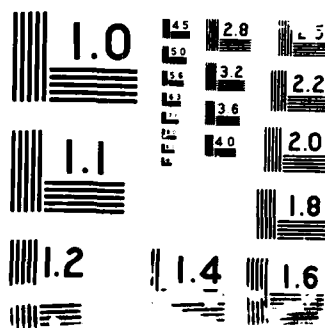


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K^{th} -Order Extended Linearization

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Abstract: For a given nonlinear system, we consider the design of a nonlinear control law such that the following properties hold. First, as in the extended linearization method, linearizations of the closed-loop system about constant operating points of the closed-loop system achieve specified, linear design objectives. Second, the Taylor series expansion of the closed-loop state equation about any constant operating point is such that terms of order 2, 3, ..., k are zero, or at least are minimized in a certain sense. Conditions under which this can be achieved, while simple to state, are restrictive.

1. Introduction

We consider generalizing the extended-linearization design approach for nonlinear systems that has been developed in recent years. [1 - 3] The approach can be described briefly as follows. The nonlinear system to be controlled is represented by its family of linearizations about a family of constant operating points (equilibrium points). Then a family of linear control laws, parameterized by the constant operating point, is computed so that the design objectives are satisfied by the closed-loop linearization family at each operating point. Finally, a nonlinear control law is computed for the original nonlinear system so that the resulting closed-loop system has the designed closed-loop linearization at each operating point. Relying on the accuracy of the linearized model, this nonlinear closed-loop system should satisfy the design objectives in a neighborhood of the family of constant operating points. Indeed, for stable closed-loop systems with slowly-varying inputs, this intuition is supported by the results in [4].

Our generalization of this approach, called k^{th} -order extended linearization, involves requiring that the Taylor series expansion of the closed-loop state equation about any constant operating point in the family satisfies an additional property; namely, that terms of order 2, 3, ..., k are zero, or at least are minimized in a certain sense.

There are at least two motivations for pursuing such a generalization. The first is that the selection of nonlinear control laws corresponding to a parameterized linear control law in an extended-linearization design is highly nonunique, and a natural method of restricting the choice would be valuable. Second, a closed-loop system with, say, second-order terms zero is more accurately described by its linearization (first-order terms) in a sufficiently-small neighborhood of the constant operating point family.

In Section 2, necessary and sufficient conditions on a parameterized linear control law are given for the existence of such a nonlinear control law, and a construction is given for the affirmative case. Related results for nonlinear observers are discussed in Section 3. Both sections refer to the Appendix where existence results and solution constructions for a special type of partial differential equations are given.

The following differentiation notation will be used. If $f(x):R^n \rightarrow R^p$, then $\partial f/\partial x$ denotes the $p \times n$ Jacobian matrix whose (i,j) -entry is the partial derivative $\partial f_i/\partial x_j$. For $n = 1$, ∂ will be replaced by d , and an overdot often will be used if the independent variable is time t . If $f(x,y):R^m \times R^n \rightarrow R^p$, then $\partial f/\partial x$ denotes the $p \times m$ matrix with (i,j) -entry $\partial f_i/\partial x_j$, and $\partial f/\partial y$ is the $p \times n$ matrix with (i,j) -entry $\partial f_i/\partial y_j$. Evaluation of a derivative is indicated in the customary fashion, for example, $(\partial f/\partial x)(x,y)$.

To avoid counting the order of continuous differentiability, all functions will be assumed sufficiently smooth that indicated partial derivatives are continuous.

2. State Feedback

We consider a nonlinear system of the form

$$\dot{x} = f(x, u), \quad x \in R^n, u \in R^m, \quad (2.1)$$

$$y = h(x), \quad y \in R^p \quad (2.2)$$

where $f:R^n \times R^m \rightarrow R^n$, $h:R^n \rightarrow R^p$, $1 \leq m \leq p \leq n < \infty$ with $f(0, 0) = 0$, $h(0) = 0$. Suppose that this system has a con-

start operating point family

$$\{[u(\alpha), x(\alpha), y(\alpha)], \alpha \in \Gamma\} \quad (2.3)$$

where Γ is an open set in R^q , $1 \leq q \leq m$, $0 \in \Gamma$, and $u: \Gamma \rightarrow R^m$, $x: \Gamma \rightarrow R^n$, $y: \Gamma \rightarrow R^p$ with $u(0) = 0$, $x(0) = 0$, $y(0) = 0$. That is, for $\alpha \in \Gamma$,

$$0 = f(x(\alpha), u(\alpha))$$

$$y(\alpha) = h(x(\alpha))$$

We will consider nonlinear feedback control laws of the form

$$u = k(x, w) \quad (2.4)$$

where $k: R^n \times R^m \rightarrow R^m$ with $k(0, 0) = 0$, and $w \in R^m$ is the new input with constant operating point family $w(\alpha)$, $\alpha \in \Gamma$, $w(0) = 0$, such that $u(\alpha) = k(x(\alpha), w(\alpha))$. For $\alpha \in \Gamma$, let

$$\frac{\partial f}{\partial x}(x(\alpha), u(\alpha)) = F(\alpha), \quad \frac{\partial f}{\partial u}(x(\alpha), u(\alpha)) = G(\alpha)$$

$$\frac{\partial h}{\partial x}(x(\alpha)) = H(\alpha)$$

$$\frac{\partial k}{\partial x}(x(\alpha), w(\alpha)) = K(\alpha), \quad \frac{\partial k}{\partial w}(x(\alpha), w(\alpha)) = M(\alpha) \quad (2.5)$$

