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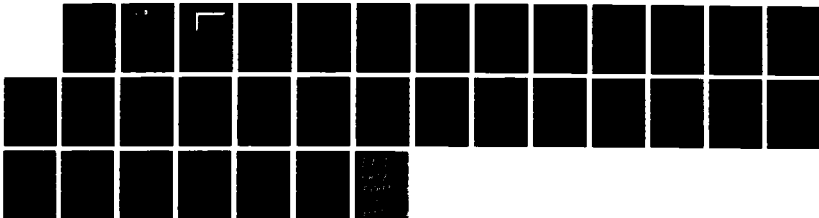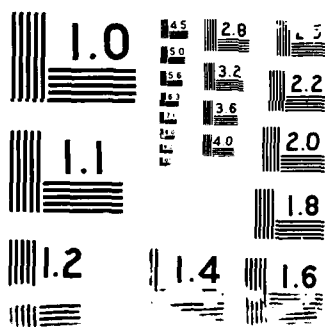AERONAUTICS AND A.. N M WERELEY ET AL. 22 FEB 88
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Approximate Evaluation of Semi-Markov

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Abstract

A property observed in high reliability fault tolerant control systems is the relatively rare occurrence of component failures compared to the frequent occurrence of redundancy management decision events. This property leads to a temporal decomposition of the semi-Markov chain reliability model into two time scales: a slow time scale for failure events, a fast time scale for FDI events. Conditions are described under which a perturbed semi-Markov chain can be approximated by an enlarged Markov process, the parameters of which are derived from the parameters of the semi-Markov chain.

1 Introduction

A typical fault-tolerant control system (FTCS) is composed of many highly reliable redundant components including sensors, actuators, power supplies and computers. These components are networked in a hierarchical architecture, and their use is governed by a redundancy management (RM) policy. Failure detection and isolation (FDI) logic is implemented to indicate to the RM system which components are no longer safely usable.

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APPROXIMATE EVALUATION OF SEMI-MARKOV

CHAIN RELIABILITY MODELS

by

Norman M. Wereley

Bruce K. Walker

Contract: AFOSR-84-0160

It has been demonstrated [1,2] that the reliability and availability of an FTCS can be computed using a finite-state generalized Markov (that is, Markov or semi-Markov) reliability model. These calculations are often difficult or impossible to accomplish by classical combinatorial methods due to time-ordered event sequences that are a consequence of the RM policy and FDI logic. If sequential tests are used to detect failures [3], then a semi-Markov chain reliability model must be used to predict the system reliability.

Many methods exist for the simplified analysis of the steady state behavior of generalized Markov chain models. However, generalized Markov chains model of FTCS invariably contain one or more trapping states that represent system loss. Thus, the steady state behavior is of no interest because the steady state condition will certainly be system loss. It is the transient behavior of these models that is of interest.

A generalized Markov chain is characterized by a discrete set of states and an arbitrary distribution of the holding or sojourn time for each transition. The semi-Markov chain specializes to a Markov chain when the holding times are geometrically distributed and identically distributed for all transitions exiting a particular state.

The result that must be routinely computed in analyzing the reliability model is the interval transition probability, $\phi_{ij}(n)$, which is the probability that the model occupies state j at time n given that it entered state i at the initial time. For FTCS, the states represent a complete characterization of the condition of the system. Thus, if all of the $\phi_{ij}(n)$ that correspond to system loss configurations for j can be computed for n corresponding to the finite duration of the mission, then the probability of an unsuccessful mission can be computed.

Once the interval transition probabilities have been determined for a particular time n, the probability of occupying each state can be determined if the initial state occupancy probabilities are known. Let $\underline{\pi}(n)$ be the state probability distribution at time m. If $\underline{\pi}(0)$ is known, then

$$\underline{\pi}(n) = \underline{\pi}(0)\boldsymbol{\Phi}(n) \tag{1}$$

In the context of the FTCS, the first state is routinely chosen to represent the situation where all components are working. Usually, the system occupies the first state with probability one at the initial time.

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The interval transition probabilities are generated by the semi-Markov chain recursion formula [4]:

$$\Phi(n) = {}^>\mathbf{W}(n) + \sum_{m=0}^{n} \mathbf{G}(n)\Phi(n-m); \quad IC: \quad \Phi(0) = \mathbf{I} \tag{2}$$

Taking z-transforms of both sides of (2) and solving for $\Phi(z)$:

$$\Phi(z) = [\mathbf{I} - \mathbf{G}(z)]^{-1>}\mathbf{W}(z) \tag{3}$$

The z-transform of the state occupancy probability distribution is

$$\underline{\pi}(z) = \underline{\pi}(0)[\mathbf{I} - \mathbf{G}(z)]^{-1>}\mathbf{W}(z) \tag{4}$$

which follows directly from (2). The inverse matrix of $[\mathbf{I} - \mathbf{G}(z)]$ always exists for a semi-Markov chain. The inverse transform of either $\underline{\pi}(z)$ or $\Phi(z)$ can be found using standard partial fraction expansion techniques. However, for all but the simplest of situations, transform methods are useless in a practical sense.

In practice, the interval transition probability matrix is nearly always found by performing the semi-Markov recursion numerically. For a model with N states, computation of $\Phi(n)$ requires storage of $2nN^2$ values because both $\Phi(n)$ and $\mathbf{G}(n)$ must be stored for all times prior to and including time n. A reliability model for a typical inertial navigation system might have twenty states, a sampling period of 200ms, and a two hour mission time. This would require storage of 2.88×10^7 single precision values and require 230 megabytes of storage. Moreover, the number of floating point multiplications required to compute $\underline{\pi}(n)$ from $\underline{\pi}(0)$ is about n^2N^2 - which is 2.59×10^{11} for the example described above. Thus, the computational burden and memory requirements are tremendous even for a simple system.

The problem to be addressed in this paper is to substantially reduce the computational burden while preserving the accuracy of reliability and availability calculations.

One possible means for doing this is direct Monte Carlo techniques. If a sufficient number of Monte Carlo simulations are made of system operations to account correctly for all possible random events that bear on the reliability calculation, then any aspect of system performance can be evaluated. To obtain meaningful results for high reliability systems with events that occur with probabilities as low as 1×10^{-9} (typical of the probability of a component failure over a single time step), over one billion simulations must be performed.

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This task is as formidable as evaluating the semi-Markov chain recursion for large values of the time index. Consequently, reliability calculations via direct Monte Carlo methods also have prohibitive computational costs.

Lewis suggested in [5,6] that a modified Monte Carlo approach be used for high reliability systems. Again, failure events are assumed to be extremely rare relative to other events that occur in the system. Thus, the vast majority of simulations will be those for which no failures occur. Lewis assumes that all events have exponentially distributed times of occurrence and can be modeled by a Markov chain. It is possible to sample the failure distributions before a simulation is initiated to determine if any failures will occur during the mission. If all failures occur after the mission has been completed (which is usually the case), then a normal simulation results. If a failure occurs during the mission, then the complete simulation must be performed including FDI decisions, decision errors, and repairs. However, this approach does not apply to semi-Markov chains because FDI events arising from a sequential FDI test are not exponentially distributed. In these cases, a complete simulation must always be run and no benefits are derived from the modified technique.

Another approach that exploits the rare occurrence of failure events is suggested by Trivedi in [7,8]. The model is based upon a time-scale decomposition of the system into virtually disjoint fault-occurrence and fault handling submodels. The fault-handling submodels represent aggregated states and the failure occurrence submodels dictate the behavior between these aggregated states. The reliability of the system predicted by the aggregated model is then computed using Markov or Monte Carlo techniques. However, the only fault-handling events that are accounted for are detections and missed detections following actual faults. A common FDI event that cannot be treated by these hybrid models is the false alarm, which occurs in the absence of a fault. Therefore, this approach is limited to systems where false alarms cannot occur.

In this paper, the relatively rare occurrence of component failures relative to RM decision events will be exploited in the development of an approximate method for evaluating semi-Markov chain reliability models of fault tolerant control systems.

2 A Limit Theorem for Semi-Markov Chains

Theorem 1 describes how a perturbed semi-Markov chain, which is dependent on a small parameter ϵ in a certain way, can be described asymptotically by an enlarged Markov process as $\epsilon \to 0$. This theorem is an extension of the results for discrete parameter semi-Markov processes stated in [9].

The semi-Markov chain depends on a small parameter ϵ such that the entire state space of the semi-Markov chain can be decomposed into disjoint classes of states where the probabilities of departure from each class tend to zero with ϵ. Also, the total sojourn in each class is assumed to have a non-degenerate distribution in the limit as $\epsilon \to 0$. (When $\epsilon = 0$, the chain will be referred to as the unperturbed semi-Markov chain while the ϵ–dependent chain will be referred to as the perturbed semi-Markov chain.)

