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WRIGHT-PATTERSON AFB OH SCHOOL OF ENGI.. R N PASCHALL

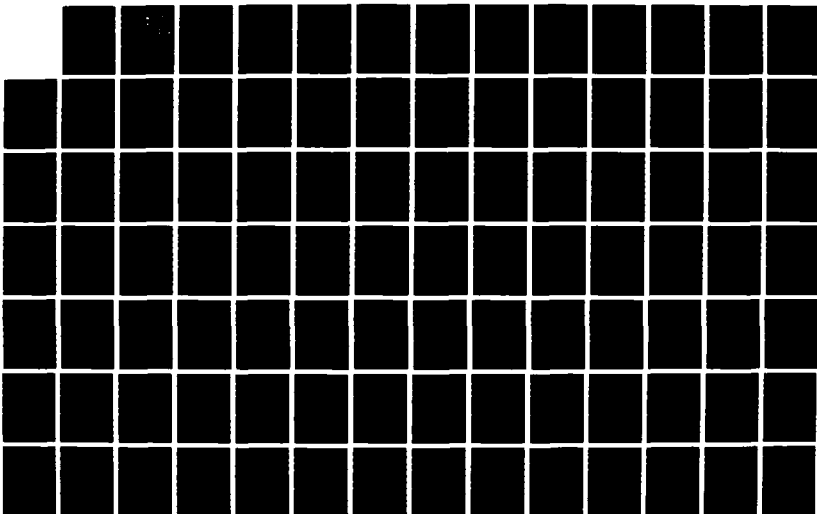
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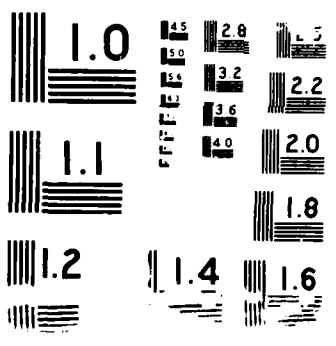
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Robust Finite-Dimensional LQG-Based Controllers
for a Class of Distributed Parameter Systems

DISSERTATION

Randall N. Paschall
Captain

AFIT/DS/ENG/88J-1

DEPARTMENT OF THE AIR FORCE
AIR UNIVERSITY

AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

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Robust Finite-Dimensional LQG-Based Controllers
for a Class of Distributed Parameter Systems

DISSERTATION

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology

Air University

In Partial Fulfillment of the
Requirements for the Degree of
Doctor of Philosophy

Randall N. Paschall, B.S.E.E., M.S.E.E.

Captain

June 1988

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Robust Finite-Dimensional LQG-Based Controllers
for a Class of Distributed Parameter Systems

Randall N. Paschall, B.S.E.E., M.S.E.E.

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TEACHING	<input type="checkbox"/>
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Randall N. Paschall

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List of Symbols

Symbol	page defined
A	5
\hat{A}	89
a	50
A_{cl}	99
A_e	36
A_s	21
A_u	21
$A(\sigma)$	29
$A(\sigma)^{n \times n}$	31
$A_-(\sigma)$	29
$\hat{A}(\sigma)$	29
$\hat{A}_-(\sigma)$	29
$\hat{A}_-^\infty(\sigma)$	30
ΔA	48
$\Delta \tilde{A}$	89
A^k	76
A_c	83
A_δ	97
A_{nom}	97
B	5

$B(\sigma)$	31
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\mathcal{B}	50
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ΔB	48
B^k	76
B_c	83
B_{cl}	98
B_e	36
B_{nom}	97
B_δ	97
C	5
\mathcal{C}	12
ΔC	48
C_o	16
C_c	83
C_{cl}	98
C_δ	97
C_{nom}	97
C^k	76
C^{nzn}	104
C^+	109
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Ψ	88
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Π	75
θ	71
Ξ	68
β	60
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λ_{ij}	39
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LQG	2
LQG/LTR	2
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SISO	1
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Abstract

This dissertation considers the problem of robustly stabilizing systems modeled on infinite-dimensional state spaces using finite-dimensional controllers. In particular, the controllers considered in this research are assumed to be linear quadratic Gaussian (LQG) based controllers.

This research first uses a direct approach to demonstrate the existence of finite-dimensional LQG based controllers that stabilize the nominal system. Once the existence is proven, the rest of the research focuses on ways to analyze the robustness of the controller. It is pointed out that the exponential growth constant of the semigroup generated by the system A operator is not the only measure of robustness, nor is it the best one.

Several types of perturbations are considered, including bounded, relatively bounded, additive, and multiplicative. As a result, several approaches to analyzing robustness are developed. Direct analysis using results from functional analysis is accomplished, followed by a recent approach called the optimal projection equation approach, and then H_∞ techniques are used to develop a sufficient condition for robustness in the presence of multiplicative perturbations of the plant transfer function. It is pointed out that each approach can be used to account for different types of perturbations. No one approach seems able to deal with all perturbation types.

A major development in this research is the new interpretation of the linear quadratic Gaussian / loop transfer recovery technique (LQG/LTR) for the case of reduced order controllers. It is demonstrated that the technique can be interpreted as modeling system uncertainty through the added noise term rather than tuning to recover a desired loop transfer function.

Also contained in this research is a sufficient condition for which the LQG/LTR technique can be extended to the entire class of problems considered. The devel-

opment of the sufficient condition is different than approaches taken by others, and may provide the needed insight to extend the LQG/LTR technique to the class of problems considered without any added assumption being required. A way to approximate the LQG/LTR technique is also given using the results of H. T. Banks.

Robust Finite-Dimensional LQG-Based Controllers for a Class of Distributed Parameter Systems

I. Introduction

1.1 Background

A constant concern of engineers designing control systems is whether or not a feedback controller will maintain stability of a closed-loop system in spite of all possible plant variations, i.e. stability robustness. Plant variations may be the result of modeling errors, disturbances, component failures, or changes in operating conditions. A system that is designed to be stable around a set of nominal conditions may or may not remain stable with plant variations present. It is uncertainty about the plant that forces one to use feedback control instead of open-loop control [43].

Classical stability margins such as gain and phase margins provide a partial description of how much variation a system can tolerate before instability occurs. However, they do not provide a complete description of a system's robustness, as is available through the characterization of the closest distance to the critical point of a Nyquist diagram or Nichols chart, or similar graphic descriptions. For the case of single-input single-output (SISO) systems, several useful design tools exist [25, 49] that enable the designer to determine stability robustness of a control system prior to its implementation. For multiple-input multiple-output (MIMO) systems the tools are not quite as well developed. One measure of stability robustness that has been widely used is the singular value [30, 35]. Though the singular value often leads to overly conservative robustness characteristics, attempts have been made to reduce its conservatism by the use of structured singular values [32]. Nevertheless,

the singular value does provide a designer with a tool by which to evaluate stability robustness of a controller for a MIMO system.