We assume that, for $\alpha \in \Gamma$, $M(\alpha)$ is nonsingular, $\text{rank } G(\alpha) = m$, and

$$\text{rank} \begin{bmatrix} \frac{\partial x}{\partial \alpha}(\alpha) \\ \frac{\partial u}{\partial \alpha}(\alpha) \end{bmatrix} = \text{rank} \begin{bmatrix} \frac{\partial x}{\partial \alpha}(\alpha) \\ \frac{\partial w}{\partial \alpha}(\alpha) \end{bmatrix} = q \quad (2.6)$$

For simplicity and concreteness, we first consider second-order extended linearization. The parameterized linear gains $K(\alpha)$ and $M(\alpha)$ in (2.5) are specified by the design objectives for the closed-loop linearization, and conditions for the existence of a corresponding nonlinear control law (2.4) are given in [6]. However, we want to impose further conditions on the nonlinear control law in relation to second-order terms in the Taylor series expansion of the closed-loop system state equation (2.1), (2.4) about $[x(\alpha), w(\alpha)]$, $\alpha \in \Gamma$. Using the partitioning notation

$$K(\alpha) = [K_1(\alpha) \cdots K_n(\alpha)]$$

$$K_i(\alpha) = \begin{bmatrix} K_{i1}(\alpha) \\ \vdots \\ K_{im}(\alpha) \end{bmatrix}, \quad i = 1, \dots, n \quad (2.7)$$

and letting, for example

$$K^{x,x}(\alpha) = \frac{\partial^2 k}{\partial x_i \partial x_j}(x(\alpha), w(\alpha))$$

$$K^{w,w}(\alpha) = \frac{\partial^2 k}{\partial w_i \partial w_j}(x(\alpha), w(\alpha))$$

$$K^{x,w}(\alpha) = \frac{\partial^2 k}{\partial w_j \partial x_i}(x(\alpha), w(\alpha)) \quad (2.8)$$

second-order terms can be computed as follows. (For convenience we make use of the notation $u = k(x, w)$, $u(\alpha) = k(x(\alpha), w(\alpha))$.)

$$\begin{aligned} \frac{\partial^2 f(x, k(x, w))}{\partial x_i \partial x_j} \bigg|_{\substack{x(\alpha) \\ w(\alpha)}} &= \frac{\partial f(x, u)}{\partial x_i \partial x_j} \bigg|_{\substack{x(\alpha) \\ u(\alpha)}} \\ &+ \frac{\partial f(x, u)}{\partial u \partial x_i} \bigg|_{\substack{x(\alpha) \\ u(\alpha)}} K_j(\alpha) + \frac{\partial f(x, u)}{\partial u \partial x_j} \bigg|_{\substack{x(\alpha) \\ u(\alpha)}} K_i(\alpha) \\ &+ \sum_{l=1}^m \frac{\partial f(x, u)}{\partial u_l \partial u} \bigg|_{\substack{x(\alpha) \\ u(\alpha)}} K_{li}(\alpha) K_j(\alpha) + G(\alpha) K^{x,x}(\alpha) \end{aligned}$$

Upon setting

$$\frac{\partial^2 f(x, k(x, w))}{\partial x_i \partial x_j} \bigg|_{\substack{x(\alpha) \\ w(\alpha)}} = 0, \quad \alpha \in \Gamma \quad (2.9)$$

there may or may not be a solution for $K^{x,x}(\alpha)$, but the following is either a solution, or is such that the Euclidean norm of the left-hand side of (2.9) is minimized at each $\alpha \in \Gamma$. (The superscript T denotes transpose.)

$$\begin{aligned} K^{x,x}(\alpha) &= -[G^T(\alpha)G(\alpha)]^{-1}G^T(\alpha) \left[\frac{\partial f(x, u)}{\partial x_i \partial x_j} \bigg|_{\substack{x(\alpha) \\ u(\alpha)}} \right. \\ &+ \frac{\partial f(x, u)}{\partial u \partial x_j} \bigg|_{\substack{x(\alpha) \\ u(\alpha)}} K_i(\alpha) + \frac{\partial f(x, u)}{\partial u \partial x_i} \bigg|_{\substack{x(\alpha) \\ u(\alpha)}} K_j(\alpha) \\ &\left. + \sum_{l=1}^m \frac{\partial f(x, u)}{\partial u_l \partial u} \bigg|_{\substack{x(\alpha) \\ u(\alpha)}} K_{li}(\alpha) K_j(\alpha) \right], \quad i, j = 1, \dots, n \quad (2.10) \end{aligned}$$

