $s = s/\sqrt{R} = 0.6061s$

$$\begin{aligned} H_{LP_P}(s) &= \frac{0.1(0.6061s)^2 + 0.5338}{(0.6061s)^2 + 0.8094(0.6061s) + 0.5667} \\ &= \frac{0.0367s^2 + 0.5338}{0.3674s^2 + 0.4906s + 0.5667} \end{aligned} \tag{2.45}$$

where now, $w_{1_P} = 1$. and $w_{2_P} = R$, since $R = w_{2_P}/w_{1_P}$.

The normalized analog prototype was then converted to a digital prototype through the use of bilinear transformation.

$$\begin{aligned} H_{LP_P}(z) &= H_{LP_P}(s)\Big|_{s=\frac{z-1}{z+1}} \\ &= \frac{0.0367\left(\frac{z-1}{z+1}\right)^2 + 0.5338}{0.367\left(\frac{z-1}{z+1}\right)^2 + 0.4906\left(\frac{z-1}{z+1}\right) + 0.5667} \\ &= \frac{1.575z^2 - 2.75z + 1.575}{3.91z^2 + 1.13z + 1.22} \end{aligned} \tag{2.46}$$

The digital passband critical frequency is $\theta_{1_P} = \pi/2$, since an analog frequency of $w_{1_P} = 1$ corresponds to digital frequency of $\pi/2$ when using the bilinear transformation.

Table 2.9 contains the compiled results of applying this process to some of the analog prototype transfer functions of Table 2.4. As with analog elliptic filters, special relationships exist between the passband critical frequency, θ_{1_P}, and the stopband frequency, θ_{2_P}, of digital elliptic filters.

Using the digital frequency transformations of Table 2.5, the following values for θ_{1_P} and θ_{2_P} are found based on the analog prototype values for w_{1_P} and w_{2_P}.

For θ_{1_P},

$$\begin{aligned} e^{j\theta_{1_P}} &= \frac{j1+1}{-j1+1} = \frac{1e^{j\pi/4}}{1e^{-j\pi/4}} = 1e^{j\pi/2} \\ \Longrightarrow \theta_{1_P} &= \pi/2 \end{aligned} \tag{2.47}$$

For θ_{2_P},

$$\begin{aligned} e^{j\theta_{2_P}} &= \frac{jR+1}{-jR+1} = \frac{1e^{j\tan^{-1}R}}{1e^{-j\tan^{-1}(-R)}} = 1e^{j2\tan^{-1}R} \\ \Longrightarrow \theta_{2_P} &= 2\tan^{-1}R \end{aligned} \tag{2.48}$$

Again, R is the transition region parameter described by the ratio of w_{2_P} to w_{1_P} of the analog version of the filter. Table 2.9 presents a summary of the values for θ_{1_P}, θ_{2_P}, and R for some of the digital lowpass prototype elliptic filters.

Example 2.6

Once again the design example of the previous section will be used to illustrate the new direct design procedure. The problem statement is:

Design an elliptic lowpass filter that meets the following specifications:

- passband ripple of 0.5 dB
- passband ripple-edge frequency of 2 kHz
- stopband gain of at most −20 dB for frequencies greater than 6 kHz
- sampling frequency is 20 kHz.

TABLE 2.9

ELLIPTIC LOWPASS PROTOTYPE FILTERS

Passband ripple = 0.5 dB ; Stopband gain = −20 dB

(due to reference 5)

N	θ_{1P}	θ_{2P}	R
2	1.5708	2.447	2.7626
3	1.5708	1.9157	1.9157
4	1.5708	1.6145	1.0447
5	1.5708	1.5862	1.0155

Transfer Functions

N	$H(z)$
2	$\frac{1.575z^2+2.75z+1.575}{3.91z^2+1.13z+1.22}$
3	$\frac{1.276z^3+2.367z^2+2.376z+1.276}{4.626z^3+0.008z^2+3.030z-0.352}$
4	$\frac{1.4179z^4+2.366z^3+3.4362z^2+2.336z+1.4179}{5.885z^4-1.0958z^3+6.5858z^2-1.0954z+1.338}$
5	$\frac{1.794z^5+2.9512z^4+4.8532z^3+4.8532z^2+2.9512z+1.794}{7.9516z^5-2.314z^4+12.7192z^3-3.1856z^2+4.9276z-0.902}$

Step 1: Find the critical digital design frequencies.

$$\theta'_{1_P} = \frac{2\pi w_1}{f_s} = \frac{2\pi(2\times 10^3)}{20\times 10^3} = 0.2\pi \text{ rad } = \theta'_C$$
$$\theta'_{2_P} = \frac{2\pi w_2}{f_s} = \frac{2\pi(6\times 10^3)}{20\times 10^3} = 0.6\pi \text{ rad}$$

Step 2: To determine the filter order of the prototype filter, find α and θ_{2_P}, which are defined as follows from Table 2.5.

$$\alpha = \frac{\sin\left(\theta_C/2 - \theta'_{1_P}/2\right)}{\sin\left(\theta_C/2 + \theta'_{1_P}/2\right)}$$

where θ_C is the critical frequency of the prototype filter i.e., $\theta_C = \pi/2$. Therefore,

$$\begin{aligned}\alpha &= \frac{\sin(0.5\pi/2 - 0.2\pi/2)}{\sin(0.5\pi/2 + 0.2\pi/2)} \\ &= \frac{\sin(0.7854 - 0.31416)}{\sin(0.7854 + 0.31416)} = \frac{0.45399}{1.09956} \\ &= 0.5095\end{aligned} \tag{2.49}$$

Find θ_{2_P}, the stopband frequency for the lowpass prototype filter.

$$\begin{aligned}e^{j\theta_{2_P}} &= \frac{e^{j\theta'_{2_P}} - \alpha}{1 - \alpha e^{j\theta'_{2_P}}} \\ &= \frac{e^{j1.855} - 0.5095}{1 - (0.5095)e^{j1.855}} \\ &= \frac{1.255e^{j2.282}}{1.255e^{-j0.3965}} = 1e^{j2.6785} \\ &\Longrightarrow \theta_{2_P} = 2.6785 \text{ rad}\end{aligned} \tag{2.50}$$

Step 3: From the design chart (Table 2.9), determine the lowest order elliptic lowpass filter prototype. Table 2.9, indicates that a second-order filter has a stopband frequency θ_{2_P} of 2.447 rad, which meets the design specifications. Thus, the lowpass prototype transfer function is:

$$H_{LP_P}(z) = \frac{1.575z^2 + 2.75z + 1.575}{3.91z^2 + 1.13z + 1.22} \tag{2.51}$$

Step 4: Find the actual filter transfer function, using the prototype transfer function and the digital filter frequency transformations of Table 2.5.

$$\begin{aligned} H_{LP}(z) &= H_{LP_P}(z)\Big|_{z=\frac{z-\alpha}{1-\alpha z}} \\ &= \frac{1.575z^2+2.75z+1.575}{3.91z^2+1.13z+1.22}\Bigg|_{z=\frac{z-0.5095}{1-(0.5095)z}} \\ &= \frac{0.5829z^2+0.2541z+0.5829}{3.651z^2-3.8042z+1.6593} \end{aligned} \tag{2.52}$$

or

$$H_{LP}(z) = \frac{0.1596z^2+0.0696z+0.1596}{z^2-1.042z+0.4545} \tag{2.53}$$

The transfer function of Equation (2.52) is identical to the transfer function obtained using the traditional design technique (see previous section).

Step 5: Verify the design by obtaining a computer generated frequency response plot (see Figure 2.16).

To further illustrate this direct design method, an additional example follows involving the design of a bandpass filter.

Example 2.7

A digital elliptic bandpass filter is to be designed to meet the following specifications:

- 0.5 dB ripple in the frequency range 600 Hz $\leq f \leq$ 900 Hz,
- sampling frequency of 3 kHz, and
- maximum gain of -20 dB for $0 \leq f \leq$ 200 Hz.

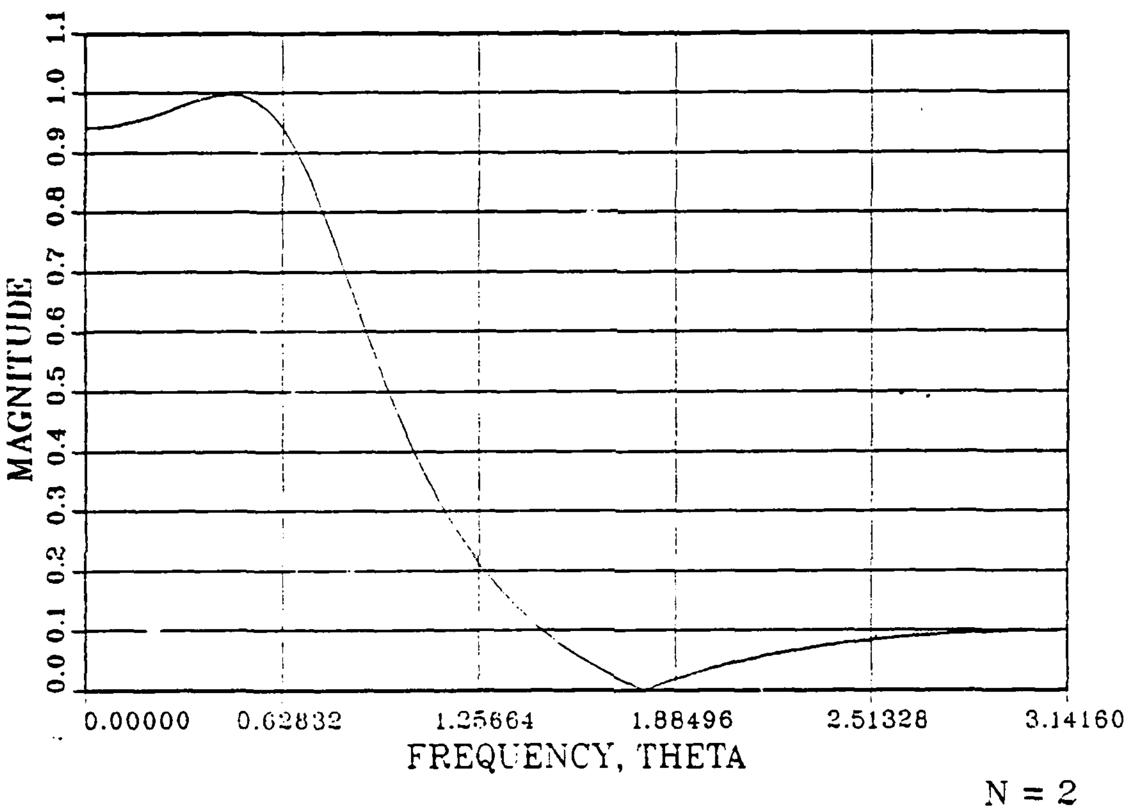


Figure 2.16. Frequency Response for Elliptic Lowpass Filter

Step 1: Find the critical digital frequencies from the design specifications:

$$\theta'_\ell = \frac{2\pi f_\ell}{f_s} = \frac{2\pi(600)}{3000} = 0.4\pi = 1.25 \text{ rad}$$
$$\theta'_u = \frac{2\pi f_u}{f_s} = \frac{2\pi(900)}{3000} = 0.6\pi = 1.88 \text{ rad}$$
$$\theta'_{2_P} = \frac{2\pi f_2}{f_s} = \frac{2\pi(200)}{3000} = 0.133\pi = 0.418 \text{ rad}$$
$$\theta'_0 = \sqrt{\theta_\ell \theta_u} = \sqrt{(1.25)\,(1.88)} = 1.53 \text{ rad}$$

Step 2: Again, since the design chart is based on a cutoff frequency of $\theta_C = \pi/2$, the design specifications need to be normalized to determine the lowpass prototype filter from Table 2.9.

From the Table of Digital Frequency Transformations (Table 2.5), the following frequency transformations apply:

Prototype filter from desired filter:

$$e^{j\theta_{2_P}} = -\frac{e^{j2\theta'_{2_P}} - \frac{2\alpha k}{k+1} e^{j\theta'_{2_P}} + \frac{k-1}{k+1}}{1 - \frac{2\alpha k}{k+1} e^{j\theta'_{2_P}} + \frac{k-1}{k+1} e^{j\theta'_{2_P}}} \tag{2.53}$$

where:

$$\alpha = \frac{\cos(\theta'_u/2 + \theta'_\ell/2)}{\cos(\theta'_u/2 - \theta'_\ell/2)}$$
$$k = \tan(\theta_C/2)\cot(\theta'_u/2 - \theta'_\ell/2)$$

θ'_{2_P} is the stopband frequency used to determine the filter order, N. θ_C is the cutoff frequency of the prototype filter, $\theta_C = \pi/2$.

For this problem:

$$\theta_{2_P} = ?$$

$$\theta'_{2_P} = 0.133\pi$$

$$\theta'_u = 0.6\pi$$

$$\theta'_\ell = 0.4\pi$$

$$\theta_C = 0.5\pi$$

$$\alpha = \frac{\cos(0.3\pi + 0.2\pi)}{\cos(0.3\pi - 0.2\pi)} = \frac{\cos(0.5)\pi}{\cos(0.1\pi)}$$

$$\alpha = 0$$

$$k = \tan(0.5\pi/2)\cot(0.6\pi/2 - 0.4\pi/2) = 3.07$$

Step 3: Find θ_{2_P} to determine the lowest order prototype filter that may be used. Since $\alpha = 0$

$$\begin{aligned} e^{j\theta_{2_P}} &= -\frac{e^{j2\theta'_{2_P}} + \frac{k-1}{k+1}}{1 + \frac{k-1}{k+1}e^{j2\theta'_{2_P}}} = -\frac{e^{j\theta'_{2_P}} + 0.509}{1 + (0.509)e^{j2\theta'_{2_P}}} \\ &= -\frac{e^{j0.836} + 0.509}{1 + (0.509)e^{j0.836}} = \frac{1.394e^{-j2.58}}{1.394e^{j0.282}} = 1e^{-j2.86} \end{aligned} \tag{2.55}$$

$$\Longrightarrow \theta_{2_P} = 2.86 \text{ rad}$$

Step 4: From Table 2.9, determine the filter order of the prototype filter that corresponds to the above value of θ_{2_P}. It can be seen from the table that a second-order filter fulfills the design specifications.

Again, the lowpass prototype transfer function is:

$$H_{LP_P}(z) = \frac{1.575z^2 + 2.75z + 1.575}{3.91z^2 + 1.13z + 1.22} \tag{2.56}$$

Step 5: As in previous design problems, the prototype filter transfer function is converted to the actual filter transfer function through the use of the digital frequency transformations of Table 2.5.

$$\begin{aligned} H_{BP}(z) &= H_{LP_P}(z)\Big|_{z=-\frac{z^2-\frac{2\alpha k}{k+1}z+\frac{k-1}{k+1}}{1-\frac{2\alpha k}{k+1}z+\frac{k-1}{k+1}z^2}} \\ &= \frac{1.575z^2+2.75z+1.575}{3.91z^2+1.13z+1.22}\Big|_{z=\frac{z^2+0.509}{-0.509z^2-1}} \\ &= \frac{0.5833z^4-0.2558z^2+0.5833}{3.6509z^4+3.7996z^2+1.6579} \end{aligned} \tag{2.57}$$

Step 6: Obtain the frequency response plot for design verification (see Figure 2.17).

In summary then, this chapter has presented three commonly used recursive filter types: Butterworth, Chebyshev and elliptic. The type chosen by the designer is dependent on the requirements of his/her particular design problem.

Butterworth filters exhibit a flat frequency response characteristic, but do not have as steep a transition region for the same order filter as do Chebyshev and elliptic filters. With Chebyshev and elliptic filters the steepness of the transition region is attained by accepting a degree of ripple in the passband.

All three filter types may be designed using either the traditional approach, involving an analog prototype filter transfer function, or the direct design approach that uses digital prototype filter transfer functions.

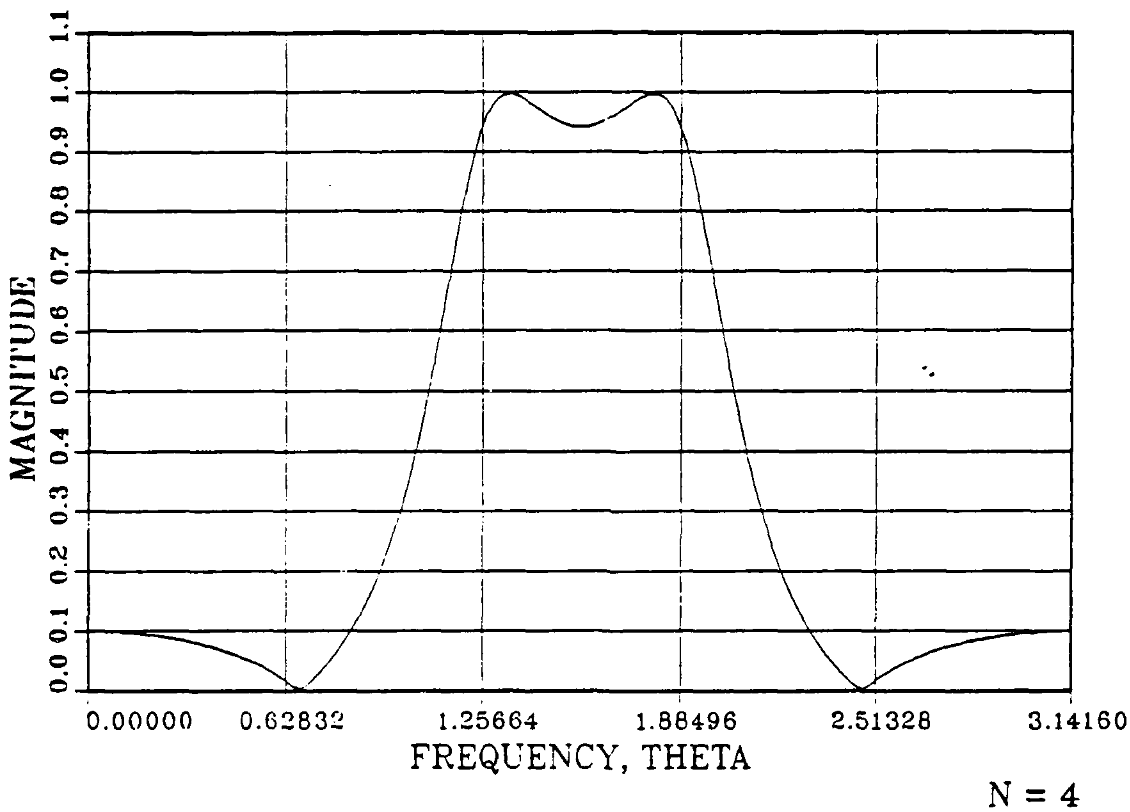


Figure 2.17. Frequency Response for Elliptic Bandpass Filter

III. NONRECURSIVE FILTER DESIGN

A. INTRODUCTION

The nonrecursive realization of digital filters is desirable because it can result in two very attractive features: linear phase and the absence of stability problems because all of the poles are at $z = 0$.

Nonrecursive filters are normally characterized by a finite duration impulse response; consequently, historical methods of design involve the analytical determination of the filter coefficients by expanding the desired frequency response in a Fourier series and then truncating the series to the desired filter length (order). A disadvantage inherent in this design method is the presence of an overshoot that occurs at discontinuities in the desired frequency response (known as Gibbs' phenomenon). The use of window functions was devised as a remedy to this problem; examples are the Kaiser, Hamming and von Hann windows [14].

Another popular method of FIR filter design is frequency sampling, whereby the desired filter frequency response is sampled, and then the inverse discrete Fourier transform of these sample values is determined to find the filter impulse response.

In this chapter both of these methods will be presented, and several design examples given, but first a review of the theoretical background of nonrecursive digital filter design by Fourier methods is in order.

B. BACKGROUND

The design of nonrecursive digital filters hinges on the following relationships:

A causal nonrecursive system can be described by a difference equation of the form [1]

$$y(n) = \sum_{k=0}^{k=L} b_k x(n-k). \tag{3.1}$$

For an input $x(n) = e^{jn\theta}$ the steady - state system output is

$$y_{ss}(n) = e^{jn\theta} H(e^{j\theta}) \tag{3.2}$$

where $H(e^{j\theta})$ is the system's frequency response.

Expanding the right side of Equation (3.1) for the same input $x(n) = e^{jn\theta}$, yields the following steady - state output

$$\begin{aligned} y_{ss}(n) &= \sum_{k=0}^{k=L} b_k e^{j(n-k)\theta} = \sum_{k=0}^{k=L} b_k e^{jn\theta} e^{-jk\theta} \\ &= b_0 e^{jn\theta} + b_1 e^{jn\theta} e^{-j\theta} + b_2 e^{jn\theta} e^{-j2\theta} + \ldots + b_L e^{jn\theta} e^{-jL\theta} \\ &= e^{jn\theta} \left[b_0 + b_1 e^{-j\theta} + b_2 e^{-j2\theta} + \ldots + b_L e^{-jL\theta} \right]. \end{aligned} \tag{3.3}$$

Equating Equations (3.2) and (3.3) gives

$$e^{jn\theta} H(e^{j\theta}) = e^{jn\theta} \left[b_0 + b_1 e^{-j\theta} + b_2 e^{-j2\theta} + \ldots + b_L e^{-jL\theta} \right] \tag{3.4}$$

which implies

$$H(e^{j\theta}) = \sum_{n=0}^{n=L} b_n e^{-jn\theta}. \tag{3.5}$$

But, by definition the frequency response of an LTI system is

$$H(e^{j\theta}) = \sum_{n=\infty}^{n=\infty} h(n) e^{-jn\theta}. \tag{3.6}$$

For a causal filter ($h(n) = 0, n < 0$), with a finite number of delays ($n = L$), however, the frequency response definition is

$$H(e^{j\theta}) = \sum_{n=0}^{n=L} h(n) e^{-jn\theta}. \tag{3.7}$$

Comparing Equations (3.5) and (3.7) produces the following relationship that is the crux of nonrecursive filter design

$$h(n) = b_n. \tag{3.8}$$

Equation (3.8) establishes a direct relationship between the unit impulse response of an FIR filter and the coefficients, b_n, in the system difference equation.

The goal of the design techniques presented in this chapter is to determine the filter coefficients (or weights), $b_0, b_1, \ldots, b_L$ for a desired frequency response characteristic, $H_D(e^{j\theta})$. These techniques include: Fourier Coefficient Design for lowpass, highpass, bandpass and bandstop filters; Windowing; and Frequency Sampling.

C. FOURIER COEFFICIENT DESIGN

As stated in the introduction we need to determine the filter coefficients, which, in the case of nonrecursive filters, are also the coefficients of the unit impulse response, $h(n)$. Thus, a relationship between the desired frequency response and the impulse response needs to be established.

Beginning with the desired frequency response

$$H_D(e^{j\theta}) = \sum_{n=-\infty}^{n=\infty} h(n)e^{-jn\theta}. \tag{3.9}$$

It can be shown, [1], that $h(n)$ of Equation (3.9) may be written,

$$h(n) = (1/2\pi)\int_{\theta_0}^{2\pi+\theta_0} H_D(e^{j\theta})e^{jn\theta}d\theta. \tag{3.10}$$

Equation (3.9) is recognized as the Fourier series expansion of the function $H_D(e^{j\theta})$, where the $h(n)$ are the Fourier coefficients (hence the source of the name for this design technique).

Depending on the type of filter that is desired, this expression can be reduced to the following forms:

Lowpass Filters:

$$h_{LP}(n) = (K/\pi n)\sin(n\theta_c)$$

$$K = \text{ desired magnitude in the passband}$$

$$n = 0, \pm 1, \pm 2, \ldots \tag{3.11}$$

$$\theta_c = \text{ the desired cutoff frequency}$$

The number of coefficients is truncated to correspond to the desired filter order; as expected, the higher the order, the better the approximation to the desired frequency response.

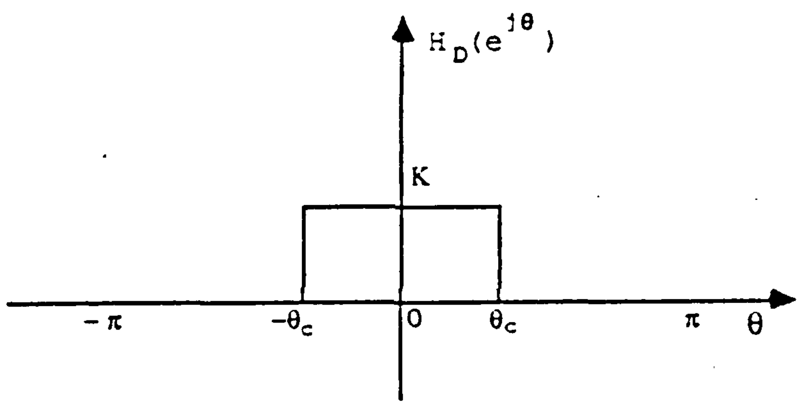


Figure 3.1. Ideal Lowpass Filter Frequency Response

Highpass Filters:

$$h_{HP}(n) = h_{LP}(n)(-1)^n \tag{3.12}$$

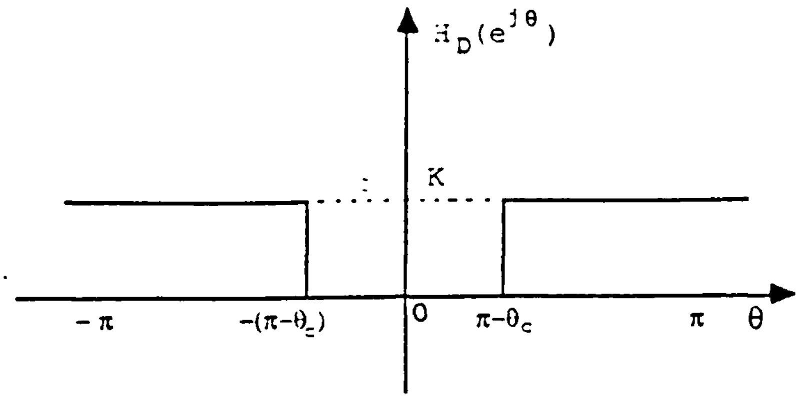


Figure 3.2. Ideal Highpass Filter Frequency Response

<u>Bandpass Filters</u>:

$$h_{BP}(n) = [2\cos(n\theta_0)]\, h_{LP}(n) \tag{3.13}$$

where,

$$\theta_c = \frac{\theta u - \theta \ell}{2} \quad ; \quad \theta_u = \text{ upper passband frequency}$$
$$\theta_0 = \frac{\theta u + \theta \ell}{2} \quad ; \quad \theta_\ell = \text{ lower passband frequency}$$

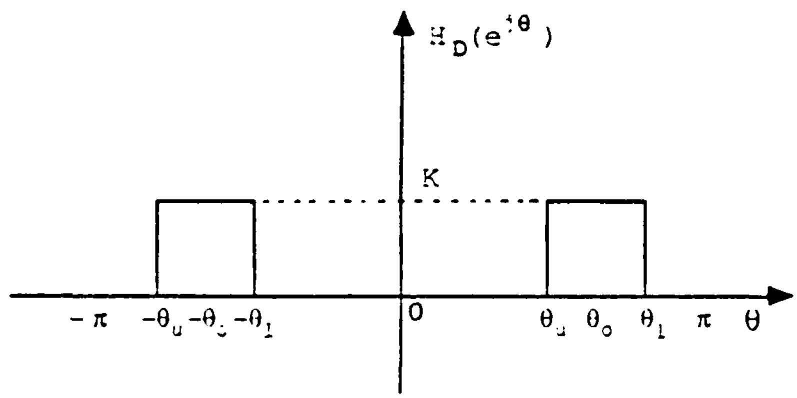


Figure 3.3. Ideal Bandpass Filter Frequency Response

<u>Bandstop Filters</u>:

$$h_{BS}(0) = K - h_{BP}(0) \tag{3.14}$$

$$h_{BS}(n) = -h_{BP}(n); n = \pm 1, \pm 2, \ldots$$

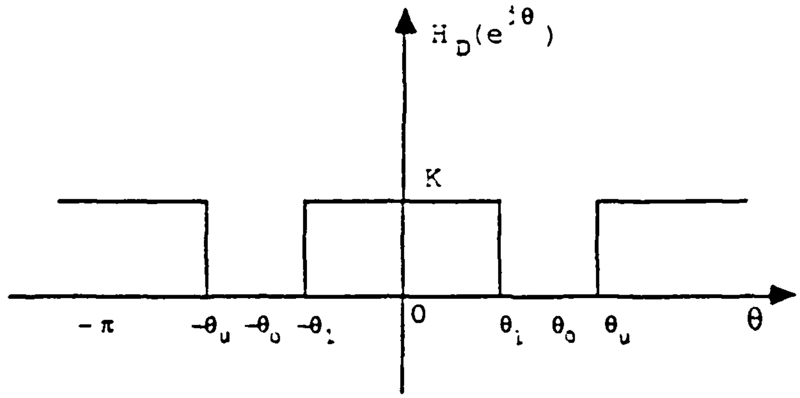


Figure 3.4. Ideal Bandstop Filter Frequency Response

As can be seen in the previous impulse response expressions for the various filter types, they are all based on a lowpass filter prototype, $h_{LP}(n)$. Thus, the design steps for the ideal lowpass prototype filter can be summarized as follows, bearing in mind that once the lowpass filter coefficients are determined they may be transformed to the coefficients for a highpass, bandpass, or bandstop filter, if desired.