In the case of distributed parameter systems (DPS), very little is known about how to define stability robustness. Most of the current research tries to design controllers that will achieve exponential stability, and then define stability robustness in terms of the magnitude of the exponential time constant [20, 21, 55, 23, 24, 58, 59, 60]. However, as pointed out by example [51], the classical stability margins of a SISO DPS can be improved, but the exponential time constant simultaneously may not change. Thus, it would seem that the exponential time constant is not the best measure of stability robustness. Also, it is not easy to relate model uncertainty to the exponential time constant, but for SISO systems with rational transfer functions, gain and phase margins can be related to model uncertainty. Some recent results [47] have shown that, for exponentially stabilizable and detectable systems, it is possible to equate exponential stability to a Banach algebra type of input-output stability which can be related to the singular value concept [16]. For MIMO systems, the singular value can be related to model uncertainty, and may be a better measure of stability robustness than the exponential time constant.

Much work has been done in trying to develop stabilizing finite-dimensional controllers for infinite-dimensional systems [2, 3, 4, 5, 6, 20, 55, 23, 58, 59, 60], but in every case the stability margin used is the exponential time constant. Recent results [51] have shown how to extend the Linear Quadratic Gaussian/Loop Transfer Recovery (LQG/LTR) technique to a class of DPS. However, the result is an infinite-dimensional controller which in general is difficult to construct unless the controller is a time-delay system. It is the goal of this research to develop a procedure by which finite-dimensional linear quadratic Gaussian (LQG) based controllers can be constructed which will asymptotically achieve stability robustness.

1.2 Scope

This research will look at four techniques to design robust finite-dimensional controllers for infinite-dimensional systems. First, Schumacher's [58, 59, 60] direct approach will be used to design a finite-dimensional LQG-based controller to achieve a desired exponential stability. Schumacher's work developed conditions that, if satisfied, are sufficient for the existence of a finite-dimensional controller that exponentially stabilizes an infinite-dimensional system. This research extends his work by showing that LQG controllers satisfy these sufficient conditions under the assumption that the eigenfunctions of $(A - BK_c)$ are complete (see Chapter 3 for details). Also, Schumacher did not provide a proof that the approximation of the infinite-dimensional controller obtained using his approach, converges to the desired infinite-dimensional controller as the dimension of the approximation increases toward infinity. This dissertation provides that proof for completeness. It will be shown that the finite-dimensional controller asymptotically converges to the controller that would result from solving the infinite-dimensional algebraic Riccati equations (A.R.E.).

Schumacher's approach does not incorporate robustness into the design, so that one must analyze the robustness of the resulting finite-dimensional controller by using direct analysis of the resulting closed-loop semigroup. It is demonstrated that not only can the exponential time constant be considered a measure of robustness with respect to bounded perturbations, but the semigroup gain constant can also be used as a measure of robustness in the presence of relative bounded perturbations (see Chapter 3, Section 4). The use of the semigroup gain constant as a measure of robustness is a contribution of this research based on theory contained in Pazy's text [54].

Second, because of the limited ability to achieve robustness by analyzing the robustness after the design is accomplished, this research develops sufficient conditions for which the LQG/LTR technique is valid for the systems to be considered in

this research. By using a technique such as LQG/LTR, one is able to incorporate robustness issues into the algebraic Riccati equations which determine the LQG gain operators. The sufficient conditions found in Chapter 4 are developed by using a different approach than that taken by Matson [51] in his work. This different formulation may yield insights into the LQG/LTR technique not before available, so that the technique can be extended to the entire class of problems to be considered. As an alternative, a way to approximate the LQG/LTR technique is developed based on the work of Banks [7]. It is shown that for many approximation schemes (such as modal, spline, Ritz [36], etc.), one can approximate the solution of the infinite-dimensional A.R.E. with a sequence of finite-dimensional A.R.E.s, and therefore use a robustness recovery technique like LQG/LTR to design a robust finite-dimensional controller. However, as will be pointed out, the robustness is with respect to the finite-dimensional model used to approximate the A.R.E. solution. Because of this, this approach will be considered an approximation of the LQG/LTR technique for the infinite-dimensional system.

Third, an approach developed by Bernstein and Hyland [10, 11, 46, 9, 45, 39, 40] called the optimum projection equation (O.P.E.) technique will be extended to allow one to design a reduced order controller that will be robust and will minimize a cost functional associated with the optimal control problem. The extension is based totally on the work by Bernstein and Hyland [10]. The approach allows one to modify the A.R.E. in such a way that robustness is achieved. It will be shown that this technique gives a new insight into the LQG/LTR technique, and therefore allows one to achieve robustness without having to recover a desired transfer function. This new interpretation of the LQG/LTR technique is a contribution of this dissertation.

The fourth approach to designing robust finite-dimensional controllers is the use of H_∞ techniques to develop sufficient conditions for robustness. In this way one will have a tool available by which to evaluate the robustness of a proposed controller, or a way to determine if a controller is acceptable given some knowledge

of the uncertainty one can expect. The work of Curtain [22] will be extended so that one can consider not only additive perturbations as she did, but also multiplicative perturbations as well. Multiplicative perturbations are preferred since the compensated plant transfer function will have the same uncertainty associated with it as the uncompensated system does (see Chapter 6). Multiplicative perturbations are also preferred because they correspond to the type of changes one considers when establishing the classical concepts of gain and phase margins. The sufficient condition developed can be used as an analysis tool by which the finite-dimensional controller can be evaluated, or one can use the approach to design finite-dimensional controllers that are not LQG-based, as demonstrated by Curtain [22].