Adopting a partitioning notation for $M(\alpha)$ similar to (2.7), and setting

$$\frac{\partial^2 f(x, k(x, w))}{\partial x_i \partial w_j} \bigg|_{\substack{x(\alpha) \\ w(\alpha)}} = 0, \quad \alpha \in \Gamma \quad (2.11)$$

$$\frac{\partial^2 f(x, k(x, w))}{\partial w_i \partial w_j} \bigg|_{\substack{x(\alpha) \\ w(\alpha)}} = 0, \quad \alpha \in \Gamma \quad (2.12)$$

gives

$$\begin{aligned} K^{x,w}(\alpha) &= -[G^T(\alpha)G(\alpha)]^{-1}G^T(\alpha) \left[\frac{\partial^2 f(x, u)}{\partial u \partial x_i} \bigg|_{\substack{x(\alpha) \\ u(\alpha)}} M_j(\alpha) \right. \\ &+ \sum_{l=1}^m \frac{\partial^2 f(x, u)}{\partial u_l \partial u} \bigg|_{\substack{x(\alpha) \\ u(\alpha)}} K_l(\alpha) M_{lj}(\alpha) \left. \right], \quad i = 1, \dots, n \\ &\quad j = 1, \dots, m \quad (2.13) \end{aligned}$$

$$K^{w,w}(\alpha) = -[G^T(\alpha)G(\alpha)]^{-1}G^T(\alpha)$$

$$\sum_{l=1}^m \frac{\partial^2 f(x, u)}{\partial u_l \partial u} \bigg|_{\substack{x(\alpha) \\ u(\alpha)}} M_{li}(\alpha) M_j(\alpha), \quad i, j = 1, \dots, m \quad (2.14)$$

Equations (2.10), (2.13) and (2.14) specify $K^{x,x}(\alpha)$, $K^{x,w}(\alpha)$ and $K^{w,w}(\alpha)$ to cancel, or minimize, second-order terms in the Taylor series expansion of the closed-loop nonlinear state equation (2.1), (2.4) about $[x(\alpha), w(\alpha)]$. Now the question is whether there exists a $k(x, w)$ with such second-order partial derivatives. That is, we need to consider the existence of a $k(x, w)$ that satisfies the set of partial differential equations for $\alpha \in \Gamma$:

$$k(x(\alpha), w(\alpha)) = u(\alpha) \quad (2.15)$$

$$\frac{\partial k}{\partial x}(x(\alpha), w(\alpha)) = K(\alpha) \quad (2.16)$$

$$\frac{\partial k}{\partial w}(x(\alpha), w(\alpha)) = M(\alpha) \quad (2.17)$$

$$\frac{\partial^2 k}{\partial x_i \partial x_j}(x(\alpha), w(\alpha)) = K^{x_i x_j}(\alpha), \quad i, j = 1, \dots, n \quad (2.18)$$

$$\frac{\partial^2 k}{\partial w_j \partial x_i}(x(\alpha), w(\alpha)) = K^{x_i w_j}(\alpha), \quad i = 1, \dots, n, j = 1, \dots, m \quad (2.19)$$

$$\frac{\partial^2 k}{\partial w_i \partial w_j}(x(\alpha), w(\alpha)) = K^{w_i w_j}(\alpha), \quad i, j = 1, 2, \dots, m \quad (2.20)$$

with

$$u(0) = 0, \quad x(0) = 0, \quad w(0) = 0$$

By Lemma A.1 in the Appendix, and Corollary 3.1 in [6], we can state the following result.

Proposition 2.1: There exists a solution $k(x, w)$ for the set of partial differential equations (2.15)-(2.20) if and only if, for $\alpha \in \Gamma$,

$$\frac{\partial u}{\partial \alpha}(\alpha) = K(\alpha) \frac{\partial x}{\partial \alpha}(\alpha) + M(\alpha) \frac{\partial w}{\partial \alpha}(\alpha) \quad (2.21)$$

$$K^{x_i x_j}(\alpha) = K^{x_j x_i}(\alpha), \quad i, j = 1, \dots, n \quad (2.22)$$

$$K^{w_i w_j}(\alpha) = K^{w_j w_i}(\alpha), \quad i, j = 1, \dots, m \quad (2.23)$$

$$K^{x_i w_j}(\alpha) = K^{x_j w_i}(\alpha), \quad i = 1, \dots, n, j = 1, \dots, m \quad (2.24)$$

$$\frac{\partial K_i}{\partial \alpha}(\alpha) = \sum_{j=1}^n K^{x_i x_j}(\alpha) \frac{\partial x_j}{\partial \alpha}(\alpha) + \sum_{j=1}^m K^{x_i w_j}(\alpha) \frac{\partial w_j}{\partial \alpha}(\alpha) \quad (2.25)$$

