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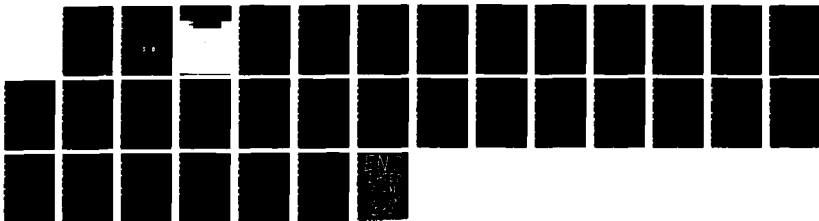
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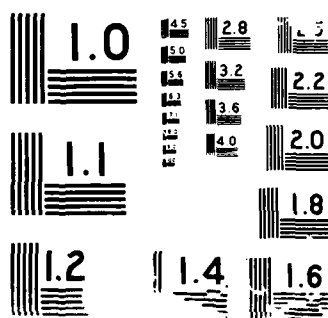
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Paul Dupuis* and Harold Kushner**

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**Lefschetz Center for Dynamical Systems
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October 1987

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ABSTRACT

We consider the problem of controlling a possibly degenerate diffusion process so as to minimize the probability of escape over a given time interval. It is assumed that the control acts on the process through the drift coefficient, and that the noise coefficient is small. By developing a large deviations type theory for the controlled diffusion, we obtain several results. The limit of the normalized log of the minimum exit probability is identified as the value I of an associated (deterministic) differential game. Furthermore, we identify a deterministic (and ϵ -independent) mapping g from the sample values $\epsilon w(s)$, $0 \leq s \leq t$, into the control space such that if we define the control used at time t by $u(t) = g(\epsilon w(s), 0 \leq s \leq t)$, then the resulting control process is progressively measurable and δ -optimal (in the sense that the limit of the normalized log of the exit probability is within δ of I).

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1. INTRODUCTION

Consider the white noise driven control system living in $\mathbb{R}^d$

$$dx^{u,\epsilon} = b(x^{u,\epsilon}, u)dt + \epsilon \sigma(x^{u,\epsilon})dw, \quad (1.1)$$

where u takes values in a compact set $K \subset \mathbb{R}^n$. There are many problems where one wants to keep $x^{u,\epsilon}(\cdot)$ in a set G until some particular job is finished. For example, in the problem of pointing a telescope on a satellite, the domain G and the duration are determined by the object to be photographed and the time required. See Meerkov and Runolfsson [6] for additional examples.

The associated control problem can be formulated in several different ways, depending on the time interval of interest. We consider two criteria. Define $\tau^{u,\epsilon} = \inf\{t: x^{u,\epsilon}(t) \in \partial G\}$. One criterion is to minimize

$$P_x\{\tau^{u,\epsilon} \leq T\}, \quad x \in G^0 = \text{interior of } G, \quad (1.2a)$$

for given T . The other criterion of interest here is the maximization of

$$E_x \tau^{u,\epsilon}, \quad x \in G^0. \quad (1.2b)$$

P_x and E_x denote the probability and expectation (resp.) given $x^{u,\epsilon}(0) = x$.

In general, it is very difficult to solve for the optimal control. However, in many problems the parameter ϵ is small. The theory of large deviations provides an alternative which can give a nearly optimal control for small ϵ , and a great deal more information and insight into the control process, likely escape routes, error bounds, etc. Take u to be a feedback function $u(x,t)$ that is smooth in x , uniformly in $t \leq T$. Define the system

$$\dot{\phi} = b(\phi, u(\phi, t)) + \alpha(\phi)v, \quad \phi(0) = x,$$

and define

$$S(x, u, T) = \inf \left\{ \frac{1}{2} \int_0^T |v(t)|^2 dt : \phi(t) \in \partial G \text{ for some } t \leq T \right\}.$$

The theory of large deviations tells us (under some other regularity conditions) that

$$S(x, u, T) = - \lim_{\epsilon} \epsilon^2 \log P_x \{ \tau^{u, \epsilon} \leq T \}.$$

Because of this result, one is tempted to try to maximize (or nearly maximize) $S(x, u, T)$, and to use the corresponding (if any) maximizing (or 'smooth' nearly maximizing) control. This approach encounters serious unresolved technical difficulties. In particular, it is not at all clear that the supremum over smooth feedback controls will be as large as that obtained over alternative classes of controls, such as those used below. Note that since we wish to supremize (over u) an infimum over v , the basic problem can be formulated as a differential game.

We mention here that calculating the limit of the normalized log of the minimum exit probability is by itself not useful in establishing the optimal performance for all small ϵ of any given control scheme. It may happen that a control that is found to be good for a small but fixed $\epsilon > 0$ actually behaves poorly in the limit $\epsilon \rightarrow 0$. Obtaining a 'good' control that depends on ϵ only through the actual driving noise process will be an important part of the development below.

Known results in this area are few in number. W. Fleming and P. Souganidis [3] consider the large deviations problem associated with the

minimization of (1.2a) over the class of feedback controls taking values in K . By use of PDE-viscosity solution techniques they calculate the asymptotics of the infimum of the exit probabilities. Their approach is restricted to the case where the diffusion is uniformly nondegenerate: $\alpha(x)\sigma'(x) \geq cI$, with $c > 0$. Furthermore they identify the limit as the value of a certain associated (deterministic) differential game. They do not deal with the uniformity issue raised previously, nor with the problem of construction of δ -optimal policies and their uniformity properties. A. D. Wentzell and M. I. Freidlin [5] consider the optimization problem associated with (1.2b) for a wide class of processes that includes (1.1) in the uniformly nondegenerate case. However, in order to obtain a solution with the desired properties, they restrict the class of available controls in a way that is probably not natural for these types of problems. For example, they consider feedback controls that are continuous, except possibly at one point. Simple examples in dimension greater than one show the 'best' control may have discontinuities along manifolds of dimension one less.

The objective of this paper is to extend the conclusions of [3]. By use of probabilistic arguments (as opposed to PDE), we recover the results presented there. The probabilistic arguments allow us to extend these results to the important degenerate case, which is in fact more natural in applications. We also address the uniformity issue raised above. The results in this direction are not completely satisfactory, in that the exhibited control is not of the simple feedback form, but depends on the 'full information' of the past. However, they do suggest that feedback

controls are available which do not depend on ϵ explicitly, and which are nearly optimal for small ϵ .