1. <u>Lowpass Prototype Filter Design Procedure</u>

<u>Step 1</u>: Translate the design specifications to those of a lowpass prototype. Determine the desired critical frequency, θ_c, and the passband magnitude, K.

<u>Step 2</u>: Find the lowpass filter coefficients given by Equation (3.8), that is,

$$h_{LP}(n) = b_n = (K/\pi n)\sin(n\theta_c); \quad n = 0, \pm 1, \pm 2, \ldots, \pm L \qquad (3.15)$$

with $L = (N-1)/2$.

<u>Step 3</u>: Shift $h_{LP}(n)$ to the right by L terms to make the filter causal.

$$h_{LP}(n) = [K/\pi(n-L)]\sin[(n-L)\theta_c]; \quad n = 0, 1, 2, \ldots, 2L \qquad (3.16)$$

Step 4: Transform the coefficients, $h_{LP}(n)$, to lowpass, highpass, bandpass or bandstop, as desired. Implement the design, and compare the frequency response obtained with the original specifications to ensure that these specifications are met.

As an example, suppose a highpass filter with a passband of unity gain for frequencies greater than 10 kHz is desired. The system sampling frequency is 50 kHz.

Step 1: Cutoff frequency, $\theta'_c = wT = \frac{2\pi f}{f_s} = \frac{2\pi(10^4)}{50(10^3)} = 0.4\pi$

Passband gain, $K = 1.0$

Translating to lowpass prototype specifications:

$\theta'_c = \pi - \theta_c$; where θ_c is the cutoff frequency of the lowpass prototype and θ'_c is the cutoff frequency of the highpass filter

$$\Longrightarrow \theta_c = \pi - \theta'_c = \pi - 0.4\pi = 0.6\pi$$

Step 2: From Equation (3.11), the lowpass prototype filter coefficients are:

$$h_{LP}(n) = \sin(0.6\pi n)/(\pi n); n = 0, \pm 1, \pm 2, \ldots, \pm 10$$

for a 21-coefficient filter, while the highpass filter coefficients are obtained from Equation (3.12). The resulting frequency response is illustrated in Figure 3.5.

n	h_{LP}	h_{HP}
0	0.600	0.600
±1	0.303	-0.303
±2	-0.094	-0.094
±3	-0.062	0.062
±4	-0.076	0.076
±5	0.0	0.0
±6	-0.050	-0.050
±7	0.027	-0.027
±8	0.023	0.023
±9	-0.034	0.034
±10	0.0	0.0

D. WINDOWS

It has been shown that the expression for the desired frequency response, $H_D(e^{j\theta})$, can be written in terms of a Fourier series. Equations (3.9) and (3.10) are rewritten below for convenience.

$$H_D(e^{j\theta}) = \sum_{n=-\infty}^{n=\infty} h(n)e^{-jn\theta}d\theta \tag{3.9}$$

where

$$h(n) = (1/2\pi)\int_{\theta_0}^{2\pi+\theta_0} H_D(e^{j\theta})e^{jn\theta}d\theta. \tag{3.10}$$

In actual implementation, however, the infinite sum in Equation (3.9) must necessarily be truncated, since a filter cannot have an infinite number of delays. Also, at points of discontinuity in the ideal frequency response (where the magnitude abruptly changes from 1.0 to 0.0, or vice versa), the Fourier series approximation cannot match the desired frequency response exactly, even if an infinite number of terms were possible. There is always an overshoot of about nine percent near

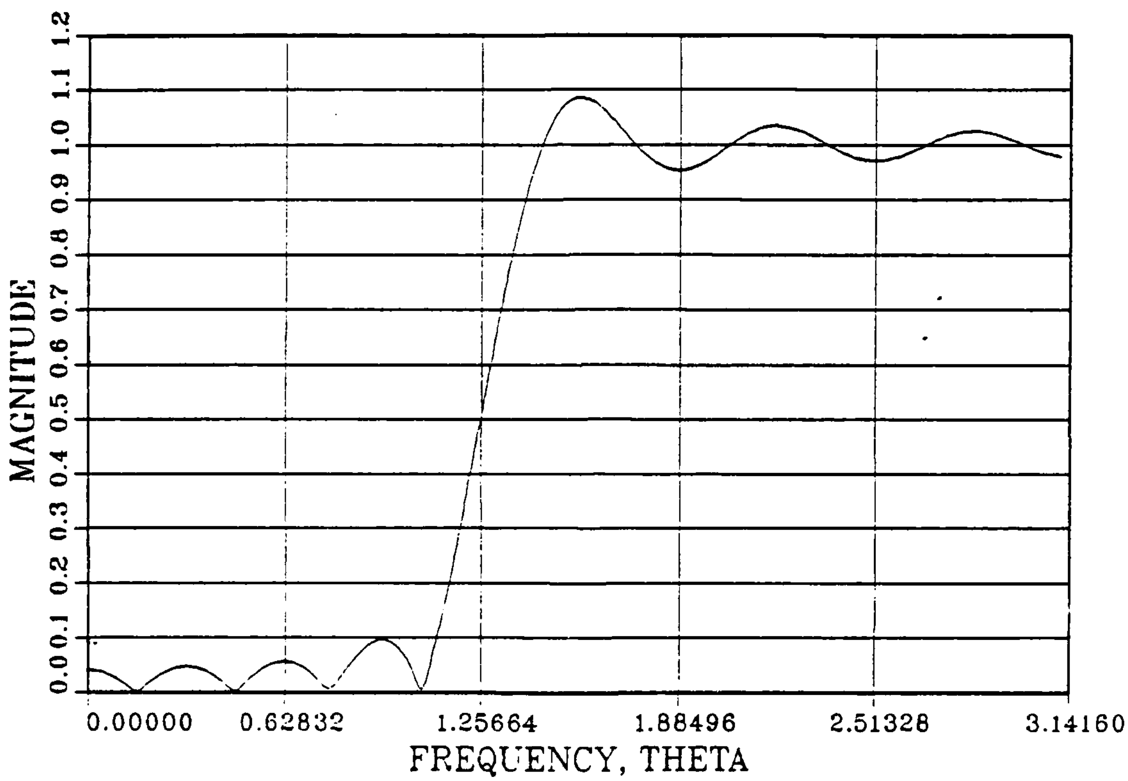


Figure 3.5. Frequency Response for Fourier Coefficient Highpass Filter

discontinuities. This overshoot is the well known Gibbs' phenomenon, and is alleviated through the use of window functions. The filter weights, $h(n)$, are modified using one of several available analytical expressions (Hamming, Von Hann, etc.) [14].

$$\hat{h}(n) = h(n) \cdot w(n) \tag{3.17}$$

$\hat{h}(n)$ = modified unit sample response

$h(n)$ = original unit sample response

$w(n)$ = window function

The modified value of the desired frequency response is therefore

$$\begin{aligned} \hat{H}_D(e^{j\theta}) &= \sum_{n=-L}^{n=L} \hat{h}(n)e^{-jn\theta} \\ &= \sum_{n=-L}^{n=L} h(n) \cdot w(n)e^{-jn\theta} \end{aligned} \tag{3.18}$$

again where,

$$h(n) = (1/2\pi) \int_{-\pi}^{\pi} H_D(e^{j\theta})e^{jn\theta} d\theta$$

which exhibits a gradual roll-off, rather than the steep slope characteristic of the ideal frequency response. Thus, the tradeoff involved when using windows in Fourier design is that a reduction in the overshoot caused by Gibbs' phenomenon, is achieved at the expense of decreasing the sharpness of the frequency response cutoff. That is to say, the passband and stopband ripples are suppressed at the expense of a wider transition from passband to stopband; specifically, there is a less steep transition. Figures 3.6 and 3.7 illustrate these points; they depict an unwindowed and windowed lowpass filter design, respectively. For convenience, some of the more popular window functions are summarized [14]:

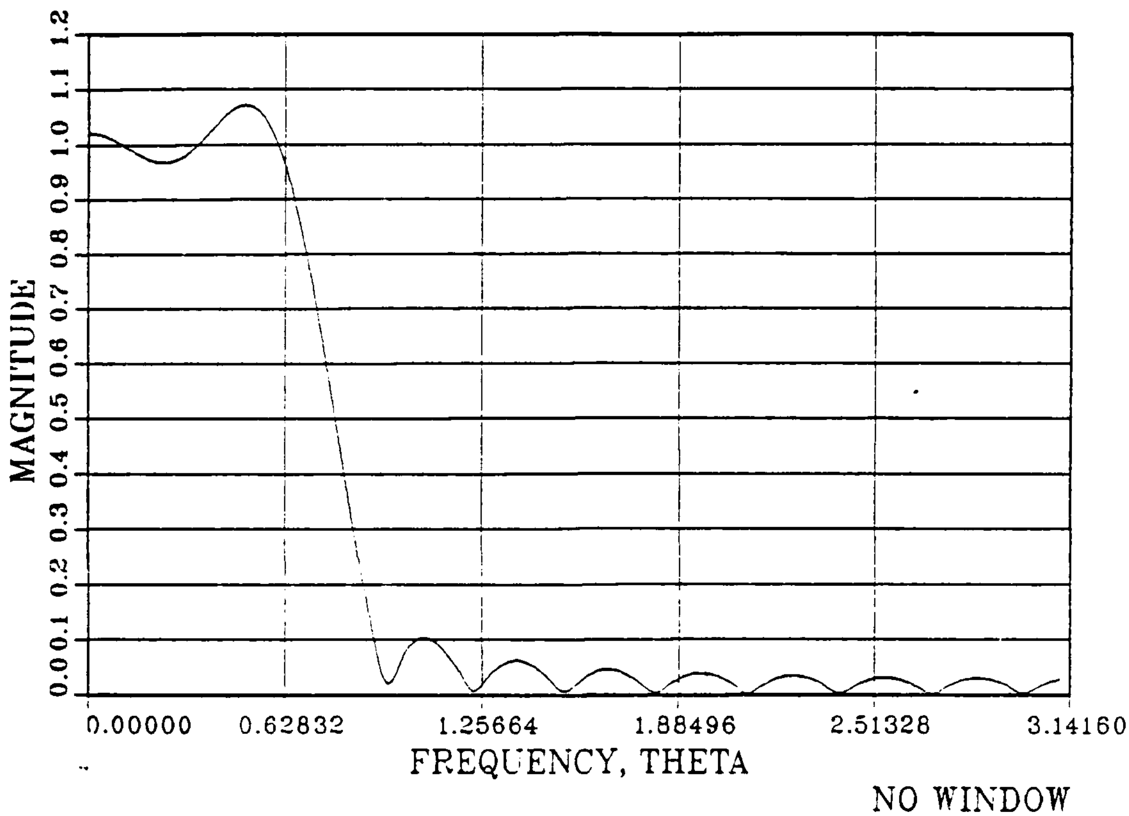


Figure 3.6. Unwindowed Lowpass Filter Frequency Response

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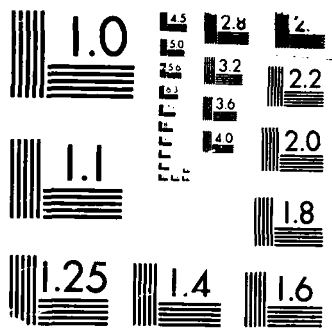

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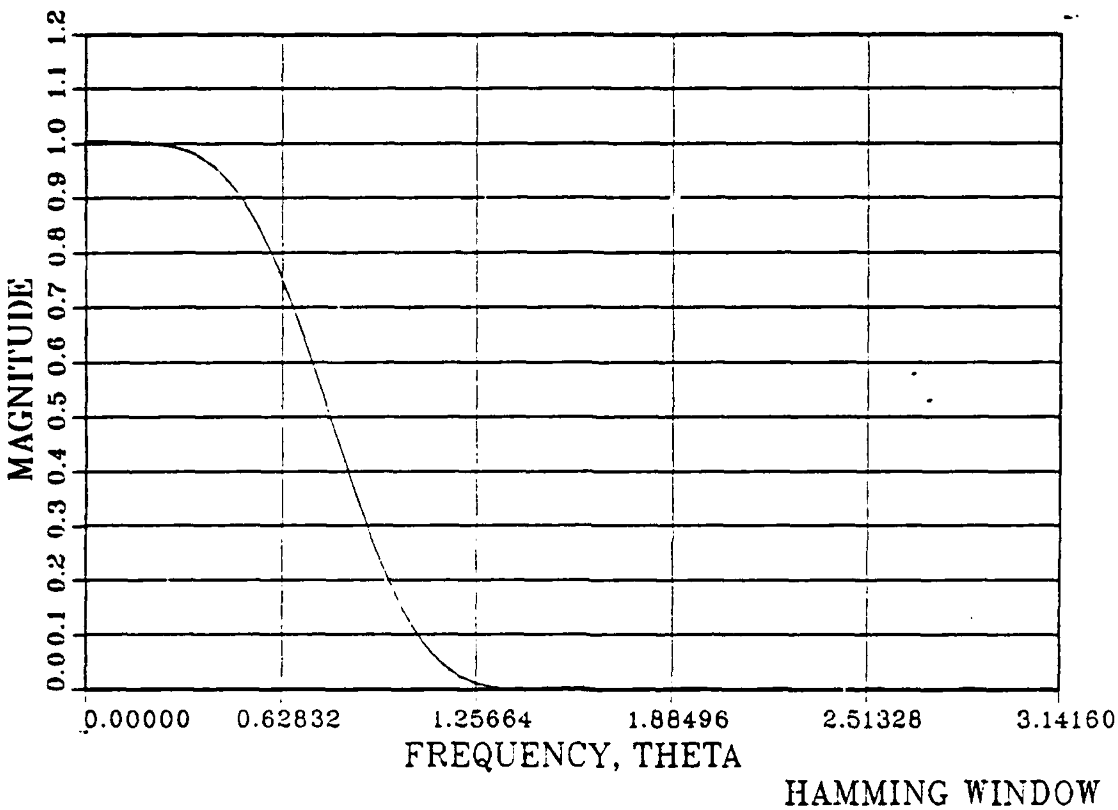


Figure 3.7. Windowed Lowpass Filter Frequency Response

- **Hamming Window**

$$w(n) = 0.54 + 0.46\cos(n\pi/L) \qquad n = 0, \pm 1, \pm 2, \ldots, \pm L \tag{3.19}$$

- **Von Hann Window**

$$w(n) = 0.50 + 0.50\cos(n\pi/L) \qquad n = 0, \pm 1, \pm 2, \ldots, \pm L \tag{3.20}$$

- **Kaiser Window**

$$w(n) = \frac{I_0(\beta[1 - (2n/L^2)])^{1/2}}{I_0\beta} \qquad n = 0, \pm 1, \pm 2, \ldots, \pm L \tag{3.21}$$

where $I_0(\cdot)$ is the modified zeroth-order Bessel function, and β is a constant that specifies a frequency response tradeoff between the peak height of the sidelobe ripples, and the width of the main lobe.

E. DESIGN OF A DIFFERENTIATOR

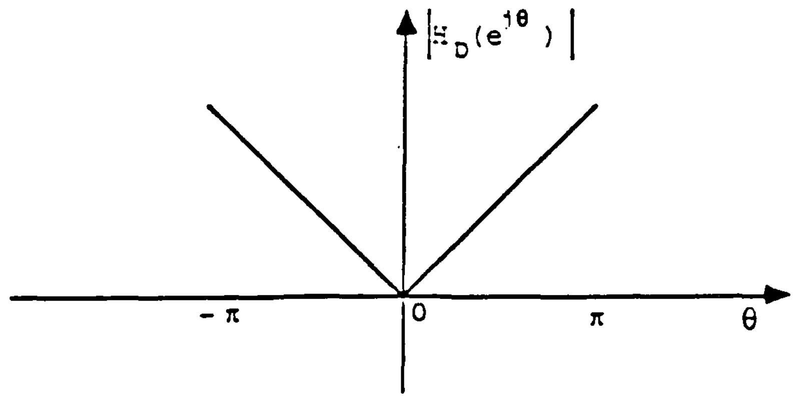
The Fourier series design procedure is easily applied to the design of a differentiator, which is often used in signal processing applications to track the rate of change of a signal. The desired analog frequency response for a differentiator is

$$H_D(jw) = jw. \tag{3.22}$$

Its counterpart in the digital frequency domain is

$$H_D(e^{j\theta}) = j\theta/T; \theta = wT. \tag{3.23}$$

Figure 3.8 illustrates the ideal magnitude and phase characteristic.



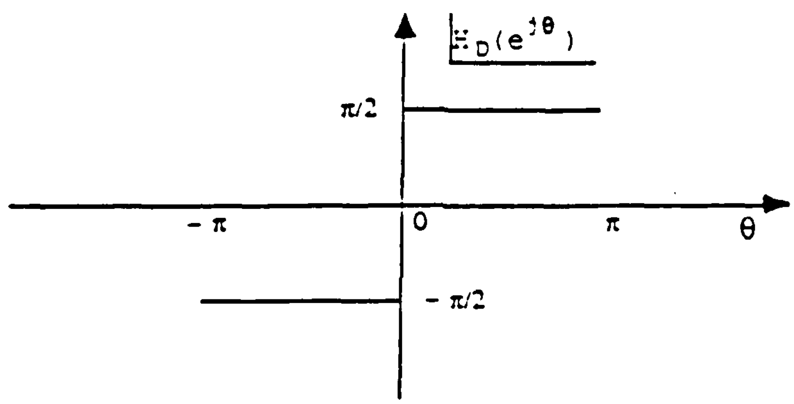


Figure 3.8. Ideal Differentiator Frequency Response Characteristic

The Fourier series design approach requires the determination of the Fourier series that approximates this ideal frequency response characteristic.

Substituting the desired frequency response (Eqn. 3.23) into Equation (3.10) yields

$$\begin{aligned} h(n) &= (1/2\pi)\int_{-\pi}^{\pi}(j\theta/T)e^{jn\theta}d\theta \\ &= (j/2\pi T)\int_{-\pi}^{\pi}\theta e^{jn\theta}d\theta. \end{aligned} \tag{3.24}$$

Evaluating this integral, the following expression is obtained for the unit sample response of a differentiator.

$$\begin{aligned} &h(n) = (-1)^n/(nT), n = \pm 1, \pm 2, \ldots \\ &h(n) = 0 \qquad , n = 0 \end{aligned} \tag{3.25}$$

Figure 3.9 depicts the frequency response of a 20th-order differentiator whose coefficients for $T = 1$ are as follows:

n	$h(n)$	n	$h(n)$
0	-1/10	11	-1
1	1/9	12	1/2
2	-1/8	13	-1/3
3	1/7	14	1/4
4	-1/6	15	-1/5
5	1/5	16	1/6
6	-1/4	17	-1/7
7	1/3	18	1/8
8	-1/2	19	-1/9
9	1	20	1/10
10	0		

Also, included is a 20th-order differentiator with a Hamming window, which, as can be seen in Figure 3.10, is a very good approximation.

In summary then, the Fourier series design technique is very effective, and can be used in many applications such as the design of a differentiator illustrated here. The use of window functions is necessary, however, to reduce the Gibbs' phenomenon effect, which introduces a sacrifice in the designed filter's transition region.

F. FREQUENCY SAMPLING

The window design technique introduced in the previous section has a distinct disadvantage in that the computation of Fourier series coefficients for the desired frequency response can be impossible when an analytical expression for the desired filter's frequency response is unknown or extremely difficult to determine.

Frequency sampling is a technique originally proposed by Gold and Jordan and later developed by Rabiner et al. as a solution to this problem [9]. This method has two distinct advantages:

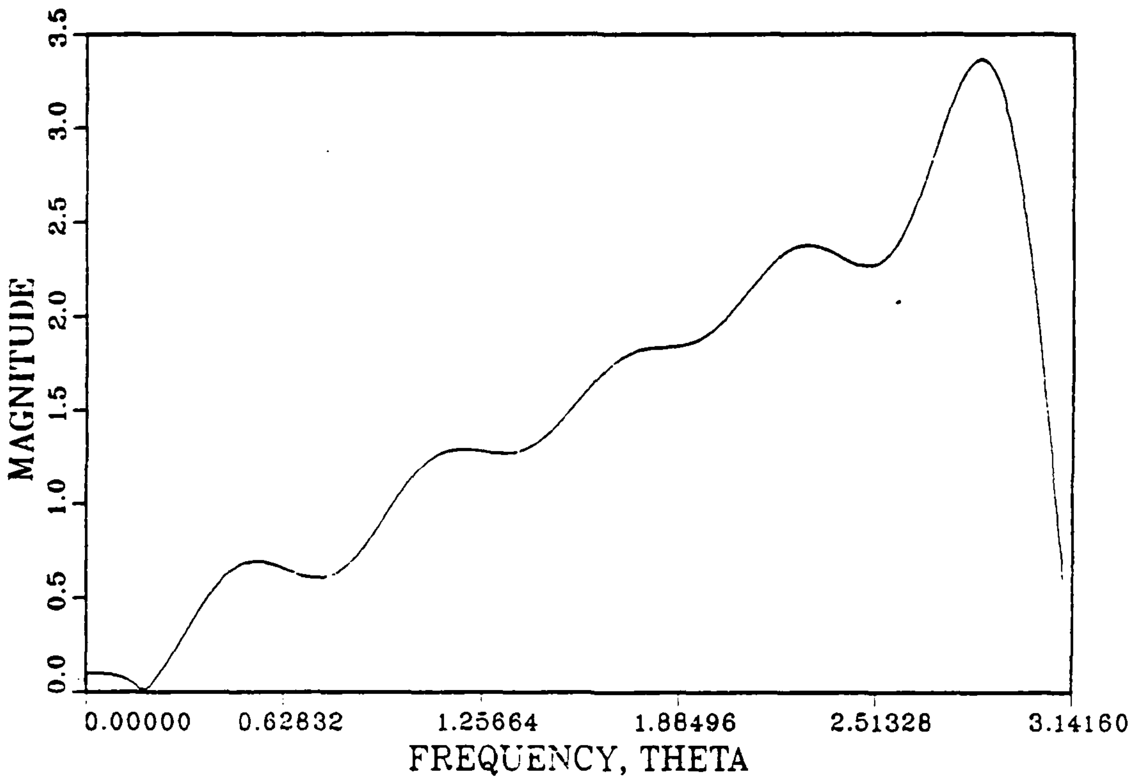


Figure 3.9. Unwindowed 21 Coefficient Bandpass Differentiator

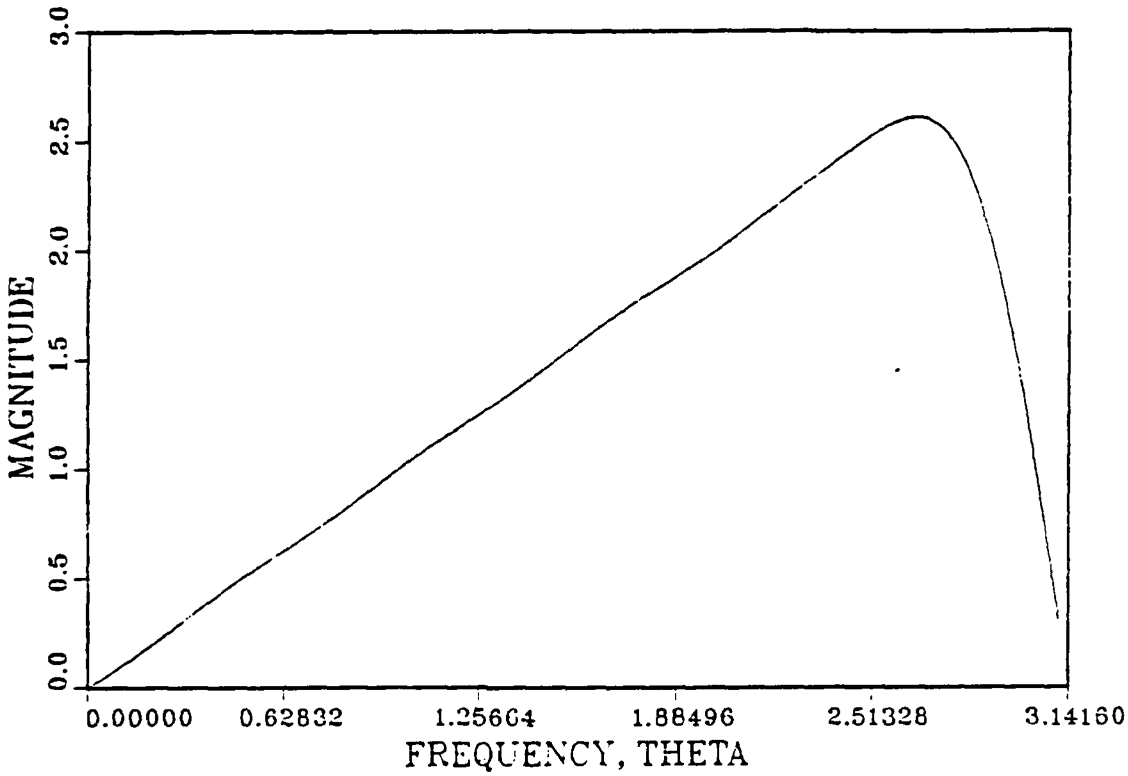


Figure 3.10. Windowed 21 Coefficient Bandpass Differentiator

- It is possible to design a filter that approximates any desired frequency domain specifications, without ever having to determine an analytical expression for the filter frequency response, $H(e^{j\theta})$.
- The filter coefficients, $h(n)$, can be determined using the IDFT algorithm.