$$\frac{\partial M_i}{\partial \alpha}(\alpha) = \sum_{j=1}^n K^{x_j w_i}(\alpha) \frac{\partial x_j}{\partial \alpha}(\alpha) + \sum_{j=1}^m K^{w_j w_i}(\alpha) \frac{\partial w_j}{\partial \alpha}(\alpha) \quad (2.26)$$

where $x_j(\alpha)$ and $w_j(\alpha)$ are the j^{th} entries of $x(\alpha)$ and $w(\alpha)$, respectively. Furthermore if these conditions are satisfied, one solution is given by

$$\begin{aligned} k(x, w) = & u(z^{-1}(z)) \\ & + K(z^{-1}(z))[x - x(z^{-1}(z))] + M(z^{-1}(z))[w - w(z^{-1}(z))] \\ & + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n K^{x_i x_j}(z^{-1}(z))[x_i - x_i(z^{-1}(z))][x_j - x_j(z^{-1}(z))] \\ & + \sum_{i=1}^n \sum_{j=1}^m K^{x_i w_j}(z^{-1}(z))[x_i - x_i(z^{-1}(z))][w_j - w_j(z^{-1}(z))] \\ & + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m K^{w_i w_j}(z^{-1}(z))[w_i - w_i(z^{-1}(z))][w_j - w_j(z^{-1}(z))] \end{aligned} \quad (2.27)$$

where $z(\alpha)$, $z: \Gamma \rightarrow R^q$ is an invertible function which is formed by selecting the components of $x(\alpha)$ and $w(\alpha)$ corresponding to the q linearly independent rows in (2.6), and $z \in R^q$ consists of corresponding components of x and w .

Note that (2.21) is the condition that $K(\alpha)$ and $M(\alpha)$

yield a feedback linearization family, [6], and (2.22), (2.23), (2.24) are satisfied by the smoothness assumption on $k(x, w)$. So the only conditions related to the cancellation, or minimization of second-order terms are (2.25) and (2.26). To consider these further, substitute (2.10), (2.13) and (2.14) into (2.25) and (2.26). After some manipulations, (2.25) and (2.26) can be simplified to, respectively,

$$G^T(\alpha) \frac{\partial}{\partial \alpha_s} [F(\alpha) + G(\alpha)K(\alpha)] = 0, \quad s = 1, \dots, q \quad (2.28)$$

$$G^T(\alpha) \frac{\partial}{\partial \alpha_s} [G(\alpha)M(\alpha)] = 0, \quad s = 1, \dots, q \quad (2.29)$$

If there exists a $k(x, w)$ such that second-order terms in the Taylor series expansion of the closed-loop state equation about $[x(\alpha), w(\alpha)]$ are zero, then (2.9), (2.11) and (2.12) hold. Therefore

$$\begin{aligned} \frac{\partial}{\partial \alpha_s} [F(\alpha) + G(\alpha)K(\alpha)] &= \frac{\partial}{\partial \alpha_s} \left[\frac{\partial}{\partial x} f(x, k(x, w)) \right]_{x(\alpha)} \\ &= \sum_{i=1}^n \left[\frac{\partial}{\partial x_i} \frac{\partial f(x, k(x, w))}{\partial x} \right]_{x(\alpha)} \frac{\partial x_i}{\partial \alpha_s}(\alpha) \\ &\quad + \sum_{k=1}^m \left[\frac{\partial}{\partial w_k} \frac{\partial f(x, k(x, w))}{\partial x} \right]_{x(\alpha)} \frac{\partial w_k}{\partial \alpha_s}(\alpha) \\ &= 0, \quad s = 1, \dots, q \end{aligned}$$

and similarly,

$$\frac{\partial}{\partial \alpha_s} [G(\alpha)M(\alpha)] = 0, \quad s = 1, \dots, q$$

Corollary 2.2: Suppose that there exist solutions $K^{x_i x_j}(\alpha)$, $K^{x_i w_j}(\alpha)$ and $K^{w_i w_j}(\alpha)$ to the linear algebraic equations (2.9), (2.11) and (2.12), respectively. Then there exists a $k(x, w)$ such that second-order terms vanish in the Taylor series expansion of the closed-loop state equation (2.1), (2.4) about $[x(\alpha), w(\alpha)]$ if and only if, for $\alpha \in \Gamma$,

$$\frac{\partial u}{\partial \alpha}(\alpha) = K(\alpha) \frac{\partial x}{\partial \alpha}(\alpha) + M(\alpha) \frac{\partial w}{\partial \alpha}(\alpha) \quad (2.30)$$

$$F(\alpha) + G(\alpha)K(\alpha) = \text{constant matrix} \quad (2.31)$$

$$G(\alpha)M(\alpha) = \text{constant matrix} \quad (2.32)$$

It is straightforward to derive conditions, similar to (2.21) through (2.26), for canceling, or minimizing all expansion terms of order 2 through k ($k \geq 2$) by using Lemma A.2. Of course, the conditions become more restrictive as k increases. Solving algebraic equations similar to those in (2.9), (2.11) and (2.12), and substituting into the conditions for canceling or minimizing all expansion terms of order 2 through k , yields the following observation.