1.3 Problem Class

The class of problems addressed in this dissertation is the set of infinite-dimensional systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + Gw(t), \quad x_0 = x(0) \in \mathcal{D}(A) \quad (1)$$

$$y(t) = Cx(t) + \eta(t) \quad (2)$$

where the control vector u is in the input space $U = L^2\{[0, \infty); \mathfrak{R}^N\}$, x is an element of a Hilbert space \mathcal{H} , y is an observation vector which is an element the output space $Y = \mathfrak{R}^N$, and w and η are white Gaussian noise terms with realizations in the spaces Y and \mathcal{H} respectively. The strength of the dynamics noise term w is described by the positive semi-definite operator Q_0 , and the strength of the measurement noise term η is described by the strictly positive operator R_f (these operators will be discussed more in Chapter 2). The operator Q_f used in the Kalman filter design will be chosen so that $Q_f = GQ_0G^*$, where G^* denotes the adjoint of G . $x(t)$ will be denoted simply as x (and similarly for the other functions), and the following assumptions are made:

1. A is the infinitesimal generator of a C_0 semigroup (i.e. strongly continuous) $T(t)$ on a real separable Hilbert space (Hilbert space with a countable orthonormal basis) \mathcal{H} [24]. Separable Hilbert spaces are needed in Chapter 3 to prove the existence of finite-dimensional LQG-based controllers, and in Chapter 4 to establish sufficient conditions for extending the LQG/LTR technique.
2. B is a bounded linear operator from \mathbb{R}^N to \mathcal{H} .
3. C is a bounded linear operator from \mathcal{H} to \mathbb{R}^N .
4. G is a bounded linear operator from the Hilbert space \mathcal{H} to \mathcal{H} .
5. The spectrum of A (denoted $\sigma(A)$) is discrete.
6. The system is exponentially stabilizable and detectable [24].
7. The eigenvectors of A are complete.
8. The system is minimum phase (i.e. no transmission zeros in the right-half plane).
9. A satisfies the spectrum decomposition assumption [59].
10. The restriction of A to the stable subspace \mathcal{H}_s satisfies the spectrum determined growth assumption [24] and generates an exponentially stable semigroup.

The first eight assumptions were made by Matson in his research [51]. The last two assumptions are added by Schumacher [59] to allow finite-dimensional compensators to be designed. Therefore, this research has made these same assumptions. Assumption (1) is a standard assumption made in the semigroup approach to optimal control for infinite-dimensional systems. The fact that the Hilbert space \mathcal{H} is real is not restrictive since every real Hilbert space has a complex extension, say \mathcal{H}_c , that is related to \mathcal{H} in the same way that \mathbb{C}^N is related to \mathbb{R}^N (where \mathbb{C}^N is an

N -dimensional complex space, and \mathfrak{R}^N is an N -dimensional space over the real numbers). Also, most of the system models for physical systems can be well modeled on real Hilbert spaces. The results of this dissertation are based on real Hilbert spaces. Although they are true for real Hilbert spaces, it is believed that they are true, or can be easily extended, to complex Hilbert spaces. Assumption (2) requires that the control enters the system in a distributed way and not via boundary control. Though the infinite-dimensional phenomenon of boundary control is interesting, in many real situations control is applied through a finite number of actuators acting at specific points in the system. Thus, distributed control, described using a bounded B operator on a finite-dimensional input space, models many real problems. Similarly, assumption (3) requires that observations of the system state occur in a distributed way and not via point observations. Since observations normally occur through a finite number of sensors, which typically are not point sensors, this is not a restrictive assumption. Assumptions (2) and (3) are both commonly made in control theory. Assumption (4) is made in order to apply results available in LQG control theory. One would not expect noise to enter a system in an unbounded fashion.

Assumption (5) is another standard assumption in LQG theory and is satisfied when the resolvent $(\lambda I - A)^{-1}$ is a compact operator for some $\lambda \in \rho(A)$ (where $\rho(A)$ denotes the resolvent set of the operator A). This includes a large class of systems described by partial differential equations on a bounded domain or by functional differential operators describing delay equations [59]. Within this class there are generators with only finitely many eigenvalues to the right of a vertical line in the complex plane, and there are generators with infinitely many eigenvalues to the right of any vertical line. As a rule, the operator A is an elliptic operator and has eigenvalues whose real parts tend to $-\infty$, for parabolic and retarded systems, and so these types of equations will generally satisfy assumption (9). On the other hand, hyperbolic equations and equations of neutral type have infinitely many eigenvalues

in a vertical strip , and so they will satisfy assumption (9) only if this strip is to the left of a desired ω before compensation is applied [59].

Assumption (6) is a standard assumption and includes a large number of linear time-invariant systems [51]. The same assumption is usually made in finite-dimensional situations. Assumption (7) will be necessary to prove the existence of a finite-dimensional controller. This assumption is quite common in partial differential equations, and solving equations using the method of eigenfunction expansion is based on it [59]. Assumption (8) is made because LQG/LTR theory has only been established for minimum phase systems. Finally, assumption (10) is necessary to use results developed by Schumacher in proving the existence of finite-dimensional controllers. Also, this assumption allows one to determine stabilizability and detectability of the entire system by restricting attention to a subspace of the entire state space. The next section gives an example of a problem that satisfies the assumptions.

1.4 Example

Ignoring the noise terms, an example of a DPS that satisfies these assumptions is the one dimensional heat equation [59], which is a parabolic equation. Let the heat equation with distributed control and observation have the following form:

$$\frac{\partial}{\partial t}\Theta(x,t) = \frac{1}{\pi^2} \frac{\partial^2}{\partial x^2}\Theta(x,t) + \sum_{i=1}^m b_i(x)u_i(t) \quad t \geq 0; \quad 0 \leq x \leq 1 \quad (3)$$

$$y_i(t) = \int_0^1 c_i(x)\Theta(x,t)dx \quad i = 1, 2, \dots, n \quad (4)$$

where $b_i(x)$ and $c_i(x)$ are known functions describing the influence of the actuators and sensors respectively, and with boundary and initial conditions given by:

$$\frac{\partial}{\partial x}\Theta(0,t) = \frac{\partial}{\partial x}\Theta(1,t) = 0 \quad (5)$$

$$\Theta(x,0) = \Theta_o(x) \quad (6)$$

For the state space, let $\mathcal{H} = L^2(0, 1)$. The domain of the operator A is defined by

$$D(A) = \{\Theta \in \mathcal{H} \mid \frac{\partial^2}{\partial x^2} \Theta \in \mathcal{H}; \frac{\partial}{\partial x} \Theta(0) = \frac{\partial}{\partial x} \Theta(1) = 0\} \quad (7)$$

A generates a C_0 semigroup and has a discrete spectrum with simple eigenvalues at $-i^2$ where $i = 0, 1, 2, \dots$. The corresponding normalized eigenvectors are given by

$$\phi_i(x) = \begin{cases} 1 & i = 0 \\ \sqrt{2} \cos i\pi x & i = 1, 2, \dots \end{cases} \quad (8)$$

and they are a complete basis for $L^2(0, 1)$.