Before proceeding with an explicit summary of the steps involved, and a detailed example, a discussion of the theoretical background will be presented.

In many filter applications a sharp cutoff amplitude characteristic and linear phase are desired. To ensure that these objectives are met when designing a filter, requires consideration of the number N and location of the equispaced frequency samples. Discussion of these parameters relative to their impact on the filter's frequency response involves consideration of four separate cases [9].

Case 1 N odd, frequency samples at

$$\theta_k = 2\pi k/N, k = 0, 1, 2, \ldots, N-1.$$

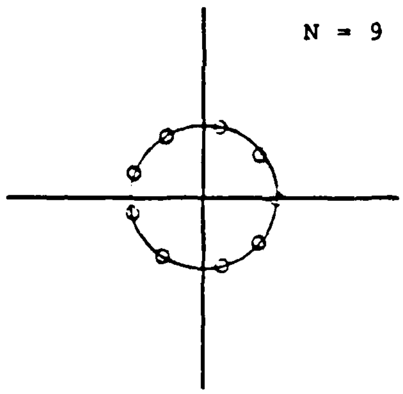


Figure 3.11a. Frequency Samples for N Odd

Case 2 N even, frequency samples at

$$\theta_k = 2\pi k/N, k = 0, 1, 2, \ldots, N-1.$$

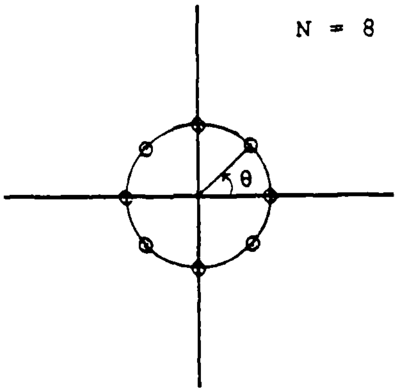


Figure 3.11b. Frequency Samples for N Even

Case 3 N odd, frequency samples at

$$\theta_k = 2\pi\frac{(k+1/2)}{N}, k = 0, 1, 2, \ldots, N-1.$$

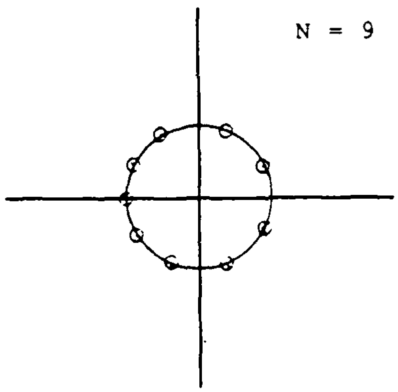


Figure 3.11c. Frequency Samples for N Odd

Case 4 N even, frequency samples at

$$\theta_k = 2\pi\frac{(k+1/2)}{N}, k = 0, 1, 2, \ldots, N-1.$$

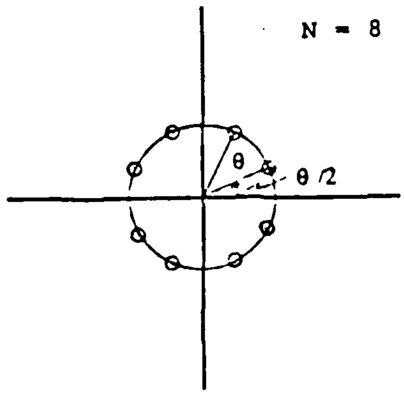


Figure 3.11d. Frequency Samples for N Even

It should be noted that Cases 3 and 4 do not readily lend themselves to inverse DFT computation using the FFT algorithm, since the first frequency sample is not at 0 Hz. For this reason a detailed discussion of these two cases will be omitted, however, information regarding these cases can be found in [9]. A discussion of frequency sampling considerations for Cases 1 and 2 follows.

Cases 1 & 2: Sampling Theorem

Recall that for an FIR filter with impulse response, $h(n) = \{h(0), h(1), \ldots, h(N-1)\}$, the transfer function is

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}. \tag{3.26}$$

As discussed earlier, the impulse response, $h(n)$, can also be represented in terms of its Discrete Fourier Transform (DFT), since it is of finite duration [9]

$$h(n) = (1/N) \sum_{k=0}^{N-1} H_k e^{j2\pi kn/N} \tag{3.27}$$

$$\text{where} \quad H_k = H(z)\Big|_{z=e^{j2\pi k/N}}.$$

Substituting the expression for $h(n)$ into Equation (3.26) gives [9]

$$
\begin{aligned}
H(z) &= \sum_{n=0}^{N-1}\left[(1/N)\sum_{k=0}^{N-1} H_k e^{j2\pi kn/N}\right] z^{-n} \\
&= (1/N)\sum_{n=0}^{N-1}\left[\sum_{k=0}^{N-1} H_k e^{j2\pi kn/N}\right] z^{-n} \\
&= (1/N)\sum_{k=0}^{N-1}\sum_{n=0}^{N-1} H_k e^{j2\pi kn/N} z^{-n} \\
&= (1/N)\sum_{k=0}^{N-1} H_k \sum_{n=0}^{N-1} e^{j2\pi kn/N} z^{-n}.
\end{aligned}
\tag{3.28}
$$

Applying the finite geometric sum property to the inner summation

$$
\sum_{n=0}^{N-1} e^{j2\pi kn/N} z^{-n} = \frac{1-\left[\frac{e^{j2\pi k/N}}{z}\right]^N}{1-\left[\frac{e^{j2\pi k/N}}{z}\right]}
$$

$$
\begin{aligned}
\sum_{n=0}^{N-1} e^{j2\pi kn/N} z^{-n} &= \frac{1-\left[\frac{e^{j2\pi k}}{z^N}\right]}{1-\left[\frac{e^{j2\pi k/N}}{z}\right]} \\
&= \frac{1-e^{j2\pi k} z^{-N}}{1-e^{j2\pi k/N} z^{-1}};\, e^{j2\pi k} = 1, k = 0, 1, 2, \ldots.
\end{aligned}
$$

Substituting back into Equation (3.28) yields [9]

$$
\begin{aligned}
H(z) &= (1/N)\sum_{k=0}^{N-1} \frac{H_k(1-z^{-n})}{1-e^{j2\pi k/N} z^{-1}} \\
&= \frac{(1-z^{-N})}{N}\sum_{k=0}^{N-1} \frac{H_k}{1-e^{j2\pi k/N} z^{-1}}.
\end{aligned}
$$

Let $z = e^{j\theta}$, to determine the interpolated frequency response

$$
\begin{aligned}
H\left(e^{j\theta}\right) &= \frac{e^{-jN\theta/2}\left(e^{jN\theta/2} - e^{-jn\theta/2}\right)}{N}\sum_{k=0}^{N-1} \frac{H_k}{1-(e^{j2\pi k/N})(e^{-j\theta})} \\
&= \frac{e^{-jN\theta/2}}{N}\sum_{k=0}^{N-1} \frac{H_k\left(e^{jN\theta/2} - e^{-j\theta N/2}\right)}{1-(e^{j2\pi k/N})(e^{-j\theta})}
\end{aligned}
$$

$$= \frac{e^{-jN\theta/2}}{N} \sum_{k=0}^{N-1} \frac{H_k \left(e^{jN\theta/2} - e^{-jN\theta/2}\right) \cdot e^{-j\pi k/N} \cdot e^{j\theta/2}}{e^{-j\pi k/N} \cdot e^{j\theta/2} - e^{j\pi k/N} \cdot e^{-j\theta/2}}$$

$$= \frac{e^{-jN\theta/2} \cdot e^{j\theta/2}}{N} \sum_{k=0}^{N-1} \frac{H_k \left(e^{jN\theta/2} - e^{-jN\theta/2}\right) \cdot e^{-j\pi k/N}}{e^{j(\theta/2-\pi k/N)} - e^{j(\theta/2-\pi k/N)}}$$

$$= \frac{e^{-jN\theta} \cdot e^{j\theta/2}}{N} \sum_{k=0}^{N-1} e^{-j\pi k/N} \cdot \frac{H_k \sin\left(\frac{\theta N}{2}\right)}{\sin\left(\frac{\theta}{2} - \frac{\pi k}{N}\right)}. \tag{3.29}$$

Thus, Equation (3.29) expresses the interpolated frequency response in terms of the sample values, H_k, of the desired frequency response [9].

In Case 1, where the number of frequency samples, N, is odd, choosing the frequency samples, H_k, to be real and symmetric yields a real and symmetric impulse response

$$h(n) = \frac{H_0}{N} + \sum_{k=1}^{(N-1)/2} \frac{2H_k}{N} \cos\left(\frac{2\pi}{N} nk\right). \tag{3.30}$$

The interpolated frequency response, $H\left(e^{j\theta}\right)$, is also purely real which, as stated in the beginning of this section, is highly desirable for most filter applications

$$H\left(e^{j\theta}\right) = \sum_{n=-\frac{(N-1)}{2}}^{(N-1)/2} 2h(n)\cos(n\theta). \tag{3.31}$$

Case 2, however, presents a problem. Looking at Equation (3.29), it can be seen that for N even, the term $e^{-j\pi k/N}$ inside the summation introduces an imaginary component into the interpolated frequency response, $H\left(e^{j\theta}\right)$.

For example, suppose the following lowpass filter frequency response is sampled from $-\pi \leq 0 \leq \pi$ with an even number of frequency samples, $N = 4$. From Equation (3.27), the following noncausal impulse response is obtained

$$\begin{aligned} h(n) &= (1/4) \sum_{k=-2}^{1} H_k e^{j2\pi nk/4} \\ &= (1/4)\left[0 + (1)e^{-j\frac{2\pi n}{4}} + 1 + (1)e^{j\frac{2\pi n}{4}}\right] \end{aligned} \tag{3.32}$$

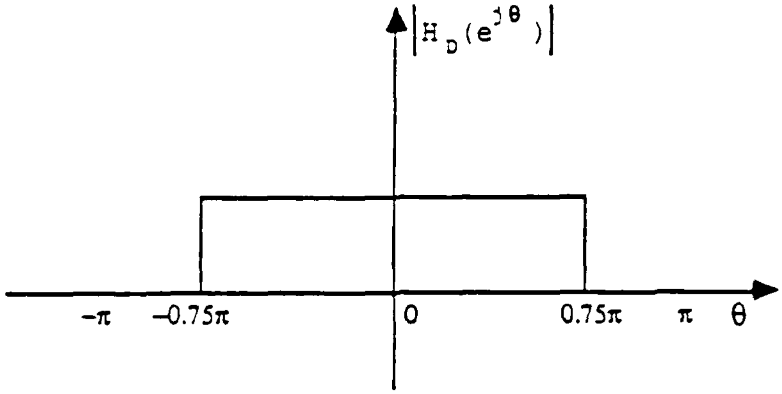


Figure 3.12. Desired Lowpass Filter Frequency Response

therefore,

$$h(-2) = (1/4)[0 + e^{j\pi} + 1 + e^{-j\pi}] = -0.25$$

$$h(-1) = (1/4)[0 + e^{j\pi 2} + 1 + e^{-j\pi/2}] = 0.25$$

$$h(0) = (1/4)[0 + 1 + 1 + 1] = 0.75$$

$$h(1) = (1/4)[0 + e^{-j\pi/2} + 1 + e^{j\pi/2}] = 0.25$$

and the frequency response is

$$\begin{aligned} H(e^{j\theta}) &= \sum_{n=-2}^{1} h(n)e^{-jn\theta} \\ &= (-0.25)e^{j2\theta} + (0.25)e^{j\theta} + (0.75) + (0.25)e^{-j\theta} \\ &= (-0.25)[\cos(2\theta) + j\sin(2\theta)] + 0.25[\cos(\theta) + j\sin(\theta)] + (0.75) \\ &+ (0.25)[\cos(\theta) - j\sin(\theta)] \\ &= (0.75) + (0.5)\cos\theta + (-0.25)[\cos(2\theta) + j\sin(2\theta)]. \end{aligned} \tag{3.33}$$

The imaginary component introduced is

$$-0.25j\sin(2\theta).$$

According to Reference 9, the amplitude of this component should be,

$$\begin{aligned} A &= (1/N)\sum_{k=-2}^{1} H_k(-1)^k \\ &= (1/4)[0 - 1 + 1 - 1] = -0.25 \end{aligned}$$

which is indeed the case.

If an odd number of samples, $N = 5$, were taken of the same frequency response, the impulse response would be

$$\begin{aligned} h(n) &= (1/5) \sum_{k=-2}^{2} H_k e^{j2\pi nk/5} \\ &= (1/5)[0 + (1)e^{-j\frac{2\pi n}{5}} + 1 + (1)e^{j\frac{2\pi n}{5}} + 0] \\ &= (1/5)[1 + 2\cos(2\pi n/5)] \end{aligned} \tag{3.34}$$

therefore,

$$h(0) = 0.6$$

$$h(\pm 1) = (1/5)[1 + 2\cos 2\pi/5] = 0.324$$

$$h(\pm 2) = (1/5)[1 + 2\cos 4\pi/5] = 0.124.$$

The frequency response is thus

$$\begin{aligned} H(e^{j\theta}) &= \sum_{n=-2}^{2} h(n)e^{-jn\theta} \\ &= (0.324)e^{j2\theta} + (0.124)e^{j\theta} + 0.6 + (0.124)e^{-j\theta} + (0.324)e^{-j2\theta} \\ &= (0.324)[\cos(2\theta) + j\sin(2\theta)] + (0.124)[\cos\theta + j\sin\theta] + \\ &\quad 0.6 + (0.124)[\cos\theta - j\sin\theta] + (0.324)[\cos(2\theta) - j\sin(2\theta)] \\ &= 0.6 + (0.648)\cos\theta + (0.248)\cos(2\theta) \end{aligned} \tag{3.35}$$

and there is no imaginary component.

To remedy the imaginary component problem for the even case, the following substitution is made for the frequency sample values H_k [9]

$$H_k = G_k e^{j\pi k/N}. \tag{3.36}$$

The set of G_k's is selected so that $G(0) = H(0)$, $G(N/2) = 0$, and $G(k) = -G(N-k)$, $k = 1, 2, \ldots, (N/2) - 1$.

The resultant impulse response is both real and symmetric, with $h(n) = h$

$(N-1-n)$; $n = 0, 1, 2, \ldots, (N/2)-1$. The values of $h(n)$ are given by,

$$h(n) = \frac{G_0}{N} + \sum_{k=1}^{(\frac{N}{2})-1} \frac{2G_k}{N} \cos\left[\left(\frac{\pi}{N}\right)k + \left(\frac{2\pi}{N}\right)nk\right] \tag{3.37}$$

and the interpolated frequency response $H(e^{j\theta})$ is,

$$H(e^{j\theta}) = \sum_{n=-\frac{(N-1)}{2}}^{(N-1)/2} 2h(n)\cos(n\theta). \tag{3.38}$$

The steps involved, when designing a filter using the frequency sampling method, are as follows:

<u>**Step 1**</u>: Given the continuous frequency response specifications for a desired filter, N samples are taken at equispaced frequencies over one period of the desired response

$$H(k) = H(e^{j\theta}); \theta = 2\pi k/N. \tag{3.39}$$

If the number of samples, N, is even, they need to be transformed using the following relationship, prior to proceeding with Step 2

$$H(k) = G(k)e^{j2\pi nk/N} \tag{3.40}$$

such that

$$\begin{aligned} G(0) &= H(0) \\ G(N/2) &= 0 \\ G(k) &= -G(N-k); k = 1, 2, \ldots, (N/2)-1. \end{aligned}$$

<u>**Step 2**</u>: Determine the Fourier coefficients of the desired filter by finding the IDFT of these sample values. In other words, determine the impulse response.

$$h(n) = (1/N)\sum_{N} H(k)e^{j2\pi nk/N} \tag{3.41}$$

Step 3: Again, it may be necessary to use an appropriate window function,

$$\hat{h}(n) = h(n) \cdot w(n). \tag{3.42}$$

Step 4: The Fourier coefficients thus determined correspond to the weighted filter impulse response, $\hat{h}(n)$, and the filter is realized nonrecursively by the difference equation

$$y(n) = \sum_{k=0}^{N-1} b_k x(n-k) \tag{3.43}$$

where

$$b_k = \hat{h}(n).$$

Step 5: Verification of the filter design is accomplished by determining the interpolated frequency response, $H(e^{j\theta})$, resulting from the use of the above filter coefficients. This frequency response is compared to the original desired frequency response, $H_D(e^{j\theta})$, to see if it is a reasonable approximation.

This procedure is illustrated by the design of a bandpass filter. First, an odd number of frequency samples will be used and then an even number.

Example 3.1 Frequency Sampling for N Odd

A bandpass filter is desired with the ideal frequency response characteristic illustrated below. The number of frequency samples to be used is $N = 255$.

Sampling gives

$$\theta_k = 2\pi k/N$$

$$\begin{aligned} &\text{where } N = 255 \\ &k = 0, 1, \ldots, 254 \\ &\text{or } \Delta\theta = 2\pi/N = 0.0246. \end{aligned} \tag{3.44}$$

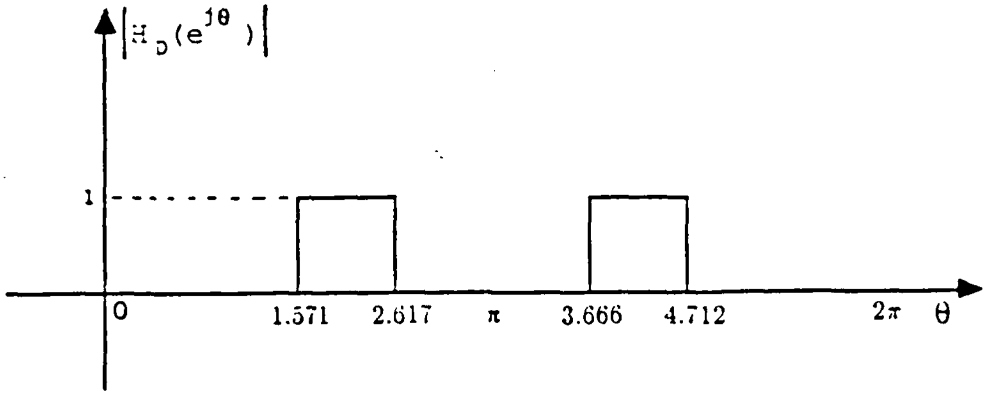


Figure 3.13. Desired Bandpass Filter Frequency Response

Thus,

$$H(k) = \begin{cases} 1.0; & k = 64, \ldots, 106 \quad \text{and } k = 149, \ldots, 191 \\ 0.0; & k = 0, \ldots, 63 \quad \text{and } k = 107, \ldots, 148 \\ & \qquad\qquad\qquad\ \ \text{and } k = 192, \ldots, 254 \ . \end{cases}$$

Taking the IDFT of the above frequency sample values and truncating $h(n)$ to 51 terms with a rectangular window, produces the interpolated frequency response of Figure 3.14a, while Figure 3.14b illustrates the response obtained using a Hamming window.

<u>Example 3.2</u> Frequency Sampling for N Even

Repeat the previous example, with $N = 256$ frequency samples.

Sampling gives

$$\theta_k = 2\pi k/N$$

where

$$N = 256$$

$$k = 0, 1, \ldots, 255$$

or

$$\Delta\theta = 2\pi/N = 0.0245.$$

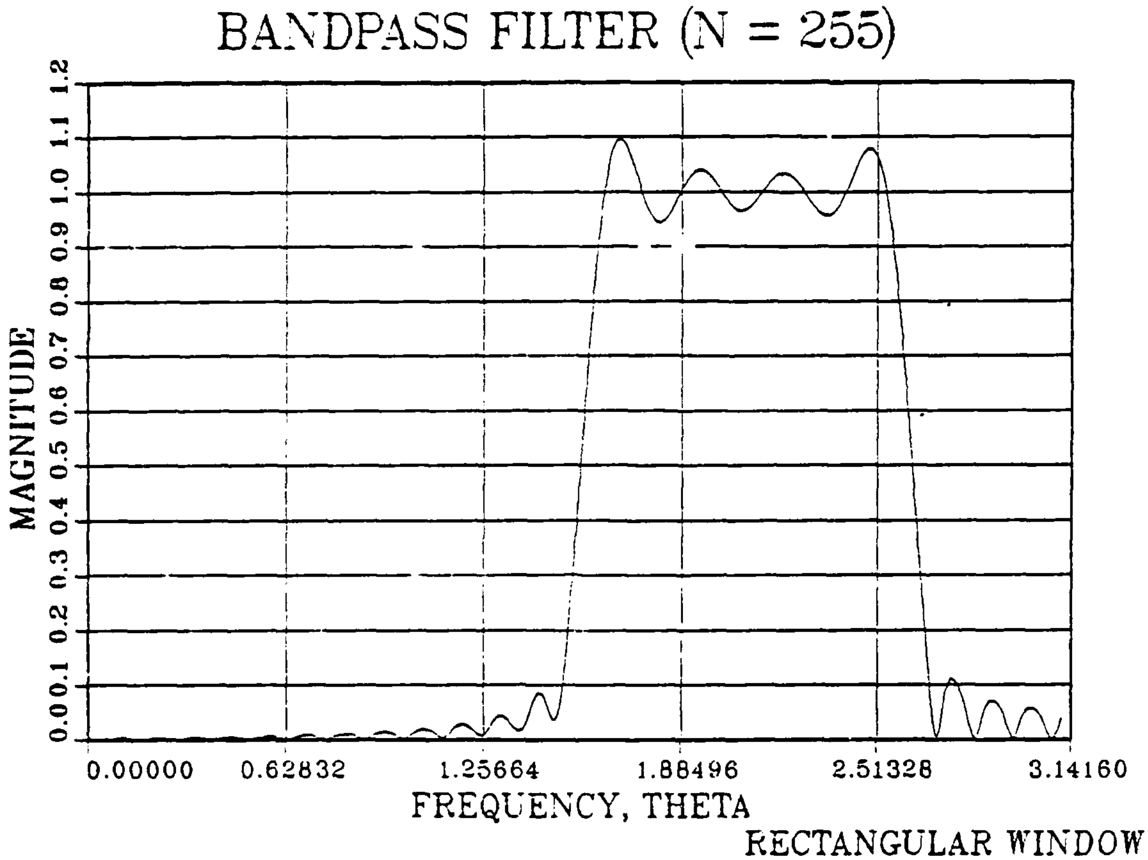


Figure 3.14a. Unwindowed Bandpass Filter Frequency Response

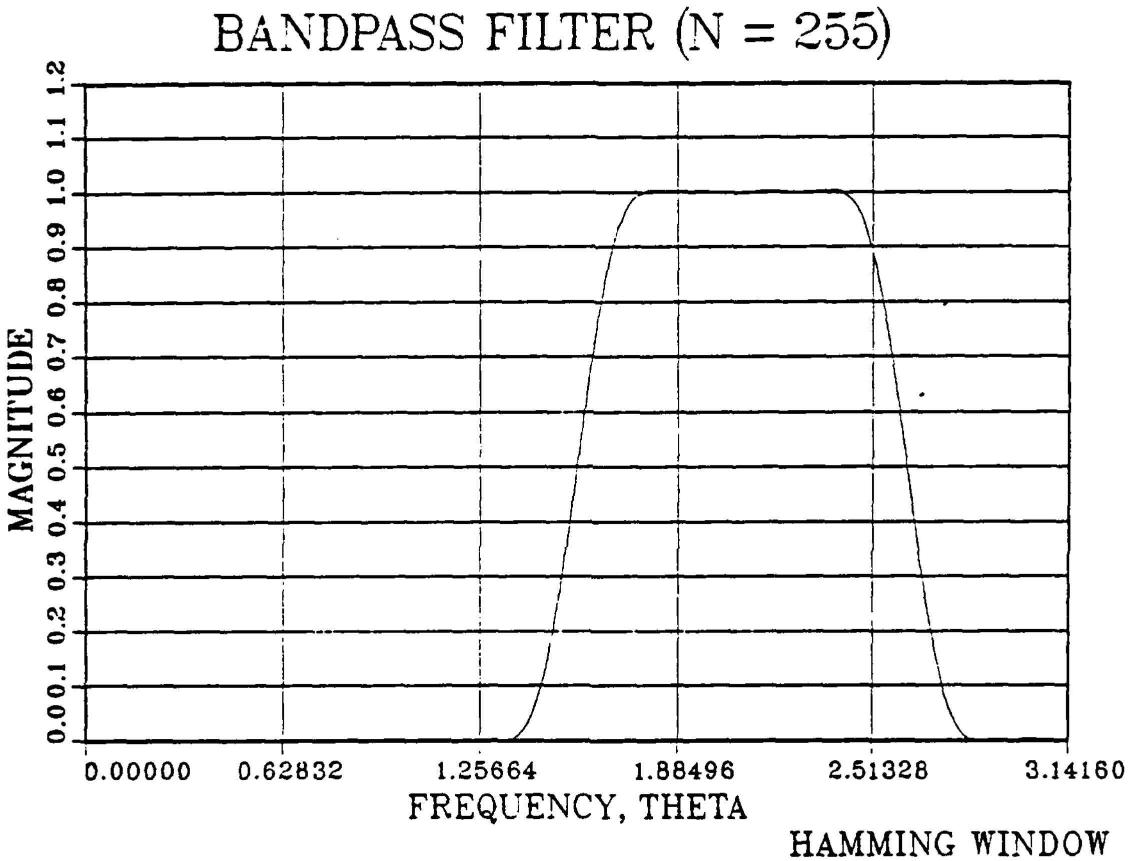


Figure 3.14b. Windowed Bandpass Filter Frequency Response

Thus,

$$H(k) = \begin{cases} 1.0; & k = 64, \ldots, 106 \quad \text{and } k = 150, \ldots, 192 \\ 0.0; & k = 0, \ldots, 63 \quad \text{and } k = 107, \ldots, 149 \\ & \qquad\qquad\qquad \text{and } k = 193, \ldots, 255\ . \end{cases}$$

But recall for N even, the sample values $H(k)$, must be transformed to eliminate the unwanted imaginary component. Applying Equation (3.40) gives

$$H(k) = G(k)e^{j2\pi k/256}$$

such that,

$$G(0) = H(0) = 0$$

$$G(256/2) = G(128) = 0$$

$$G(k) = -G(256 - k); k = 1, 2, \ldots, 127.$$

For example

$$G(65) = H(65) = 1.0 = -G(191) \Longrightarrow G(191) = -1.0$$

and the transformed $H(65)$ and $H(191)$ are

$$H(65) = 1.0e^{j2\pi(65)/256}$$

$$H(191) = -1.0e^{j2\pi(191)/256}.$$

Using the transformed $H(k)$ values, the IDFT is determined to find the impulse response, which is again truncated to 51 terms. Figures 3.15a and 3.15b show the unwindowed and windowed frequency responses, respectively. It should be noted that when using a Hamming window for an even number of terms (in this case 52) the equation for the window function (Equation 3.19) should be modified as follows

$$w(n) = 0.54 + 0.46 \cos\left[\frac{\left(n - \frac{N-1}{2}\right)\pi}{\frac{N-1}{2}}\right]. \tag{3.45}$$