Remark 2.3: Suppose there exists a $k(x, w)$ such that all terms of order 2 through k ($k \geq 2$) in the Taylor series expansion of the closed-loop state equation (2.1), (2.4)

about $[x(\alpha), w(\alpha)]$ are zero. Then there exists another $k(x, w)$ such that, in addition, the $k+1^{\text{st}}$ -order terms are zero, or at least are minimized. Whether the $k+1^{\text{st}}$ -order terms are zero or minimized solely depends on whether there exist solutions to linear algebraic equations analogous to those in (2.9), (2.11) and (2.12).

3. State Observation

Using the idea of extended linearization, various forms of nonlinear observers have been proposed for the nonlinear system (2.1), (2.2). [1,7] These nonlinear observers are obtained through different methods of constructing nonlinear systems corresponding to the parameterized linear system

$$\begin{aligned} \frac{d}{dt}[\hat{x} - \hat{x}(\alpha)] &= [F(\alpha) + N(\alpha)H(\alpha)][\hat{x} - x(\alpha)] \\ &+ G(\alpha)[u - u(\alpha)] - N(\alpha)[y - y(\alpha)], \quad \hat{x} \in R^n \\ \hat{y} - \hat{y} &= H(\alpha)[\hat{x} - x], \quad \hat{y} \in R^p \end{aligned} \quad (3.1)$$

where $\hat{x}(\alpha) = x(\alpha)$, $\hat{y}(\alpha) = y(\alpha)$, and $N: \Gamma \rightarrow R^{n \times p}$ is chosen such that $F(\alpha) + N(\alpha)H(\alpha)$ has desired eigenvalues. Clearly (3.1) is precisely the form of a parameterized linear observer for the linearization family of the nonlinear system (2.1), (2.2). It is not hard to show that a nonlinear system corresponding to (3.1) always exists. [7]

We consider the problem of constructing a nonlinear system from (3.1), that is, a nonlinear observer for (2.1), in the general form

$$\begin{aligned} \dot{\hat{x}} &= a(\hat{x}, y, u) \\ \hat{y} &= h(\hat{x}) \end{aligned} \quad (3.2)$$

with the property that second-order terms vanish, or at least are minimized, in the Taylor series expansion of the error equation

$$\frac{d}{dt}(x - \hat{x}) = f(x, u) - a(\hat{x}, h(\hat{x}), u) \quad (3.3)$$

about $\hat{x} = \hat{x}(\alpha) = x(\alpha)$, $x = x(\alpha)$, $u = u(\alpha)$, $\alpha \in \Gamma$. Assuming that $H(\alpha)$ is of full rank p , a straightforward computation shows that setting second-order terms to zero in the Taylor expansion of (3.3) about $[x(\alpha), \hat{x}(\alpha), u(\alpha)]$ gives

$$A^{\hat{x}\hat{x}}(\alpha) = 0, \quad i, j = 1, \dots, n \quad (3.4)$$

$$A^{\hat{x}y}(\alpha) = 0, \quad i = 1, \dots, n, \quad j = 1, \dots, p \quad (3.5)$$

$$A^{\hat{x}u}(\alpha) = 0, \quad i = 1, \dots, n, \quad j = 1, \dots, m \quad (3.6)$$

$$A^{uu}(\alpha) = \frac{\partial^2 f}{\partial u_i \partial u_j}(x(\alpha), u(\alpha)), \quad i, j = 1, \dots, m \quad (3.7)$$

$$\begin{aligned} \sum_{j=1}^p A^{y\mu_j}(\alpha) H_{jk}(\alpha) - \frac{\partial^2 f}{\partial x_k \partial u_i}(x(\alpha), u(\alpha)) &= 0 \\ k = 1, \dots, n, \quad i = 1, \dots, p \end{aligned} \quad (3.8)$$

$$\begin{aligned} \sum_{i,j=1}^p A^{y\mu_j}(\alpha) H_{ik}(\alpha) H_{jl}(\alpha) - \frac{\partial^2 f}{\partial x_k \partial x_l}(x(\alpha), u(\alpha)) \\ - N(\alpha) \frac{\partial^2 h}{\partial x_k \partial x_l}(x(\alpha)) = 0, \quad k, l = 1, \dots, n \end{aligned} \quad (3.9)$$

where $H_{ij}(\alpha)$ is the (i, j) -entry of $H(\alpha)$ and, similar to the notation $K^{x\mu_j}(\alpha)$ in last section, $A^{**}(\alpha)$'s are second-order partial derivatives of $a(\hat{x}, y, u)$ evaluated at $[\hat{x}(\alpha), y(\alpha), u(\alpha)]$. Applying Lemma A.1, necessary and sufficient conditions for the existence of a $a(\hat{x}, y, u)$ such that (3.4), (3.5), (3.6), (3.7) are satisfied are

$$A^{y\mu_j}(\alpha) = A^{y\mu_k}(\alpha), \quad i, j = 1, \dots, p \quad (3.10)$$