Note that the systems under consideration are models involving process noise w and measurement noise η which are assumed to be zero mean, white Gaussian noises with strengths $Q_0 \in \mathcal{L}(Y)$ and $R_f \in \mathcal{L}(\mathbb{R}^N)$ respectively (i.e. they are bounded linear operators). Q_0 is a self-adjoint positive operator and R_f is a self-adjoint, strictly positive operator. By including these noise terms, one can account for the fact that the system state is never known exactly, and so an estimator or observer will be needed since our methodology is tied to state feedback.

This research will assume that a Kalman filter is used to estimate the system state. When a Kalman filter is used to estimate the system state and provide that estimate as the input to the LQ controller, the stability margins of the resulting closed-loop system may become arbitrarily small [31]. The LQG/LTR technique gives a designer one way to regain desired stability robustness of the corresponding full-state feedback controller. However, this technique has not been extended to the entire class of problems considered in this research. Therefore, other approaches will be considered as well.

1.5 Summary of Remaining Chapters

This dissertation is organized as follows. Chapter 2 contains the mathematical background theory that is applicable to this research. Definitions and theory are presented that are used in chapters to follow.

Chapter 3 demonstrates that the LQG solution satisfies the hypothesis of Schumacher's approach [59] and therefore finite-dimensional LQG-based controllers exist for the class of problems considered assuming that the eigenfunctions of $(A - BK_c)$ are complete. Schumacher did not consider a specific form for the controller as is done in this research. It is also shown that the finite-dimensional controller converges to the infinite-dimensional LQG solution as the dimension of the controller increases toward ∞ . This proof was not given by Schumacher and is given in this dissertation for completeness. Chapter 3 demonstrates the limited robustness analysis that can be accomplished once a finite-dimensional controller has been designed. It is shown that the exponential time constant can be used as a measure of robustness for bounded perturbations, and the semigroup gain factor can be used as a measure of robustness when relative bounded perturbations are considered. This last fact has not been considered before, and it is a contribution of this dissertation. Chapter 3 also has a simple example problem that demonstrates how to apply Schumacher's approach. It is pointed out in this problem that one must be concerned with the resulting location of the closed-loop poles.

Chapter 4 develops sufficient conditions for extending the LQG/LTR technique to the entire class of problems described in Section 3. The approach taken is different from the one taken by Matson [51], and it is hoped this new approach will yield insights that will allow the LQG/LTR technique to be extended to the entire problem class. Chapter 4 also looks at the approximation procedure of Banks [7]. His approach is used as a basis for an approximation of the LQG/LTR technique for those problems to which Matson [51] could not extend the LQG/LTR technique.

Chapter 5 will consider the optimal projection equation (O.P.E.) approach that has been developed by [10, 46, 9, 39, 40]. The approach will be extended based on the work of Bernstein and Hyland [10] to allow one to put robustness consideration into the design equations. As a result, a broader interpretation of LQG/LTR will be gained. This new interpretation is a contribution of this research.

Chapter 6 will extend the work of Curtain [22] so that one can use H_∞ techniques to evaluate the robustness of a proposed controller in the presence of uncertainty that is modeled as a multiplicative perturbation of the plant transfer function rather than an additive perturbation. Since multiplicative perturbations are often preferred, this extension is seen as a meaningful contribution.

Chapter 7 will contain a simple example problem that will demonstrate the four approaches discussed in this dissertation. The problem will be simple so that a solution can be obtained and is not meant to be representative of a realistic problem. It will demonstrate some of the advantages and disadvantages of each technique.

Finally, Chapter 8 will contain conclusions drawn from this research, and will provide recommendations for future research in the area of finite-dimensional control of infinite-dimensional systems. There are still many areas to study that could produce good fruits. The chapter also contains a proposed design procedure based on the developments of the preceding chapters.

II. Background Material

2.1 Introduction

This chapter contains background material that is needed in the chapters to follow. Section 2.2 contains basic theory from functional analysis as found in [41, 24, 48, 64]. Section 2.3 presents aspects of semigroup theory and is taken from [24, 54]. In Section 2.4 is the material dealing with optimal control and estimator theory. The last section, Section 2.5, presents material dealing with an algebra of transfer functions that was developed by Callier and Desoer [12, 13]. The material in Section 2.5 will be coupled with recent developments by Curtain [22] so that a sufficient condition for robustness, in the presence of multiplicative perturbations of the transfer function, can be obtained in Chapter 6.

2.2 Definitions

Let A and B be arbitrary function spaces. An operator F is any mapping whose domain, denoted $D(F) \subset A$, or codomain (or both) is a space of functions. An operator is said to map its domain into its range, denoted $R(F)$. The notation used to denote this mapping is $F : A \rightarrow B$. Implied in this notation is that $D(F) = A$, and $R(F)$ is a subset of the codomain B . The norm of a linear operator in a normed space is denoted $\| F \|$, and is defined following Definition 2.2.1 [64]. The inner product of two functions X and Y in an inner product space is denoted $\langle X, Y \rangle$. The space of all real numbers is denoted \mathfrak{R} (and \mathcal{C} for the space of complex numbers), and \mathfrak{R}^N denotes the N dimensional Cartesian product space over \mathfrak{R} . Normed spaces of particular interest are Hilbert spaces denoted \mathcal{H} . Hilbert spaces are complete inner product spaces.

The definition of a bounded linear operator is [64]:

Definition 2.2.1: A linear operator $L : A \rightarrow B$ where A and B are Banach spaces (see [64]), is called a *bounded linear operator* if there exists a nonnegative constant M such that

$$\| Lx \| \leq M \| x \| \quad \forall x \in A \quad (9)$$

The space of all bounded linear operators from A into B is denoted by $\mathcal{L}(A, B)$. In this case where $A = B$, this is denoted $\mathcal{L}(A)$. The smallest M that satisfies Equation (9) is called the operator norm of L .

An operator L is said to be continuous at a point $x \in D(L)$ if for any real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that

$$\| Ly - Lx \| < \epsilon \quad (10)$$

for all $y \in D(L)$ such that

$$\| y - x \| < \delta \quad (11)$$

If L is linear operator from a normed linear space A into another normed linear space B , then [52] demonstrates that the following are true:

1. If L is continuous at any point in $D(L)$, it is continuous on all of $D(L)$.
2. L is bounded if and only if L is continuous.

This research will also consider unbounded linear operators. Specifically closed unbounded linear operators are considered which are important in semigroup theory. These operators are defined as follows [64].