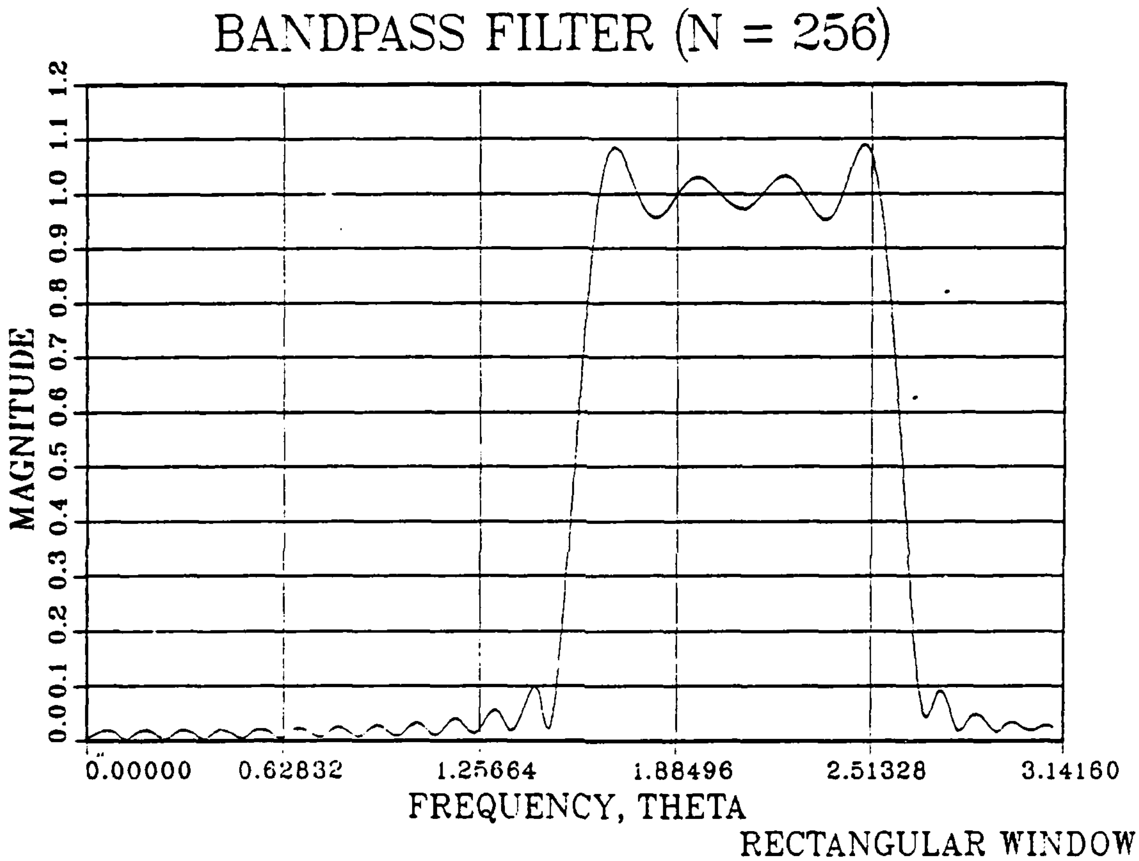


Figure 3.15a. Unwindowed Bandpass Filter Frequency Response

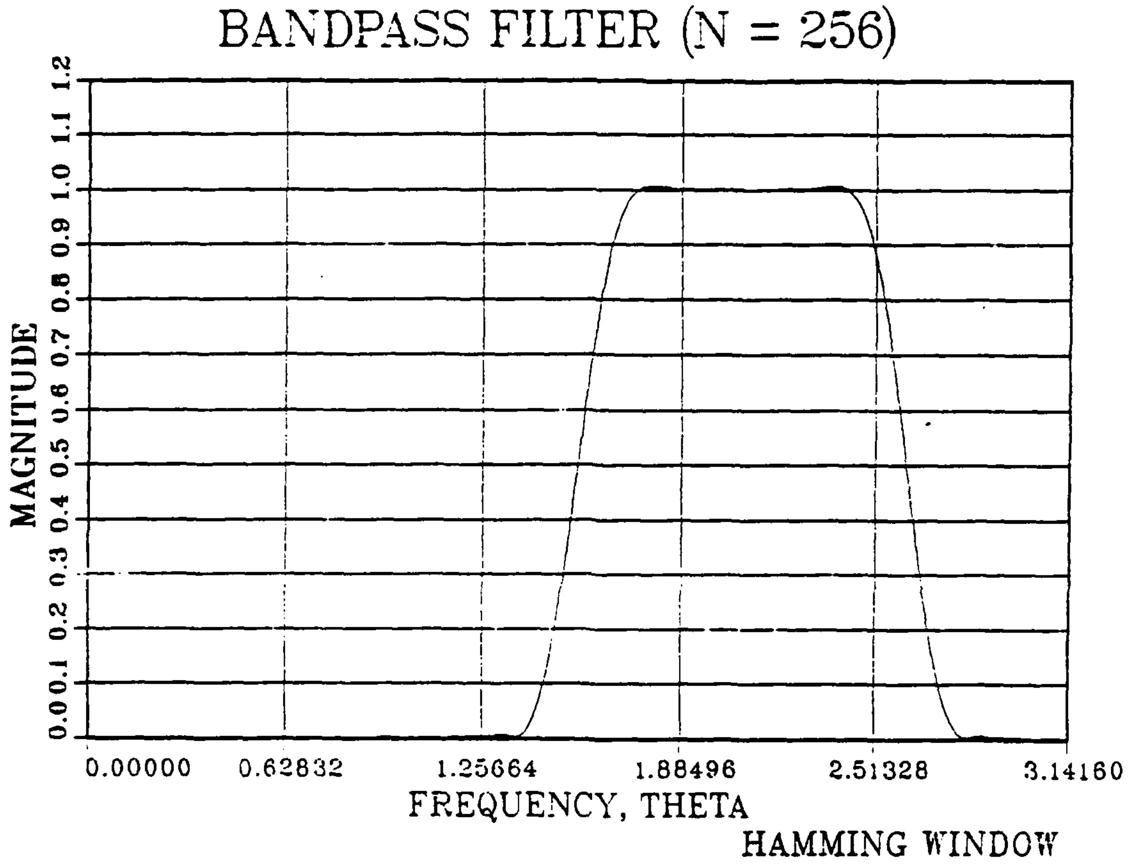


Figure 3.15b. Windowed Bandpass Filter Frequency Response

G. TRANSITION POINTS

As discussed in the previous section, frequency sampling involves taking N equispaced samples of a desired filter frequency response, $H_D(e^{j\theta})$. The IDFT algorithm is used to approximate the unit sample response, $h(n)$, which, for the case of nonrecursive filters, is equivalent to the Fourier series coefficients. The coefficients, in turn, are used to design a filter that approximates the original desired continuous frequency response.

A filter designed in this manner has a frequency response that is equal to the desired frequency response at the frequencies of the sampled values, however, the response can deviate significantly from the desired response at frequencies in between these sample values.

A method that can be used to smooth the frequency response involves the use of transition samples between the passband and stopband of the desired frequency response [9]. The values of these transition samples are selected based on minimization of the ripple in the passband and/or stopband, or minimization of the maximum sidelobe of the frequency response. In other words, a minimization algorithm is used to find the optimum values for the transition samples based on the selected minimization criterion.

As before, an example will be given to illustrate this technique, but first a summary of the steps involved:

Step 1: Sample the desired continuous frequency response at N equispaced frequencies where N is the number of Fourier coefficients that will be used to approximate the desired filter response. The spacing between the frequencies is $\Delta\theta = 2\pi/N$.

Step 2: Determine the impulse response $h(n)$ (Fourier coefficients) by finding the Inverse Discrete Fourier Transform, IDFT, of the sample values.

Step 3: Using the coefficient values determined in the previous step, find the continuous frequency response and compare with the desired filter response.

Step 4: Use the minimization algorithm to adjust the transition coefficient values, in order to obtain as close a match as possible between the desired filter response and that obtained through the design procedure.

The minimization algorithm as used in [9], is based on minimizing the maximum sidelobe of the frequency response. Tables were generated for lowpass filters, bandpass filters and wide-band differentiators of varying bandwidths; 1, 2, or 3 transition points; and odd and even values of the number of frequency samples, N.

Table 3.1 is a reprint of subsets of these tables (due to Reference 9) for lowpass filter design, using 1, 2, or 3 transition points. The number of frequency samples is $N = 15$, the column labeled minimax refers to the maximum sidelobe, and Type-1 data means that the first frequency sample is taken at $\theta = 0$.

Design Example

In this example the goal is to design a lowpass filter with three frequency samples in the passband symmetric about the origin ($BW = 3$), and a total of 15 frequency samples ($N = 15$). The effects of using 0, 1, 2, and 3 transition points will be investigated.

The values of the frequency samples, transition point values, and impulse responses, $h(n)$, are summarized below for all four cases; the transition point values were obtained from Table 3.1.

TABLE 3.1
LOWPASS FILTER DESIGN
(TYPE-1 DATA, N ODD)

ONE TRANSITION COEFFICIENT

(due to reference [9])

BW	Minimax	T_1
	$N = 15$	
1	-42.30932283	0.43378296
2	-41.26299286	0.41793823
3	-41.25333786	0.41047363
4	-41.94907713	0.40405884
5	-44.37124538	0.39268189
6	-56.01416588	0.35766525

TWO TRANSITION COEFFICIENTS

BW	Minimax	T_1	T_2
	$N = 15$		
1	-70.60540585	0.09500122	0.58995418
2	-69.26168156	0.10319824	0.59347118
3	-69.91973495	0.10083618	0.58594327
4	-75.51172256	0.08407593	0.55715312
5	-103.46078300	0.05180206	0.49917424

THREE TRANSITION COEFFICIENTS

BW	Minimax	T_1	T_2	T_3
		$N = 15$		
1	-94.61166191	0.01455078	0.18457882	0.66897613
2	-104.99813080	0.01000977	0.17360713	0.65951526
3	-114.90719318	0.00873413	0.16397310	0.64711264
4	-157.29257584	0.00378799	0.12393963	0.60181154

Case 1 - 0 Transition Points

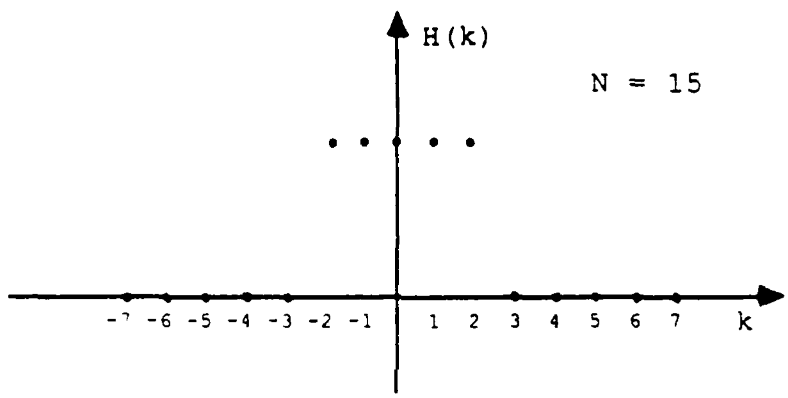


Figure 3.16a. Case 1 - Frequency Samples

k	$\rightarrow$	0	± 1	± 2	± 3	± 4	± 5	± 6	± 7
$H(k)$	$\rightarrow$	1	1	1	0	0	0	0	0
$h(n)$	$\rightarrow$	0.333	0.278	0.142	0.0	-0.077	-0.067	0.0	0.058

Case 2 - 1 Transition Point

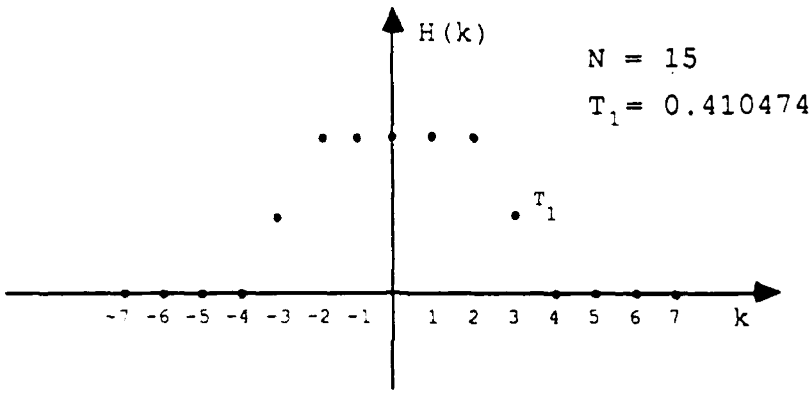


Figure 3.16b. Case 2 - Frequency Samples

k	$\rightarrow$	0	± 1	± 2	± 3	± 4	± 5	± 6	± 7
$H(k)$	$\rightarrow$	1	1	1	0.410	0	0	0	0
$h(n)$	$\rightarrow$	0.388	0.295	0.098	-0.044	-0.061	-0.012	0.017	0.014

<u>**Case 3**</u> - 2 Transition Points

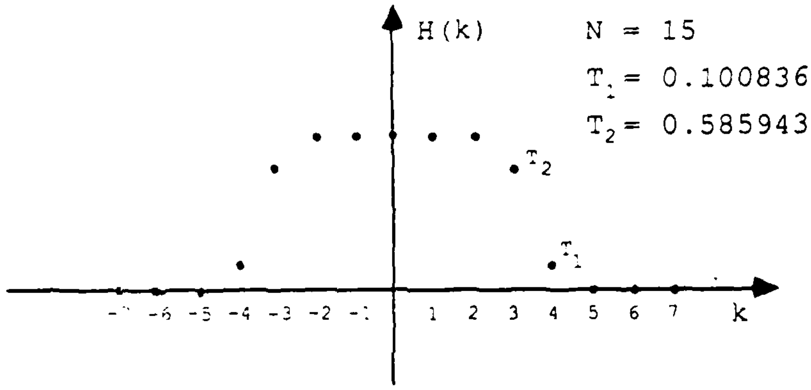
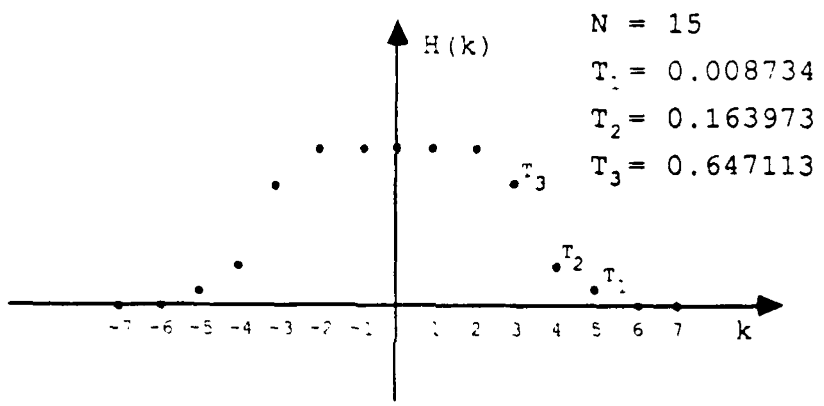


Figure 3.16c. Case 3 - Frequency Samples

k	$\rightarrow$	0	± 1	± 2	± 3	± 4	± 5	± 6	± 7
$H(k)$	$\rightarrow$	1	1	1	0.586	0.101	0	0	0
$h(n)$	$\rightarrow$	0.425	0.300	0.066	−0.059	−0.041	0.005	0.013	0.004

Case 4 - 3 Transition Points



k	$\rightarrow$	0	± 1	± 2	± 3	± 4	± 5	± 6	± 7
$H(k)$	$\rightarrow$	1	1	1	0.647	0.164	0.009	0	0
$h(n)$	$\rightarrow$	0.443	0.301	0.050	-0.062	-0.032	0.008	0.010	0.002

Figure 3.16d. Case 4 - Frequency Samples

Figure 3.17 depicts the results of frequency sampling design using transition points. As expected, the amplitude of the passband and stopband ripple is reduced with an increasing number of transition points; however, the tradeoff is that the sharpness of the cutoff is reduced. This example illustrates the usefulness of minimization algorithms in smoothing interpolated filter frequency responses obtained with the frequency sampling technique. As stated earlier, Reference 9 contains tables for lowpass and bandpass filter design, as well as wideband differentiators. While quite extensive, a designer may want parameters that have not been tabulated. In this case, approximate values of the transition coefficients can be derived from the tabulated values given, using linear interpolation.

H. DESIGN OF AN INTEGRATOR

In Section A analytical methods were used to design a differentiator. The use of these methods to design an integrator, however, does not yield an easily obtainable solution. As will be shown shortly, frequency sampling solves this dilemma by providing a straightforward design technique that produces extremely good results.

Bandpass Integrator Design Example

A bandpass integrator with the following frequency response characteristic is desired. Both of the aforementioned design methods will be applied to the design problem to illustrate the advantage of the frequency sampling technique.

a. Method 1 - Analytical

In the s-domain an integrator is described by the transfer function $H(s) = 1/s$. The frequency response is thus: $H(jw) = 1/jw = (jw)^{-1}$. Converting the frequency response to digital frequencies

$$H(jw) = 1/jw \Longrightarrow H(e^{j\theta}) = -jT/\theta; T = \text{ period.} \tag{3.46}$$

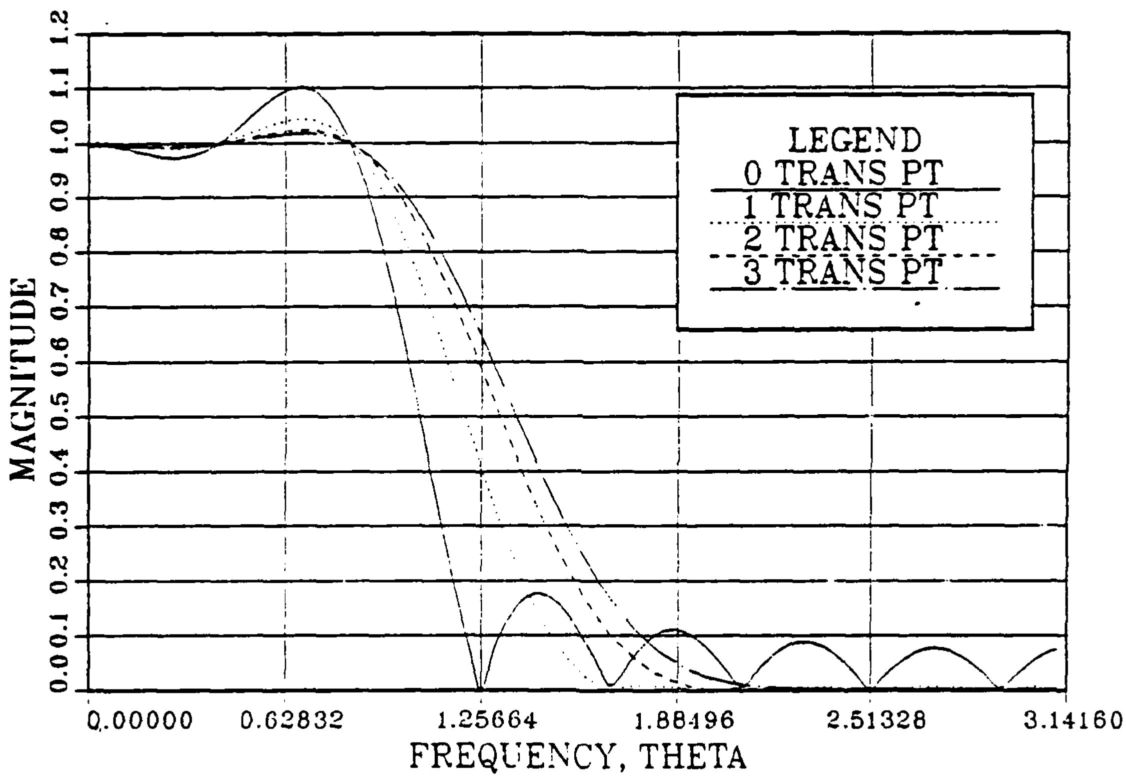


Figure 3.17. Frequency Responses for Varying Numbers of Transition Points

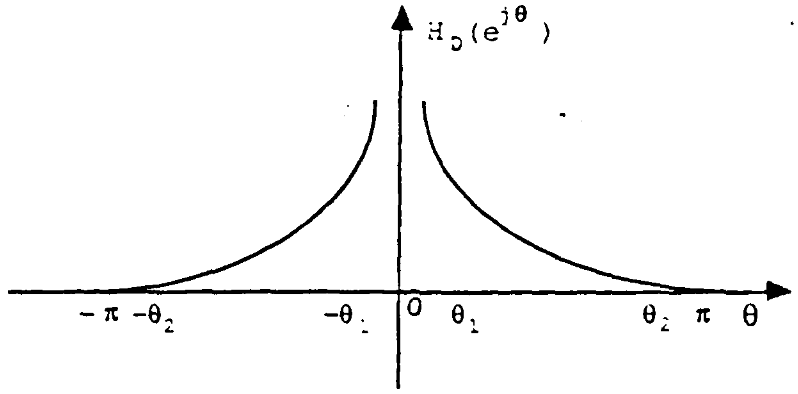


Figure 3.18. Ideal Integrator Frequency Response Characteristic

To determine the impulse response, $h(n)$,

$$h(n) = (1/2\pi)\int_{-\theta_2}^{-\theta_1} H_D\left(e^{j\theta}\right) e^{j\theta_n} d\theta + 1/2\pi \int_{\theta_1}^{\theta_2} H_D\left(e^{j\theta}\right) e^{j\theta_n} d\theta$$

$$h(n) = (1/2\pi)\int_{-\theta_2}^{-\theta_1} (-jT/\theta)\, e^{j\theta_n} d\theta + 1/2\pi \int_{\theta_1}^{\theta_2} 1/\theta e^{j\theta_n} d\theta$$

$$\text{Since}: \int (1/x)e^{ax}dx = \ln x + \frac{ax}{1!} + \frac{a^2x^2}{2\cdot 2!} + \frac{a^3x^3}{3\cdot 3!} + \ldots$$

$$h(n) = -jT/2\pi\left[\ln\theta + \frac{jn\theta}{1!} + \frac{(jn)^2\theta^2}{2\cdot 2!} + \frac{(jn)^3\theta^3}{3\cdot 3!} + \ldots\right]_{-\theta_2}^{-\theta_1}$$

$$+\left(\frac{-jT}{2\pi}\right)\left[\ln\theta + \frac{jn\theta}{1!} + \frac{(jn)^2\theta^2}{2\cdot 2!} + \frac{(jn)^3\theta^3}{3\cdot 3!} + \ldots\right]_{\theta_1}^{\theta_2}.$$

A considerable number of steps later yields

$$h(n) = T/\pi\left[n(\theta_2 - \theta_1) + \frac{n^2}{2\cdot 2!}\left(\theta_2^2 - \theta_1^2\right) + \ldots + \frac{n^n}{n\cdot n!}\left(\theta_2^n - \theta_1^n\right)\right] \qquad (3.47)$$

where n is the number of coefficients desired for the implementation of the bandpass integrator.

The analytical expression for the impulse response of the bandpass integrator, Equation (3.40), is cumbersome, especially when the integrator design is of a high order. Therefore, an alternative method is desirable.

b. Method 2 - Frequency Sampling

Recall that the expression for the desired frequency response is

$$H_D\left(e^{j\theta}\right) = -jT/\theta. \tag{3.48}$$

Letting $T = 1$, one hundred and one frequency samples will be taken of the desired bandpass integrator with a bandwidth of 0.1π to 1.6π. In other words, 101 frequency samples, $H(k)$, of the magnitude and phase characteristic (Figure 3.19) will be taken using the following increment of the digital frequency θ

$$\begin{aligned} \Delta\theta &= 2\pi k/N = 0.062k \text{ for } N = 101 \\ H(k) &= H\left(e^{j\theta}\right)\Big|_{\theta=0.062k} \quad \text{for } k = -50, \ldots, +50. \end{aligned} \tag{3.49}$$

The 101-coefficient impulse response is obtained by determining the IDFT of these frequency sample values, and a 21-coefficient rectangular window is applied yielding the following bandpass integrator coefficients.

n	$h(n)$	n	$h(n)$
0	0.0974	11	0.4813
1	0.0793	12	0.2391
2	0.0946	13	0.2248
3	0.0439	14	0.0818
4	0.0362	15	0.0562
5	−0.0562	16	−0.0362
6	−0.0818	17	−0.0439
7	−0.2248	18	−0.0946
8	−0.2391	19	−0.0793
9	−0.4813	20	−0.0974
10	0.000		

Finally, the interpolated frequency response is obtained. As can be seen in Figure 3.20, the results are good. (Solid dots indicate the ideal frequency response.) Also included in Figure 3.21 are the results obtained when a Hamming window is used, while Figures 3.22 and 3.23 are the results of using 41 coefficients.

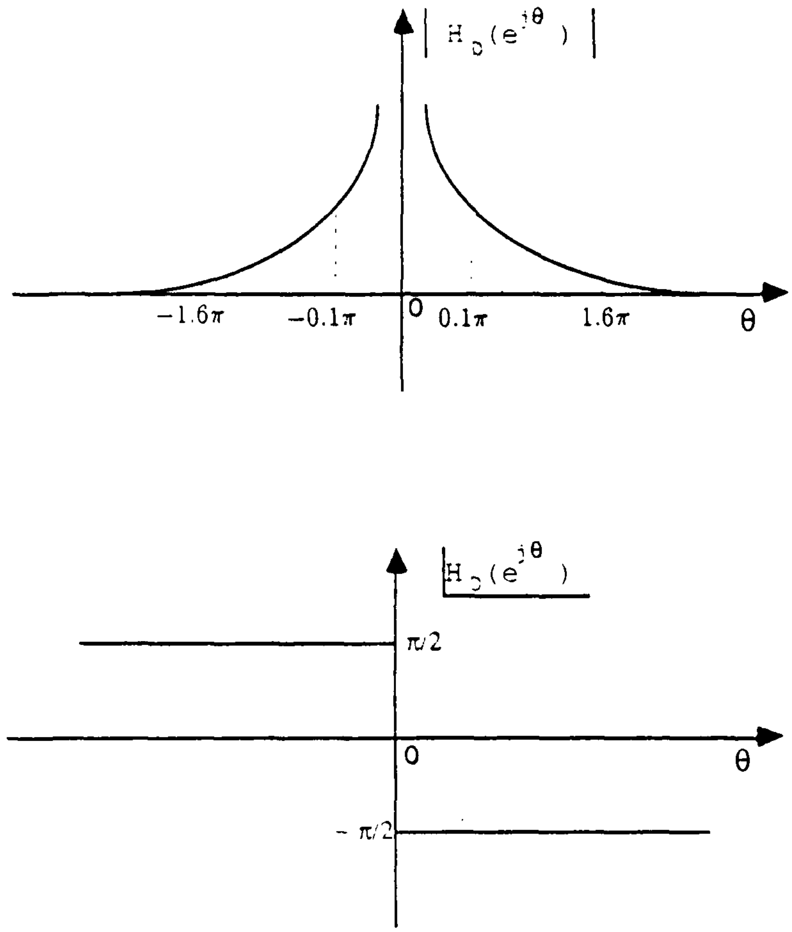


Figure 3.19. Ideal Bandpass Integrator Frequency Response Characteristic

This example further demonstrates the versatility and usefulness of the frequency sampling technique. It allows for the design of filters that may be unattainable using traditional analytic methods, and is thus an indispensable tool.