$$F(\alpha) + N(\alpha)H(\alpha) = \text{a constant matrix} \quad (3.11)$$

$$\begin{aligned} \frac{\partial N_i}{\partial \alpha}(\alpha) &= -[A^{y\mu_1}(\alpha) \cdots A^{y\mu_p}(\alpha)]H(\alpha) \frac{\partial x}{\partial \alpha}(\alpha) \\ &- [A^{y\mu_1}(\alpha) \cdots A^{y\mu_p}(\alpha)] \frac{\partial u}{\partial \alpha}(\alpha), \quad i = 1, \dots, p \end{aligned} \quad (3.12)$$

where $N_i(\alpha)$ is the i^{th} column of $N(\alpha)$. Obviously, these conditions are restrictive. In addition, the algebraic equations (3.8) and (3.9) must be satisfied. This situation indicates that second-order extended linearization is more restrictive in the observer case than in the state feedback case. Of course, the usual approach is to use the plant dynamics $f(x, u)$ in forming $a(\hat{x}, y, u)$, as in [1,7], and this further restricts the possibility for second-order extended linearization.

4. Conclusion

In the state feedback case, a nonlinear control law that yields zero second-order terms in the closed-loop state equation, for example, must be such that the closed-loop linearization family is independent of the closed-loop operating point. This is a stringent requirement, and it indicates that a different approach is needed to further specify desirable properties of nonlinear control laws arising from the extended-linearization design method. For observer design, the requirements are even more restrictive. When second-order terms can only be minimized, then the conditions are not so simple to state, but are restrictive none the less.

5. Appendix

Suppose we are given a set of functions $g^0(\alpha); g^i(\alpha)$, $i = 1, \dots, n$; $g_{ij}^2(\alpha)$, $i, j = 1, \dots, n$, from the open set $0 \in \Gamma \subset R^q$ to R such that $g^0(0) = 0$. Suppose also that $x: \Gamma \rightarrow R^n$ is given with $x(0) = 0$ and $\text{rank } \partial x / \partial \alpha(\alpha) = q$, for $\alpha \in \Gamma$. We consider the following partial differential equation for $v(x)$, $v: R^n \rightarrow R$,

$$v(x(\alpha)) = g^0(\alpha) \quad (A.1)$$

$$\frac{\partial v}{\partial x_i}(x(\alpha)) = g_i^1(\alpha), \quad i = 1, \dots, n \quad (A.2)$$

$$\frac{\partial^2 v}{\partial x_i \partial x_j}(x(\alpha)) = g_{ij}^2(\alpha), \quad i, j = 1, \dots, n \quad (\text{A.3})$$

Letting

$$g^1(\alpha) = [g_1^1(\alpha) \ g_2^1(\alpha) \ \dots \ g_n^1(\alpha)] \quad (\text{A.4})$$

$$g_i^2(\alpha) = [g_{i1}^2(\alpha) \ g_{i2}^2(\alpha) \ \dots \ g_{in}^2(\alpha)], \quad i = 1, \dots, n \quad (\text{A.5})$$

for notational convenience, we have the following result.

Lemma A.1: There exists a $v(x)$ satisfying (A.1) - (A.3) if and only if, for $\alpha \in \Gamma$,

$$g_{ij}^2(\alpha) = g_{ji}^2(\alpha), \quad i, j = 1, \dots, n \quad (\text{A.6})$$

$$\frac{\partial g^0}{\partial \alpha}(\alpha) = g^1(\alpha) \frac{\partial x}{\partial \alpha}(\alpha) \quad (\text{A.7})$$

$$\frac{\partial g_i^1}{\partial \alpha}(\alpha) = g_i^2(\alpha) \frac{\partial x}{\partial \alpha}(\alpha), \quad i = 1, \dots, n \quad (\text{A.8})$$

Proof: (Necessity) The conditions (A.7) and (A.8) follow directly from calculating $\partial g^0 / \partial \alpha(\alpha)$ and $\partial g_i^1 / \partial \alpha(\alpha)$ using (A.1), (A.2), (A.3), and the chain rule, and (A.6) follows from the smoothness assumption on $v(x)$.

(Sufficiency) Since $\text{rank } \partial x / \partial \alpha(\alpha) = q$ we can form an invertible function $z(\alpha)$, $z: R^q \rightarrow R^q$ by choosing the components of $x(\alpha)$ corresponding to the q linearly independent rows in $\partial x / \partial \alpha(\alpha)$. Let z be the vector consisting of corresponding components of x . We claim that

$$v(x) = g^0(z^{-1}(z)) + g^1(z^{-1}(z))[x - x(z^{-1}(z))] + \frac{1}{2} \sum_{i_1, i_2=1}^n g_{i_1 i_2}^2(z^{-1}(z))[x_{i_1} - x_{i_1}(z^{-1}(z))][x_{i_2} - x_{i_2}(z^{-1}(z))] \quad (\text{A.9})$$

is a solution. Obviously (A.1) is satisfied, and (A.2) can be shown by using (A.7). To show that (A.3) is true, let $O(\epsilon)$ denote terms of the same order as ϵ as $\epsilon \rightarrow 0$, and compute as follows.