Definition 2.2.2: Let $L : D(L) \rightarrow B$ where $D(L)$ is a subset of a Banach space A , and B is also a Banach space. Let $\{x_n\}$ be a sequence in $D(L)$ such that

x_n converges strongly to x (i.e. $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$). Let Lx_n converge strongly to $y \in B$. Then the operator L is said to be *closed* if $x \in D(L)$ and $Lx = y$ for every such sequence in $D(L)$. L is *densely defined* if the closure of $D(L)$ (denoted $\overline{D(L)}$) is A .

An important property of a linear operator is its spectral property. Let L be a linear operator with $D(L)$ and $R(L)$ contained in a complex linear space C . Consider the operator $(sI - L)$ where s is a complex number and I is the identity operator. Then the following taken from Naylor and Sell [52] page 412 defines the different parts of the spectrum and resolvent of L (denoted $\sigma(L)$ and $\rho(L)$, respectively).

Definition 2.2.3: Let L be a linear operator whose domain and range are contained in a complex linear space C . If the complex number s_0 is such that the range of $(s_0I - L)$ is dense in C and $(s_0I - L)$ has a continuous inverse, then s_0 is in the *resolvent set* $\rho(L)$. The operator $(s_0I - L)^{-1}$ is called the *resolvent operator*. Those complex numbers not in $\rho(L)$ are said to be in the *spectrum* $\sigma(L)$.

The spectrum of L can be divided into three disjoint sets.

1. The *point spectrum* of L is the set of complex numbers, s , for which $(sI - L)$ is not a one-to-one operator. This set is denoted by $P_o(L)$. Elements of the point spectrum are called the eigenvalues of L .

2. The *continuous spectrum* is the set of complex numbers, s , for which $(sI - L)$ is a one-to-one operator with its range dense in C , and for which the inverse defined on the range is discontinuous. This set is denoted by $C_o(L)$.

3. The *residual spectrum* is the set of complex numbers, s , for which $(sI - L)$ is a one-to-one operator, but whose range is not dense in C . This set is denoted by $R_o(L)$.

If C is a finite dimensional space, the sets $C_\sigma(L)$ and $R_\sigma(L)$ are both empty.

An operator may be positive, strictly positive, negative, or strictly negative (or none of these). The following defines what is meant by positive and strictly positive [64] (the definitions for negative and strictly negative follow directly from these).

Definition 2.2.4: An operator $P \in \mathcal{L}(\mathcal{H})$, where \mathcal{H} is a Hilbert space, is said to be *positive* if $\langle Ph, h \rangle \geq 0$ for all $h \in \mathcal{H}$. If $\langle Ph, h \rangle > 0$ for all nonzero $h \in \mathcal{H}$, then P is said to be *strictly positive*.

Another term for positive is positive semi-definite (and similarly for negative). Positive operators can be factored into the product of two operators called square root operators as defined in the following definition [41].

Definition 2.2.5: Let P be a bounded linear positive operator and let P be self-adjoint (i.e. $P = P^*$ where P^* is the adjoint of P). A self-adjoint operator, denoted $P^{1/2}$, such that $P^{1/2} \in \mathcal{L}(\mathcal{H})$ and $P^{1/2}P^{1/2} = P$, is called the *square root* of P . There exists a unique positive operator $P^{1/2}$ for any such P .

The existence of $P^{1/2}$ is proven by Naylor and Sell in problem 15, page 377 [52] so that one does not have to assume that the square root operator exists.

2.3 Semigroup Theory

The systems under consideration are assumed to be of the form

$$\dot{x} = Ax + Bu \quad x_0 \in D(A) \quad (12)$$

or by the associated integral expression

$$x = T(t)x_0 + \int_0^t T(t-s)Bu(s)ds \quad x_0 \in \mathcal{H} \quad (13)$$

along with the output relation

$$y = Cx \quad (14)$$

where $A : D(A) \rightarrow \mathcal{H}$, A is a closed, densely defined operator on a Hilbert space \mathcal{H} which generates the C_0 semigroup $T(t)$ (to be defined later along with $D(A)$), and B and C are bounded linear operators as given in the assumptions in Section 3 of Chapter 1. x is the state variable and is an element of the real Hilbert space \mathcal{H} . u is the input control vector, and y is the observation vector.

Equation (12) is a time invariant evolution equation [24] and Equation (13) is the solution of Equation (12) under the conditions described in this section. A semigroup approach will be taken since this is a standard approach toward infinite-dimensional systems [18].

The definition of a C_0 semigroup is [54]:

Definition 2.3.1: A strongly continuous semigroup of operators (a C_0 semigroup) is a one-parameter family $T(t)$, $0 \leq t < \infty$, of bounded linear operators from a Banach space X into X which satisfies the following properties:

$$T(t+s) = T(t)T(s) \quad \forall s, t \geq 0 \quad (15)$$

$$T(0) = I \quad (16)$$

$$\|T(t)x - x\| \rightarrow 0 \quad \text{as } t \rightarrow 0^+ \quad \forall x \in X \quad (17)$$

The generator of a C_0 semigroup is defined as [54]:

Definition 2.3.2: The operator A is the infinitesimal generator of a strongly continuous semigroup $T(t)$ on a Banach space X if and only if

$$Ax = \lim_{t \rightarrow 0^+} \frac{T(t)x - Ix}{t} \quad \forall x \in D(A) \quad (18)$$

where $D(A) = \{x \in X: \text{the limit exists}\}$.

When considering the stability of infinite-dimensional systems, many people use the concept of exponential stability. Consider the homogeneous equation corresponding to Equations (12) and (13):

$$\dot{x} = Ax \quad x_0 \in D(A) \quad (19)$$

$$x = T(t)x_0 \quad (20)$$

The set of operators $\{T(t)\}$ for $t \geq 0$, is a semigroup of operators, and A is the infinitesimal generator of $T(t)$ [54]. The following defines exponential stability [54].

Definition 2.3.3: A C_0 -semigroup $T(t)$ is said to be *exponentially stable* if there exist constants $M \geq 1$ and $\omega < 0$ such that $\|T(t)\| \leq Me^{\omega t} \quad \forall t \in [0, \infty)$.

Hence, when one says that a system is exponentially stable, one means that the semigroup generated by the operator A in Equation (19) is exponentially stable. This makes sense when one considers that the solution to Equation (19) is given by Equation (20). Thus, when a system is exponentially stable, as $t \rightarrow \infty$ the system state will approach zero from any initial value of $x(0) \in D(A)$ with an exponential decay rate of ω .