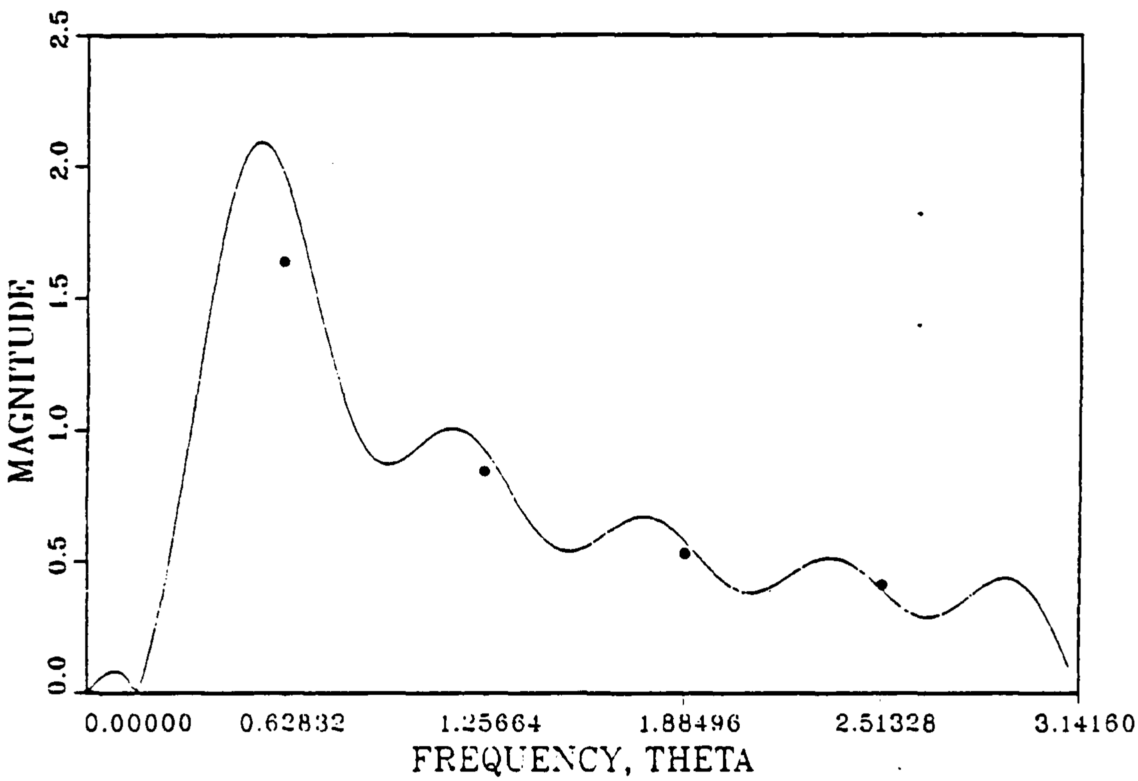


Figure 3.20. Unwindowed 21-Coefficient Bandpass Integrators

(Solid dots indicate ideal frequency response)

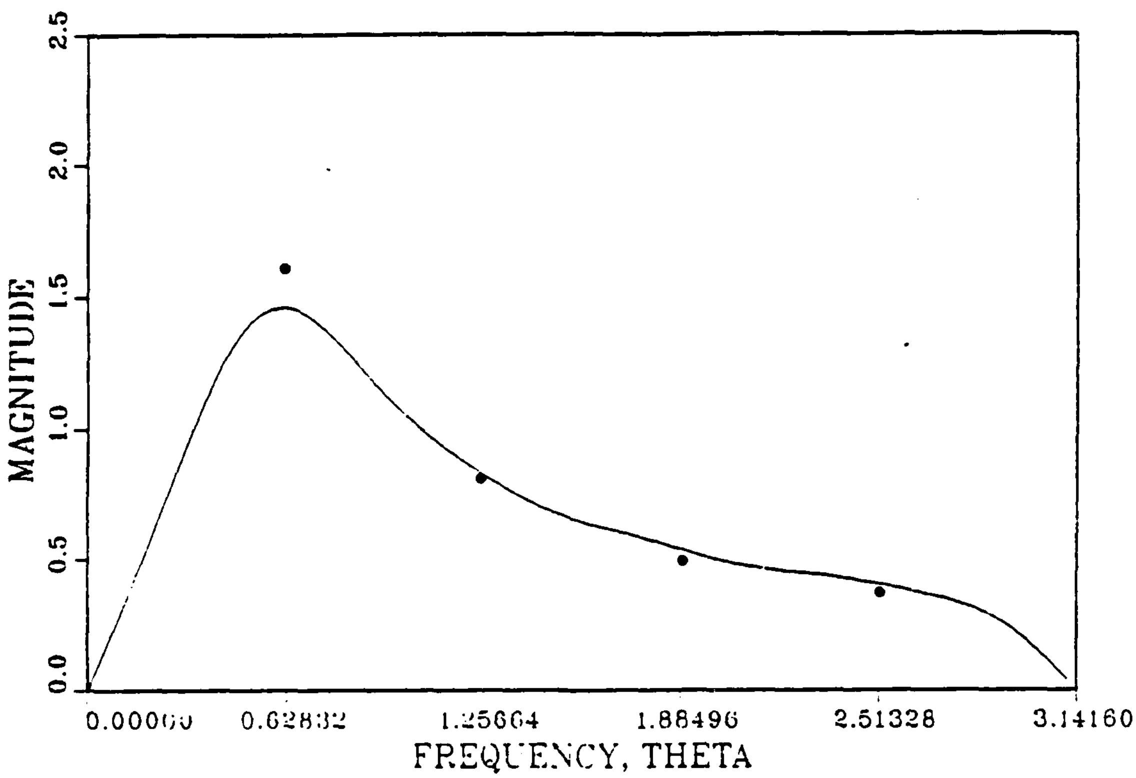


Figure 3.21. Windowed 21-Coefficient Bandpass Integrator

(Solid dots indicate ideal frequency response)

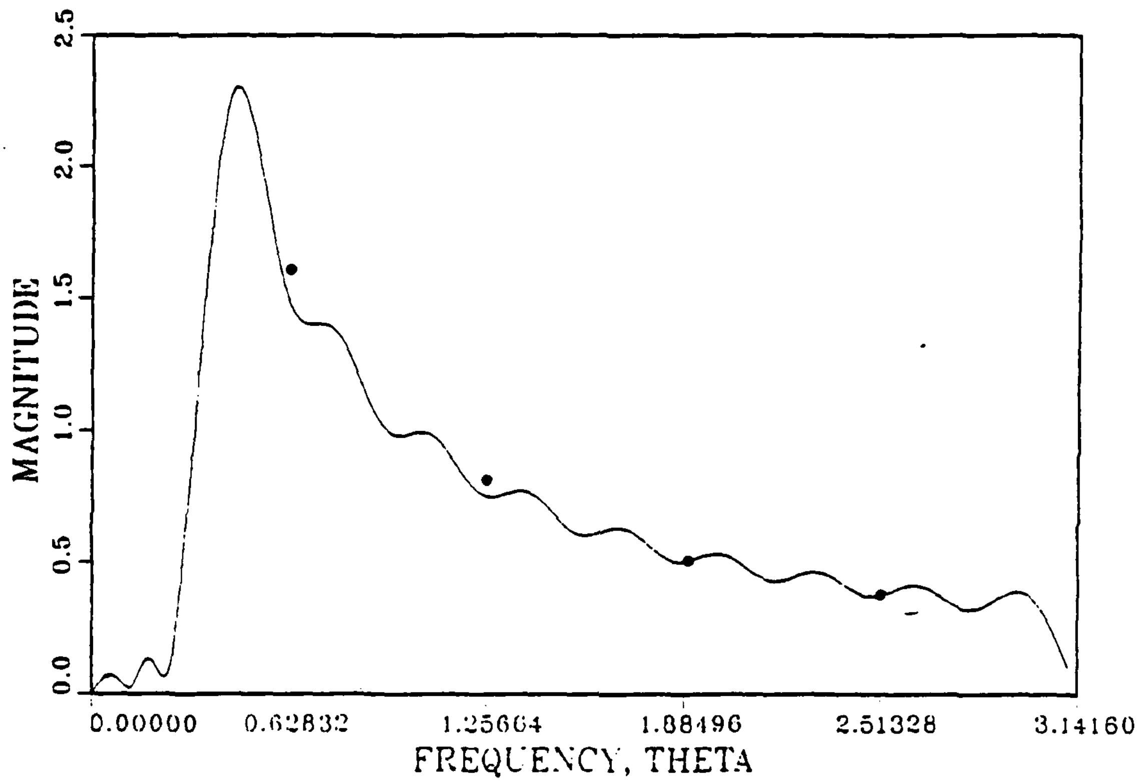


Figure 3.22. Unwindowed 41-Coefficient Bandpass Integrator

(Solid dots indicate ideal frequency response)

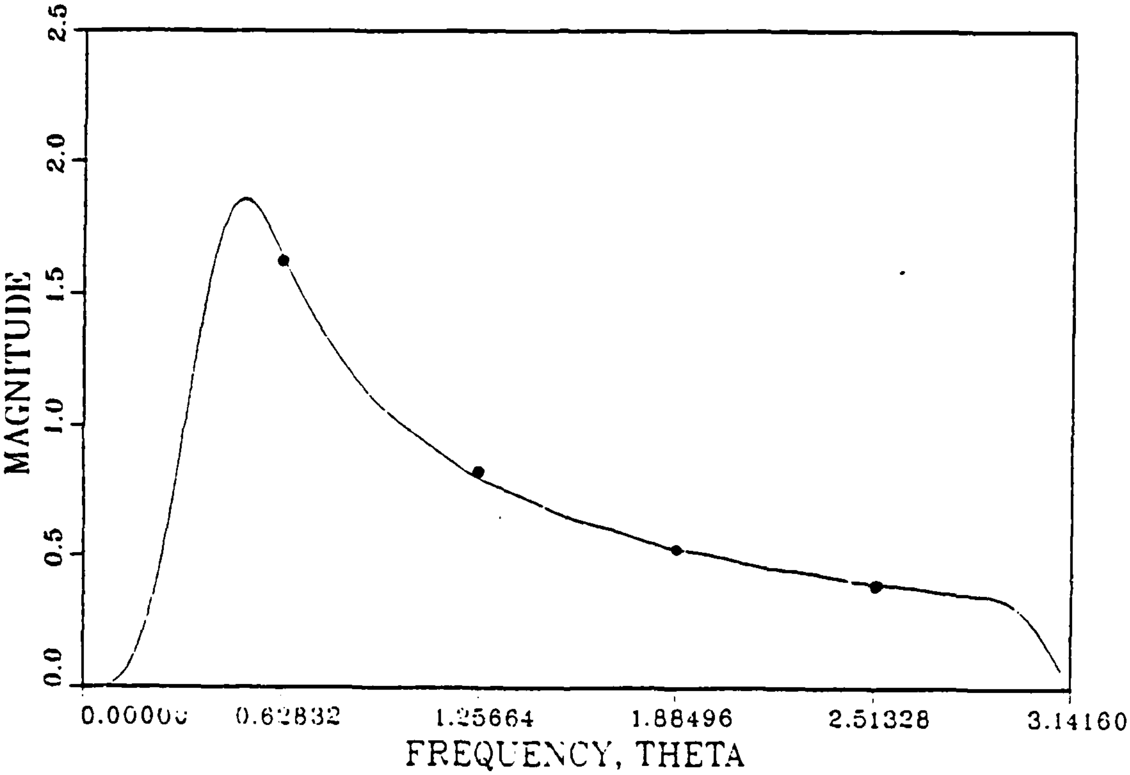


Figure 3.23. Windowed 41-Coefficient Bandpass Integrator

(Solid dots indicate ideal frequency response)

IV. COMPUTER-AIDED DESIGN

The basis of all computer-aided design (CAD) techniques is optimization. This is to say, a desired filter frequency response is approximated by a particular filter whose coefficients are to be determined. The accuracy of the approximation is evaluated according to some criterion, usually an error function, that indicates how large a disparity exists between the desired filter frequency response and the approximated filter frequency response. Variable parameters of the approximating function are then "adjusted" to optimize the filter design in terms of this criterion.

A. REMEZ EXCHANGE ALGORITHM

An extensively used computer-aided design technique for linear phase FIR filters is the Remez exchange algorithm. The mathematical basis for this algorithm is the weighted Chebyshev approximation.

A summary of the approximating and error functions for this algorithm follows. It has been shown in [1] and [13], that the frequency responses for the four cases of linear phase filters - i.e., even or odd symmetry with an even or odd number of terms - can be written in the form:

$$H\left(e^{j\theta}\right) = e^{-j\theta(N-1)/2} e^{j(\pi/2)L} \tilde{H}\left(e^{j\theta}\right) \tag{4.1}$$

where $\tilde{H}\left(e^{j\theta}\right)$ is a real-valued function used to approximate the desired filter's magnitude specifications, and the remaining terms approximate the desired phase.

Table 4.1 gives the values L, and the form of $\tilde{H}\left(e^{j\theta}\right)$ for all four cases.

TABLE 4.1

FREQUENCY RESPONSES FOR LINEAR PHASE FILTERS

(due to reference 13)

	L	$\tilde{H}\left(e^{j\theta}\right)$
Case 1 - N odd Symmetrical Impulse Response	0	$\sum_{n=0}^{(N-1)/2} a(n)\cos(n\theta)$
Case 2 - N even Symmetrical Impulse Response	0	$\sum_{n=1}^{N/2} b(n)\cos\left[\theta(n-1/2)\right]$
Case 3 - N odd Anti-symmetric Impulse Response	1	$\sum_{n=1}^{(N-1)/2} c(n)\sin(n\theta)$
Case 4 - N even Anti-symmetric Impulse Response	1	$\sum_{n=1}^{N/2} d(n)\sin[\theta(n-1/2)]$

Using trigonometric identities, the expressions for $\tilde{H}\left(e^{j\theta}\right)$ in Table 4.1 can be rewritten in the following form:

$$\tilde{H}\left(e^{j\theta}\right) = Q\left(e^{j\theta}\right) P\left(e^{j\theta}\right) \tag{4.2}$$

where $Q\left(e^{j\theta}\right)$ is a fixed function of frequency, θ, and $P\left(e^{j\theta}\right)$ consists of a sum of weighted cosine terms (see Table 4.2).

TABLE 4.2

FREQUENCY RESPONSES FOR LINEAR PHASE FILTERS

(due to reference 13)

	$Q\left(e^{j\theta}\right)$	$P\left(e^{j\theta}\right)$
Case 1	1	$\sum_{n=0}^{(N-1)/2} \tilde{a}(n)\cos(n\theta)$
Case 2	$\cos(\theta/2)$	$\sum_{n=0}^{(N/2)-1} \tilde{b}(n)\cos(n\theta)$
Case 3	$\sin(\theta)$	$\sum_{n=0}^{(N-1)/2} \tilde{c}(n)\cos(n\theta)$
Case 4	$\sin(\theta/2)$	$\sum_{n=0}^{(N/2)-1} \tilde{d}(n)\cos(n\theta)$

where

$$\tilde{a}(n) = \begin{cases} a(0) = h\left(\frac{N-1}{2}\right) \\ a(n) = 2h\left(\frac{N-1}{2} - n\right) \quad \text{for } n = 1, 2, \ldots, \frac{N-1}{2} \end{cases}$$

$$\tilde{b}(n) = \begin{cases} b(1) = \tilde{b}(0) + 1/2\tilde{b}(1) \\ b(n) = 1/2\left[\tilde{b}(n-1) + \tilde{b}(n)\right] \quad \text{for } n = 2, 3, \ldots, \frac{N}{2} - 1 \\ b(N/2) = 1/2\tilde{b}\left(\frac{N}{2} - 1\right) \end{cases}$$

$$\tilde{c}(n) = \begin{cases} c(1) = \tilde{c}(0) - 1/2\tilde{c}(2) \\ c(n) = 1/2\left[\tilde{c}(n-1) - \tilde{c}(n+1)\right] \quad \text{for } n = 2, 3, \ldots, \frac{N-1}{2} \\ c\left(\frac{N-1}{2} - 1\right) = 1/2\tilde{c}\left(\frac{N-1}{2} - 2\right) \\ c\left(\frac{N-1}{2}\right) = 1/2\tilde{c}\left(\frac{N-1}{2} - 1\right) \end{cases}$$

$$\tilde{d}(n) = \begin{cases} d(1) = \tilde{d}(0) - 1/2\tilde{d}(1) \\ d(n) = 1/2\left[\tilde{d}(n-1) - \tilde{d}(n)\right] \quad \text{for } n = 2, 3, \ldots, \frac{N}{2} - 1 \\ d\left(\frac{N}{2}\right) = 1/2\tilde{d}\left(\frac{N}{2} - 1\right) \end{cases}$$

Since $Q\left(e^{j\theta}\right)$ is a function of frequency only, it can be seen that the approximating function, $\tilde{H}\left(e^{j\theta}\right)$, can only be optimized in terms of $P\left(e^{j\theta}\right)$; that is, the dependency of $\tilde{H}\left(e^{j\theta}\right)$ on the filter coefficients is contained in $P\left(e^{j\theta}\right)$. Thus, the coefficients of $P\left(e^{j\theta}\right)$ can be varied to achieve an optimum filter design, and, the <u>true</u> approximating function can be generalized for all cases as,

$$P\left(e^{j\theta}\right) = \sum_{n=0}^{L} \alpha(n)\cos(n\theta) \tag{4.3}$$

where $L = (N-1)/2$ or $(N/2)-1$ depending on which case is being considered, and the $\alpha(n)$ are the weights from which the filter coefficients can be determined.

As stated in the introduction to this chapter, a measure of how well the designed filter frequency response approximates the desired filter frequency response is required. The weighted Chebyshev approximation uses an error function defined as follows:

$$E(\theta) = W(\theta)\left[H_D\left(e^{j\theta}\right) - \tilde{H}\left(e^{j\theta}\right)\right] \tag{4.4}$$

where

$$\begin{aligned} H_D\left(e^{j\theta}\right) &= \text{the desired frequency response} \\ \tilde{H}\left(e^{j\theta}\right) &= \text{the designed frequency response} \\ W(\theta) &= \text{weighting factor} \\ E(\theta) &= \text{error} \end{aligned}$$

In order to see the relationship between the filter coefficients and the error, Equation (4.4) can be rewritten in terms of the functions $Q\left(e^{j\theta}\right)$ and $P\left(e^{j\theta}\right)$.

$$\begin{aligned} E(\theta) &= W(\theta)\left[H_D\left(e^{j\theta}\right) - Q\left(e^{j\theta}\right)P\left(e^{j\theta}\right)\right] \\ &= W(\theta)Q\left(e^{j\theta}\right)\left[\frac{H_D\left(e^{j\theta}\right)}{Q\left(e^{j\theta}\right)} - P\left(e^{j\theta}\right)\right] \end{aligned} \tag{4.5}$$

Letting

$$\hat{W}(\theta) = W(\theta)Q\left(e^{j\theta}\right) \text{ and } \hat{H}_D\left(e^{j\theta}\right) = H_D\left(e^{j\theta}\right)/Q\left(e^{j\theta}\right) \tag{4.6}$$

yields

$$E(\theta) = \hat{W}(\theta)\left[\hat{H}_D\left(e^{j\theta}\right) - P\left(e^{j\theta}\right)\right] \tag{4.7}$$

Thus, the Chebyshev approximation that is performed using the Remez exchange algorithm can be stated as follows:

Find the set of filter coefficients (determined from the values of $\alpha(n)$) that minimizes the maximum absolute value of the error, $E(\theta)$, over the frequency range of interest.

$$E_{optimum} = \min_{(\text{coefficients})}\left[\max_{\theta} |E(\theta)|\right] \tag{4.8}$$

At this point, a discussion of the weighting function, $W(\theta)$, is in order. The purpose of this weighting function is to ensure a small tolerance for error in critical frequency ranges.

If $W(\theta)$ is large, this means a large deviation between the desired frequency response, $H_D\left(e^{j\theta}\right)$, and the designed frequency response, $\tilde{H}\left(e^{j\theta}\right)$, cannot be tolerated. Looking at Equation (4.4) we see that if $W(\theta)$ is large, the difference between the desired and designed frequency responses, $\left[H_D\left(e^{j\theta}\right) - \tilde{H}\left(e^{j\theta}\right)\right]$ has to be small to keep the weighted error small.

Conversely, if $W(\theta)$ is small, the difference $\left[H_D\left(e^{j\theta}\right) - \tilde{H}\left(e^{j\theta}\right)\right]$ can be larger and still meet the error criterion. Small values for $W(\theta)$ would be used in frequency bands where close approximation to the desired frequency response is not critical.

A copy of a program employing the Remez exchange algorithm [16] has been included, and its use is best illustrated with an example. Before proceeding with the example, the salient features of the FIR Linear Phase Filter Design program can be summarized as follows.

1. Main Program Functions

- Manage the Input -

NFILT = filter length in terms of samples, $3 \leq \text{NFILT} \leq \text{NFMAX}$

NFMAX = 128, but can made greater or smaller by the user

JTYPE = filter type

1 = Multi-Passband/Stopband

2 = Differentiator

3 = Hilbert Transformer

EDGE = number of frequency bands, specified by θ_{lower} and θ_{upper}

(maximum allowed is 10)

FX = desired frequency response magnitude in each band, $|H_D\left(e^{j\theta}\right)|$

WTX = positive weight function, $W(\theta)$, in each band

LGRID = Grid Density (the grid density, default value is 16)

- Set-up appropriate approximation problem, based on the desired frequency response, $\hat{H}_D\left(e^{j\theta}\right)$, and the weights, $W(\theta)$.
- Yield Output -
  - The coefficients of the best impulse response obtained from the best cosine approximation.
  - The optimal error $\left(\min\left[\max_{\theta} |E(\theta)|\right]\right)$
  - The extremal frequencies where $E(\theta) = \max_{\theta} |E(\theta)|$

2. Most Important Subroutines

EFF - defines the desired frequency response, $H_D\left(e^{j\theta}\right)$

WATE - defines the weight function, $W(\theta)$

REMEZ - calculates the best Chebyshev approximation of the desired frequency response

3. **Important Note**

It should be noted that the program output is the impulse response, $h(n)$. The user must supply his/her own program to calculate and plot the filter frequency response, which is necessary to determine the suitability of the filter design.

4. **Input Format**

NFILT JTYPE # Bands LGRID

EDGE (Band Edges)

FX (Magnitude in Bands)

WTX (Weights in Bands)

5. **Bandpass Filter Design Example**

We wish to design a bandpass filter that meets the following specifications:

Passband with a gain of one for the frequency interval 10 kHz to 15 kHz, with a system sampling frequency of 50 kHz.

- The digital frequency band is:

$$\theta_\ell = \frac{2\pi\left(10^4\right)}{5\left(10^4\right)} = 0.4\pi$$

$$\theta_u = \frac{2\pi(1.5)\left(10^4\right)}{5\left(10^4\right)} = 0.5\pi$$

- Normalizing so that $\theta = \pi$ corresponds to a normalized frequency of 0.5.

$$\frac{0.4\pi}{\pi} = \frac{\theta_\ell}{0.5} \Longrightarrow \theta_\ell = 0.2$$

$$\frac{0.6\pi}{\pi} = \frac{\theta_u}{0.5} \Longrightarrow \theta_u = 0.3$$

- Designate Band Edges, Magnitudes, Weights: (the weights were arbitrarily made equal for this example)

  Band 1

  Edges - 0.0 to 0.19

  Magnitude - 0.0

  Weight - 10.0

  Band 2

  Edges - 0.2 to 0.3

  Magnitude - 1.0

  Weight - 10.0

  Band 3

  Edges - 0.31 to 0.5

  Magnitude - 0.0

  Weight - 10.0

- Select Filter Length (number of coefficients)

  $N = 21$

6. Sample Input File

00021	00001	00003	0	
0.0	0.19	0.2	0.3	Band Edges
0.31	0.5			
0.0	1.0	0.0		Magnitudes
10.0	10.0	10.0		Weights

7. Sample Output File

The output file below was generated using the above input file. Note the output consists of the filter impulse response, the input values used and the deviation between the desired and approximated frequency responses.

```
                 FINITE IMPULSE RESPONSE (FIR)
               LINEAR PHASE DIGITAL FILTER DESIGN
                   REMEZ EXCHANGE ALGORITHM

                        BANDPASS FILTER

                     FILTER LENGTH =   21

               ***** IMPULSE RESPONSE *****
               H( 1) =   0.11599314E+00 = H( 21)
               H( 2) =  -0.10373381E-03 = H( 20)
               H( 3) =   0.11897093E+00 = H( 19)
               H( 4) =  -0.23107301E-03 = H( 18)
               H( 5) =  -0.14565307E+00 = H( 17)
               H( 6) =  -0.22831746E-03 = H( 16)
               H( 7) =   0.19631864E+00 = H( 15)
               H( 8) =  -0.12815343E-03 = H( 14)
               H( 9) =  -0.22079921E+00 = H( 13)
               H(10) =  -0.96616903E-04 = H( 12)
               H(11) =   0.23344022E+00 = H( 11)

                          BAND  1          BAND  2          BAND  3
LOWER BAND EDGE        0.0000000        0.2000000        0.3100000
UPPER BAND EDGE        0.1000000        0.3000000        0.5000000
DESIRED VALUE          0.0000000        1.0000000        0.0000000
WEIGHTING             10.0000000       10.0000000       10.0000000
DEVIATION              0.3446794        0.3446794        0.3446794
DEVIATION IN DB       -9.2516947        2.5723715       -9.2516947

EXTREMAL FREQUENCIES--MAXIMA OF THE ERROR CURVE
    0.0511363    0.1051131    0.1562489    0.1900000    0.2000000
    0.2511358    0.3000000    0.3100000    0.3440905    0.3952263
    0.4492030    0.5000000
```

8. Results

Figures 4.1 to 4.3, illustrate the frequency responses for filter lengths $N = 21, 41$ and 61, respectively.

As expected, the frequency response improves with an increasing number of coefficients. The cutoff is sharper and the magnitude more closely approximates 0 dB in the passband. As with other design methods, windows may also be used to improve the frequency response characteristic. Figures 4.4 to 4.6, show the effect of changing the band edges as follows, for $N = 21$ coefficients.

Figure 4.4 -	Band 1	0.0 to 0.19
	Band 2	0.2 to 0.3
	Band 3	0.31 to 0.5
Figure 4.5 -	Band 1	0.0 to 0.15
	Band 2	0.2 to 0.3
	Band 3	0.35 to 0.5
Figure 4.6 -	Band 1	0.0 to 0.1
	Band 2	0.2 to 0.3
	Band 3	0.4 to 0.5

Figure 4.4 represents the original cutoff frequencies, while Figures 4.5 and 4.6 show the results of not selecting the stopband frequency cutoff close enough to the passband frequency starting point, i.e., the passband is broadened beyond the desired 0.2 to 0.3 specifications.