Our basic assumptions and definitions are as follows.

Assumption A1.

- (1) $b(\cdot, \cdot)$ and $\alpha(\cdot)$ are Lipschitz with constant $\bar{K}$ and bounded with constant B on an open set containing $\bar{G}$, the closure of G .
- (2) The control space K is compact and independent of time.
- (3) G is an open set in $\mathbb{R}^d$.
- (4) Either (i) $\alpha(\cdot)$ is a square matrix and uniformly nondegenerate, or else (ii) we can partition b and σ in the form

$$b(x, u) = \begin{bmatrix} b_1(x, u) \\ b_2(x) \end{bmatrix}, \quad \alpha(x) = \begin{bmatrix} \sigma_1(x) \\ 0 \end{bmatrix}$$

where $\sigma_1(\cdot)$ is a square matrix and uniformly nondegenerate.

Throughout the paper we shall assume that we are given a probability space $(\Omega, \mathcal{F}, \mathcal{F}(t), P)$ and a Wiener process $w(\cdot)$ on $[0, 1]$ with respect to $\mathcal{F}(t)$. We then take as our class of admissible controls the set of K -valued progressively measurable processes. We denote the set of all such processes by F . For convenience we recall the definition of a progressively measurable process (with respect to $\mathcal{F}(t)$).

Definition. A stochastic process $\xi(t)$ on the sample space Ω and time interval $[0, 1]$ is $\mathcal{F}(t)$ -progressively measurable if the mapping $[0, t] \times \Omega \ni (s, \omega) \rightarrow \xi(s)(\omega)$ is $B(t) \times \mathcal{F}(t)$ measurable for every $0 \leq t \leq 1$, where $B(t)$ is the Borel σ -algebra of $[0, t]$.

Remark. The symbol u will be used to represent two different types of control processes, depending on the context. At times it will be a deterministic process used in the differential game, and at times it will denote a stochastic process used to control the diffusion. Likewise v will be used to represent both stochastic and deterministic processes, depending on the context. In all cases the intended use should be clear.

The organization of the remainder of the paper is as follows. In Section 2 we give a precise definition of the associated differential game in terms of an adaptation of the Elliott-Kalton [4] formulation, and discuss how the existence of value for this differential game relates to our problem. The only difference between our definition and the usual Elliott-Kalton definition is the added requirement that the maps α and β defined below must be measurable. The additional requirement of measurability is due to the fact that several uses are made of stochastic processes defined by composing α (or β) with a given progressively measurable process. Measurability of α (or β) ensures that the resulting process is adapted. The addition of this condition does not change the resulting 'value' of the game. Section 3 contains the statement and proof of the main theorem. The proofs of several technical lemmas make up a concluding appendix. For notational simplicity, we shall consider the problem on the interval $[0,1]$. The results carry over to an arbitrary interval in the obvious way.

Notation. We use $C_x[0,1]$ to denote the set of continuous functions taking values in $\mathbb{R}^k$ (with k depending on the context) and starting at x , and take $d(\cdot, \cdot)$ as the sup norm metric in this space.

2. THE ASSOCIATED DIFFERENTIAL GAME

Define

$$M = \{u: [0,1] \rightarrow K: u \text{ is measurable}\},$$

$$N = \left\{v: [0,1] \rightarrow \mathbb{R}^r: \int_0^1 |v| dt < \infty \right\}.$$

We identify any two functions which agree a.e. and consider M and N as metric spaces with the L^1 metric. A mapping $\alpha: N \rightarrow M$ is called a strategy for the maximizing player if α is measurable and if whenever $0 \leq s \leq 1$ and

$$v(t) = \hat{v}(t) \quad \text{for a.e. } 0 \leq t \leq s,$$

then

$$\alpha[v](t) = \alpha[\hat{v}](t) \quad \text{for a.e. } 0 \leq t \leq s.$$

A strategy for the minimizing player is defined in an analogous way, and such a strategy will be denoted by the symbol β . The set of all minimizing (respectively maximizing) strategies will be denoted by Δ (resp. Γ).

Next define $\chi(x)$ to be zero if $x \in \partial G$ and $+\infty$ if $x \in G^0$. The definition of the differential game (DG) is then given in terms of the following dynamical equation and cost:

Dynamics.

$$\dot{\phi} = b(\phi, u) + \alpha(\phi)v, \quad \phi(0) = x. \quad (2.1)$$

Let $\tau_x = \inf\{t: \phi(t) \in \partial G\} \wedge 1$.

Cost. For $\phi(\cdot)$ defined through (2.1), set

$$C(u,v) = \frac{1}{2} \int_0^{\tau_x} |v(t)|^2 dt + \chi(\phi(\tau_x)).$$

We then define the lower value of the DG by

$$I^-(x) = \inf_{\beta \in \Delta} \sup_{u \in M} C(u, \beta[u]).$$

The upper value is defined by

$$I^+(x) = \sup_{\alpha \in \Gamma} \inf_{v \in N} C(\alpha[v], v).$$

Remarks. The terms upper and lower refer to which player has the 'information advantage'. In a heuristic sense, for the game corresponding to the lower value we allow the minimizing player (v here) to know the next move of the maximizing player (u) before choosing his own move. Although this distinction is somewhat obscured in the abstract Elliott-Kalton formulation, it is intuitively obvious in the Fleming and Friedman formulations [1], which are equivalent to the Elliott-Kalton formulation under some hypotheses. The reader is referred to [1] for further discussion. The DG we consider differs from that of [3], but it seems to be more natural for this type of problem. The remarks that follow illustrate this point.