One may consider robustness of the system by considering additive perturbations of the operator A . If B is a bounded linear time-invariant operator, then the following is true [54]:

$$\|S(t)\| \leq Me^{(\omega + M\|B\|)t} \quad (21)$$

where $S(t)$ is the semigroup generated by the operator $(A + B)$. Thus, as long as the perturbation is such that $(\omega + M\|B\|) < 0$, the system will remain exponentially stable. This result will be used in Section 4 of Chapter 3 when the robustness of a

finite-dimensional controller is analyzed. Note that an operator B is bounded if and only if there exists a constant K such that $\| B \| \leq K < \infty$. This is why authors have used the exponential time factor as a stability margin. The more negative ω becomes, the larger the perturbation $\| B \|$ can be and still retain exponential stability. However, one must question whether or not all perturbations can be modeled as the addition of a bounded linear time-invariant operator. Also, it is not clear how to relate exponential stability to model uncertainty. As demonstrated by example [51], it may be possible to adjust a SISO design for better gain and phase margins, but the exponential time factor may not change at all. One could also change both gain and phase margin, and not affect the smallest distance to the critical point on a Nyquist diagram, which demonstrates the fact that gain and phase margin do not provide a complete description of robustness. However, since they can be improved without improving the exponential stability of a system, it appears that the exponential time factor may not be the best stability margin as claimed by some [18, 20, 21, 55, 23, 58, 59, 60], unless it can be shown that there is a direct relationship between the magnitude of the exponential time factor and the minimum distance to the critical point. Chapter 3 demonstrates that the exponential time factor is not the only semigroup factor that can be used as a measure of robustness. The semigroup gain constant can also be used as a measure of robustness when relative bounded perturbations are considered (see Section 3.4).

Next the spectrum determined growth assumption (SDGA) is defined. From semigroup theory (see [51] page 18), it is known that

$$\sup \operatorname{Re}[\sigma(A)] \leq \inf\{\omega \in \mathfrak{R} \mid \| T(t) \| \leq M e^{\omega t}\} = \omega_0 \quad (22)$$

where $T(t)$ is a C_0 semigroup with infinitesimal generator A . For infinite-dimensional systems, there may be strict inequality, so that the $\sigma(A)$ does not completely determine the growth constant of $T(t)$ [42]. This gives rise to the following definition.

Definition 2.3.4: An operator A is said to satisfy the SDGA if

$$\sup \operatorname{Re}[\sigma(A)] = \omega_0 \quad (23)$$

where ω_0 is defined in Equation (22).

This condition holds for generators of the following types [65]:

- (i) A is a bounded operator
- (ii) A generates an analytic semigroup
- (iii) A generates a compact semigroup for $t > 0$
- (iv) A generates a strongly differentiable semigroup for $t > t_0 > 0$

Condition (i) holds if A is a continuous operator or a finite-dimensional operator. Condition (ii) holds if the spectrum of the operator A is contained in a closed sector $\Delta = \{\lambda : \arg(\lambda - a) \leq \phi; a \in \mathfrak{R}; \frac{\pi}{2} < \phi < \pi\}$ (see [65] page 388). Condition (iii) generally is satisfied by A operators associated with delay equations or if the Banach space containing $D(A)$ is finite dimensional. Condition (iv) is typically satisfied for diffusion equations or if the generator is bounded. An important property of these operators is that if A is one of these types and B is a bounded operator, then $A + B$ is also the same type of operator [54]. Thus, if feedback is applied through bounded operators, the SDGA is retained.

Now the terms *exponentially stabilizable* and *exponentially detectable* will be defined [24].

Definition 2.3.5: Let X and U be Banach spaces and let $B \in \mathcal{L}(U, X)$. Also, let $A : D(A) \subseteq X \rightarrow X$ generate a C_0 semigroup $T(t)$. Then the pair (A, B) is *exponentially stabilizable* if there exists an operator $D \in \mathcal{L}(X, U)$ such that the operator $A + BD$ generates an exponentially stable semigroup $S(t)$ with

$\| S(t) \| \leq M e^{\omega t}$, $\omega < 0$. $A + BD$ generates a semigroup because A generates a semigroup, and BD is a bounded linear operator [54].

Definition 2.3.6: Let X and Y be Banach spaces and let $C \in \mathcal{L}(X, Y)$. Also, let $A : D(A) \subseteq X \rightarrow X$ generate a C_0 semigroup $T(t)$. Then the pair (A, C) is *exponentially detectable* if there exists an operator $G \in \mathcal{L}(Y, X)$ such that the operator $A + GC$ generates an exponentially stable semigroup $S(t)$ with $\| S(t) \| \leq \tilde{M} e^{\tilde{\omega} t}$, $\tilde{\omega} < 0$. $A + GC$ generates a semigroup because A generates a semigroup, and GC is a bounded linear operator [54].

Another definition that will be needed is the definition of the spectrum decomposition assumption (SDA).

Definition 2.3.7 Let $\delta > 0$ be given, and consider the partitions of the spectrum given by

$$\sigma_u = \{ \lambda \in \sigma(A) : R\epsilon\lambda \geq -\delta \} \quad (24)$$

$$\sigma_s = \{ \lambda \in \sigma(A) : R\epsilon\lambda < -\delta \} \quad (25)$$

where the subscripts u and s denote “unstable” and “stable” respectively, relative to the line at $-\delta$. Then $\sigma(A) = \sigma_u(A) \cup \sigma_s(A)$, and for A the generator of a C_0 semigroup, it can be shown that (see [24] page 75):

$$\sigma(A) \subset \{ \lambda : R\epsilon\lambda \leq \omega_o \} \quad (26)$$

where ω_o is defined in Equation (22).

Then the following definition can be made:

Definition 2.3.8 : If the set $\sigma_u(A)$ is bounded and separated from $\sigma_s(A)$ in such a way that a rectifiable, simple closed curve can be drawn to enclose an open

set containing $\sigma_u(A)$ in its interior and $\sigma_s(A)$ in its exterior, then A is said to satisfy the *spectrum decomposition assumption* [24].

If A satisfies the SDA, then it can be decomposed into the form (see [59] page 93):

$$A = \begin{pmatrix} A_u & 0 \\ 0 & A_s \end{pmatrix} \quad (27)$$

which corresponds to the state space decomposition given by $\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_u$, where \mathcal{H}_s is stable subspace of \mathcal{H} and \mathcal{H}_u is the unstable subspace of \mathcal{H} . Note that stability is defined by the choice of δ in Definition 2.3.7. Thus, the operators A_s and A_u are the restrictions of A onto \mathcal{H}_s and \mathcal{H}_u , respectively. In an obvious way, one can also write the input and output operators as

$$B = \begin{pmatrix} B_u \\ B_s \end{pmatrix} \quad (28)$$

$$C = (C_u \ C_s) \quad (29)$$

By stabilizing the system partition corresponding to the triple (A_u, B_u, C_u) one is then able to stabilize the entire system [59]. Thus, the problem becomes a restriction to the space \mathcal{H}_u . If \mathcal{H}_u is finite-dimensional and A and A_s satisfy the assumptions of Chapter 1, then [59] (A, C) is detectable if and only if (A_u, C_u) is observable, and (A, B) is stabilizable if and only if (A_u, B_u) is controllable [59].