Theorem 1 (Limit Theorem for Semi-Markov Chains) *Let the set E of states of the semi-Markov chain be expressible as a union of disjoint classes:*

$$E = \sum_{k=1}^{N^e} E_k \quad k \in M \equiv \{1, 2, \ldots, N^e\}. \tag{5}$$

Let $\tau_{kr}^{(i)}$ be the sojourn of the semi-Markov chain in class E_k when it starts from state $i \in E_k$ and moves to class E_r where $r \neq k$. If the following two conditions hold for the semi-Markov chain E:

1. The elements of the core matrix sequence $\{g_{ij}^\epsilon(n) \mid i, j \in E\}$ specifying the semi–Markov chain depend as follows on the small parameter ϵ:

$$^{\leq}g_{ij}^\epsilon(n) = p_{ij}^\epsilon {}^{\leq}h_{ij}\left(\frac{n}{\epsilon}\right) \tag{6}$$

where $^{\leq}h_{ij}(0) = 0$. The p_{ij}^ϵ can be expanded in a Taylor series about $\epsilon{=}0$. Retaining terms that are linear in ϵ:

$$p_{ij}^\epsilon = \begin{cases} p_{ij}^{(k)} - \epsilon q_{ij}^{(k)} + O(\epsilon) & \text{if } i, j \in E_k \\ \epsilon q_{ij}^{(k)} + O(\epsilon) & \text{if } i \in E_k \text{ and } j \notin E_k \end{cases} \tag{7}$$

The embedded Markov chain for $\epsilon{=}0$ obeys the usual Markov chain properties:

$$\sum_{j \in E_k} p_{ij}^{(k)} = 1; \text{ and } p_{ij}^{(k)} \in [0, 1]; \forall k \in M \tag{8}$$

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2. The embedded Markov chains defined by the matrices $\left\{ p_{ij}^{(k)} \mid i,j \in E_k \; \forall k \in M \right\}$ **are ergodic with stationary distributions** $\left\{ \pi_i^{(k)} \mid i \in E_k \; \forall k \in M \right\}$.

Then:

$$\lim_{n \to \infty} Pr\left\{ \tau_{kr} \leq t \right\} = \gamma_{kr} \left\{ 1 - \exp\left[\frac{-\Lambda_k t}{T} \right] \right\} \tag{9}$$

where:

$$\gamma_{kr} \equiv \frac{\sum_{i \in E_k} \pi_i^{(k)} q_i^{(kr)}}{\sum_{i \in E_k} \pi_i^{(k)} q_i^{(k)}} \tag{10}$$

$$\Lambda_k \equiv \frac{\sum_{i \in E_k} \pi_i^{(k)} q_i^{(k)}}{\sum_{i \in E_k} \pi_i^{(k)} a_i^{(k)}} \tag{11}$$

Here:

$$q_i^{(kr)} \equiv \sum_{j \in E_r} q_{ij}^{(k)} \tag{12}$$

$$q_i^{(k)} \equiv \sum_{j \in E_k} q_{ij}^{(k)} \tag{13}$$

$$a_i^{(k)} \equiv \sum_{j \in E_k} p_{ij}^{(k)} \bar{\tau}_{ij} \tag{14}$$

$$\bar{\tau}_{ij} \equiv \sum_{n=0}^{\infty} n \, h_{ij}(n) \tag{15}$$

PROOF: Let $\epsilon_{\varsigma_{ij}}$ denote the integer valued sojourn of the semi-Markov chain in state i with next transition to state j with the holding time distribution $\leq h_{ij}(n/\epsilon)$ while the δ_{ij}^{ϵ} are the transition indicators from state i to state j. The probability distribution of the random quantities $\tau_{kr}^{(i)}$ can be expressed in terms of total probability as

$$Pr\left\{ \tau_{kr}^{(i)} \leq n \right\} = \sum_{j \in E_k} Pr\left\{ \delta_{ij}^{\epsilon} = 1; \; \epsilon_{\varsigma_{ij}} + \tau_{kr}^{(j)} \leq n \right\} + \sum_{j \in E_r} Pr\left\{ \delta_{ij}^{\epsilon} = 1; \; \epsilon_{\varsigma_{ij}} \leq n \right\} \tag{16}$$

Defining the interval transition CDF as

$$\leq \phi_{kr}^{(i)}(n) = Pr\left\{ \tau_{kr}^{(i)} \leq n \right\} \tag{17}$$

then

$$\leq \phi_{kr}^{(i)}(n) = \sum_{j \in E_k} \sum_{m=0}^{n} g_{ij}^{\epsilon}(m) \leq \phi_{kr}^{(j)}(n-m) + \sum_{j \in E_r} \leq g_{ij}^{\epsilon}(n) \tag{18}$$

Taking z-transforms of both sides yields:

$$\leq \phi_{kr}^{(i)}(z) = \sum_{j \in E_k} g_{ij}^{\epsilon}(z) \leq \phi_{kr}^{(j)}(z) + \left\{ \frac{z}{z-1} \right\} \sum_{j \in E_r} g_{ij}^{\epsilon}(z). \tag{19}$$

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The z-transforms of the $g_{ij}^\epsilon(z)$ must be evaluated to first order in ϵ. From (6) and the definition of the z-transform [10]:

$$g_{ij}^\epsilon(z) = p_{ij}^\epsilon \sum_{n=0}^\infty h_{ij}\left(\frac{n}{\epsilon}\right) z^{-n} \tag{20}$$

Note that p_{ij}^ϵ has been moved in front of the summation sign because it does not depend on time. Let $m = n/\epsilon$ and expand $z^{-\epsilon m}$ in a Taylor series about $\epsilon=0$. Then:

$$g_{ij}^\epsilon(z) = p_{ij}^\epsilon \sum_{m=0}^\infty \left\{1 - \epsilon m \log z\right\} h_{ij}(m) + O(\epsilon) \tag{21}$$

where $O(\epsilon)$ represents terms such that in the limit as $\epsilon \to 0$, the quantity $O(\epsilon)/\epsilon$ approaches zero. Noting that:

$$\sum_{n=0}^\infty h_{ij}(n) = 1 \tag{22}$$

$$\sum_{n=0}^\infty n\, h_{ij}(n) = \bar{\tau}_{ij}. \tag{23}$$

and substituting p_{ij}^ϵ from (7) and combining terms of $O(\epsilon)$ yields:

$$g_{ij}^\epsilon(z) = \begin{cases} p_{ij}^{(k)}\left\{1 - \epsilon\bar{\tau}_{ij}\log z\right\} - \epsilon q_{ij}^{(k)} O(\epsilon) & \text{if } i,j \in E_k \\ \epsilon q_{ij}^{(k)} + O(\epsilon) & \text{if } i \in E_k \text{ and } j \notin E_k \end{cases} \tag{24}$$

Incorporating these results into (19) and placing all terms proportional to ϵ on the RHS:

$$
{}^\le\phi_{kr}^{(i)}(z) - \sum_{j \in E_k} p_{ij}^{(k)}\, {}^\le\phi_{kr}^{(j)}(z) = -\epsilon \sum_{j \in E_k} {}^\le\phi_{kr}^{(j)}(z) \left\{ q_{ij}^{(k)} + p_{ij}^{(k)}\bar{\tau}_{ij}\log z \right\}
$$
$$
+ \epsilon \left\{ \frac{z}{z-1} \right\} \sum_{j \in E_r} q_{ij}^{(k)} + O(\epsilon) \tag{25}
$$

Now, passing to the limit as $\epsilon \to 0$, the RHS vanishes and the ${}^\le\phi_{kr}(z)$ are found to satisfy the system of equations below:

$${}^\le\phi_{kr}^{(i)}(z) - \sum_{j \in E_k} p_{ij}^{(k)}\, {}^\le\phi_{kr}^{(j)}(z) = 0 \tag{26}$$

Let $\mathbf{P}_k = \left[p_{ij}^{(k)}\right]$ represent the embedded Markov chain operator in class E_k of the unperturbed semi–Markov chain E. The system of equations in (26) can be expressed as:

$${}^\le\phi_{kr}^{(i)}(z)^T = \mathbf{P}_k\, {}^\le\phi_{kr}^{(j)}(z)^T \tag{27}$$

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After successive premultiplication by $\mathbf{P}_k$, and taking the limit as $n \to \infty$:

$$^{\leq}\phi_{kr}(z)^T = \left\{ \lim_{n \to \infty} \mathbf{P}_k^n \right\} {}^{\leq}\phi_{kr}(z)^T \tag{28}$$

Under Condition 2, the ergodic theorem for Markov chains [11] implies that:

$$\lim_{n \to \infty} \mathbf{P}_k^n = \mathbf{P}_k^\infty = \begin{bmatrix} \pi_M^{(k)} \\ \vdots \\ \pi_M^{(k)} \end{bmatrix} \tag{29}$$

so that the solution to (27) is independent of the superscript:

$$^{\leq}\phi_{kr}^{(i)}(z) = {}^{\leq}\phi_{kr}(z) \ \forall \, i \in E_k, \ \forall \, k \in M \tag{30}$$