The Elliott-Kalton definitions of upper and lower values in terms of strategies have interesting interpretations in terms of the large deviations properties of the controlled diffusion. First note that the v -control in the DG plays the role of the small noise ϵw in the diffusion. Let small $\epsilon > 0$ be given. Consider the upper value $I^+(x)$, and let α be a 'nearly' optimizing strategy for the maximizing player. Let $v \in N$ be given. Then

The 'nearly' supremizing α gives us a strategy that accomplishes one of two things. Either $\chi(\phi(\tau_x)) = \infty$ (ϕ never escapes from G) or

$$\frac{1}{2} \int_0^{\tau_x} |v(t)|^2 dt \geq I^+(x) - \delta$$

(ϕ escapes from G , but at a 'cost' of not less than $I^+(x) - \delta$). Large deviations theory for the process $\epsilon \dot{w}$ then suggests that when ϵ is small the probability of $\epsilon \dot{w}$ 'tracking' one of the v functions that corresponds to escape from G (in the sense that $x^{u, \epsilon}$ is near to the corresponding ϕ associated with $\alpha[v]$, v) is no greater than $\exp - (I^+(x) - 2\delta)/\epsilon^2$. This suggests that we can obtain a progressively measurable control u_0 from the 'nearly' supremizing α so that when ϵ is small

$$P_x(\tau^{u_0, \epsilon} \leq 1) \leq \exp - (I^+(x) - 2\delta)/\epsilon^2.$$

On the other hand, consider the lower value $I^-(x)$, and let β be 'nearly' infimizing. Then, no matter what progressively measurable control strategy $u(t)$ is used, β describes a path for the noise to follow whose 'action' or 'cost' is no greater than $I^-(x) + \delta$, and which leads to escape. The large deviations properties of $\epsilon \dot{w}$ now suggest that no matter what control is used the probability of escape should (roughly) be bounded below by $\exp - (I^-(x) - 2\delta)/\epsilon^2$.

We thus have (roughly)

$$\exp - (I^-(x) - 2\delta)/\epsilon^2 \leq P_x(\tau^{u_0, \epsilon} \leq 1) \leq \exp - (I^+(x) - 2\delta)/\epsilon^2,$$

with the conclusion that $I^-(x) \geq I^+(x)$. From the definition of the game it is possible to show $I^-(x) \leq I^+(x)$, which implies that the game has a value.

3. THE MAIN THEOREM

Before stating the main theorem, we introduce a 'continuity' assumption on the domain G . Define G^δ for small δ as follows: if $\delta \geq 0$, then

$$G^\delta = \{x \in \mathbb{R}^d: \inf\{|x-y|: y \in G\} \leq \delta\},$$

if $\delta < 0$, then

$$G^\delta = \{x \in \mathbb{R}^d: \inf\{|x-y|: y \notin G\} \geq -\delta\}.$$

Next define $I^+(x, \delta)$, $I^-(x, \delta)$ as the upper and lower values of the DG defined in Section 2, but with G^δ replacing G there. Since $I^+(x, \delta)$ (respectively $I^-(x, \delta)$) is monotone nondecreasing in δ , the set of discontinuities of $I^+(x, \cdot)$ (resp., $I^-(x, \cdot)$) is countable. (Note that x is fixed here.)

Assumption A2. $I^+(x, \delta)$ and $I^-(x, \delta)$ are continuous at $\delta = 0$.

Remarks. It is simple to prove in the uniformly nondegenerate case that $I^+(x, \cdot)$ and $I^-(x, \cdot)$ are in fact continuous functions. This follows from the fact that $b(\cdot, \cdot)$ is bounded on $G \times K$, while v is allowed to 'push' the state in any direction. In the degenerate case it can happen that $I^+(x, \cdot)$ (or $I^-(x, \cdot)$) is in fact discontinuous at $\delta = 0$, but even then Assumption A2 is not very restrictive, since it is satisfied for an arbitrarily small perturbation of G . A consequence of the theorem stated below is that at points at which both $I^+(x, \cdot)$ and $I^-(x, \cdot)$, are continuous, we have $I^+(x, \delta) = I^-(x, \delta)$. Monotonicity then implies $I^+(x, \cdot)$ and $I^-(x, \cdot)$ have the same set of discontinuity points. It should also be noted that in order to obtain the result analogous to the main

theorem in the simpler case of uncontrolled diffusion processes:

$$\lim_{\epsilon} \epsilon^2 \log P_x \{ \tau^\epsilon \leq 1 \} = -I(x),$$

the assumption obtained from Assumption A2 when the set K contains only one element is also required.

Theorem. Assume A1 and A2, and let $I^+(x)$ and $I^-(x)$ be the upper and lower values of the DG described in Section 2. For any $u \in F$, let $x^{u,\epsilon}(\cdot)$ be the solution of

$$dx^{u,\epsilon} = b(x^{u,\epsilon}, u)dt + \epsilon \alpha(x^{u,\epsilon})dw, \quad x^{u,\epsilon}(0) = x, \quad (3.1)$$

and define

$$\tau^{u,\epsilon} = \inf\{t: x^{u,\epsilon}(t) \in \partial G\}. \quad (3.2)$$

Then

$$(1) \quad \lim_{\epsilon} \epsilon^2 \log \inf_{u \in F} P_x(\tau^{u,\epsilon} \leq 1) \geq -I^-(x), \quad (3.3)$$

(2) given $c > 0$ there exists a measurable function $g: C_0[0,1] \rightarrow M$ with the following properties:

(i) if $0 \leq s \leq 1$ and $f(t) = \hat{f}(t)$ for $0 \leq t \leq s$, then $g[f](t) = g[\hat{f}](t)$ for a.e. $0 \leq t \leq s$,

(ii) if we define $u = g[\epsilon w]$ then $u \in F$ and

$$\overline{\lim}_{\epsilon} \epsilon^2 \log P_x(\tau^{u,\epsilon} \leq 1) \leq -I^+(x) + c, \quad (3.4)$$

(3) $I^+(x) = I^-(x)$.

Remarks. Part (2) of the theorem gives the existence of a c -optimal (in

the asymptotic sense) control u that depends on $x^{u,\epsilon}(s)$, $0 \leq s \leq t$, at time t . Part (1) yields an important uniformity property. For any given $c > 0$ and any (possibly ϵ -dependent) progressively measurable control u_ϵ , there is $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$,

$$P_x\{\tau^{u_\epsilon, \epsilon} \leq 1\} \geq P_x\{\tau^{u, \epsilon} \leq 1\} \exp - c/\epsilon^2.$$

Proof of (1). For $c > 0$ there exists $\delta > 0$ such that $I^-(x, \delta) \leq I^-(x) + c$. Consider now the DG with domain G^δ and let $C^\delta(u, v)$ denote the cost associated with the domain G^δ . Then there exists a minimizing strategy $\beta \in \Delta$ such that

$$\sup_{u \in M} C^\delta(u, \beta[u]) \leq I^-(x) + 2c. \quad (3.5)$$

If we redefine $\beta[u](t)$ to be zero when $t \geq \tau_x$ (given by (2.1)) then β is still a strategy and obviously still satisfies (3.5).