One type of normed space of particular interest is a Lebesgue space, which is denoted $L^p\{(a, b); X\}$ where $p \in [1, \infty)$, and X is a normed linear space consisting of functions f such that $\|f\|^p$ is Lebesgue integrable (see [52] page 589). This type of space is used in the next definition.

The following defines what is meant by the *mild* solution for a class of nonhomogeneous system equations [24]. The control and estimation problems will assume that the systems under consideration can be modeled as described in this definition.

Definition 2.3.9: Let $A : D(A) \subseteq X \rightarrow X$ where X is a Banach space. Also, let A generate a semigroup $T(t)$ and let $f \in L^p\{[0, \infty); X\}$, where L^p denotes a Lebesgue space for $1 \leq p < \infty$. Then

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds \quad (30)$$

is called the *mild solution* to the evolution equation

$$\dot{x}(t) = Ax(t) + f(t) \quad x_0 \in D(A) \quad (31)$$

The mild solution may not be differentiable, and therefore would not strictly satisfy Equation (31). If the solution to Equation (30) is continuously differentiable for $t \geq 0$, and if it satisfies Equation (31) for $t \geq 0$, then it is referred to as the *classical solution*. It is clear that not every mild solution is also a classical solution. Notice that when the input vector $u \in L^2\{[0, \infty); \mathfrak{R}^N\}$, then Bu is also in $L^2\{[0, \infty); \mathfrak{R}^N\}$ when B is a bounded linear operator. Thus, one can set $f = Bu$ in Definition 2.3.9. This will be done in the next section. The optimal control and estimation problems can now be discussed.

2.4 Optimal Control and Estimation Theory

This section presents relevant results from optimal control theory and estimation theory for abstract evolution equations. The results are taken from Curtain and Pritchard [24]. This section will focus on the infinite time problems in order to present results applicable to steady state time invariant controllers which will be used in later chapters.

The system to be controlled will be assumed to be modeled by the mild solution given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)ds \quad x_0 \in \mathcal{H} \quad (32)$$

where \mathcal{H} and U are assumed to be real separable Hilbert spaces, B is a bounded linear operator from U to \mathcal{H} , $x_o \in \mathcal{H}$, and $T(t)$ is a C_o semigroup with generator A that satisfies the assumptions of Chapter 1. Also (A, B) is assumed exponentially stabilizable.

The optimal control problem is to find a control $u \in L^2\{[0, \infty); U\}$ which minimizes the cost functional

$$J(u; x_o) = \int_0^\infty \langle x(s), Q_c x(s) \rangle + \langle u(s), R_c u(s) \rangle ds \quad (33)$$

where the operator $Q_c \in \mathcal{L}(\mathcal{H})$ is positive semi-definite self-adjoint, and the operator $R_c \in \mathcal{L}(U)$ is strictly positive self-adjoint.

Under these constraints, there exists a unique optimal control u^* which minimizes the cost functional $J(u; x_o)$. The following theorem from [24] page 109 describes the optimal control input.

THEOREM 2.4.1: Let \mathcal{H} and U be real separable Hilbert spaces and let (A, B) be stabilizable. Let Q_c be positive semi-definite self-adjoint, and let R_c be strictly positive self-adjoint. Let $B \in \mathcal{L}(U, \mathcal{H})$, $x_o \in \mathcal{H}$, and let A generate a C_o semigroup $T(t)$. Then there exists a unique control $u^* \in L^2\{[0, \infty); U\}$ which minimizes the cost functional $J(u; x_o)$, and u^* is given by

$$u^*(t) = -K_c x(t)$$

where $K_c = R_c^{-1} B^* P_c$ (B^* denotes the adjoint of the operator B). $P_c \in \mathcal{L}(\mathcal{H})$ is a positive, self-adjoint operator which satisfies

$$P_c h = \int_0^\infty [T^*(s)(Q_c + P_c B R_c^{-1} B^* P_c) T(s) h] ds$$

where $T(t)$ is the semigroup generated by the operator $(A - BK_c)$ and $T^*(t)$ denotes the adjoint of $T(t)$.

Proof: See [24] page 109.

P_c is also the unique solution of an A.R.E. when the added assumption that $(A, Q_c^{1/2})$ is detectable is made. The following theorem from Curtain and Pritchard [24] gives the result.

THEOREM 2.4.2: If $(A, Q_c^{1/2})$ is detectable and (A, B) is exponentially stabilizable, then the following is true:

- a. The operator $(A - BK_c)$ generates an exponentially stable semigroup.
- b. The operator P_c is the unique positive, self-adjoint solution of the following A.R.E.

$$\langle Ah, P_c k \rangle + \langle P_c h, Ak \rangle + \langle (Q_c - P_c B R_c^{-1} B^* P_c) h, k \rangle = 0 \quad (34)$$

for all $h, k \in D(A)$.

Proof: See [24] page 111.

This control theory is referred to as the Linear Quadratic (LQ) control theory because of the quadratic cost functional and the linear models. The optimal control law given in Theorem 2.4.1 is called the LQ regulator.

The optimal control development assumes that the state of the system is known exactly and it is available to generate the control input. However, since the state is usually estimated from measured outputs of the system, and these measurements are often incomplete and noise-corrupted, it is usually necessary to use an estimator. The Kalman filter is one estimator that can estimate the system state from noisy measurements. It is the optimum estimator with respect to minimum mean squared error, and it is based on the following model of the system.

$$\dot{x} = Ax + Bu + Gw \quad x_0 \in D(A) \quad (35)$$

$$y = Cx + \eta \quad (36)$$

where \mathcal{H} and Y are real separable Hilbert spaces (Hilbert spaces with countable orthonormal bases), $G \in \mathcal{L}(Y, \mathcal{H})$, and the other assumptions of Chapter 1 apply. Also, the operator $Q_o \in \mathcal{L}(Y)$ is positive and self-adjoint and the operator $R_f \in \mathcal{L}(\mathbb{R}^N)$ is strictly positive and self-adjoint. Q_o is the “strength” of the white Gaussian zero-mean dynamics noise term w , and R_f is the “strength” of the white Gaussian zero-mean measurement noise term η . It is assumed that x_o, w and η are mutually independent. Since the observation space Y is assumed to be finite-dimensional, the operator R_f is nuclear and thus compact (see [70] page 279 for definitions) and R_f^{-1} will exist (see [24] page 158).