$$\begin{aligned} \frac{\partial^2 v}{\partial x_i \partial x_j}(x) &= \frac{\partial}{\partial x_i} \left[\frac{\partial g^0}{\partial z^{-1}(z)}(z^{-1}(z)) \frac{\partial z^{-1}}{\partial x_i}(z) \right. \\ &+ \sum_{i_1=1}^n \frac{\partial g_{i_1}^1}{\partial z^{-1}(z)}(z^{-1}(z)) \frac{\partial z^{-1}}{\partial x_i}(z) [x_{i_1} - x_{i_1}(z^{-1}(z))] \\ &+ \sum_{i_1=1}^n g_{i_1}^1(z^{-1}(z)) \left[\frac{\partial x_{i_1}}{\partial x_i} - \frac{\partial x_{i_1}}{\partial z^{-1}(z)}(z^{-1}(z)) \frac{\partial z^{-1}}{\partial x_i}(z) \right] \\ &+ \frac{1}{2} \sum_{i_1, i_2=1}^n \frac{\partial g_{i_1 i_2}^2}{\partial z^{-1}(z)}(z^{-1}(z)) \frac{\partial z^{-1}}{\partial x_i}(z) \\ &\quad [x_{i_1} - x_{i_1}(z^{-1}(z))][x_{i_2} - x_{i_2}(z^{-1}(z))] \\ &+ \frac{1}{2} \sum_{i_1, i_2=1}^n g_{i_1 i_2}^2(z^{-1}(z)) \left[\frac{\partial x_{i_1}}{\partial x_i} - \frac{\partial x_{i_1}}{\partial z^{-1}(z)}(z^{-1}(z)) \frac{\partial z^{-1}}{\partial x_i}(z) \right] \\ &\quad [x_{i_2} - x_{i_2}(z^{-1}(z))] \\ &+ \frac{1}{2} \sum_{i_1, i_2=1}^n g_{i_1 i_2}^2(z^{-1}(z)) [x_{i_1} - x_{i_1}(z^{-1}(z))] \end{aligned}$$

$$\begin{aligned} &\left[\frac{\partial x_{i_2}}{\partial x_i} - \frac{\partial x_{i_2}}{\partial z^{-1}(z)}(z^{-1}(z)) \frac{\partial z^{-1}}{\partial x_i}(z) \right] \\ &= \sum_{i_1=1}^q \sum_{i_2=1}^q \frac{\partial^2 g^0}{\partial z_{i_1}^{-1} \partial z_{i_2}^{-1}}(z^{-1}(z)) \frac{\partial z_{i_1}^{-1}}{\partial x_i}(z) \frac{\partial z_{i_2}^{-1}}{\partial x_j}(z) \\ &+ \frac{\partial g^0}{\partial z^{-1}(z)}(z^{-1}(z)) \frac{\partial^2 z^{-1}}{\partial x_j \partial x_i}(z) \\ &+ \sum_{i_1=1}^n \frac{\partial g_{i_1}^1}{\partial z^{-1}(z)}(z^{-1}(z)) \frac{\partial z^{-1}}{\partial x_i}(z) \left[\frac{\partial x_{i_1}}{\partial x_j} - \frac{\partial x_{i_1}}{\partial z^{-1}(z)}(z^{-1}(z)) \frac{\partial z^{-1}}{\partial x_j}(z) \right] \\ &+ \sum_{i_1=1}^n \frac{\partial g_{i_1}^1}{\partial z^{-1}(z)}(z^{-1}(z)) \frac{\partial z^{-1}}{\partial x_j}(z) \\ &\quad \left[\frac{\partial x_{i_1}}{\partial x_i} - \frac{\partial x_{i_1}}{\partial z^{-1}(z)}(z^{-1}(z)) \frac{\partial z^{-1}}{\partial x_i}(z) \right] \\ &+ \sum_{i_1=1}^n g_{i_1}^1(z^{-1}(z)) \left[\sum_{i_1=1}^q \sum_{i_2=1}^q \frac{\partial^2 x_{i_1}}{\partial z_{i_1}^{-1} \partial z_{i_2}^{-1}}(z^{-1}(z)) \right. \\ &\quad \left. \frac{\partial z_{i_1}^{-1}}{\partial x_j}(z) \frac{\partial z_{i_2}^{-1}}{\partial x_i}(z) - \frac{\partial x_{i_1}}{\partial z^{-1}(z)}(z^{-1}(z)) \frac{\partial^2 z^{-1}}{\partial x_j \partial x_i}(z) \right] \\ &+ \sum_{i_1=1}^n \sum_{i_2=1}^n g_{i_1 i_2}^2(z^{-1}(z)) \left[\frac{\partial x_{i_1}}{\partial x_i} - \frac{\partial x_{i_1}}{\partial z^{-1}(z)}(z^{-1}(z)) \frac{\partial z^{-1}}{\partial x_i}(z) \right] \\ &\quad \left[\frac{\partial x_{i_2}}{\partial x_j} - \frac{\partial x_{i_2}}{\partial z^{-1}(z)}(z^{-1}(z)) \frac{\partial z^{-1}}{\partial x_j}(z) \right] \\ &+ O(x - x(z^{-1}(z))) \end{aligned}$$