Thus, when selecting the stopband cutoff frequencies one should choose values as close to the passband frequencies as possible. It is of importance to note that the program <u>does</u> <u>not</u> allow for the upper stopband frequency equaling the lower passband frequency. In other words the following parameters would not be acceptable:

Band 1	0.0 to 0.2
Band 2	0.2 to 0.3
Band 3	0.3 to 0.5

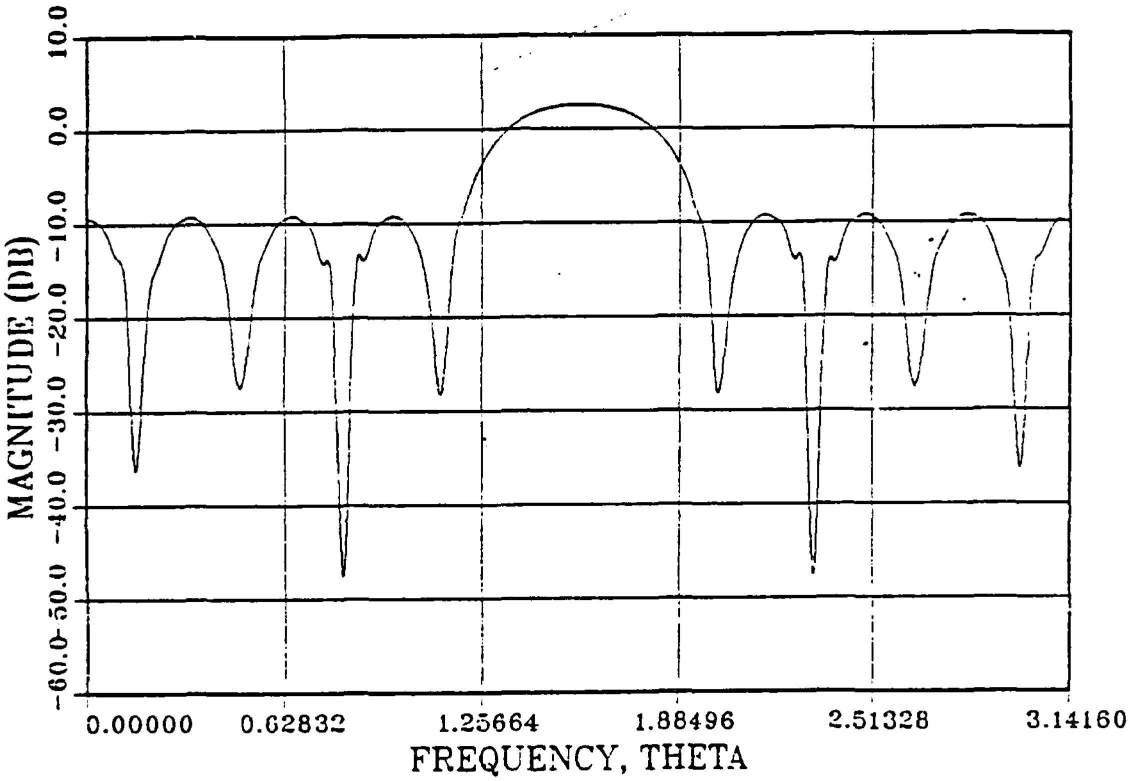


Figure 4.1. 21-Coefficient Remez Bandpass Filter Design

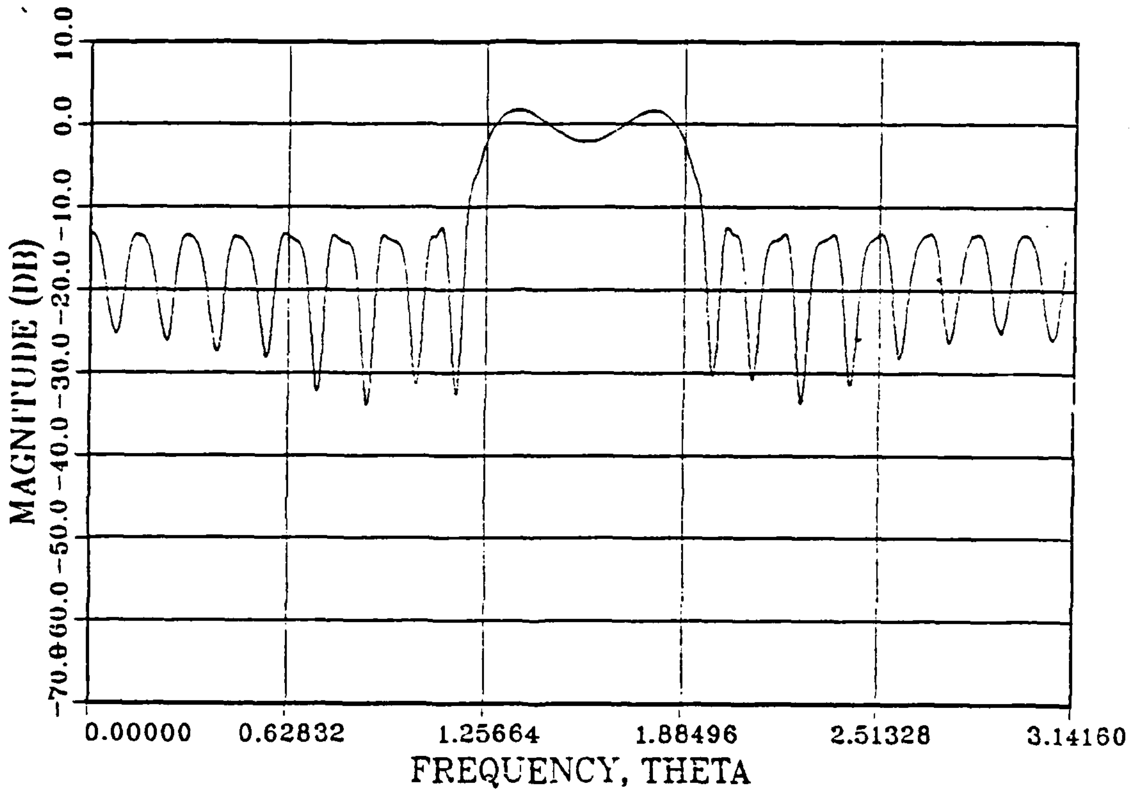


Figure 4.2. 41-Coefficient Remez Bandpass Filter Design

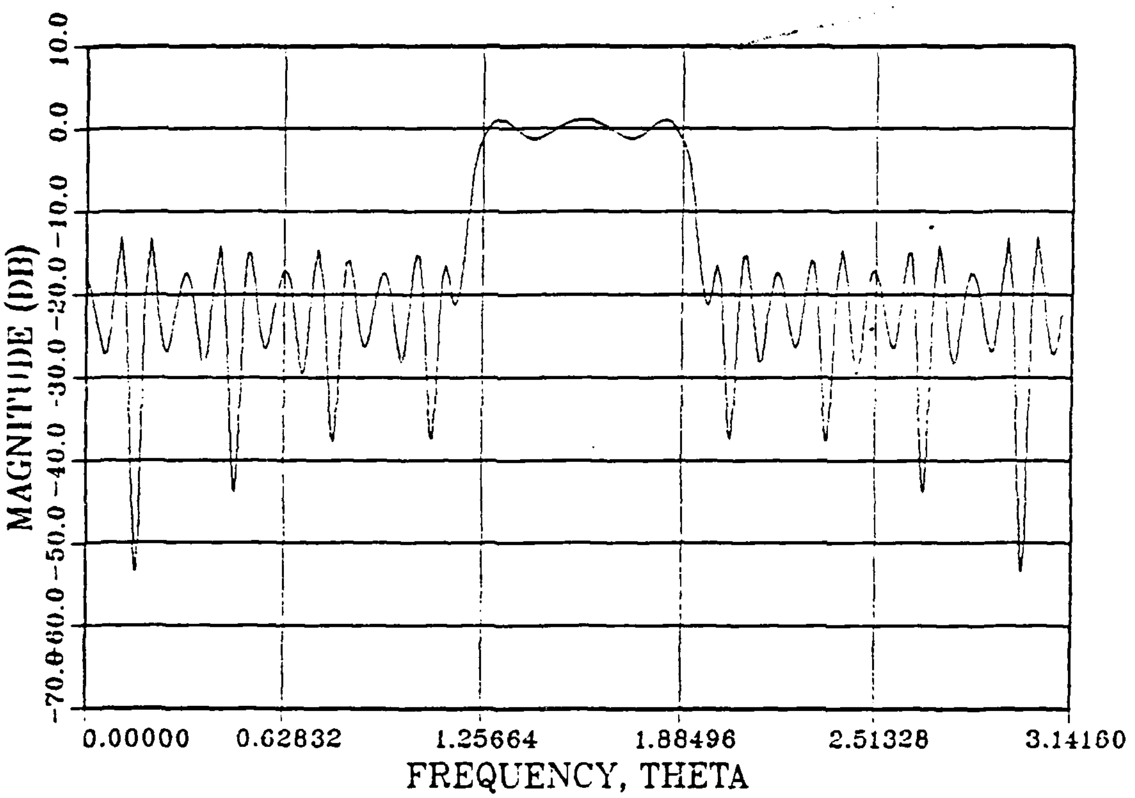


Figure 4.3. 61-Coefficient Remez Bandpass Filter Design

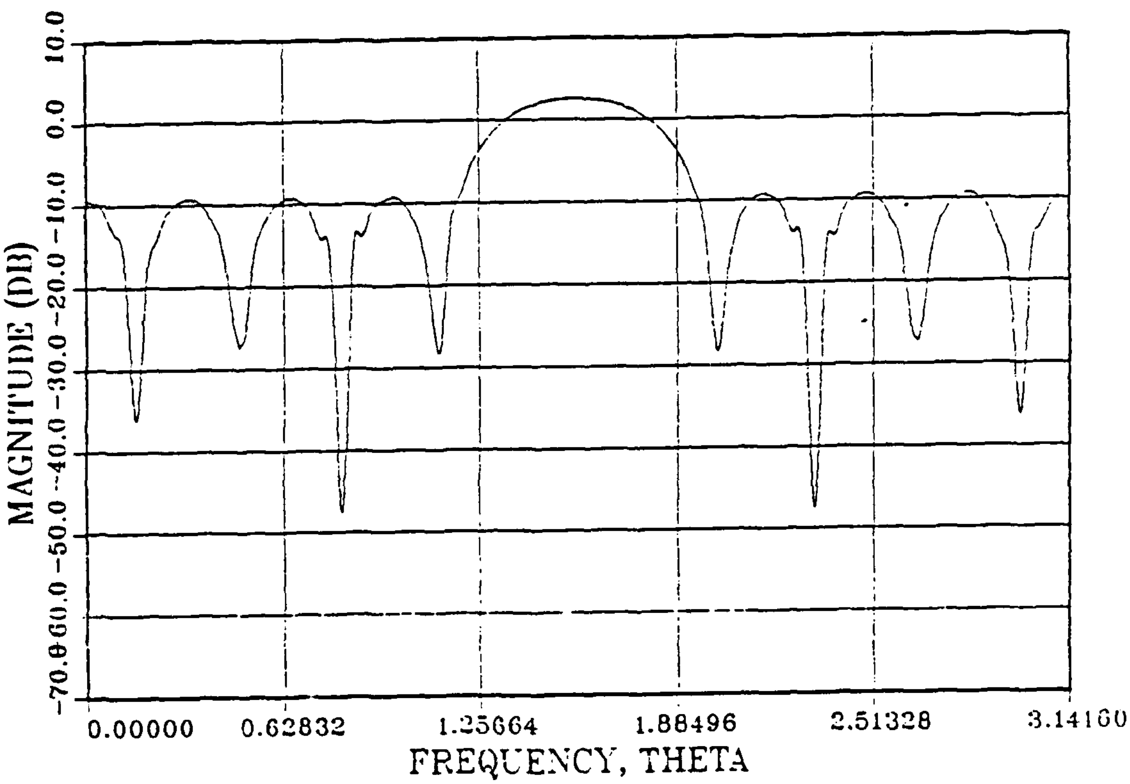


Figure 4.4. 21-Coefficient Remez Bandpass Filter Design
(Band edge spacing 0.01)

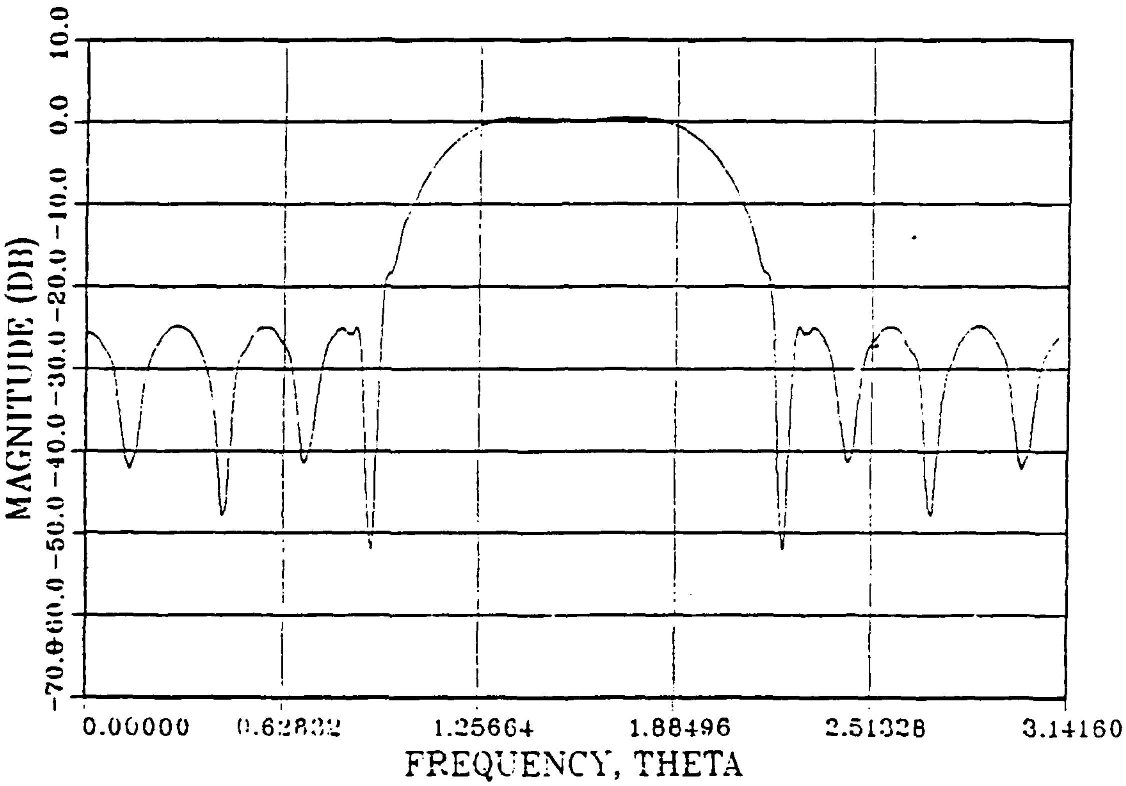


Figure 4.5 21-Coefficient Remez Bandpass Filter Design

(Band edge spacing 0.05)

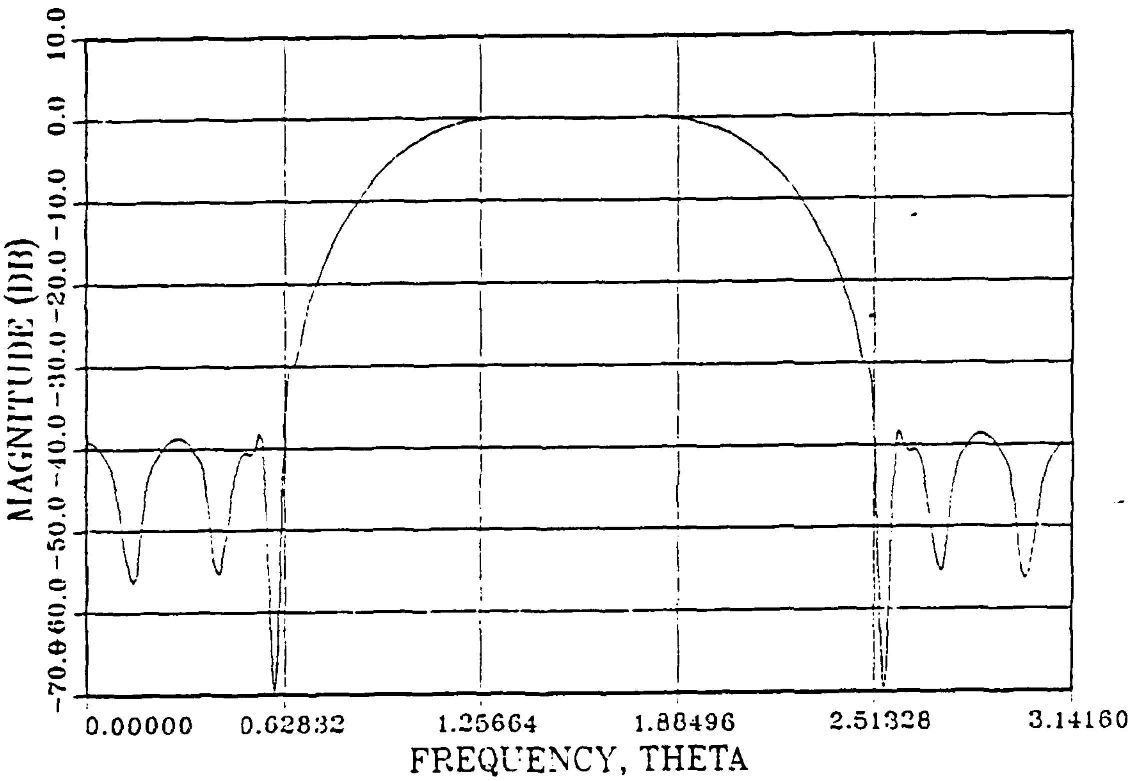


Figure 4.6. 21-Coefficient Remez Bandpass Filter Design

(Band edge spacing 0.1)

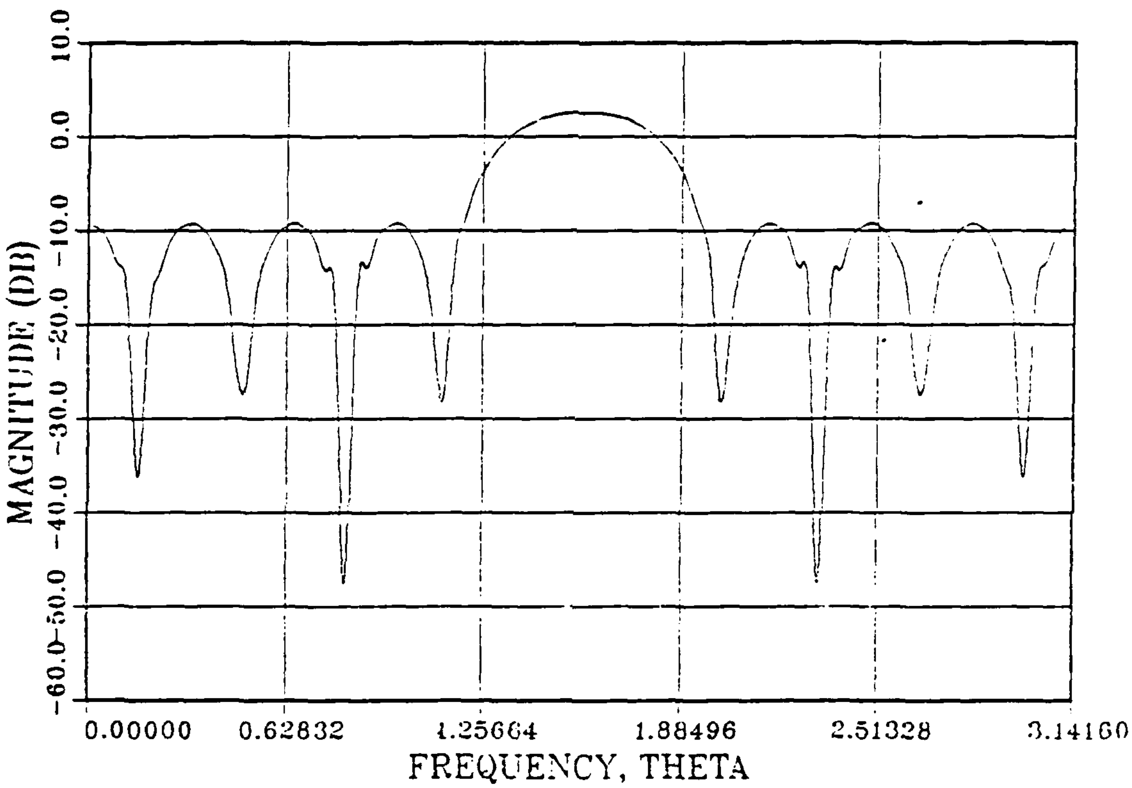


Figure 4.7. 21-Coefficient Remez Bandpass Filter Design

(Band weighting 1 : 1 : 1)

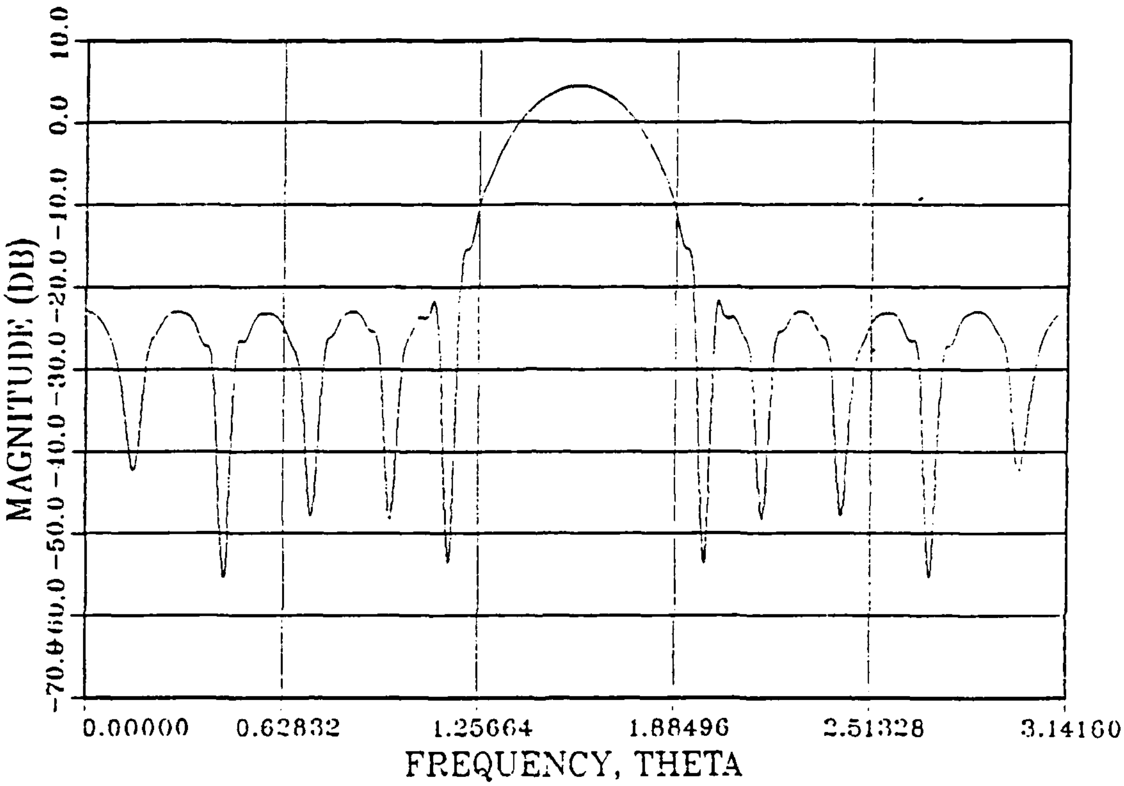


Figure 4.8. 21-Coefficient Remez Bandpass Filter Design

(Band weighting 10 : 1 : 10)

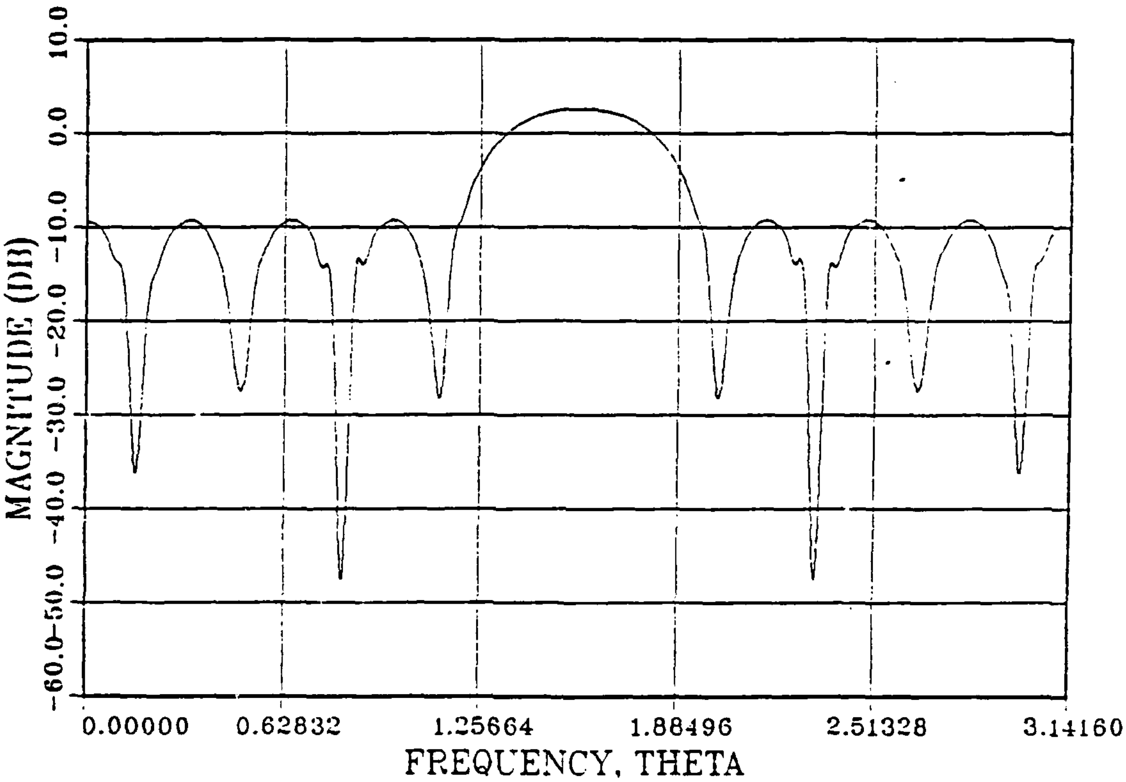


Figure 4.9. 21-Coefficient Remez Bandpass Filter Design

(Band weighting 10 : 10 : 10)

Finally, Figures 4.7 to 4.9 are the result of varying the weights of the original filter specifications as follows for $N = 21$.

Figure 4.7 -	Band 1	weight = 1.0
	Band 2	weight = 1.0
	Band 3	weight = 1.0
Figure 4.8 -	Band 1	weight = 10.0
	Band 2	weight = 1.0
	Band 3	weight = 10.0
Figure 4.9 -	Band 1	weight = 10.0
	Band 2	weight = 10.0
	Band 3	weight = 10.0

Again, as expected, the frequency bands with the heavier weight assigned more closely approximated the desired frequency response, Figure 4.8. Here it should be noted that the weights are taken into account relative to one another. Figures 4.7 and 4.9 are identical because in both cases the relationships between the weights is 1:1:1.