Now, (25) is of the form $f(x) = g(x, \epsilon)$, that is, the LHS is not a function of ϵ and is therefore constant with respect to ϵ. However, as $\epsilon \to 0$, the RHS approaches zero so that the LHS must be zero for all values of ϵ. Canceling ϵ from the result, multiplying by the stationary probabilities of the unperturbed semi-Markov chain in class k, $\pi_i^{(k)}$, and summing over $i \in E_k$ yields:

$$\sum_{i \in E_k} \pi_i^{(k)} \sum_{j \in E_k} {}^{\leq}\phi_{kr}^{(j)}(z) \left\{ q_{ij}^{(k)} + p_{ij}^{(k)} \bar{r}_{ij} \log z \right\} + \left\{ \frac{z}{z-1} \right\} \sum_{i \in E_k} \pi_i^{(k)} \sum_{j \in E_r} q_{ij}^{(k)} + O(\epsilon) \tag{31}$$

On passing again to the limit as $\epsilon \to 0$, noting that all of the ${}^{\leq}\phi_{kr}^{(i)}(z)$ have the limit function ${}^{\leq}\phi_{kr}(z)$, and solving for ${}^{\leq}\phi_{kr}(z)$, the z-transform of the class-to-class transition ΓMF becomes:

$$\phi_{kr}(z) = \gamma_{kr} \Lambda_k \frac{1}{\log z + \Lambda_k} \tag{32}$$

The mapping from the z domain to the s domain (Laplace) is given by $s = (\log z)/T$. Dividing top and bottom by the sampling period T, and applying the transformation concludes the proof. □

In summary, Theorem 1 describes the conditions under which a perturbed semi-Markov chain can be approximated by an enlarged Markov process that evolves in the slow time–scale, and also states how the parameters of the Markov process are determined from the parameters of the semi-Markov chain. In the context of FTCS, the fast time scale behavior within a class would represent FDI decision and RM events while the slower class–to–class

behavior would represent the occurrence of failures. The class–to–class interval transition CDF $\Phi_{kr}(t)$ that results is a continuous time envelope of the behavior between the classes. This interpretation is intuitively satisfying since failures are invariably assumed to have exponentially distributed times of occurrence over continuous time.

However, two problems occur in the application of Theorem 1 to FTCS models: (1) the embedded Markov chains for each class of the unperturbed model are rarely ergodic, and (2) the holding time PMFs are usually functions of n, not n/ϵ, that is, the holding times are typically not on the order of the mean time to a component failure. The requirement that the embedded Markov chains of the unperturbed classes be ergodic is important in producing (26) and guarantees the existence of the stationary probabilities $\left\{ \pi_i^{(k)} \mid i \in E_k \, \forall \, k \in M \right\}$. The ergodicity condition can be relaxed in much the same way as was done in [12] for semi-Markov processes. This will be accomplished in Lemma 2 and Lemma 3. The second problem can be mitigated by introducing time-scaling into Theorem 1, as will be done in Theorem 4.

3   Relaxation of the Ergodicity Condition

Lemma 2 discusses how the existence of the Caesaro limit of the embedded Markov chain operator leads to a relaxation of the ergodicity condition.

Lemma 2 *Consider a semi-Markov chain state space E that can be expressed as a sum of disjoint classes according to (5) and (7). Let $\mathbf{P}_k = \left[p_{ij}^{(k)} \right]$ represent the embedded Markov chain operator for class E_k. The solution of (26) is independent of the superscript (and the results of Theorem 1 hold), if the Caesaro limit exists:*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} \mathbf{P}_k^l = \Pi_k = \begin{bmatrix} \pi_c^{(k)} \\ \vdots \\ \pi_c^{(k)} \end{bmatrix} \tag{33}$$

PROOF: The system of equations in (26) can be expressed in matrix form as is done in (27). Successively premultiplying both sides by $\mathbf{P}_k$, and averaging an infinite number of these terms:

$$\stackrel{\leq}{} \phi_{kr}(z)^T = \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} P_k^l \stackrel{\leq}{} \phi_{kr}(z)^T \tag{34}$$

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Because the operator P_k satisfies the Caesaro limit from (33), the solution of (26) is independent of the superscript. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The relaxation due to Lemma 2 demonstrates that the ergodicity condition of Theorem 1 was sufficient, but not necessary. Thus, the conditions under which the Caesaro limit exists should be determined in hopes of finding a necessary condition.

Lemma 3 *Consider a semi-Markov chain state space E that can be expressed as a sum of disjoint classes according to (5) and (7). Let* $\mathbf{P}_k = \left[p_{ij}^{(k)}\right]$ *represent the embedded Markov chain operator of the unperturbed chain for class* E_k. *If the embedded Markov chain represented by the operator* $\mathbf{P}_k$ *is: 1) ergodic, or 2) non-ergodic with one and only one unit eigenvalue, then the Caesaro limit in (34) exists.*

Proof: The proof of this lemma is essentially similar to that in [12]. For details of this proof, see [13]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4 Limit Theorem with Time Scaling

In FTCS with small single step component failure probabilities, the holding time PMFs associated with the core matrix sequence elements do not depend on ϵ but only on the FDI decision delay. If a semi–Markov chain is observed in another time scale that is $1/\delta$ times that of the original time scale, then the PMF $h_{ij}(n)$ will be affected but the eventual transition probabilities, p_{ij}^ϵ, will remain the same because they characterize the transition probability from state i to state j regardless of when the transition takes place. However, the holding time PMFs in the new time scale are not obtained by simply changing the argument of $h_{ij}(\cdot)$ from n to n/δ. This is because the summation of $h_{ij}(n/\delta)$ for all non-negative values of the time index would not be unity and so would not yield a proper holding time function. The CDF $^\le h_{ij}(n)$ associated with the PMF $h_{ij}(n)$ must be determined and the argument of the CDF replaced by n/δ. The new PMF $h'_{ij}(n)$ observed in the new time scale would have most of its probability mass close to the origin. The statistics of the process in the new time scale will depend on the small parameter δ - the time scaling factor.

Theorem 4 (Limit Theorem With Time Scaling) *Let the set E of states of the semi-Markov chain be expressible as a sum of disjoint classes as in (5). Let* $\tau_{kr}^{(i)}$ *be the sojourn*

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of the semi-Markov chain in class E_k when it starts from state $i \in E_k$ and moves to class E_r for $r \neq k$. If the following two conditions hold for the semi-Markov chain E:

1. The elements of the core matrix sequence $\{g_{ij}^{\epsilon}(n) \mid i, j \in E\}$ specifying the semi-Markov chain depend as follows on the small parameters δ and ϵ:

$$^{\leq}g_{ij}^{\epsilon}(n) = p_{ij}^{\epsilon} {}^{\leq}h_{ij}(\frac{n}{\delta}) \tag{35}$$

Here, $^{\leq}h_{ij}(\cdot)$ is the transition CDF of the semi- Markov chain in the original time scale and $^{\leq}h_{ij}(0) = 0$. The p_{ij}^{ϵ} can be expanded in a Taylor series about $\epsilon=0$ as in (7). The embedded Markov chain obeys the usual Markov chain properties described in (8).

2. The embedded Markov chains defined by the matrices $\left\{p_{ij}^{(k)} \mid i, j \in E_k \forall k \in M\right\}$ are ergodic or non-ergodic with one and only one unit eigenvalue with the stationary probabilities (in the Caesaro limit sense) $\left\{\pi_i^{(k)} \mid i \in E_k \forall k \in M\right\}$.

Then:

$$\lim_{n \to \infty} Pr\left\{\tau_{kr} \leq t'\right\} = \gamma_{kr}\left\{1 - \exp\left[\frac{-\Lambda_k t'}{\alpha T}\right]\right\} \tag{36}$$

where the parameters of the enlarged Markov process were defined in Theorem 1 and $\alpha = \delta/\epsilon$.

PROOF: The proof of this theorem is essentially identical to that of Theorem 1. For details of this proof, see [13]. □

It should be noted that an explicit analytical expression of the core matrix sequence, $\mathbf{G}^{\epsilon}(n)$, is not required to expand the eventual transition probabilities of the perturbed semi-Markov chain, p_{ij}^{ϵ} in a Taylor series about $\epsilon=0$. The eventual transition probabilities may be evaluated numerically, which is what would be done in practice. This is fortunate because the direct form of the core matrix is not always available [3]. In many cases, the decision time PMFs are tabulated numerically and no functional form is available.

Also, the time scale decomposition of the semi-Markov chain is crucial to the use of this technique. A simple way of characterizing each class is as follows: the first class contains states for which no failures have occurred, the second class contains states for which a single failure has occurred, the third class contains states for which two failures have occurred, etc. These classes arise by setting $\epsilon=0$ and observing which groups of states of the unperturbed semi-Markov chain do not communicate.

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Finally, estimates of the original semi-Markov chain state probabilities can be recovered from the enlarged Markov process. The asymptotic behavior of the unperturbed semi-Markov chains in each class are the stationary probabilities (or Caesaro limit probabilities) for that class. The class–to–class behavior is determined by the enlarged process. The approximate state probabilities in each class are:

$$\hat{\pi}_i^{(k)}(n) = \pi_i^k \hat{\pi}_k^e(n) \tag{37}$$

where the approximate class probabilities of the enlarged process are found from its interval transition probability matrix.