Without loss of generality we may assume the following property of the chosen strategy β : $(d/dt)\beta[u](t)$ exists for all $u \in M$ (a.s. in t) and furthermore there is $C_1 < \infty$ such that

$$\left| \frac{d}{dt} \beta[u](t) \right| \vee |\beta[u](t)| \leq C_1$$

(a.s. in t) for all $u \in M$. This fact follows from Assumption A2 and Lemma A1 of the appendix.

Take any control process $u \in F$, and define the processes

$$\begin{aligned} v(t) &= \beta[u](t) \\ \dot{\phi}^\epsilon &= b(x^{u, \epsilon}, u) + \alpha(\phi^\epsilon)v, \quad \phi^\epsilon(0) = x. \end{aligned}$$

We then have

$$|\dot{v}(t)| \vee |v(t)| \leq C_1 \text{ (a.s. in } t)$$

for every ω . It follows from the definition of a strategy that $v(t)$ is $\mathcal{F}(t)$ measurable. Since the $\mathcal{B}$ under consideration has the property that $\mathcal{B}[u](\cdot)$ is continuous for every $u \in M$, $v(\cdot)$ and $\phi^\epsilon(\cdot)$ are $\mathcal{F}(t)$ -progressively measurable processes [7; Theorem 1.5.1].

Now define $y^\epsilon = x^{u, \epsilon} - \phi^\epsilon$. Then y^ϵ satisfies the stochastic equation

$$dy^\epsilon = \alpha(x^{u, \epsilon})\epsilon dw - \alpha(\phi^\epsilon)v dt, \quad y^\epsilon(0) = 0 \quad (3.6a)$$

Let P_1 denote the measure induced on $C_0[0,1]$ by the solution to (3.6a). By Girsanov's theorem there is a Brownian motion $\bar{w}(\cdot)$ (with respect to the same filtration $\mathcal{F}(\cdot)$ as $w(\cdot)$) such that

$$dy^\epsilon = \alpha(x^{u, \epsilon})\epsilon d\bar{w}, \quad y^\epsilon(0) = 0, \quad (3.6b)$$

and such that if P_0 is the measure induced on $C_0[0,1]$ by (3.6b), then

$$\frac{dP_1}{dP_0} = \exp \left[\frac{1}{\epsilon^2} \int_0^1 \langle \alpha(\phi^\epsilon)v, \alpha(x^{u, \epsilon})\epsilon d\bar{w} \rangle - \frac{1}{2\epsilon^2} \int_0^1 |\sigma^{-1}(x^{u, \epsilon})\alpha(\phi^\epsilon)v|^2 dt \right]$$

(In the degenerate case replace σ by σ_1 in the above.)

Define $\Omega_{\delta_2}^\epsilon = \{\omega: \sup_{0 \leq t \leq 1} |y^\epsilon(t)| \leq \delta_2\}$. We will use the equality

$$P_1(\Omega_{\delta_2}^\epsilon) = \int_{\Omega_{\delta_2}^\epsilon} \frac{dP_1}{dP_0} dP_0.$$

First note that for any $\delta_2 > 0$, $P_0(\Omega_{\delta_2}^\epsilon) \rightarrow 1$, as $\epsilon \rightarrow 0$. Using the nondegeneracy and the Lipschitz continuity of $\alpha(\cdot)$ (or of $\sigma_1(\cdot)$ in the degenerate case), for given $\delta' > 0$ there is $\delta'' > 0$ such that $|x - y| \leq \delta''$ implies $|\sigma^{-1}(x)\alpha(y) - \sigma^{-1}(y)\alpha(x)| \leq \delta'$. This, together with (3.5) yields

$$\frac{1}{2} \int_0^1 |\sigma^{-1}(x^{u, \epsilon})\alpha(\phi^\epsilon)v|^2 dt \leq I^-(x) + 3c$$

on $\Omega_{\delta_2}^\epsilon$, if δ_2 is small enough.

Finally we consider the term

$$\left| \int_0^1 \langle \alpha(\phi^\epsilon)v, dy^\epsilon \rangle \right|.$$

Since $(d/dt)\alpha(\phi^\epsilon(t))v(t)$ is bounded, an integration by parts yields the bound $\delta_2 C_2$ for some fixed finite constant C_2 , when on the set $\Omega_{\delta_2}^\epsilon$.

Assembling these estimates, we have (for small enough δ_2)

$$P_1(\Omega_{\delta_2}^\epsilon) \geq \exp - (I^-(x) + 5c)/\epsilon^2 \quad (3.8)$$

when ϵ is small. We now pick δ_2 small enough so that the event $\sup_{0 \leq t \leq 1} |y^\epsilon(t)| \leq \delta_2$ implies $x^{u,\epsilon}(t)$ exits G before $t = 1$. The Lipschitz condition on $b(\cdot, \cdot)$ implies that on $\Omega_{\delta_2}^\epsilon$,

$$\dot{\phi}^\epsilon = b(\phi^\epsilon, u) + \gamma + \alpha(\phi^\epsilon)v, \quad \phi^\epsilon(0) = x,$$

where $\sup_{0 \leq t \leq 1} |\gamma(t)| \leq \bar{K}\delta_2$. We compare ϕ^ϵ to the solution of

$$\dot{\psi} = b(\psi, u) + \alpha(\psi)v, \quad \psi(0) = x.$$

By Gronwall's lemma, and the various Lipschitz and boundedness conditions, we can pick $\delta_2 \leq \delta/2$ so that $d(\phi^\epsilon, \psi) \leq \delta/2$ on $\Omega_{\delta_2}^\epsilon$. By the definition of β , $\psi(\cdot)$ must exit G^δ before time $t = 1$. Hence on $\Omega_{\delta_2}^\epsilon$ it must happen that $x^{u,\epsilon}(\cdot)$ exists G before $t = 1$. This combined with (3.8) finishes the proof.