In this example, we have seen that the Remez exchange algorithm provides an effective way to design linear phase FIR filters. However, a primary drawback to this procedure is that the CPU time required grows quite rapidly (on the order of N) with the filter order N.

For example, fifteen CPU minutes are required for the design of a lowpass filter with $N = 512$ using a CDC 6500 [15]. This is the time required for one iteration of the procedure. Typically, more than one iteration is required to find technique to reduce the CPU time required is desirable. Reference 15 presents a method whereby the properties of a high-order filter can be extrapolated from a lower-order filter.

B. METHOD FOR THE DESIGN OF HIGH-ORDER LINEAR PHASE FIR FILTERS BASED ON A LOW-ORDER PROTOTYPE

A detailed explanation of this method will not be presented, however, a summary of the general idea behind this technique follows. The interested reader is directed to Reference 15 for details of its application. It should be noted that the term "high-order" refers to filters with orders approaching N = 2048.

The underlying basis for the high-order filter design technique is the observation that in high-order, multi-passband/stopband filters, extremal frequencies (i.e., ripple frequencies) in the broad part of these bands are more evenly distributed than in the transition regions.

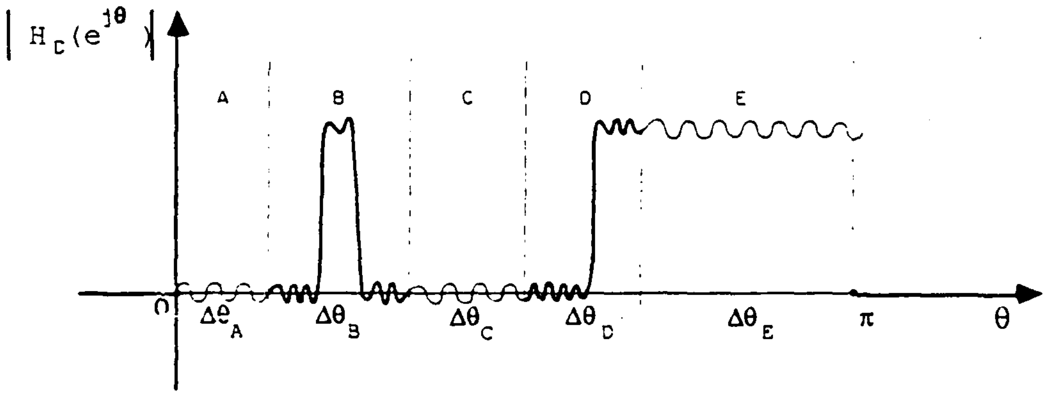


Figure 4.10. Desired High-Order Filter

Figure 4.10 represents the desired frequency response of a high-order multi-passband/stopband filter; and it is noted that in regions A, C, and E there is a uniform distribution of extremal frequencies, while in regions B and D the distribution is non-uniform.

The design of this filter using a lower order prototype involves the following steps:

- To obtain the low-order prototype, the uniform extremal frequency regions, A, C and E are "cut-out". The remaining non-uniform regions B and D are then merged to form the lower order filter whose frequency response $H_D\left(e^{j\theta'}\right)$ is shown in Figure 4.11.

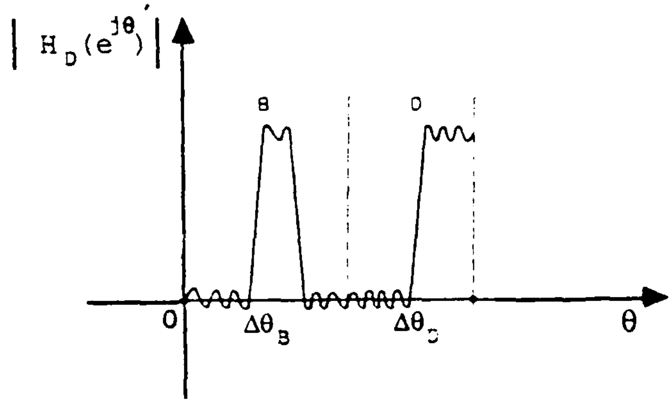


Figure 4.11. Low-Order Filter Prototype

- The Remez exchange algorithm is then used to obtain the filter order, N, and the coefficients required that will meet the specifications of $H_D\left(e^{j\theta'}\right)$. This is done by varying the order and the weights, until the order and weights that yield the desired passband ripple and stopband attenuation are found.

- Once the low-order prototype filter is obtained, its extremal frequencies are plugged back into the appropriate regions of the original desired high-order filter, Figure 4.12.

- The extremal frequencies are used as initial values, and a final run of the Remez Exchange Algorithm is then made to obtain the filter order and impulse response of the high-order filter design. Again, the filter order and weights are varied until an optimum design is obtained.

In conclusion it should be mentioned that in Reference 21 a program is outlined that automatically designs high-order FIR filters using this procedure. Approximations of filters with orders up to $N = 4096$ have been achieved using this program.

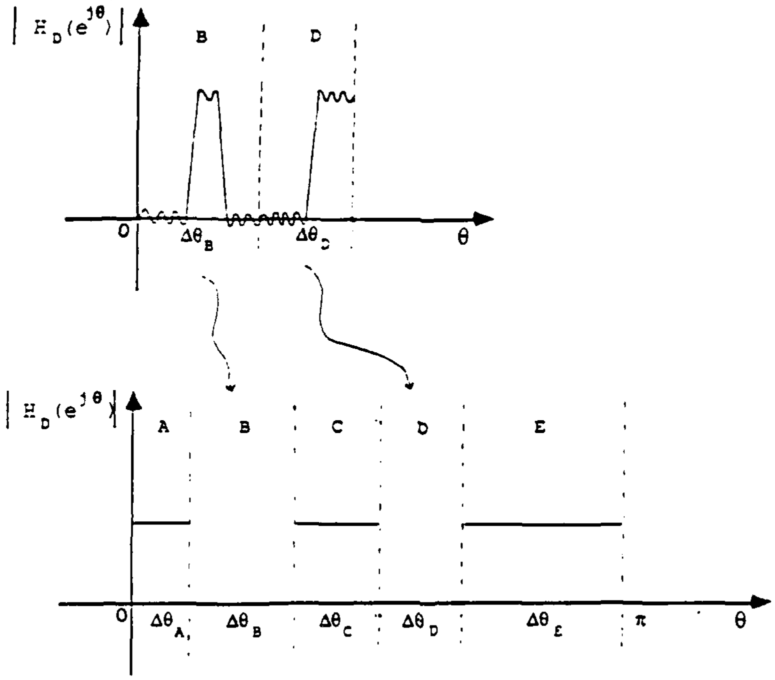


Figure 4.12. Low-order filter prototype being placed back into the appropriate regions of the desired high-order filter

In conclusion it should be mentioned that in Reference 21 a program is outlined that automatically designs high-order FIR filters using this procedure. Approximations of filters with orders up to $N = 4096$ have been achieved using this program.

This section has illustrated the usefulness and relative ease of application of Computer-Aided-Design (CAD) techniques, such as the Remez exchange algorithm.

Since this algorithm is only applicable to linear phase FIR filters, it is worthwhile at this point to diverge to a discussion of a popular algorithm for the design of recursive IIR filters, the Minimum p-Error Design Method.

C. THE MINIMUM p-ERROR DESIGN METHOD

Before proceeding with a discussion of this technique, it should be mentioned that, in the interest of brevity, the pertinent equations and relationships used are presented without elaboration. This is because the mathmatical basis for the minimum p-error criterion is quite extensive and beyond the scope of this thesis. The interested reader may find details of its derivation in Reference 19.

The minimum p-error design technique consists of a generalization of the minimum mean-squared error design method, wherein the optimum filter coefficients are determined by minimizing the following error function [18].

$$E_{2_p}(A) = \sum_{k=1}^{k} w(\theta_k)\left[H(\mathbf{A}, \theta_k) - H_D(\theta_k)\right]^{2_p} \tag{4.9}$$

where

$w(\theta_k) =$ weighting factor

$H(\mathbf{A}, \theta_k) =$ approximating function (actual response)

$H_D(\theta_k) =$ desired frequency response

$\theta_k =$ digital frequencies over the range of interest, k

$\mathbf{A} =$ vector containing the k independent parameters (i.e., the filter coefficients)

Note: If the value for p in Equation (4.9) is unity, this relationship describes the mean-square error.

The filter approximation problem can thus be stated as follows: Given an amount of error that can be tolerated, E_{2_p}, find the set of k parameters (filter coefficients) such that the error between the approximate frequency response and the desired frequency response is within the stated tolerance.

Again, looking at Equation (4.9) it can be seen that for large values of p, the approximate frequency response, $H(\mathbf{A}, \theta_k)$, has to be very close to the desired frequency response, $H_D(\theta_k)$, to remain within the pre-selected error tolerance value of E_{2_p}. Furthermore, a sufficiently large value of p results in an optimal solution that is very close to the optimal Chebyshev (or minimax) solution [19]. The method of solution of this approximation problem, as will be shown, is dependent on the error tolerance value selected, E_{2_p}, the form of the transfer function, $H(z)$, and the parameter vector, $\mathbf{A}$. First a discussion of the transfer function is presented because its form has a direct impact on several important features in digital filter design [18].

As is well known, three possible forms of the filter transfer function are direct, cascade and parallel realizations. With respect to errors directly related to the physical structure of the filter, i.e., quantization effects and finite coefficient size, the use of the direct form of the transfer function in filter realization is undesirable. This leaves the parallel and cascade forms, with the cascade form being selected because the zeros of the transfer function remain unaltered in the process of cascading, resulting in well defined stopbands. Conversion of a filter transfer function to the parallel form, on the other hand, involves partial fraction expansion, which results in the zeros being less well defined. Theoretically this should not be the case, but, when the parallel form is implemented digitally, the zeros are slightly different than those of the original direct form transfer function due to the finite precision of the computer.

Using the cascade form of the filter transfer function has other advantages including:

- stability tests of the filter can be accomplished without lengthy calculations,

- the frequency response is of a simple functional form that readily lends itself to insights as to how the poles and zeros (hence, the filter coefficients) impact the error function.

Thus, the first step in the use of this design procedure involves decomposing the proposed approximating transfer function, $H(z)$, into cascaded second-order sections.

$$H(z) = k_0 \prod_{i=1}^{N} H_i(z) = k_0 \prod_{i=1}^{N} \frac{z^2 + a_{1i}z + a_{2i}}{z^2 + b_{1i}z + b_{2i}} \tag{4.10}$$

where

$k_0 =$ a positive normalizing constant, and

$N =$ filter order /2.

The parameter vector, $\mathbf{A}$, therefore, consists of the multipliers used in the cascade structure.

$$\mathbf{A} = [a_{11}, a_{12}, b_{11}, b_{12}, \ldots, a_{1i}, a_{2i}, b_{1i}, b_{2i}, \ldots, k_0] \tag{4.11}$$

The following expressions for the frequency response and group delay of the filter incorporate the cascade structure and parameter vector.

- <u>Frequency Response</u>

$$\alpha(\mathbf{A}, \theta) = H(e^{j\theta}) = k_0 \prod_{i=1}^{N} \frac{(1 + a_{2i})\cos\theta + a_{1i} + j(1 - a_{2i})\sin\theta}{(1 + b_{2i})\cos\theta + b_{1i} + j(1 - b_{2i})\sin\theta} \tag{4.12}$$

- <u>Group Delay</u>

$$\tau(\mathbf{A}, \theta) = -\frac{d}{d\theta}H(e^{j\theta}) = \sum_{i=1}^{N} \left[Re\left(\frac{2\cos\theta + b_{1i} + j2\sin\theta}{(1 + b_{2i})\cos\theta + b_{1i} + j(1 - b_{2i})\sin\theta} \right) - Re\left(\frac{2\cos\theta + a_{1i} + j2\sin\theta}{(1 + a_{2i})\cos\theta + a_{1i} + j(1 - a_{2i})\sin\theta} \right) \right] \tag{4.13}$$

Equations (4.12) and (4.13) describe the frequency response and group delay of the approximating function, $H(\mathbf{A}, \theta_k)$. To determine if the coefficients of the

A vector yield an optimum filter design based on the desired frequency response $H_D(\theta_k)$ and the error that will be tolerated $E_{2_p}(\mathbf{A})$ involves finding the partial derivatives of the frequency response and group delay portions of the error function, in terms of the filter coefficients, and minimizing them.

In other words, the filter coefficients comprising the A vector are to be selected so as to minimize the following partial derivatives:

$$\frac{\partial E_{2_p}\alpha(\mathbf{A})}{\partial a_{1i}} = \sum_{k=1}^{k} pw_\alpha(\theta_k) \frac{\partial \alpha}{\partial a_{1i}} (\alpha(\mathbf{A}, \theta_k) - H_D(\theta_k))^{2p-1} \tag{4.14}$$

$$\frac{\partial E_{2_p}\tau(\mathbf{A})}{\partial a_{1i}} = \sum_{k=1}^{k} pw_\tau(\theta_k) \frac{\partial \tau}{\partial a_{1i}} (\tau(\mathbf{A}, \theta_k) - H_D(\theta_k))^{2p-1} \tag{4.15}$$

and similarly for $\partial E_{2p}\alpha(\mathbf{A})/\partial b_{1i}$ and $\partial E_{2p}\tau(\mathbf{A})/\partial b_{1i}$, etc.

Upon examining Equations (4.12) and (4.13), it is that apparent these partial derivatives are not easily attainable because complex expressions are involved. To remedy this the polar coordinates of the poles and zeros are used for parameters, rather than the original rectangular form. Thus, the parameter vector and expressions for the amplitude and group delay are modified as follows,

- A Parameter

$$\mathbf{A} = [r_{01}, \theta_{01}, r_{p1}, \theta_{p1}, \ldots, r_{0i}, \theta_{0i}, r_{pi}, \theta_{pi}, \ldots, k_0] \tag{4.16}$$

- Frequency Response

$$\alpha(\mathbf{A}, \theta) = k_0 \prod_{i=1}^{N} \frac{\left\{1 - 2r_{0i}\cos(\theta - \theta_{oi}) + r_{0i^2}\right\}^{1/2} \left\{1 - 2r_{0i}\cos(\theta + \theta_{0i}) + r_{0i^2}\right\}^{1/2}}{\left\{1 - 2r_{pi}\cos(\theta - \theta_{pi}) + r_{pi^2}\right\}^{1/2} \left\{1 - 2r_{pi}\cos(\theta + \theta_{pi}) + r_{pi^2}\right\}^{1/2}} \tag{4.17}$$

• <u>Group Delay</u>

$$\tau(\mathbf{A},\theta) = \sum_{i=1}^{N} \left[\frac{1 - r_{pi}\cos(\theta - \theta_{pi})}{1 - 2r_{pi}\cos(\theta - \theta_{pi}) + r_{pi^2}} + \frac{1 - r_{pi}\cos(\theta + \theta_{pi})}{1 - 2r_{pi}\cos(\theta + \theta_{pi}) + r_{pi^2}} \right. \\ \left. - \frac{1 - r_{oi}\cos(\theta - \theta_{oi})}{1 - 2r_{oi}\cos(\theta - \theta_{oi}) + r_{oi^2}} - \frac{1 - r_{oi}\cos(\theta + \theta_{oi})}{1 - 2r_{oi}\cos(\theta + \theta_{oi}) + r_{oi^2}} \right] \tag{4.18}$$

resulting in the following partial derivatives of the error function:

$$\frac{\partial E_{2_p}\tau(\mathbf{A})}{\partial r_{oi}} = \sum_{k=1}^{k} p w_\tau(\theta_k) \frac{\partial \tau}{\partial r_{oi}} (\tau(\mathbf{A},\theta_k) - H_D(\theta_k))^{2p-1} \tag{4.19}$$

and for $\partial E_{2_p}\alpha(A)/\partial\theta_{oi}$ and $\partial E_{2_p}\tau(A)/\partial\theta_{0i}$, etc. where

$$\begin{aligned}
\frac{\partial\alpha}{\partial r_{oi}} &= \alpha \left[\frac{r_{oi} - \cos(\theta - \theta_{oi})}{1 - 2r_{oi}\cos(\theta - \theta_{oi}) + r_{oi^2}} \right. \\
&\qquad \left. + \frac{r_{oi} - \cos(\theta + \theta_{oi})}{1 - 2r_{oi}\cos(\theta + \theta_{oi}) + r_{oi^2}} \right] \\
\frac{\partial\alpha}{\partial\theta_{oi}} &= \alpha \left[\frac{r_{oi}\sin(\theta - \theta_{oi})}{1 - 2r_{oi}\cos(\theta - \theta_{oi}) + r_{oi^2}} \right. \\
&\qquad \left. - \frac{r_{oi}\sin(\theta + \theta_{oi})}{1 - 2r_{oi}\cos(\theta + \theta_{oi}) + r_{oi^2}} \right] \\
\frac{\partial\tau}{\partial r_{pi}} &= \frac{(1 + r_{pi^2})\cos(\theta - \theta_{pi}) - 2r_{pi}}{(1 - 2r_{pi}\cos(\theta - \theta_{pi}) + r_{pi^2})^2} \\
&\qquad + \frac{(1 + r_{pi^2})\cos(\theta + \theta_{pi}) - 2r_{pi}}{(1 - 2r_{pi}\cos(\theta + \theta_{pi}) + r_{pi^2})^2} \\
\frac{\partial\tau}{\partial\theta_{oi}} &= \frac{r_{pi}(1 - r_{pi^2})\sin(\theta - \theta_{pi})}{(1 - 2r_{pi}\cos(\theta - \theta_{pi}) + r_{pi^2})^2} \\
&\qquad - \frac{r_{pi}(1 - r_{pi^2})\sin(\theta + \theta_{pi})}{(1 - 2r_{pi}\cos(\theta + \theta_{pi}) + r_{pi^2})^2}.
\end{aligned} \tag{4.20}$$

The partial derivatives $\partial\alpha/\partial r_{pi}, \partial\alpha/\partial\theta_{pi}, \partial\tau/\partial r_{0i}$, and $\partial\tau/\partial\theta_{oi}$, are the same as the above but with changed signs.

Thus, the approximation problem amounts to the minimization of nonlinear functions (Equations (4.19) and (4.20)), of n variables (the poles and zeros). This problem is readily solved through the use of the Fletcher-Powell algorithm, the details of which can be found in Reference 20.

A program that performs the synthesis of recursive digital filters using the minimum p-error criterion and the method of Fletcher and Powell for function minimization is contained in Reference 16. Included in this reference are an extensive program description, input requirements, dimension restrictions, and examples to illustrate how the program is used for various types of filter design. Also included is a copy of the actual program.

The reader is reminded, however, that to make use of the program, the transfer function must be in the cascade form mentioned previously. The program output consists of an argument vector, $\underline{X}$, and a gradient vector, $\underline{G}$, corresponding to the minimization of the functions F1, F2, and F3 which are used by the program for the magnitude approximation, the group delay approximation, and the combined magnitude and group delay approximation, respectively. Based on $\underline{X}$, the poles, zeros and coefficients of the cascade realization of the filter are computed. If desired by the user, the frequency response is also given [16].

D. SUMMARY

In summary then it is appropriate to cite a recent paper by Little and Gowdy [17], that evaluates both the optimum FIR design method (that employs the Remez exchange algorithm), and the minimum p-error method used in IIR design. Their evaluation specifically investigates convergence problems encountered when using these iterative techniques, and considers the design of lowpass, highpass, bandpass and bandstop filters.

The following is a brief summary of the more important points mentioned in this article, that should be considered when using these iterative techniques.

1. Minimum p-Error Design Method

   a. Advantages:

- Extremely flexible - can be used to design filters with arbitrary magnitude and/or phase characteristics, as opposed to being limited to lowpass, highpass, bandpass, or bandstop designs.

   b. Disadvantages:

- Nonconvergence problems due to:
  - a poor guess for the initial parameter vector $\underline{X}$
  - finite wordlength
- Uses large amounts of CPU (IBM 3081) time (up to 2 minutes for higher order filters, greater than $N = 12$)
- Requires a considerable amount of input to specify one filter

2. Optimum FIR Filter Design Program

   a. Advantages:

- The program is easy to use
- Consumes relatively little CPU (IBM 3081) time, provided the filter is not of extremely high order.

   b. Disadvantages:

- Restricted to the design of the more common filter types: multi-band, bandpass, Hilbert transform, and differentiators
- Highpass and bandpass filter designs were poor when even filter lengths were used, but good for odd filter lengths
- The user must supply an FFT program to obtain frequency response data to verify the filter design. Convergence of the program does not guarantee an acceptable design.

Keeping these considerations in mind, both programs can be used effectively to design a wide variety of FIR and IIR filters. It is suggested that if a designer wishes to use either of these programs this paper is a valuable reference to assist in identifying problems that may be encountered.

V. CONCLUSION

The areas of recursive (IIR) and nonrecursive (FIR) filter design have been investigated, while the more predominant design methods have been extensively discussed and exemplified. Additionally, the prevailing computer-aided-design algorithms (Remez exchange and Fletcher-Powell) were also presented.

In conclusion, points in these three areas that merit special emphasis are summarized below:

A. RECURSIVE FILTER DESIGN

- The desired frequency response may be obtained using a lower order filter than if a nonrecursive realization were used, however, filter stability must be considered, and a linear phase characteristic is not guaranteed.

- The traditional design methods involving Butterworth, Chebyshev or elliptic analog prototypes, and the bilinear transformation are algebraically intensive. The direct design method presented eliminates the need for determining an analog prototype and for prewarping, thereby reducing the number of calculations required.

B. NONRECURSIVE FILTER DESIGN

- Although a higher order filter is required than in recursive realizations to obtain the same frequency response; nonrecursive filters are always stable due to the all zero nature of their transfer functions, and can exhibit a linear phase characteristic.

- The filter coefficients, $h(n)$, may be determined analytically by expanding the desired frequency response in a Fourier series, because the Fourier coefficients correspond to the filter coefficients.

- Windows may be used to eliminate the overshoot in the frequency response (Gibbs' phenomenon) caused by truncating the number of terms in the Fourier series to a finite value. However, windowing decreases the sharpness of the filter's cutoff region.

- Frequency sampling can be used when an analytical expression for the desired frequency response cannot be found. The IDFT is then used to determine the filter's impulse response from the these sample values.

C. COMPUTER-AIDED DESIGN

- For FIR filters, programs using the Remez Exchange algorithm are the most popular, while for IIR filters, the Fletcher-Powell algorithm is in predominate use.

- CAD is especially advantageous in the design of extremely high-order filters, or filters with arbitrary frequency response characteristics.

As stated in the introduction, the design methods presented here are by no means the only ones available to the filter design engineer; however, due to the quantity of information available, one is easily overwhelmed. A need exists for a concise source of information on the popular design methods available, and how they are used. This has been the intent of this thesis.