5 Performance Evaluation of the SCMS

Two simple semi-Markov reliability models of a single component monitoring system (SCMS) will be developed. The SCMS uses a sequential FDI test to monitor the status (failed or working) of a single component. The two models will differ in monitoring policy. The first example, SCMS-I, models an FDI test that operates continuously over the entire mission duration. The second example, SCMS-II, models an FDI test that is discontinued after the first failure indication (namely, abbreviated monitoring).

In this section, the performance of the SCMS will be evaluated through application of the approximate method to a semi-Markov model. The procedure follows: (1) semi-Markov transition diagrams are constructed describing all of the random events that can take place, (2) the direct form of the core matrix sequence is derived, (3) the core matrix is placed in standard form, (4) the performance is evaluated through application of Theorem 4.

In addition, z-transforms will be used to determine an analytical expression for the state and class occupancy probabilities, $\underline{\pi}(n)$ and $\underline{\pi}^e(n)$ respectively. The results of the z-transform analysis will be used to evaluate the accuracy of the approximate method. This is possible here because the models are relatively simple. In more general cases, this would not be practical.

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Table 1: State definitions and class decompositions for SCMS-I,II

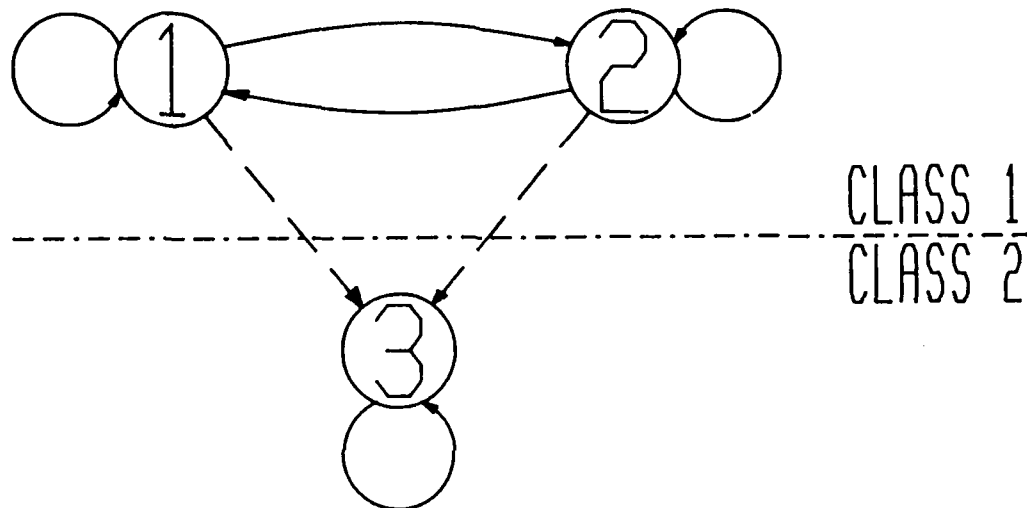| State	State Definition	Class
1	Component is working	1
2	Component has a false alarm	1
3	System loss - component failed	2

Figure 1: Semi-Markov transition diagram for SCMS-I

5.1 SCMS with continuous monitoring

Table 1 enumerates and defines the states of a semi-Markov chain reliability model of the SCMS-I. The dashed line in the table distinguishes the class decomposition of the model: class 1 contains states 1 and 2, class 2 contains only state 3.

The semi-Markov transition diagram for the SCMS-I is presented in Figure 1. Two aspects of this diagram should be noted. Given that the chain has entered a state, the lines directed out of that state represent transitions after the chain has remained in that state for a period of time, namely, the holding time. Secondly, the dashed lines represent transitions whose transition PMFs are proportional to ϵ. Thus, a dashed line represents the condition that no such transition occurs when $\epsilon = 0$. This is a convenient way of depicting the class decomposition of a semi-Markov chain reliability model.

A complete statistical description of the sequential test used in the FDI process requires knowledge of the conditional PMFs of the time to decision of the test. The following two functions are required:

$f_{\bar{D}}^0(n)$ PMF of time to a decision that no failure is present when no failure is present.

$f_D^0(n)$ PMF of time to a failure indication when no failure is present (false alarm).

In these PMFs, the fault monitoring event at time n must be conditioned on the failure events that take place prior to and including time $n - 1$. Thus, it is assumed that there is a delay of at least a single time step between when a failure takes place and when it can be detected.

Another necessary function is the sum of all probabilities of all possible test outcomes - nominal decision, failure indication, and decision not yet available - at a given time n. Itcan be specified in terms of the decision time PMFs as:

$$Q_0(n) = 1 - \sum_{k=1}^{n-1} \left\{ f_{\bar{D}}^0(k) + f_D^0(k) \right\}; \quad n \geq 1 \tag{38}$$

Note that $Q_0(n)$ is defined only for positive values of the time index n and is defined to be zero for $n = 0$. Thus, one of the necessary criteria for a permissible holding time PMF is maintained - there is no probability mass at the initial time.

The core matrix sequence, $\mathbf{G}^\epsilon(n)$, for SCMS-I can be expressed in matrix form as:

$$\mathbf{G}^\epsilon(n) = \begin{bmatrix} (1-\epsilon)^n f_{\bar{D}}^0(n) & (1-\epsilon)^n f_D^0(n) & \epsilon(1-\epsilon)^{n-1} Q_0(n) \\ (1-\epsilon)^n f_{\bar{D}}^0(n) & (1-\epsilon)^n f_D^0(n) & \epsilon(1-\epsilon)^{n-1} Q_0(n) \\ 0 & 0 & \delta(n-1) \end{bmatrix} \tag{39}$$

Any reasonable PMF may be used for the decision time PMFs. However, a closed form solution for $\underline{\pi}(n)$ is desired. A simple but realistic choice for the decision time PMFs is the hypergeometric PMF [13]. This PMF is a good approximation to the holding time behavior of many sequential tests, as demonstrated by Table 6.6 of [3]. Choosing an appropriate eventual transition probability yields the hypergeometric decision time PMFs below:

$$f_{\bar{D}}^0(n) = A_{\bar{D}}^0 \left(a^n - b^n \right) \quad ; \quad A_{\bar{D}}^0 = (1 - P_{fa}) \frac{(1-a)(1-b)}{(a-b)} \tag{40}$$

$$f_D^0(n) = A_D^0 \left(a^n - b^n \right) \quad ; \quad A_D^0 = P_{fa} \frac{(1-c)(1-d)}{(c-d)} \tag{41}$$

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where $0 < b < a < 1$ and $0 < d < c < 1$. The parameter P_{fa} is the eventual false alarm probability of the sequential test. The core matrix can now be expressed in terms of these PMFs.

A z-transform analysis of the semi-Markov recursion formula using the above core matrix sequence yields the state occupancy probability vector $\underline{\pi}(n)$:

$$\pi_1(n) = (1 - P_{fa}) R^n + P_{fa} \frac{c(1-d)}{(c-d)} (cR)^n - P_{fa} \frac{(1-c)d}{(c-d)} (dR)^n \qquad (42)$$

$$\pi_2(n) = P_{fa} R^n - P_{fa} \frac{c(1-d)}{(c-d)} (cR)^n + P_{fa} \frac{(1-c)d}{(c-d)} (dR)^n \qquad (43)$$

$$\pi_3(n) = 1 - R^n \qquad (44)$$

and the class occupancy probability vector $\underline{\pi}^\epsilon(n)$:

$$\underline{\pi}^\epsilon(n) = [(1-\epsilon)^n , \, 1 - (1-\epsilon)^n] \qquad (45)$$

Availability of these analytical results permits comparisons to be made with the approximate results that exploit the class decomposition to be described below. It should be emphasized again that the existence of analytical is rare, and occurs only because the system is very simple.

In order to derive the enlarged Markov process for this model, $\mathbf{G}^\epsilon(n)$ must be placed in standard form. For an in–class transition, the decomposition is obtained from the first two terms of the Taylor series expansion of the eventual transition probability about $\epsilon = 0$. In addition, the mean waiting times, $\bar{\tau}_{ij}$, must be derived. For an out– of–class transition, the decomposition is obtained by dividing the eventual transition probability by ϵ and then taking the zeroth order term in the Taylor series expansion about $\epsilon = 0$.

Consider an in–class transition from state 1 to state 1. First, the eventual transition probability is found:

$$p_{11}^\epsilon = A_D^0 \left\{ \frac{aR}{(1-aR)} - \frac{bR}{(1-bR)} \right\} \qquad (46)$$

The decomposition for the transition is:

$$p_{11}^\epsilon = 1 - P_{fa} + \epsilon(1 - P_{fa}) \frac{(1-ab)}{(1-a)(1-b)} \qquad (47)$$

To satisfy the requirements for a permissible holding time function, the holding time function for this transition must be expressed as:

$$h_{11}(n) = \frac{(1 - aR)(1 - bR)}{(a - b)R} \{(aR)^n - (bR)^n\} \tag{48}$$

From (15), the mean holding time can be found:

$$\bar{\tau}_{11} = \frac{(1 - abR^2)}{(1 - aR)(1 - bR)} \tag{49}$$

Thus, all of the parameters required to place this in–class transition PMF in standard form have been derived.