Proof of (2). Now consider the upper value of the differential game:

$$I^+(x) = \sup_{\alpha \in \Gamma} \inf_{v \in V} C(\alpha[v], v).$$

Fix $c > 0$, and pick $\delta > 0$ so that $I^+(x, -\delta) \geq I^+(x) - c$. Let α be a 'nearly' maximizing strategy for the differential game with domain $G^{-\delta}$, in the sense that

$$\inf_{v \in N} C^{-\delta}(\alpha[v], v) \geq I^+(x, -\delta) - c. \quad (3.9)$$

We next describe how we use α to control the diffusion process. Let the Wiener process $w(\cdot)$ be given, and define (for $\Delta > 0$)

$$v^\Delta(t) = \begin{cases} 0 & \text{for } t \in [0, \Delta) \\ [w(n\Delta) - w(n\Delta - \Delta)]/\Delta & \text{for } t \in [n\Delta, n\Delta + \Delta), n \geq 1. \end{cases} \quad (3.10)$$

We then define our control process by

$$u(t) = \alpha[v^\Delta](t). \quad (3.11)$$

From Assumption A2 and Lemma A2 of the appendix it follows that we may assume without loss of generality that the strategy α has been chosen so that $\alpha[v](\cdot)$ is a piecewise constant function for every $v \in N$. As was the case previously, the definition of a strategy implies $u(t)$ is $\mathcal{F}(t)$ measurable. Hence $u(t)$ is an $\mathcal{F}(t)$ -progressively measurable process [7; Theorem 1.5.1].

The controlled diffusion is therefore

$$dx^{u, \epsilon} = b(x^{u, \epsilon}, u)dt + \epsilon \alpha(x^{u, \epsilon})dw, \quad x^{u, \epsilon}(0) = x. \quad (3.12)$$

In order to prove the desired result it is convenient to compare $x^{u, \epsilon}(\cdot)$ with the solution to

$$\dot{x}^{\epsilon, \Delta} = b(x^{\epsilon, \Delta}, u) + \epsilon \alpha(x^{\epsilon, \Delta})v^\Delta, \quad x^{\epsilon, \Delta}(0) = x.$$

Assume that for any given $\rho > 0$ and $M < \infty$ one could show the existence of

$\epsilon_0 > 0$ and $\Delta_0 > 0$ so that for $\Delta \leq \Delta_0$, $\epsilon \leq \epsilon_0$

$$P_x\{d(x^{u,\epsilon}, x^{\epsilon,\Delta}) \geq \rho\} \leq \exp - M/\epsilon^2. \quad (3.13)$$

Then by taking $M = I^+(x) + 1$ and $\rho = \delta$, it is obvious that the upper bound is proved if we can show

$$\overline{\lim} \epsilon^2 \log P_x\{x^{\epsilon,\Delta}(t) \in \partial G^{-\delta} \text{ for some } t < 1\} \leq -I^+(x, -\delta) + 2c. \quad (3.14)$$

However this follows from our choice of α . Since (3.9) holds, there are only two possibilities for each $v \in N$. Either

$$\frac{1}{2} \int_0^1 |v(t)|^2 dt \geq I^+(x, -\delta) - c, \quad (3.15)$$

or the solution of (2.1) does not escape $G^{-\delta}$ by time $t = 1$. Hence $x^{\epsilon,\Delta}(\cdot)$ escapes only on the set of paths for which

$$\frac{\epsilon\Delta}{2} \sum_0^{1/\Delta-1} v^{\Delta}(i\Delta)^2 = \epsilon \sum_1^{1/\Delta-1} [w(i\Delta) - w(i\Delta - \Delta)]^2/2\Delta \geq I^+(x, -\delta) - c. \quad (3.16)$$

Standard estimates from the theory of large deviations [2] imply that there exist $\Delta_0 > 0$, $\epsilon_0 > 0$ such that for $\Delta \leq \Delta_0$, $\epsilon \leq \epsilon_0$ the probability of the event given in (3.16) is less than $\exp - (I^+(x, -\delta) + 2c)/\epsilon^2$. We are therefore finished, except for the proof of (3.13). The details of this estimate are given in Lemma A3 of the appendix.

Proof of (3). It follows from (1) and (2) that $I^-(x) \geq I^+(x)$. We give the easy proof of $I^-(x) \leq I^+(x)$ in Lemma A4 of the appendix, which completes the proof. $\square$

APPENDIX

In this appendix we prove several technical lemmas that are needed to prove the main theorem of Section 3. Before presenting the lemmas we introduce some new notation. For $-1 \leq s \leq 1$, we define $\Delta(s)$ as the set of all measurable mappings β from $M \rightarrow N$ such that

$$u(r) = \hat{u}(r) \text{ for a.e. } 0 \leq r \leq t$$

implies

$$\beta[u](r) = \beta[\hat{u}](r) \text{ for a.e. } 0 \leq r \leq \min(t+s, 1).$$

Hence β has a 'reaction time' of s , which means he anticipates if $s < 0$. The set $\Gamma(s)$ of mappings from $N \rightarrow M$ is defined in the obvious analogous way.

Lemma A1. Let $I < \infty$, $\delta > 0$, and $\beta \in \Delta$ be given such that

$$\sup_{u \in M} C(u, \beta[u]) \leq I. \tag{A.1}$$

Then there exists $\beta' \in \Delta$ and $C_1 < \infty$ such that for all $u \in M$,

$$\left| \frac{d}{dt} \beta'[u](t) \right| \vee |\beta'[u](t)| \leq C_1, \text{ (a.s.),} \tag{A.2}$$

$$C^{-\delta}(u, \beta'[u]) \leq I. \tag{A.3}$$

(As before, $C^{-\delta}$ is the cost associated with the domain $G^{-\delta}$.) Furthermore, there is $s < 0$ such that given $\beta \in \Delta(s)$ satisfying (A.1) there exists $\beta'' \in \Delta$ such that (A.3) holds for all $u \in M$ (with β'' replacing β' there).

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Proof. The cost associated with β is simply $\frac{1}{2} \int_0^1 (\beta[u](t))^2 dt \leq I$, since exit

before time $t = 1$ must occur. Define

$$S(u, C_1) = \{t: |\beta[u](t)| \geq C_1\},$$

$$\beta_1[u](t) = \begin{cases} 0 & , t \in S(u, C_1) \\ \beta[u](t), & t \notin S(u, C_1). \end{cases}$$

Then β_1 is obviously a strategy, and

$$\frac{1}{2} \int_0^1 (\beta_1[u](t))^2 dt \leq \frac{1}{2} \int_0^1 (\beta[u](t))^2 dt.$$