APPENDIX

MAIN PROGRAM

```
C
C-----------------------------------------------------------------------
C MAIN PROGRAM: FIR LINEAR PHASE FILTER DESIGN PROGRAM
C
C AUTHORS: JAMES H. MCCLELLAN
C          DEPARTMENT OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE
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C
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C          HOUSTON, TEXAS 77001
C
C          LAWRENCE R. RABINER
C          BELL LABORATORIES
C          MURRAY HILL, NEW JERSEY 07974
C
C INPUT:
C  NFILT-- FILTER LENGTH
C  JTYPE-- TYPE OF FILTER
C          1 = MULTIPLE PASSBAND/STOPBAND FILTER
C          2 = DIFFERENTIATOR
C          3 = HILBERT TRANSFORM FILTER
C  NBANDS-- NUMBER OF BANDS
C  LGRID-- GRID DENSITY, WILL BE SET TO 16 UNLESS
C          SPECIFIED OTHERWISE BY A POSITIVE CONSTANT.
C
C  EDGE(2*NBANDS) -- BANDEDGE ARRAY, LOWER AND UPPER EDGES FOR EACH BAND
C                    WITH A MAXIMUM OF 10 BANDS.
C
C  FX(NBANDS) -- DESIRED FUNCTION ARRAY (OR DESIRED SLOPE IF A
C                DIFFERENTIATOR) FOR EACH BAND.
C
C  WTX(NBANDS) -- WEIGHT FUNCTION ARRAY IN EACH BAND.  FOR A
C                 DIFFERENTIATOR, THE WEIGHT FUNCTION IS INVERSELY
C                 PROPORTIONAL TO F.
C
C  SAMPLE INPUT DATA SETUP:
C      32,1,3,0
C      0.0,0.1,0.2,0.35
C      0.425,0.5
C      0.0,1.0,0.0
C      10.0,1.0,10.0
C    THIS DATA SPECIFIES A LENGTH 32 BANDPASS FILTER WITH
C    STOPBANDS 0 TO 0.1 AND 0.425 TO 0.5, AND PASSBAND FROM
C    0.2 TO 0.35 WITH WEIGHTING OF 10 IN THE STOPBANDS AND 1
C    IN THE PASSBAND.  THE GRID DENSITY DEFAULTS TO 16.
C    THIS IS THE FILTER IN FIGURE 10.
C
C    THE FOLLOWING INPUT DATA SPECIFIES A LENGTH 32 FULLBAND
C    DIFFERENTIATOR WITH SLOPE 1 AND WEIGHTING OF 1/F.
C    THE GRID DENSITY WILL BE SET TO 20.
C      32,2,1,20
C      0,0.5
C      1.0
C      1.0
C
C-----------------------------------------------------------------------
C
      COMMON PI2,AD,DEV,X,Y,GRID,DES,WT,ALPHA,IEXT,NFCNS,NGRID
      COMMON /OOPS/NITER,IOUT
      DIMENSION IEXT(66),AD(66),ALPHA(66),X(66),Y(66)
      DIMENSION H(66)
      DIMENSION DES(1045),GRID(1045),WT(1045)
      DIMENSION EDGE(20),FX(10),WTX(10),DEVIAT(10)
      DOUBLE PRECISION PI2,PI
      DOUBLE PRECISION AD,DEV,X,Y
      DOUBLE PRECISION GEE,D
      INTEGER BD1,BD2,BD3,BD4
      DATA BD1,BD2,BD3,BD4/1HB,1HA,1HN,1HD/
C     INPUT=I1MACH(1)
```

```
C     IOUT=I1MACH(2)
      PI=4.0*DATAN(1.0D0)
      PI2=2.0D00*PI
C
C  THE PROGRAM IS SET UP FOR A MAXIMUM LENGTH OF 128, BUT
C  THIS UPPER LIMIT CAN BE CHANGED BY REDIMENSIONING THE
C  ARRAYS IEXT, AD, ALPHA, X, Y, H TO BE NFMAX/2 + 2.
C  THE ARRAYS DES, GRID, AND WT MUST DIMENSIONED
C  16(NFMAX/2 + 2).
C
      NFMAX=128
  100 CONTINUE
      JTYPE=0
C
C  PROGRAM INPUT SECTION
C
      READ(4,110) NFILT,JTYPE,NBANDS,LGRID
      IF(NFILT.EQ.0) STOP
  110 FORMAT(4(I5,3X))
      IF(NFILT.LE.NFMAX.AND.NFILT.GE.3) GO TO 115
      CALL ERROR
      STOP
  115 IF(NBANDS.LE.0) NBANDS=1
C
C  GRID DENSITY IS ASSUMED TO BE 16 UNLESS SPECIFIED
C  OTHERWISE
C
      IF(LGRID.LE.0) LGRID=16
      JB=2*NBANDS
      READ(4,120) (EDGE(J),J=1,JB)
      READ(4,120) (FX(J),J=1,NBANDS)
      READ(4,120) (WTX(J),J=1,NBANDS)
  120 FORMAT(4F15.9)
      IF(JTYPE.GT.0.AND.JTYPE.LE.3) GO TO 125
      CALL ERROR
      STOP
  125 NEG=1
      IF(JTYPE.EQ.1) NEG=0
      NODD=NFILT/2
      NODD=NFILT-2*NODD
      NFCNS=NFILT/2
      IF(NODD.EQ.1.AND.NEG.EQ.0) NFCNS=NFCNS+1
C
C  SET UP THE DENSE GRID.  THE NUMBER OF POINTS IN THE GRID
C  IS (FILTER LENGTH + 1)*GRID DENSITY/2
C
      GRID(1)=EDGE(1)
      DELF=LGRID*NFCNS
      DELF=0.5/DELF
      IF(NEG.EQ.0) GO TO 135
      IF(EDGE(1).LT.DELF) GRID(1)=DELF
  135 CONTINUE
      J=1
      L=1
      LBAND=1
  140 FUP=EDGE(L+1)
  145 TEMP=GRID(J)
C
C  CALCULATE THE DESIRED MAGNITUDE RESPONSE AND THE WEIGHT
C  FUNCTION ON THE GRID
C
      DES(J)=EFF(TEMP,FX,WTX,LBAND,JTYPE)
      WT(J)=WATE(TEMP,FX,WTX,LBAND,JTYPE)
      J=J+1
      GRID(J)=TEMP+DELF
      IF(GRID(J).GT.FUP) GO TO 150
      GO TO 145
  150 GRID(J-1)=FUP
      DES(J-1)=EFF(FUP,FX,WTX,LBAND,JTYPE)
      WT(J-1)=WATE(FUP,FX,WTX,LBAND,JTYPE)
      LBAND=LBAND+1
      L=L+2
```

```
      IF(LBAND.GT.NBANDS) GO TO 160
      GRID(J)=EDGE(L)
      GO TO 140
  160 NGRID=J-1
      IF(NEG.NE.NODD) GO TO 165
      IF(GRID(NGRID).GT.(0.5-DELF)) NGRID=NGRID-1
  165 CONTINUE
C
C  SET UP A NEW APPROXIMATION PROBLEM WHICH IS EQUIVALENT
C  TO THE ORIGINAL PROBLEM
C
      IF(NEG) 170,170,180
  170 IF(NODD.EQ.1) GO TO 200
      DO 175 J=1,NGRID
      CHANGE=DCOS(PI*GRID(J))
      DES(J)=DES(J)/CHANGE
  175 WT(J)=WT(J)*CHANGE
      GO TO 200
  180 IF(NODD.EQ.1) GO TO 190
      DO 185 J=1,NGRID
      CHANGE=DSIN(PI*GRID(J))
      DES(J)=DES(J)/CHANGE
  185 WT(J)=WT(J)*CHANGE
      GO TO 200
  190 DO 195 J=1,NGRID
      CHANGE=DSIN(PI2*GRID(J))
      DES(J)=DES(J)/CHANGE
  195 WT(J)=WT(J)*CHANGE
C
C  INITIAL GUESS FOR THE EXTREMAL FREQUENCIES--EQUALLY
C  SPACED ALONG THE GRID
C
  200 TEMP=FLOAT(NGRID-1)/FLOAT(NFCNS)
      DO 210 J=1,NFCNS
      XT=J-1
  210 IEXT(J)=XT*TEMP+1.0
      IEXT(NFCNS+1)=NGRID
      NM1=NFCNS-1
      NZ=NFCNS+1
C
C  CALL THE REMEZ EXCHANGE ALGORITHM TO DO THE APPROXIMATION
C  PROBLEM
C
      CALL REMEZ
C
C  CALCULATE THE IMPULSE RESPONSE.
C
      IF(NEG) 300,300,320
  300 IF(NODD.EQ.0) GO TO 310
      DO 305 J=1,NM1
      NZMJ=NZ-J
  305 H(J)=0.5*ALPHA(NZMJ)
      H(NFCNS)=ALPHA(1)
      GO TO 350
  310 H(1)=0.25*ALPHA(NFCNS)
      DO 315 J=2,NM1
      NZMJ=NZ-J
      NF2J=NFCNS+2-J
  315 H(J)=0.25*(ALPHA(NZMJ)+ALPHA(NF2J))
      H(NFCNS)=0.5*ALPHA(1)+0.25*ALPHA(2)
      GO TO 350
  320 IF(NODD.EQ.0) GO TO 330
      H(1)=0.25*ALPHA(NFCNS)
      H(2)=0.25*ALPHA(NM1)
      DO 325 J=3,NM1
      NZMJ=NZ-J
      NF3J=NFCNS+3-J
  325 H(J)=0.25*(ALPHA(NZMJ)-ALPHA(NF3J))
      H(NFCNS)=0.5*ALPHA(1)-0.25*ALPHA(3)
      H(NZ)=0.0
      GO TO 350
  330 H(1)=0.25*ALPHA(NFCNS)
```

```
      DO 335 J=2,NM1
      NZMJ=NZ-J
      NF2J=NFCNS+2-J
  335 H(J)=0.25*(ALPHA(NZMJ)-ALPHA(NF2J))
      H(NFCNS)=0.5*ALPHA(1)-0.25*ALPHA(2)
C
C  PROGRAM OUTPUT SECTION.
C
  350 WRITE(8,360)
  360 FORMAT(1H1, 70(1H*)//15X,29HFINITE IMPULSE RESPONSE (FIR)/
     113X,34HLINEAR PHASE DIGITAL FILTER DESIGN/
     217X,24HREMEZ EXCHANGE ALGORITHM/)
      IF(JTYPE.EQ.1) WRITE(8,365)
  365 FORMAT(22X,15HBANDPASS FILTER/)
      IF(JTYPE.EQ.2) WRITE(8,370)
  370 FORMAT(22X,14HDIFFERENTIATOR/)
      IF(JTYPE.EQ.3) WRITE(8,375)
  375 FORMAT(20X,19HHILBERT TRANSFORMER/)
      WRITE(8,378) NFILT
  378 FORMAT(20X,16HFILTER LENGTH = ,I3/)
      WRITE(8,380)
  380 FORMAT(15X,28H***** IMPULSE RESPONSE *****)
      DO 381 J=1,NFCNS
      K=NFILT+1-J
      IF(NEG.EQ.0) WRITE(8,382) J,H(J),K
      IF(NEG.EQ.1) WRITE(8,383) J,H(J),K
  381 CONTINUE
  382 FORMAT(13X,2HH(,I2,4H) = ,E15.8,5H = H(,I3,1H))
  383 FORMAT(13X,2HH(,I2,4H) = ,E15.8,6H = -H(,I3,1H))
      IF(NEG.EQ.1.AND.NODD.EQ.1) WRITE(8,384) NZ
  384 FORMAT(13X,2HH(,I2,8H) =  0.0)
      DO 450 K=1,NBANDS,4
      KUP=K+3
      IF(KUP.GT.NBANDS) KUP=NBANDS
      WRITE(8,385) (BD1,BD2,BD3,BD4,J,J=K,KUP)
  385 FORMAT(/24X,4(4A1,I3,7X))
      WRITE(8,390) (EDGE(2*J-1),J=K,KUP)
  390 FORMAT(2X,15HLOWER BAND EDGE,5F14.7)
      WRITE(8,395) (EDGE(2*J),J=K,KUP)
  395 FORMAT(2X,15HUPPER BAND EDGE,5F14.7)
      IF(JTYPE.NE.2) WRITE(8,400) (FX(J),J=K,KUP)
  400 FORMAT(2X,13HDESIRED VALUE,2X,5F14.7)
      IF(JTYPE.EQ.2) WRITE(8,405) (FX(J),J=K,KUP)
  405 FORMAT(2X,13HDESIRED SLOPE,2X,5F14.7)
      WRITE(8,410) (WTX(J),J=K,KUP)
  410 FORMAT(2X,9HWEIGHTING,6X,5F14.7)
      DO 420 J=K,KUP
  420 DEVIAT(J)=DEV/WTX(J)
      WRITE(8,425) (DEVIAT(J),J=K,KUP)
  425 FORMAT(2X,9HDEVIATION,6X,5F14.7)
      IF(JTYPE.NE.1) GO TO 450
      DO 430 J=K,KUP
  430 DEVIAT(J)=20.0*ALOG10(DEVIAT(J)+FX(J))
      WRITE(8,435) (DEVIAT(J),J=K,KUP)
  435 FORMAT(2X,15HDEVIATION IN DB,5F14.7)
  450 CONTINUE
      DO 452 J=1,NZ
      IX=IEXT(J)
  452 GRID(J)=GRID(IX)
      WRITE(8,455) (GRID(J),J=1,NZ)
  455 FORMAT(/2X,47HEXTREMAL FREQUENCIES--MAXIMA OF THE ERROR CURVE/
     1 (2X,5F12.7))
      WRITE(8,460)
  460 FORMAT(/1X,70(1H*)/1H1)
      GO TO 100
      END
C
C-----------------------------------------------------------------------
C FUNCTION: EFF
C   FUNCTION TO CALCULATE THE DESIRED MAGNITUDE RESPONSE
C   AS A FUNCTION OF FREQUENCY.
C   AN ARBITRARY FUNCTION OF FREQUENCY CAN BE
```

```
C     APPROXIMATED IF THE USER REPLACES THIS FUNCTION
C     WITH THE APPROPRIATE CODE TO EVALUATE THE IDEAL
C     MAGNITUDE.  NOTE THAT THE PARAMETER FREQ IS THE
C     VALUE OF NORMALIZED FREQUENCY NEEDED FOR EVALUATION.
C-----------------------------------------------------------------------
C
      FUNCTION EFF(FREQ,FX,WTX,LBAND,JTYPE)
      DIMENSION FX(5),WTX(5)
      IF(JTYPE.EQ.2) GO TO 1
      EFF=FX(LBAND)
      RETURN
    1 EFF=FX(LBAND)*FREQ
      RETURN
      END
C
C-----------------------------------------------------------------------
C FUNCTION: WATE
C     FUNCTION TO CALCULATE THE WEIGHT FUNCTION AS A FUNCTION
C     OF FREQUENCY.  SIMILAR TO THE FUNCTION EFF, THIS FUNCTION CAN
C     BE REPLACED BY A USER-WRITTEN ROUTINE TO CALCULATE ANY
C     DESIRED WEIGHTING FUNCTION.
C-----------------------------------------------------------------------
C
      FUNCTION WATE(FREQ,FX,WTX,LBAND,JTYPE)
      DIMENSION FX(5),WTX(5)
      IF(JTYPE.EQ.2) GO TO 1
      WATE=WTX(LBAND)
      RETURN
    1 IF(FX(LBAND).LT.0.0001) GO TO 2
      WATE=WTX(LBAND)/FREQ
      RETURN
    2 WATE=WTX(LBAND)
      RETURN
      END
C
C-----------------------------------------------------------------------
C SUBROUTINE: ERROR
C     THIS ROUTINE WRITES AN ERROR MESSAGE IF AN
C     ERROR HAS BEEN DETECTED IN THE INPUT DATA.
C-----------------------------------------------------------------------
C
      SUBROUTINE ERROR
      COMMON /OOPS/NITER,IOUT
      WRITE(6,1)
    1 FORMAT(44H ************ ERROR IN INPUT DATA **********)
      RETURN
      END
C
C-----------------------------------------------------------------------
C SUBROUTINE: REMEZ
C     THIS SUBROUTINE IMPLEMENTS THE REMEZ EXCHANGE ALGORITHM
C     FOR THE WEIGHTED CHEBYSHEV APPROXIMATION OF A CONTINUOUS
C     FUNCTION WITH A SUM OF COSINES.  INPUTS TO THE SUBROUTINE
C     ARE A DENSE GRID WHICH REPLACES THE FREQUENCY AXIS, THE
C     DESIRED FUNCTION ON THIS GRID, THE WEIGHT FUNCTION ON THE
C     GRID, THE NUMBER OF COSINES, AND AN INITIAL GUESS OF THE
C     EXTREMAL FREQUENCIES.  THE PROGRAM MINIMIZES THE CHEBYSHEV
C     ERROR BY DETERMINING THE BEST LOCATION OF THE EXTREMAL
C     FREQUENCIES (POINTS OF MAXIMUM ERROR) AND THEN CALCULATES
C     THE COEFFICIENTS OF THE BEST APPROXIMATION.
C-----------------------------------------------------------------------
C
      SUBROUTINE REMEZ
      COMMON PI2,AD,DEV,X,Y,GRID,DES,WT,ALPHA,IEXT,NFCNS,NGRID
      COMMON /OOPS/NITER,IOUT
      DIMENSION IEXT(66),AD(66),ALPHA(66),X(66),Y(66)
      DIMENSION DES(1045),GRID(1045),WT(1045)
      DIMENSION A(66),P(65),Q(65)
      DOUBLE PRECISION PI2,DNUM,DDEN,DTEMP,A,P,Q
      DOUBLE PRECISION DK,DAK
      DOUBLE PRECISION AD,DEV,X,Y
      DOUBLE PRECISION GEE,D
```

```
C
C  THE PROGRAM ALLOWS A MAXIMUM NUMBER OF ITERATIONS OF 25
C
      ITRMAX=25
      DEVL=-1.0
      NZ=NFCNS+1
      NZZ=NFCNS+2
      NITER=0
  100 CONTINUE
      IEXT(NZZ)=NGRID+1
      NITER=NITER+1
      IF(NITER.GT.ITRMAX) GO TO 400
      DO 110 J=1,NZ
      JXT=IEXT(J)
      DTEMP=GRID(JXT)
      DTEMP=DCOS(DTEMP*PI2)
  110 X(J)=DTEMP
      JET=(NFCNS-1)/15+1
      DO 120 J=1,NZ
  120 AD(J)=D(J,NZ,JET)
      DNUM=0.0
      DDEN=0.0
      K=1
      DO 130 J=1,NZ
      L=IEXT(J)
      DTEMP=AD(J)*DES(L)
      DNUM=DNUM+DTEMP
      DTEMP=FLOAT(K)*AD(J)/WT(L)
      DDEN=DDEN+DTEMP
  130 K=-K
      DEV=DNUM/DDEN
      WRITE(6,131) DEV
  131 FORMAT(1X,12HDEVIATION = ,F12.9)
      NU=1
      IF(DEV.GT.0.0) NU=-1
      DEV=-FLOAT(NU)*DEV
      K=NU
      DO 140 J=1,NZ
      L=IEXT(J)
      DTEMP=FLOAT(K)*DEV/WT(L)
      Y(J)=DES(L)+DTEMP
  140 K=-K
      IF(DEV.GT.DEVL) GO TO 150
      CALL OUCH
      GO TO 400
  150 DEVL=DEV
      JCHNGE=0
      K1=IEXT(1)
      KNZ=IEXT(NZ)
      KLOW=0
      NUT=-NU
      J=1
C
C  SEARCH FOR THE EXTREMAL FREQUENCIES OF THE BEST
C  APPROXIMATION
C
  200 IF(J.EQ.NZZ) YNZ=COMP
      IF(J.GE.NZZ) GO TO 300
      KUP=IEXT(J+1)
      L=IEXT(J)+1
      NUT=-NUT
      IF(J.EQ.2) Y1=COMP
      COMP=DEV
      IF(L.GE.KUP) GO TO 220
      ERR=GEE(L,NZ)
      ERR=(ERR-DES(L))*WT(L)
      DTEMP=FLOAT(NUT)*ERR-COMP
      IF(DTEMP.LE.0.0) GO TO 220
      COMP=FLOAT(NUT)*ERR
  210 L=L+1
      IF(L.GE.KUP) GO TO 215
      ERR=GEE(L,NZ)
```

```
      ERR=(ERR-DES(L))*WT(L)
      DTEMP=FLOAT(NUT)*ERR-COMP
      IF(DTEMP.LE.0.0) GO TO 215
      COMP=FLOAT(NUT)*ERR
      GO TO 210
  215 IEXT(J)=L-1
      J=J+1
      KLOW=L-1
      JCHNGE=JCHNGE+1
      GO TO 200
  220 L=L-1
  225 L=L-1
      IF(L.LE.KLOW) GO TO 250
      ERR=GEE(L,NZ)
      ERR=(ERR-DES(L))*WT(L)
      DTEMP=FLOAT(NUT)*ERR-COMP
      IF(DTEMP.GT.0.0) GO TO 230
      IF(JCHNGE.LE.0) GO TO 225
      GO TO 260
  230 COMP=FLOAT(NUT)*ERR
  235 L=L-1
      IF(L.LE.KLOW) GO TO 240
      ERR=GEE(L,NZ)
      ERR=(ERR-DES(L))*WT(L)
      DTEMP=FLOAT(NUT)*ERR-COMP
      IF(DTEMP.LE.0.0) GO TO 240
      COMP=FLOAT(NUT)*ERR
      GO TO 235
  240 KLOW=IEXT(J)
      IEXT(J)=L+1
      J=J+1
      JCHNGE=JCHNGE+1
      GO TO 200
  250 L=IEXT(J)+1
      IF(JCHNGE.GT.0) GO TO 215
  255 L=L+1
      IF(L.GE.KUP) GO TO 260
      ERR=GEE(L,NZ)
      ERR=(ERR-DES(L))*WT(L)
      DTEMP=FLOAT(NUT)*ERR-COMP
      IF(DTEMP.LE.0.0) GO TO 255
      COMP=FLOAT(NUT)*ERR
      GO TO 210
  260 KLOW=IEXT(J)
      J=J+1
      GO TO 200
  300 IF(J.GT.NZZ) GO TO 320
      IF(K1.GT.IEXT(1)) K1=IEXT(1)
      IF(KNZ.LT.IEXT(NZ)) KNZ=IEXT(NZ)
      NUT1=NUT
      NUT=-NU
      L=0
      KUP=K1
      COMP=YNZ*(1.00001)
      LUCK=1
  310 L=L+1
      IF(L.GE.KUP) GO TO 315
      ERR=GEE(L,NZ)
      ERR=(ERR-DES(L))*WT(L)
      DTEMP=FLOAT(NUT)*ERR-COMP
      IF(DTEMP.LE.0.0) GO TO 310
      COMP=FLOAT(NUT)*ERR
      J=NZZ
      GO TO 210
  315 LUCK=6
      GO TO 325
  320 IF(LUCK.GT.9) GO TO 350
      IF(COMP.GT.Y1) Y1=COMP
      K1=IEXT(NZZ)
  325 L=NGRID+1
      KLOW=KNZ
      NUT=-NUT1
```

```
      COMP=Y1*(1.00001)
  330 L=L-1
      IF(L.LE.KLOW) GO TO 340
      ERR=GEE(L,NZ)
      ERR=(ERR-DES(L))*WT(L)
      DTEMP=FLOAT(NUT)*ERR-COMP
      IF(DTEMP.LE.0.0) GO TO 330
      J=NZZ
      COMP=FLOAT(NUT)*ERR
      LUCK=LUCK+10
      GO TO 235
  340 IF(LUCK.EQ.6) GO TO 370
      DO 345 J=1,NFCNS
      NZZMJ=NZZ-J
      NZMJ=NZ-J
  345 IEXT(NZZMJ)=IEXT(NZMJ)
      IEXT(1)=K1
      GO TO 100
  350 KN=IEXT(NZZ)
      DO 360 J=1,NFCNS
  360 IEXT(J)=IEXT(J+1)
      IEXT(NZ)=KN
      GO TO 100
  370 IF(JCHNGE.GT.0) GO TO 100
C
C   CALCULATION OF THE COEFFICIENTS OF THE BEST APPROXIMATION
C   USING THE INVERSE DISCRETE FOURIER TRANSFORM
C
  400 CONTINUE
      NM1=NFCNS-1
      FSH=1.0E-06
      GTEMP=GRID(1)
      X(NZZ)=-2.0
      CN=2*NFCNS-1
      DELF=1.0/CN
      L=1
      KKK=0
      IF(GRID(1).LT.0.01.AND.GRID(NGRID).GT.0.49) KKK=1
      IF(NFCNS.LE.3) KKK=1
      IF(KKK.EQ.1) GO TO 405
      DTEMP=DCOS(PI2*GRID(1))
      DNUM=DCOS(PI2*GRID(NGRID))
      AA=2.0/(DTEMP-DNUM)
      BB=-(DTEMP+DNUM)/(DTEMP-DNUM)
  405 CONTINUE
      DO 430 J=1,NFCNS
      FT=J-1
      FT=FT*DELF
      XT=DCOS(PI2*FT)
      IF(KKK.EQ.1) GO TO 410
      XT=(XT-BB)/AA
      XT1=SQRT(1.0-XT*XT)
      FT=ATAN2(XT1,XT)/PI2
  410 XE=X(L)
      IF(XT.GT.XE) GO TO 420
      IF((XE-XT).LT.FSH) GO TO 415
      L=L+1
      GO TO 410
  415 A(J)=Y(L)
      GO TO 425
  420 IF((XT-XE).LT.FSH) GO TO 415
      GRID(1)=FT
      A(J)=GEE(1,NZ)
  425 CONTINUE
      IF(L.GT.1) L=L-1
  430 CONTINUE
      GRID(1)=GTEMP
      DDEN=PI2/CN
      DO 510 J=1,NFCNS
      DTEMP=0.0
      DNUM=J-1
      DNUM=DNUM*DDEN
```

```
      IF(NM1.LT.1) GO TO 505
      DO 500 K=1,NM1
      DAK=A(K+1)
      DK=K
  500 DTEMP=DTEMP+DAK*DCOS(DNUM*DK)
  505 DTEMP=2.0*DTEMP+A(1)
  510 ALPHA(J)=DTEMP
      DO 550 J=2,NFCNS
  550 ALPHA(J)=2.0*ALPHA(J)/CN
      ALPHA(1)=ALPHA(1)/CN
      IF(KKK.EQ.1) GO TO 545
      P(1)=2.0*ALPHA(NFCNS)*BB+ALPHA(NM1)
      P(2)=2.0*AA*ALPHA(NFCNS)
      Q(1)=ALPHA(NFCNS-2)-ALPHA(NFCNS)
      DO 540 J=2,NM1
      IF(J.LT.NM1) GO TO 515
      AA=0.5*AA
      BB=0.5*BB
  515 CONTINUE
      P(J+1)=0.0
      DO 520 K=1,J
      A(K)=P(K)
  520 P(K)=2.0*BB*A(K)
      P(2)=P(2)+A(1)*2.0*AA
      JM1=J-1
      DO 525 K=1,JM1
  525 P(K)=P(K)+Q(K)+AA*A(K+1)
      JP1=J+1
      DO 530 K=3,JP1
  530 P(K)=P(K)+AA*A(K-1)
      IF(J.EQ.NM1) GO TO 540
      DO 535 K=1,J
  535 Q(K)=-A(K)
      NF1J=NFCNS-1-J
      Q(1)=Q(1)+ALPHA(NF1J)
  540 CONTINUE
      DO 543 J=1,NFCNS
  543 ALPHA(J)=P(J)
  545 CONTINUE
      IF(NFCNS.GT.3) RETURN
      ALPHA(NFCNS+1)=0.0
      ALPHA(NFCNS+2)=0.0
      RETURN
      END
C
C-----------------------------------------------------------------------
C FUNCTION: D
C    FUNCTION TO CALCULATE THE LAGRANGE INTERPOLATION
C    COEFFICIENTS FOR USE IN THE FUNCTION GEE.
C-----------------------------------------------------------------------
C
      DOUBLE PRECISION FUNCTION D(K,N,M)
      COMMON PI2,AD,DEV,X,Y,GRID,DES,WT,ALPHA,IEXT,NFCNS,NGRID
      DIMENSION IEXT(66),AD(66),ALPHA(66),X(66),Y(66)
      DIMENSION DES(1045),GRID(1045),WT(1045)
      DOUBLE PRECISION AD,DEV,X,Y
      DOUBLE PRECISION Q
      DOUBLE PRECISION PI2
      D=1.0
      Q=X(K)
      DO 3 L=1,M
      DO 2 J=L,N,M
      IF(J-K)1,2,1
    1 D=2.0*D*(Q-X(J))
    2 CONTINUE
    3 CONTINUE
      D=1.0/D
      RETURN
      END
C
C-----------------------------------------------------------------------
C FUNCTION: GEE
```

```
C   FUNCTION TO EVALUATE THE FREQUENCY RESPONSE USING THE
C   LAGRANGE INTERPOLATION FORMULA IN THE BARYCENTRIC FORM
C-----------------------------------------------------------------------
C
      DOUBLE PRECISION FUNCTION GEE(K,N)
      COMMON PI2,AD,DEV,X,Y,GRID,DES,WT,ALPHA,IEXT,NFCNS,NGRID
      DIMENSION IEXT(66),AD(66),ALPHA(66),X(66),Y(66)
      DIMENSION DES(1045),GRID(1045),WT(1045)
      DOUBLE PRECISION P,C,D,XF
      DOUBLE PRECISION PI2
      DOUBLE PRECISION AD,DEV,X,Y
      P=0.0
      XF=GRID(K)
      XF=DCOS(PI2*XF)
      D=0.0
      DO 1 J=1,N
      C=XF-X(J)
      C=AD(J)/C
      D=D+C
    1 P=P+C*Y(J)
      GEE=P/D
      RETURN
      END
C
C-----------------------------------------------------------------------
C SUBROUTINE: OUCH
C   WRITES AN ERROR MESSAGE WHEN THE ALGORITHM FAILS TO
C   CONVERGE.  THERE SEEM TO BE TWO CONDITIONS UNDER WHICH
C   THE ALGORITHM FAILS TO CONVERGE: (1) THE INITIAL
C   GUESS FOR THE EXTREMAL FREQUENCIES IS SO POOR THAT
C   THE EXCHANGE ITERATION CANNOT GET STARTED, OR
C   (2) NEAR THE TERMINATION OF A CORRECT DESIGN,
C   THE DEVIATION DECREASES DUE TO ROUNDING ERRORS
C   AND THE PROGRAM STOPS.  IN THIS LATTER CASE THE
C   FILTER DESIGN IS PROBABLY ACCEPTABLE, BUT SHOULD
C   BE CHECKED BY COMPUTING A FREQUENCY RESPONSE.
C-----------------------------------------------------------------------
C
      SUBROUTINE OUCH
      COMMON /OOPS/NITER,IOUT
      WRITE(8,1)NITER
    1 FORMAT(44H ************ FAILURE TO CONVERGE **********/
     141H0PROBABLE CAUSE IS MACHINE ROUNDING ERROR/
     223H0NUMBER OF ITERATIONS =,I4/
     339H0IF THE NUMBER OF ITERATIONS EXCEEDS 3,/
     462H0THE DESIGN MAY BE CORRECT, BUT SHOULD BE VERIFIED WITH AN FFT)
      RETURN
      END
```

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