A second type of core matrix element that must be placed in standard form is one corresponding to an out–of–class transition such as a transition from state 1 to state 3. First, the eventual transition probability must be found.

$$p_{31}^{\epsilon} = \epsilon(1 - P_{fa})\frac{(1 - abR)}{(1 - aR)(1 - bR)} + \epsilon P_{fa}\frac{(1 - cdR)}{(1 - cR)(1 - dR)} \tag{50}$$

The sole parameter required for the approximation technique from this eventual transition probability is found from:

$$q_{31} = \left[\frac{1}{\epsilon}p_{31}^{\epsilon}\right]_{\epsilon=0} = (1 - P_{fa})\frac{(1 - ab)}{(1 - a)(1 - b)} + P_{fa}\frac{(1 - cd)}{(1 - c)(1 - d)} \tag{51}$$

The eventual transition probabilities of each row of $\mathbf{G}^{\epsilon}(n)$ sum to unity. Thus, this is a proper semi-Markov chain [4].

The next step in the procedure is to determine the eventual transition probability matrix of the unperturbed semi-Markov chain. This is found by setting $\epsilon = 0$ and ignoring all time varying terms in the core matrix:

$$\mathbf{P} = \begin{bmatrix} 1 - P_{fa} & P_{fa} & 0 \\ 1 - P_{fa} & P_{fa} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{52}$$

By raising $\mathbf{P}$ to successively higher powers, the stationary interval transition probability matrix is found to be identical to (52). The embedded stationary probability distribution in partitioned form is thus:

$$\underline{\pi}_M = \begin{bmatrix} 1 - P_{fa} & P_{fa} \mid 1 \end{bmatrix} \tag{53}$$

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With knowledge of this and of the mean holding times for transitions from state i to j, $\bar{\tau}_{ij}$, it is possible to determine the stationary probability distribution of the unperturbed semi-Markov chain, $\underline{\pi}^{(k)}$. This probability distribution is needed to approximate the state probability distribution of the original perturbed semi-Markov chain.

From semi-Markov theory, the stationary probability distribution for each unperturbed class E_k is given by

$$\pi_i^{(k)} = \pi_{M_i}^{(k)} \bar{\tau}_i^{(k)} / \bar{\tau}^{(k)} \tag{54}$$

where $\bar{\tau}^{(k)}$ is the mean waiting time of the chain in class E_k:

$$\bar{\tau} = \sum_{i \in E_k} \pi_{M_i}^{(k)} \bar{\tau}_i^{(k)} \tag{55}$$

$\underline{\pi}_M^{(k)}$ was determined above, and $\bar{\tau}_i^{(k)}$ is the mean holding time in state i:

$$\bar{\tau}^{(k)} = \sum_{j \in E_k} p_{ij}^{(k)} \bar{\tau}_{ij}^{(k)} \tag{56}$$

where $\bar{\tau}_{ij}^{(k)}$ is determined from the limit of $\bar{\tau}_{ij}$, defined in (15), as $\epsilon \to 0$.

The stationary probability distribution of the unperturbed semi-Markov chain will now be determined. The mean holding times of the unperturbed semi-Markov chain in the first class are:

$$\bar{\tau}_{11}^{(1)} = \bar{\tau}_{12}^{(1)} = \frac{(1 - ab)}{(1 - a)(1 - b)} \tag{57}$$

$$\bar{\tau}_{21}^{(1)} = \bar{\tau}_{22}^{(1)} = \frac{(1 - cd)}{(1 - c)(1 - d)} \tag{58}$$

The mean holding time in class 1 starting from state i is thus

$$\bar{\tau}_1^{(1)} = (1 - P_{fa}) \frac{(1 - ab)}{(1 - a)(1 - b)} + P_{fa} \frac{(1 - cd)}{(1 - c)(1 - d)} \tag{59}$$

Similarly, $\bar{\tau}_2^{(1)} = \bar{\tau}_1^{(1)}$. The mean waiting time of the semi-Markov chain in class 1 is:

$$\bar{\tau}^{(1)} = \sum_{i \in E_1} \underline{\pi}_{M_i}^{(1)} \bar{\tau}_i^{(1)} = \bar{\tau}_1^{(1)} \tag{60}$$

Hence, for this situation (but not in general): $\underline{\pi} = \underline{\pi}_M$.

The time scale factor δ is set equal to ϵ for convenience. It should be noted that δ must be of the same order as ϵ, but not necessarily equal.

All parameters required to describe the enlarged Markov process have now been stated. The parameters of the approximate class–to–class interval transition CDF can be found as described in Theorem 4: $\gamma_{21} = 1$, $\Lambda_1 = q_{31}^{(1)}/a_1$. So, the class–to–class interval transition CDF expressed in the slow time scale is:

$$\leq \hat{\phi}_{12}(t') = 1 - \exp\left\{-\frac{\Lambda_1 t'}{T}\right\} \tag{61}$$

To return to the original time scale, let $t' = \delta t$, and recall that δ was chosen to be equal to ϵ in this case. The rows of the interval transition probability matrix of the enlarged process must sum to unity. Since the semi-Markov chain is always in state 1 at the initial time, the enlarged process is always in class 1 at the initial time. Hence, approximate class occupancy probabilities can be stated directly from the first row of the interval transition probability matrix since $\hat{\underline{\pi}}^e(t) = \underline{\pi}^e(0) \leq \hat{\underline{\Phi}}(t)$:

$$\hat{\underline{\pi}}^e(t) = \left[\exp\left\{-\frac{\epsilon \Lambda_1 t}{T}\right\} \;\middle|\; 1 - \exp\left\{-\frac{\epsilon \Lambda_1 t}{T}\right\}\right] \tag{62}$$

By expanding the approximate Markov process in terms of the stationary probabilities of the unperturbed semi-Markov chain as in (48), approximate expressions for the state occupancy probabilities of the original process can be stated as follows:

$$\hat{\underline{\pi}}^e(t) = \left[(1 - P_{fa})\exp\left\{-\frac{\epsilon \Lambda_1 t}{T}\right\} \; P_{fa}\exp\left\{-\frac{\epsilon \Lambda_1 t}{T}\right\} \;\middle|\; 1 - \exp\left\{-\frac{\epsilon \Lambda_1 t}{T}\right\}\right] \tag{63}$$

The approximate expressions above will be compared to the analytical expressions derived using z-transform techniques.

5.2 Discussion of Results for SCMS-I

This section examines sources of error associated with the approximate technique for a specific set of system parameters: a=0.95, b=0.94, c=0.89, d=0.88 and P_{fa}=0.05. This set of parameters implies a time to detection in the absence of a failure of 16 time steps (3.2 seconds), and a time to a nominal decision in the absence of a failure of 36 time steps (7.2 seconds) for a sample period of 200 milliseconds.

The relative error (in percent), $\Delta_i = |\pi_i(n) - \hat{\pi}_i(n)| / \pi_i(n)$ will be used to compare the approximate and the analytical state occupancy probabilities.

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The approximate state probability time histories, $\hat{\underline{\pi}}(n)$, are compared to those obtained analytically, $\underline{\pi}(n)$, in Figure 2 for each of the three states. These results are for $\epsilon=0.00005$, implying an $MTBF$ of 20,000 time steps (4000 seconds or just over an hour). In this figure, the state probabilities are propagated for a period of one component $MTBF$. Time is normalized by the $MTBF$.

The largest error occurs early, especially in the first class. This is due to the fact that the normalized state probabilities in class 1 have not converged to the class 1 stationary probabilities of the unperturbed semi-Markov chain. For example, at the tenth time step the normalized probabilities in class 1 are

$$\underline{\pi}_N^{(1)}(10) = [0.9817, \quad 0.0183]. \tag{64}$$

These differ substantially from the class 1 stationary probabilities of the unperturbed semi-Markov chain:

$$\underline{\pi}^{(1)} = [0.9500, \quad 0.0500]. \tag{65}$$

The approximate method accurately estimates the state probabilities when the normalized probabilities have converged to the stationary probabilities in each class. This occurs as early as time step 200, and the relative errors for states 1 and 2 have dropped to $\Delta_1 = \Delta_2 = 8.62 \times 10^{-4}\%$, which indicates that the estimate is closely tracking the exact solution. Until time step 200, use of the approximate method is not valid resulting in large relative errors in the state probabilities.