In order to show $C^{-5}(u, \beta_1[u]) \leq C(u, \beta[u])$, it is sufficient to prove that if ϕ and ψ are defined by

$$\begin{aligned} \dot{\phi} &= b(\phi, u) + \alpha(\phi)\beta[u] \\ &= b(\phi, u) + \alpha(\phi)\beta_1[u] + \alpha(\phi)\beta[u]I_{S(u, C_1)}(t) \end{aligned}$$

$$\dot{\psi} = b(\psi, 0) + \alpha(\psi)\beta_1[u], \quad \phi(0) = \psi(0) = x,$$

then $d(\phi, \psi) \leq 5$. First note that

$$\left| \int_0^t \alpha(\phi(s))\beta[u](s)I_{S(u, C_1)}(s)ds \right| \leq 2BI/C_1$$

for $0 \leq t \leq 1$. Hence,

$$\begin{aligned} |\phi(t) - \psi(t)| &\leq \int_0^t \bar{K} |\phi(s) - \psi(s)| ds \\ &\quad + \int_0^t \bar{K} |\phi(s) - \psi(s)| |\beta_1[u](s)| ds + 2BI/C_1. \end{aligned}$$

Using the inequality $ab \leq (a^2 + b^2)/2$ in the second integral, and the

Gronwall inequality, we obtain

$$d(\phi, \psi) \leq 2BI[1 + \bar{K}(2 + I)e^{\bar{K}(2+I)}]/C_1.$$

By choosing C_1 large, we have

$$C^{-\delta}(u, \beta'[u]) \leq C(u, \beta[u])$$

for all $u \in M$.

Next we obtain β' by smoothing β_1 . For $\Delta > 0$, define

$$\beta'[u](t) = \frac{1}{\Delta} \int_{t-\Delta}^t \beta_1[u](s) ds$$

(we define $\beta_1[u](s) = 0$ for $s < 0$). Obviously β' satisfies (A.2). We also have

$$\frac{1}{2} \int_0^1 (\beta'[u](t))^2 dt \leq \frac{1}{2} \int_0^1 (\beta_1[u](t))^2 dt.$$

(A.3) now follows if we can show that small $\Delta > 0$ implies that the solutions of

$$\dot{\phi} = b(\phi, u) + \alpha(\phi)\beta'[u], \quad \phi(0) = x$$

$$\dot{\psi} = b(\psi, u) + \alpha(\psi)\beta_1[u], \quad \psi(0) = x$$

satisfy $d(\phi, \psi) \leq \delta$. This follows from another application of Gronwall's lemma and an integration by parts.

Finally we consider the last statement of the lemma.

Let $s < 0$ be given. By the same argument as above we may assume the existence of $\beta' \in \Delta(s)$ satisfying (A.2) and (A.3). Define

$$\beta''[u](t) = \begin{cases} 0 & 0 \leq t \leq -s \\ \beta'[u](t+s) & -s < t \leq 1. \end{cases}$$

$$\dot{\phi} = b(\phi, u) + \alpha(\phi)\beta''[u], \quad \phi(0) = x,$$

$$\dot{\psi} = b(\psi, u) + \alpha(\psi)\beta'[u], \quad \psi(0) = x.$$

Then $\beta'' \in \Delta$. Arguments such as those used above combined with the boundedness of β' , β'' imply that when $s < 0$ is sufficiently large $d(\phi, \psi) \leq \delta$. Hence we have $\beta'' \in \Delta$ such that

$$C^{-2\delta}(u, \beta''[u]) \leq C(u, \beta[u]),$$

and the lemma is proved. $\square$

Lemma A2. Let $I, \delta > 0$, and $\alpha \in \Gamma$ be given such that

$$\inf_{v \in N} C(\alpha[v], v) \geq I. \quad (A.4)$$

Then there exists $\alpha' \in \Gamma$ such that for all $v \in N$

$$\alpha'[v](\cdot) \text{ is a piecewise constant function.} \quad (A.5)$$

$$C^\delta(\alpha'[v], v) \geq I. \quad (A.6)$$

Furthermore, there is a $s < 0$ such that given $\alpha \in \Gamma(s)$ satisfying (A.4) there exists $\alpha'' \in \Gamma$ such that (A.6) holds for all $v \in N$ (where α'' replaces α' there).

Proof. N may be written as the disjoint union $N = N_1 \cup N_2 \cup N_3$ with

$$N_1 = \{v \in N: \chi(\phi(\tau_x)) = 0\},$$

$$N_2 = \left\{v \in N: \chi(\phi(\tau_x)) = \infty \text{ and } \frac{1}{2} \int_0^1 v^2 dt \geq I\right\},$$

$$N_3 = \left\{ v \in N: X(\phi(\tau_x)) = \infty \text{ and } \frac{1}{2} \int_0^1 v^2 dt < 1 \right\}$$

(here $\dot{\phi} = b(\phi, \alpha[v]) + \alpha(\phi)v$, $\phi(0) = x$, and $\tau_x = \inf\{t: \phi(t) \in \partial G\} \wedge 1$). It is clear that we may define α' in any way we like on N_1 and N_2 , as long as it is a strategy. For $\epsilon > 0$ let $\{u_i, i = 1, \dots, J\}$ be an ϵ -net of the control space K , and let $\{K_i, i = 1, \dots, J\}$ be a Borel measurable partition of K such that the Hausdorff distance between $\{u_i\}$ and K_i is less than ϵ for $i = 1, \dots, J$. For $\gamma > 0$, and $0 \leq l \leq 1/\gamma$, define

$$\tau(i, l, v) = \int_{l\gamma}^{(l+1)\gamma} 1_{\{\alpha[v](t) \in K_i\}} dt.$$

Then for all $v \in N$, l ,

$$\sum_{i=1}^J \tau(i, l, v) = \gamma.$$

We define $\alpha'[v]$ by $\alpha'[v](t) = u_1$, for $0 \leq t \leq \gamma$, and

$$\alpha'[v](t) = u_i \text{ for } t \in \left[l\gamma + \sum_{j=1}^{i-1} \tau(j, l-1, v), l\gamma + \sum_{j=1}^i \tau(j, l-1, v) \right],$$

$l = 1, \dots, 1/\gamma$.