Evaluating at $x = x(\alpha)$ and using (A.7), (A.8) will give (A.3) to complete the proof. $\square$

In fact, Lemma A.1 can be generalized easily to the case where all partial derivatives of $v(x)$, through order k , are specified. That is

$$\begin{aligned} v(x(\alpha)) &= g^0(\alpha) \\ \frac{\partial v}{\partial x_i}(x(\alpha)) &= g_i^1(\alpha), \quad i = 1, \dots, n \\ \frac{\partial^2 v}{\partial x_{i_1} \partial x_{i_2}}(x(\alpha)) &= g_{i_1 i_2}^2(\alpha), \quad i_1, i_2 = 1, \dots, n \\ &\vdots \\ \frac{\partial^k v}{\partial x_{i_1} \dots \partial x_{i_k}}(x(\alpha)) &= g_{i_1 \dots i_k}^k(\alpha) \end{aligned}$$

$$i_1, \dots, i_k = 1, \dots, n \quad (\text{A.10})$$

where the notations and assumptions are as in Lemma A.1.

Lemma A.2: There exists a $v(x)$ satisfying (A.10) if and only if, for $\alpha \in \Gamma$,

$g_{i_1 \dots i_k}^l(\alpha)$ is symmetric in $i_1, \dots, i_l$,

$$l = 1, \dots, k; \quad i_1, \dots, i_k = 1, \dots, n$$

$$\frac{\partial g^0}{\partial \alpha}(\alpha) = g^1(\alpha) \frac{\partial x}{\partial \alpha}(\alpha)$$

$$\frac{\partial g_{i_1 \dots i_{k-1}}^{k-1}}{\partial \alpha}(\alpha) = \sum_{i_k=1}^n g_{i_1 \dots i_k}^k(\alpha) \frac{\partial x_{i_k}}{\partial \alpha}(\alpha)$$

$$i_1, \dots, i_{k-1} = 1, \dots, n \quad (\text{A.11})$$

Furthermore, if these conditions are satisfied, one solution is given by

$$\begin{aligned} v(x) &= g^0(z^{-1}(z)) + g^1(z^{-1}(z))[x - x(z^{-1}(z))] \\ &+ \frac{1}{2!} \sum_{i_1=1}^n \sum_{i_2=1}^n g_{i_1 i_2}^2(z^{-1}(z))[x_{i_1} - x_{i_1}(z^{-1}(z))][x_{i_2} - x_{i_2}(z^{-1}(z))] \\ &+ \dots + \\ &+ \frac{1}{k!} \sum_{i_1=1}^n \dots \sum_{i_k=1}^n g_{i_1 \dots i_k}^k(z^{-1}(z)) \prod_{l=1}^k [x_{i_l} - x_{i_l}(z^{-1}(z))] \quad (\text{A.12}) \end{aligned}$$

6. References

- [1] W.T. Baumann, W.J. Rugh, "Feedback Control of Nonlinear Systems by Extended Linearization," *IEEE Transactions on Automatic Control*, Vol. AC-31, No. 1, pp. 40-46, 1986.
- [2] W.J. Rugh, "The Extended-Linearization Approach for Nonlinear Systems Problems," *Algebraic and Geometric Methods in Nonlinear Control Theory*, M. Fliess, M. Hazewinkel, eds., D. Reidel, Dordrecht, Holland, pp. 285-309, 1986.
- [3] C. Reboulet, P. Mouyon, C. Champetier, "About the Local Linearization of Nonlinear Systems," *Algebraic and Geometric Methods in Nonlinear Control Theory*, M. Fliess, M. Hazewinkel, eds., D. Reidel, Dordrecht, Holland, 1986.
- [4] M. Kelemen, "A Stability Property," *IEEE Transactions on Automatic Control*, Vol. AC-31, No. 8, pp. 766-768, 1986.
- [5] J. Wang, W.J. Rugh, "Parameterized Linear Systems and Linearization Families for Nonlinear Systems," *IEEE Transactions on Circuits & Systems*, to appear, 1987.
- [6] J. Wang, W.J. Rugh, "Feedback Linearization Families for Nonlinear Systems with Applications to Input-Output Decoupling," *Technical Report JHU/ECE 86-17.1*, Department of Electrical & Computer Engineering, The Johns Hopkins University, Baltimore, Maryland, 1986.
- [7] W.T. Baumann, "Feedback Control of Multi-Input Nonlinear Systems by Extended Linearization," *Proceedings of the 20th Annual Conference on Information Sciences and Systems*, Princeton University, Princeton, New Jersey, pp. 657-660, 1986.

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