Another source of error is due to non-zero value of ϵ since Theorem 4 describes $\stackrel{\leq}{\hat{\Phi}}(t)$ in the limit as $\epsilon \to 0$. Obviously, the ϵ chosen in Figure 2 was "small enough" because the state probabilities were estimated adequately. Figure 3 examines the class 2 (or state 3) probability at 100%, 50% and 25% of an MTBF for a range of values of ϵ. The relative error decreases markedly with decreasing ϵ for all three choices of mission time. For large ϵ, $(\epsilon > .01)$, the "slow" time scale represented by failure events and the "fast" time scale represented by fault monitoring events are nearly indistinguishable from each other resulting in poor estimates of the state probabilities. In contrast, for small ϵ, $(\epsilon < .001)$ the two time scales are distinct. For $\epsilon=0.00005$, the time to a decision is about 36 seconds and the $MTBF$ is 4000 seconds, or, the "slow" time scale is approximately 100 times slower than
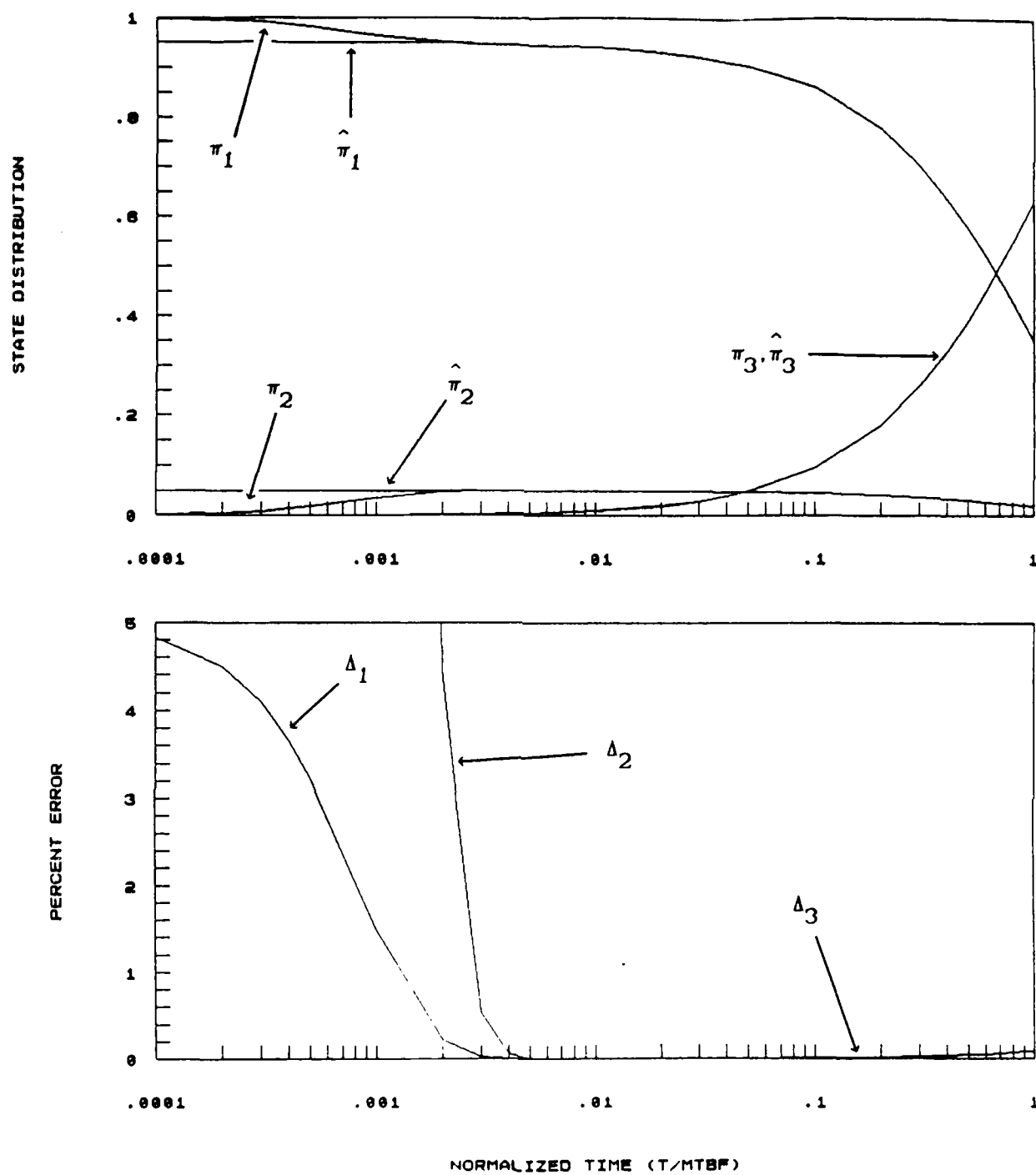
Figure 2: State probability time histories for SCMS-I. (a) Analytical and approximate solutions, (b) Relative error
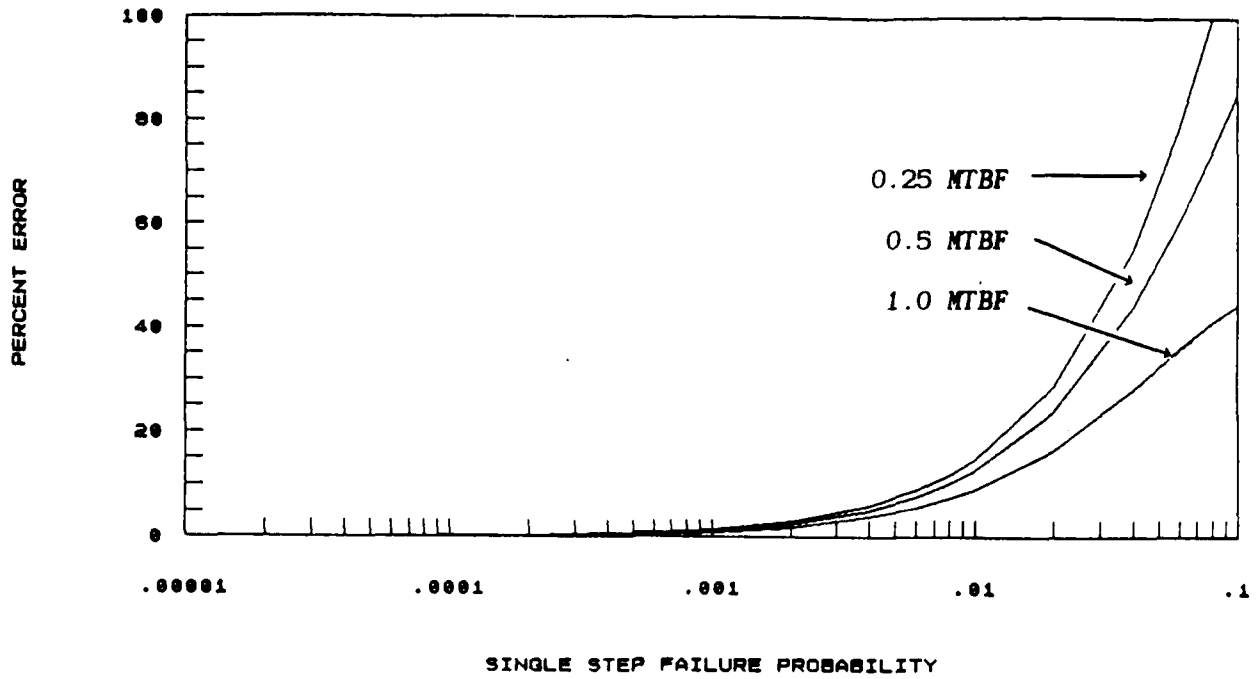
PERCENT ERROR

0.25 MTBF

0.5 MTBF

1.0 MTBF

SINGLE STEP FAILURE PROBABILITY

Figure 3: Sensitivity to ϵ for SCMS-I. The relative error is plotted versus the single-step probability, ϵ, for mission times of one $MTBF$, $0.5 \times MTBF$, and $0.25 \times MTBF$.

the fast time scale. Therefore, to obtain accurate estimates of the state probabilities, it is imperative that the fast and slow time scales be distinctly separated in terms of their mean holding times. A possible rule of thumb is suggested by these results for determining whether the time scales are distinct. That is, compute the holding time of the slowest FDI event. For the approximation to be valid, the $MTBF$ of the fastest failure should be at least 100 times longer than this calculated FDI holding time.