Owing to the definition of α' , we have

$$\sup_{0 \leq t \leq 1} \left| \int_0^t [b(\phi(r), \alpha[v](r)) - b(\phi(r), \alpha'[v](r))] dr \right| \leq \epsilon \bar{K} + \gamma B$$

for every $v \in N$ and measurable function $\phi(\cdot)$ taking values in G^6 . Define

$$\dot{\phi} = b(\phi, \alpha[v]) + \alpha(\phi)v, \quad \phi(0) = x,$$

$$\dot{\psi} = b(\psi, \alpha'[v]) + \alpha(\psi)v, \quad \psi(0) = x.$$

In order to prove (A.6) it is sufficient to prove $d(\phi, \psi) \leq \delta$ when ϵ and γ are sufficiently small, and when $v \in N_s$. Using the estimate

$$\begin{aligned} |\phi(t) - \psi(t)| &\leq \left| \int_0^t [b(\phi, \alpha[v]) - b(\phi, \alpha'[v])] ds \right| \\ &\quad + \left| \int_0^t [b(\phi, \alpha'[v]) - b(\psi, \alpha'[v]) + \alpha(\phi)v - \alpha(\psi)v] ds \right| \\ &\leq \epsilon \bar{K} + \gamma B + 3\bar{K} \int_0^t |\phi - \psi| ds / 2 + \bar{K} \int_0^t |\phi - \psi|^2 ds / 2, \end{aligned}$$

and Gronwall's lemma, we obtain

$$d(\phi, \psi) \leq (\epsilon \bar{K} + \gamma B)[1 + \bar{K}(2 + 1)\exp \bar{K}(2 + 1)]. \quad (\text{A.7})$$

Hence we obtain (A.6) for small ϵ, γ .

If we are given $\alpha \in \Gamma(s)$, and define

$$\alpha''[v](t) = \begin{cases} u_1 & \text{for } 0 \leq t \leq -s \\ \alpha[v](t+s) & \text{for } -s < t \leq 1, \end{cases}$$

then $\alpha'' \in \Gamma$, and by the same argument as above we can obtain (A.6) when $s < 0$ is sufficiently large. The only difference is that in the inequality (A.7) we replace $\epsilon \bar{K} + \gamma B$ by $-sB$. $\square$

Lemma A3. Given $\rho > 0$ and $M < \infty$, there exists $\Delta_0 > 0$ and $\epsilon_0 > 0$ such that (3.13) holds for $\epsilon \leq \epsilon_0$, $\Delta \leq \Delta_0$.

Proof. We begin by defining a stopping time (all stopping times are with respect to $w(\cdot)$) for $\rho_1 > 0$:

$$\tau_1 = \inf\{t: |x^{\epsilon, \Delta}(t) - x^{\epsilon, \Delta}([t/\Delta]\Delta)| \geq \rho_1\}, \quad \wedge 1.$$

A simple calculation shows there are $\epsilon_{0,1} > 0$ and $\Delta_{0,1} > 0$ (depending on ρ_1) such that $\epsilon \leq \epsilon_{0,1}$ and $\Delta \leq \Delta_{0,1}$ imply

$$P_x\{\tau_1 < 1\} \leq \exp - (M + 2)/\epsilon^2. \quad (A.8)$$

Next we rewrite the equation for $x^{\epsilon, \Delta}$ as

$$dx^{\epsilon, \Delta} = b(x^{\epsilon, \Delta}, u)dt + \epsilon \alpha(x^{\epsilon, \Delta})dw + d\gamma^{\epsilon, \Delta}, \quad x^{\epsilon, \Delta}(0) = x, \quad (A.9)$$

where

$$d\gamma^{\epsilon, \Delta} = \epsilon \alpha(x^{\epsilon, \Delta})[v^\Delta dt - dw]. \quad (A.10)$$

$$\int_{i\Delta}^{i\Delta+\Delta} \epsilon \alpha(x^{\epsilon, \Delta}(i\Delta))v^\Delta(t)dt = \int_{i\Delta-\Delta}^{i\Delta} \epsilon \alpha(x^{\epsilon, \Delta}(i\Delta))dw(t).$$

We therefore have the decomposition

$$\gamma^{\epsilon, \Delta}(t) = I_1(t) + I_2(t) + I_3(t) + I_4(t),$$

where (for $k = [t/\Delta] - 1$)

$$I_1(t) = -\sum_1^k \int_{i\Delta-\Delta}^{i\Delta} \epsilon [\alpha(x^{\epsilon, \Delta}(s)) - \alpha(x^{\epsilon, \Delta}(i\Delta))]dw(s),$$

$$I_2(t) = \sum_1^k \int_{i\Delta}^{i\Delta+\Delta} \epsilon [\alpha(x^{\epsilon, \Delta}(s)) - \alpha(x^{\epsilon, \Delta}(i\Delta))]v^\Delta(s)ds,$$

$$I_3(t) = \int_{k\Delta+\Delta}^t \epsilon \alpha(x^{\epsilon, \Delta}(s))v^\Delta(s)ds,$$

$$I_4(t) = \int_{k\Delta}^t \epsilon \alpha(x^{\epsilon, \Delta}(s))dw(s).$$

For $\rho_2 > 0$, define the stopping times

$$\tau_{2,i} = \inf\{t: |I_i(t)| \geq \rho_2/4\} \wedge 1.$$

The same estimates as those used to show (A.8) give the existence of $0 < \epsilon_{0,2} \leq \epsilon_{0,1}$, and $0 < \Delta_{0,2} \leq \Delta_{0,1}$ such that for $\epsilon \leq \epsilon_{0,2}$ and $\Delta \leq \Delta_{0,2}$,

$$P_x(\tau_{2,i} < 1) \leq \exp - (M + 1)/\epsilon^2$$

for $i = 3, 4$.

Next consider $\tau_{2,1}$. Using

$$P_x(\tau_{2,1} < 1) \leq P_x(\tau_{2,1} < 1, \tau_1 = 1) + P_x(\tau_1 < 1),$$

equation (A.8), and a standard estimate on stochastic integrals [8; Lemma 4.7], by picking ρ_1 small we obtain $0 < \epsilon'_{0,2} \leq \epsilon_{0,2}$ and $0 < \Delta'_{0,2} \leq \Delta_{0,2}$ such that $\epsilon \leq \epsilon'_{0,2}$ and $\Delta \leq \Delta'_{0,2}$ imply

$$P_x(\tau_{2,1} < 1) \leq \exp - (M + 1)/\epsilon^2.$$

Finally we consider $\tau_{2,2}$. Using the Lipschitz property of $\alpha(\cdot)$, we have the following bound on a typical summand in $I_2(t)$

$$\begin{aligned} & \int_{i\Delta}^{i\Delta+\Delta} \epsilon [\alpha(x^{\epsilon, \Delta}(s)) - \alpha(x^{\epsilon, \Delta}(i\Delta))] v^{\Delta}(s) ds \\ & \leq \int_{i\Delta}^{i\Delta+\Delta} \epsilon \bar{K} \left| \int_{i\Delta}^t (b(x^{\epsilon, \Delta}(s)) + \epsilon \alpha(x^{\epsilon, \Delta}(s)) v^{\Delta}(s)) ds \right| |v^{\Delta}(t)| dt \\ & \leq \epsilon \bar{K} B \Delta^2 |v^{\Delta}(i\Delta)|/2 + \epsilon^2 \bar{K} B \Delta^2 |v^{\Delta}(i\Delta)|^2/2. \end{aligned}$$