The analytical and approximate solutions of the class 2 probability can also be compared by expanding each in a Taylor series about $\epsilon = 0$. If the two are the same to first order in ϵ then the estimate is a first order perturbation solution. If they differ, this would suggest that an alternative estimate could be derived. Expanding $\pi_2^\epsilon(n)$ and $\hat{\pi}_2^\epsilon(n)$ in Taylor series about $\epsilon=0$:

$$\pi_2^\epsilon(n) = n\epsilon - \frac{1}{2}(n^2 - n)\epsilon^2 + O(\epsilon^2) \tag{66}$$

$$\hat{\pi}_2^\epsilon(n) = n\epsilon - \frac{1}{2}\left\{n^2 - 2n\left[\frac{\partial}{\partial \epsilon}\Lambda_1(\epsilon)\bigg|_{\epsilon=0}\right]\right\}\epsilon^2 + O(\epsilon^2) \tag{67}$$

To first order in ϵ:

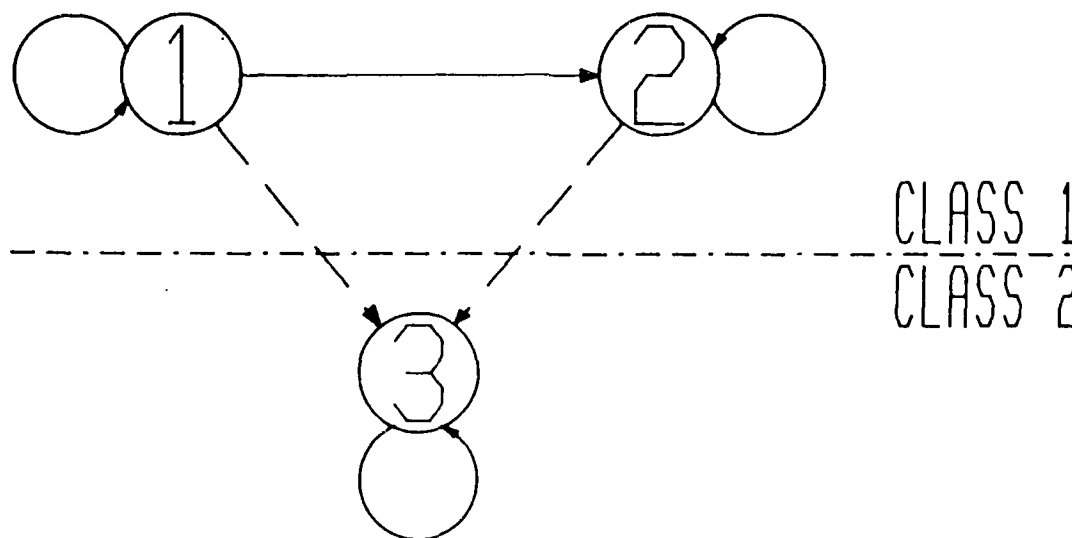$$\pi_2^\epsilon(n) = \hat{\pi}_2^\epsilon(n) = n\epsilon + O(\epsilon) \tag{68}$$

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Figure 4: Semi-Markov transition diagram for SCMS-II

So, the approximation developed in Theorem 4 produces a first order perturbation solution in ϵ for this model. Therefore, the error between the analytical and approximate class 2 probabilities begins with the order ϵ^2 terms. Note that the dominant sec nd order term $(n^2\epsilon^2)$ is also the same. It can be shown [13] that the error is due to a difference in a second order term with a small coefficient, namely a term that is proportional to elapsed time. Although this observation is strongly model dependent, it may also be true for other models as well.

5.3 The SCMS with abbreviated monitoring

A second method of fault monitoring is to deploy a sequential test that monitors the status of a component until a failure is indicated, at which point the sequential test is discontinued. An SCMS of this type will be denoted by SCMS–II.

The states for the semi-Markov model of the SCMS-II are enumerated in Table 1. The semi-Markov transition diagram of the SCMS-II is depicted in Figure 4. The class decomposition of the SCMS-II is similar to SCMS-I. However, in this case, the embedded Markov chain in class 1 is non-ergodic.

The direct form of the core matrix sequence can be developed in the same manner as for

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the SCMS-I. A notable difference is in the transition probabilities out of state 2. Because the fault monitoring test is discontinued upon a failure indication, only failure events cause such transitions. A reset of state 2 occurs when no failure occurs. A transition from state 2 to state 3 occurs only if a failure takes place. Assuming geometrically distributed failures, the core matrix can be stated:

$$
\mathbf{G}^{\epsilon}(n) = \begin{bmatrix} (1-\epsilon)^n f_D^0(n) & (1-\epsilon)^n f_D^0(n) & \epsilon(1-\epsilon)^{n-1} Q_0(n) \\ 0 & (1-\epsilon)\delta(n-1) & \epsilon\,\delta(n-1) \\ 0 & 0 & \delta(n-1) \end{bmatrix} \tag{69}
$$

As for the SCMS-I, $\underline{\pi}(n)$ can be found using z-transforms. The state probability time histories could not be obtained, however, because the partial fraction expansions could only be done numerically. These results are decribed fully in Appendix B of [13]. However, the class probabilities were found and are stated below:

$$
\pi^{\epsilon}(n) = [(1-\epsilon)^n , \ 1 - (1-\epsilon)^n] \tag{70}
$$

Again, these analytical expressions for $\underline{\pi}(n)$ and $\underline{\hat{\pi}}^{\epsilon}(n)$ will be compared to the approximate results derived using the approximate technique in the next section.

To generate the approximate solutions, the core matrix must be placed in standard form. However, all of the required quantities are known based on the manipulations performed for the SCMS-I. The eventual transition probability matrix of the unperturbed semi-Markov chain is obtained by setting $\epsilon = 0$ and ignoring the holding time PMFs:

$$
\mathbf{P} = \begin{bmatrix} 1 - P_{fa} & P_{fa} & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{71}
$$

By raising this matrix to successively higher powers, the stationary interval transition probability matrix can be found, and the embedded stationary probability distribution in partitioned form is:

$$
\underline{\pi}_M = [0 \ \ 1 \ | \ 1] \tag{72}
$$

Because of the model structure, it is clear that the stationary probabilities for each class of the unperturbed semi-Markov chain are: $\underline{\pi} = \underline{\pi}_M$. For this analysis, the time scale factor

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δ is again set equal to ϵ. Finally, $\gamma_{21} = 1$, and $\Lambda_1 = 1$, so that the approximate expressions for the class probabilities can be found:

$$\hat{\pi}^e(t) = \left[\exp(-\frac{t}{T}) , \ 1 - \exp(-\frac{t}{T}) \right].$$ (73)

By expanding the enlarged Markov process in terms of the stationary probabilities of the unperturbed semi-Markov chain, approximate expressions for the state occupancy probabilities of the original process can be stated:

$$\hat{\pi}(t) \approx \left[0 \quad \exp(-\frac{t}{T}) , \ 1 - \exp(-\frac{t}{T}) \right]$$ (74)

5.4   Discussion of results for SCMS-II

The approximate state probability time histories, $\hat{\underline{\pi}}(n)$, are compared to those obtained analytically, $\underline{\pi}(n)$, in Figure 5 for each of the three states. These results are for the same parameter set as SCMS-I. The largest absolute errors occur in estimating state 1 and do not attenuate until 50% of an $MTBF$ has passed. The approximation estimates the state 1 probability to be zero because the class 1 embedded Markov chain is non-ergodic and yields zero for the stationary state 1 probability. The estimated state probabilities in states 2 and 3 are very accurate with relative errors of less than 0.01% for all time steps.

The relative error in state 1 is 100% at all times because the normalized probabilities in class 1 cannot converge to the stationary probabilities of the unperturbed semi-Markov chain. This is because the state 1 probability will never be exactly zero. For example, at the tenth time step in class 1 the normalized state probabilities are

$$\underline{\pi}_N^{(1)}(10) = [0.981175 , \ 0.014111],$$ (75)

and the unperturbed stationary probabilities are:

$$\underline{\pi}^{(1)} = [0 , \ 1].$$ (76)

The approximate method requires that the normalized probabilities converge to the stationary probabilities for each class in order to obtain accurate state probability estimates.