We therefore have

$$\begin{aligned} P_x(\tau_{2,2} < 1) & \leq P \left\{ \epsilon \Delta^2 \bar{K} B \sum_1^{1/\Delta} |\theta_i| \geq \rho_2/4 \right\} \\ & \quad + P \left\{ \epsilon^2 \Delta^2 \bar{K} B \sum_1^{1/\Delta} |\theta_i|^2 \geq \rho_2/4 \right\}, \end{aligned} \tag{A.11}$$

where $\{\theta_i\}$ is a sequence of i.i.d. $N(0, 1/\Delta)$ random variables. For the sake of notational simplicity, we estimate these terms in the case where $\{\theta_i\}$ is a scalar valued sequence.

Using $E \exp c\theta_i^2 = (1 - 2c/\Delta)$ (for $2c/\Delta < 1$), we obtain (for any $\xi > 0$ such that $2\epsilon^2\Delta \bar{K} B\xi < 1$)

$$\begin{aligned} P\left\{\epsilon^2\Delta^2 \bar{K} B \sum_1^{1/\Delta} |\theta_i|^2 \geq \rho_2/4\right\} \\ \leq (\exp - \xi\rho_2/4)(1 - 2\epsilon^2\Delta\bar{K}B\xi)^{1/\Delta} \\ = \exp[-\xi\rho_2/4 + \frac{1}{\Delta} \log(1 - 2\epsilon^2\Delta \bar{K}B\xi)]. \end{aligned}$$

Now take $\xi = (M + 2)4/\rho_2\epsilon^2$, and use the fact that the log term $\rightarrow -8(M+2)\bar{K}B/\rho_2$ as $\Delta \rightarrow 0$ to get the estimate of the type (A.8) for the second term of (A.11).

For the first term of (A.11), we will use the fact that $E \exp c|\theta_i| \leq 2E \exp c\theta_i^2 = 2\exp c^2/2\Delta$. For $\xi > 0$ we have

$$\begin{aligned} P\left\{\epsilon\Delta^2 \bar{K} B \sum_1^{1/\Delta} |\theta_i| \geq \rho_2/4\right\} \\ \leq \exp - \xi\rho_2/4 \cdot \exp \xi^2\epsilon^2\Delta^2 \bar{K}^2B^2/2 \cdot \exp \frac{1}{\Delta} \log 2. \end{aligned}$$

Minimizing w.r.t. $\xi > 0$, we obtain the bound

$$\exp [-\rho_2^2/32\epsilon^2\Delta^2\bar{K}^2B^2 + (\log 2)/\Delta]$$

which again gives the desired bound of the type (A.8) for small Δ, ϵ .

Hence there is $0 < \epsilon_{0,2}'' \leq \epsilon_{0,2}'$ and $0 < \Delta_{0,2}'' \leq \Delta_{0,2}'$ such that for $\epsilon \leq \epsilon_{0,2}''$ and $\Delta \leq \Delta_{0,2}''$,

$$P_x(\tau_{2,2} < 1) \leq \exp - (M + 1)/\epsilon^2.$$

Now set $\tau_2 = \bigwedge_1 \tau_{2,1}$. On the set where $\tau_2 = 1$, $\sup_{0 \leq t \leq 1} |\gamma^{\epsilon, \Delta}(t)| \leq \rho_2$. We have shown that for ϵ sufficiently small, $P_x(\tau_2 < 1) \leq \exp - (M + 1)/\epsilon^2$. These facts, together with a standard estimate in large deviations theory [2; proof of Lemma 6.2], yield the lemma. $\square$

Lemma A4. Assume A1 and A2. Then $I^-(x) \leq I^+(x)$.

Proof. Let $c > 0$ be given. By A2 there is $\delta > 0$ such that $I^-(x) \leq I^-(x, -\delta) + c$, $I^+(x) \geq I^+(x, \delta) - c$. Next choose $s < 0$ such that the second statements of Lemmas A1 and A2 hold, with $I^-(x) + 1$ (resp., $I^+(x) - 1$) replacing I in Lemma A1 (resp. A2). Suppose $\beta \in \Delta(s)$ is a c -optimal solution to the problem

$$\inf_{\beta \in \Delta(s)} \sup_{u \in M} C(u, \beta[u]) \quad (\text{A.12})$$

Let $\bar{I}^-(s)$ denote the value of the expression given in (A.12). Then by Lemma A1 we may find $\beta'' \in \Delta$ such that

$$\sup_{u \in M} C^{-\delta}(u, \beta''[u]) \leq \bar{I}^-(s).$$

Hence we may conclude $I^-(x, -\delta) \leq \bar{I}^-(s)$. In an analogous manner we may prove $I^+(x, \delta) \geq \bar{I}^+(s)$, where

$$\bar{I}^+(s) = \sup_{\alpha \in \Gamma(s)} \inf_{v \in N} C(\alpha[v], v).$$

It follows that $I^-(x) - I^+(x) \leq \bar{I}^-(s) - \bar{I}^+(s) + 2c$. Since $c > 0$ is arbitrary, we are finished if we can show there is $s_0 < 0$ such that $\bar{I}^-(s) \leq \bar{I}^+(s)$ for all $s_0 < s < 0$. However, as is proved in [4; p. 17], when $-2^{-N} < s$, $\bar{I}^-(s)$ is a lower bound

for the value v_N^- defined in the sense of Friedman having step size 2^{-N} and allowing the minimizing player to move first (for the full definition of values in the sense of Friedman, see [4; Sect. 3]). An analogous statement holds for the corresponding upper values; $v_N^+ \leq \bar{I}^+(s)$. Since (as is easily proved) $v_N^- \leq v_N^+$ for every N [4; p. 11], we are finished. $\square$

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