The other source of error is due to non-zero ϵ. In Figure 5, the value of ϵ was small enough to provide accurate results because the state 2 and 3 probabilities were estimated
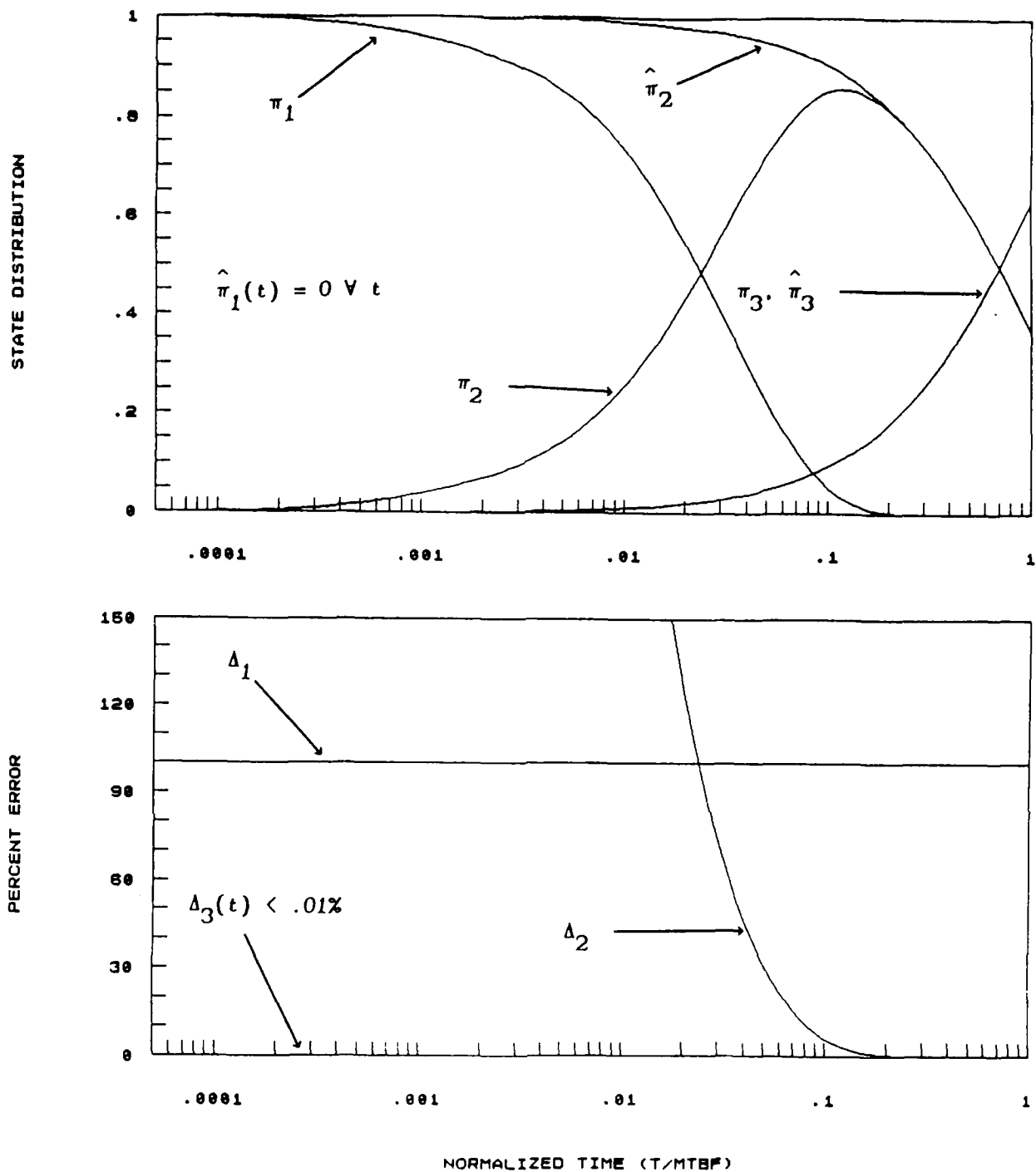
24

Figure 5: State probability time histories for SCMS-II (a) Analytical and approximate solutions. (b) Relative error.
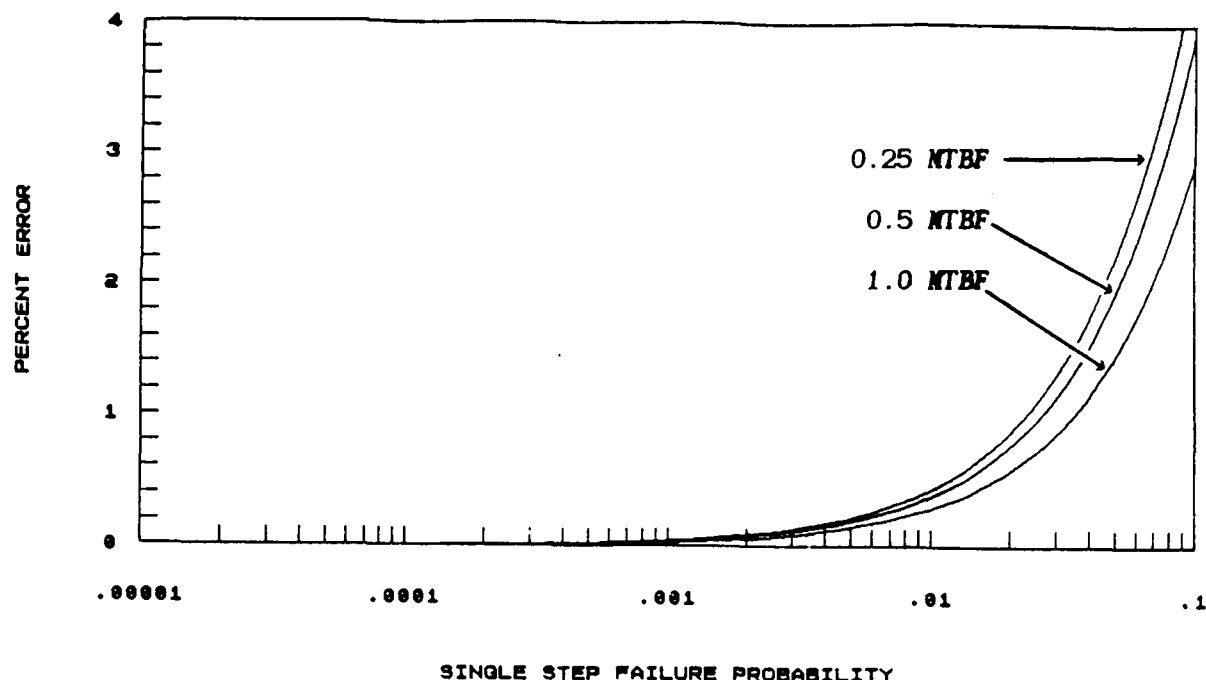
25

PERCENT ERROR (vertical axis label)

0.25 *MTBF* →

0.5 *MTBF*

1.0 *MTBF*

SINGLE STEP FAILURE PROBABILITY

Figure 6: Sensitivity to ϵ for SCMS-II. The relative error is plotted versus the single-step probability, ϵ, for mission times of 1 $MTBF$, $0.5 \times MTBF$, and $0.25 \times MTBF$.

adequately. Figure 6 presents the class 2 (or state 3) occupancy probability for mission times of 100%, 50% and 25% of an $MTBF$ for a range of values of ϵ corresponding to a component $MTBF$ ranging from 4 seconds to 5555 hours. As was the case for the SCMS-I, the relative error decreases markedly with decreasing ϵ for the three choices of mission time. This reiterates the observation that the fast and slow time scales must be distinct in terms of their mean holding times in order to obtain accurate estimates of the state probabilities. This analysis also demonstrates the usefulness of the rule of thumb suggested earlier.

The Taylor series expansions for the analytical and approximate class 2 probability will again be compared. Expanding the class 2 occupancy probability in a Taylor series about $\epsilon = 0$ yields

$$\pi_2^\epsilon(n) = n\epsilon - \frac{1}{2}(n^2 - n)\epsilon^2 + O(\epsilon^2) \tag{77}$$

$$\hat{\pi}_2^\epsilon(n) = n\epsilon - \frac{1}{2}n^2\epsilon^2 + O(\epsilon^2) \tag{78}$$

To first order, $\pi_2^\epsilon(n)$ and $\hat{\pi}_2^\epsilon(n)$ are identical. This proves that the approximate method produces a first order perturbation solution in ϵ for this model. The two expressions begin to differ starting with the ϵ^2 terms, but the dominant second order term $(n^2\epsilon^2)$ is the same.

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Hence, the error can be expressed as:

$$\tilde{\pi}_2^{\epsilon}(n) = \frac{1}{2}n\epsilon^2 + O(\epsilon^2) \tag{79}$$

which is second order in ϵ and proportional to time, which emphasizes the asymptotic nature of the approximation. Again, this observation is model dependent. However, the same behavior was found for the SCMS-I.

6   The SCDR System Model

The single-component dual-redundant (SCDR) system consists of two identical components, a primary and a backup, operating in parallel. An independent sequential test monitors the status of each component. The reliability of this system was evaluated using the approximate technique in [13]. However, in the interest of brevity and clarity, the interested reader is referred to [13].

7   Conclusions

A primary contribution of this work is the extension of Korolyuk's limit theorem for semi–Markov processes to semi– Markov chains in Theorem 1, which describes the conditions under which a perturbed semi–Markov chain can be approximated by an enlarged Markov process. Moreover, Theorem 1 describes how the parameters of the enlarged Markov process are derived from the parameters of the semi–Markov chain.

Two problems arise in applying Theorem 1 to fault tolerant control system (FTCS) models. First, the non-perturbed embedded Markov chains in each class are usually non-ergodic. This was required in Theorem 1, but was relaxed to the existence of the Caesaro limit probabilities in Lemma 2. These were found to exist in Lemma 3 if the embedded Markov chain was either ergodic, or non-ergodic with one and only one unity eigenvalue.

Second, the transition PMFs are typically not functions of the perturbation parameter ϵ. This problem was mitigated by introducing the concept of time scaling in Theorem 4. The form of the transition PMFs was generalized to include those common to FTCS reliability models. This generalization included a dependence on a time scaling factor δ and on a

small parameter ϵ that determined the state space partitioning of the original semi-Markov chain.

Use of the approximate technique was demonstrated by two simple examples. Accurate estimates of the state probabilities were determined for situations where ϵ was "small enough" and where the normalized probabilities in each class had converged to the stationary probabilities of the non-perturbed semi-Markov chain. In the two examples presented, the approximate technique yielded a first order perturbation solution in ϵ to the analytically obtained class probabilities.

The approximation error was found to be insignificant if the slow and fast time scales were distinct. Finally, a rule of thumb was suggested by the error analysis: the slow and fast time scales are distinct if the MTBF of the fastest failure is 1000 times longer than the mean decision time of the slowest FDI event.

8   Acknowledgments

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