The details of the Kalman filter theory for infinite-dimensional systems is covered in detail in [24] and so just the major results will be presented here. The filter will be treated as a compensator element without great concern for the stochastic nature of the problem underlying the filter’s development.

The steady state Kalman filter estimate \hat{x} of the system state is defined by

$$\dot{\hat{x}} = A\hat{x} + Bu + K_f(y - C\hat{x}) \quad (37)$$

where the Kalman filter gain operator $K_f = P_f C^* R_f^{-1}$, and the operator P_f exist as the unique positive, self-adjoint solution of the A.R.E. (see [24] Chapter 6)

$$\langle P_f h, A^* k \rangle + \langle A^* h, P_f k \rangle + \langle (Q_f - P_f C^* R_f^{-1} C P_f) h, k \rangle = 0 \quad (38)$$

for all $h, k \in D(A^*)$ where $Q_f = G Q_o G^*$, and it is assumed that (A, G) is stabilizable and (A, C) is detectable. By duality to the results of Theorem 2.4.2 for the LQ regulator, it can be shown that the operator $(A - K_f C)$ is the generator of a stable semigroup [24].

Using these basic ideas, one can design a finite-dimensional compensator by using dynamic feedback [58]. In the case of a LQG controller, Figure 1 shows that the output is fed back through a Kalman filter and an estimate of the system state is constructed which in turn is used as the input for the LQ controller. The loop

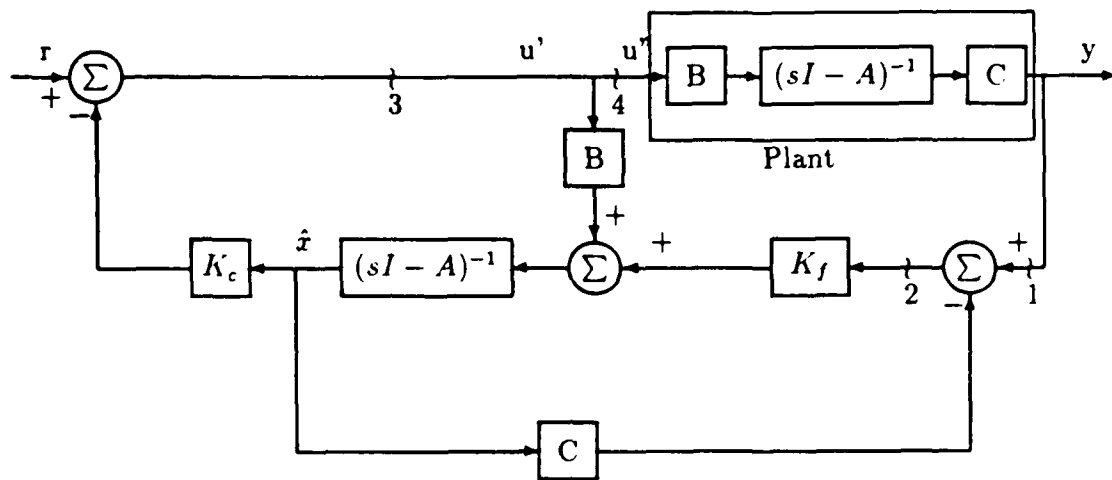


Figure 1. Observer-Based Feedback Configuration

transfer function corresponding to the loop being opened at point 3 is identical to the full-state LQ regulator transfer function. The loop transfer function obtained by opening the loop at point 2 is identical to the Kalman filter loop transfer function. Both of these transfer functions have desirable stability robustness properties as demonstrated in Chapter 2, Section 4, of Matson [51]. Unfortunately, these points in the loop are not very meaningful, since they are internal to the compensator. Since the compensator is designed by the control engineer, uncertainties are not considered to be a problem at these points. The points that are physically important are points 1 and 4. These are the points where the controller interfaces with the plant being controlled. Point 4 is the input to the plant, and point 1 is the output of the plant that is used as an input to the Kalman filter. At these points the transfer functions are not, in general, identical to or even similar to the transfer functions at points 3 and 2. Figure 1 will be referenced again in Section 2 of Chapter 3.

K_c and K_f are bounded operators [24] which can be adjusted to achieve desired performance and/or stability. The Kalman filter and LQG control law have as their

defining equations respectively

$$\dot{\hat{x}} = A\hat{x} + Bu + K_f(y - C\hat{x}) \quad (39)$$

$$u = -K_c\hat{x} \quad (40)$$

which can be equivalently expressed as

$$\dot{\hat{x}} = A\hat{x} - BK_c\hat{x} - K_fC\hat{x} + K_fy \quad (41)$$

$$\dot{\hat{x}} = (A - BK_c - K_fC)\hat{x} + K_fy \quad (42)$$

However, the gain operators K_c and K_f will not be the same as the ones that result using a finite-dimensional compensator. Specifically, $K_f \in \mathcal{L}(\mathfrak{R}^N; \mathcal{H})$ and $K_c \in \mathcal{L}(\mathcal{H}; \mathfrak{R}^N)$ for the infinite-dimensional controller, but for a finite-dimensional compensator one would have $\hat{K}_f \in \mathcal{L}(\mathfrak{R}^N; \mathfrak{R}^k)$ and $\hat{K}_c \in \mathcal{L}(\mathfrak{R}^k; \mathfrak{R}^N)$ where k is the order of the compensator and the dimension of the state model upon which the compensator is based. This research will view the LQG controller as a dynamic compensator, and apply the techniques of Schumacher [59] to obtain conditions under which a *finite-dimensional* LQG controller exists that is an approximation of the infinite-dimensional one would get by solving the infinite-dimensional A.R.Es. The two controllers will be related by an isomorphism (see Chapter 3), and the approximation will be chosen so that the norm difference between the two controllers is sufficiently small. Showing that such a finite-dimensional LQG controller exists is just one step. Next, one needs to evaluate the resulting compensator through the use of some sort of stability margin other than the exponential factor.

The work of Matson [51] extended the LQG/LTR technique to a class of DPS but not to the entire class that is considered in this research. The idea behind the LQG/LTR technique is to adjust the loop transfer function for the closed-loop system using an LQG controller so that asymptotically, the loop transfer function approaches the full-state LQ regulator transfer function (or by duality the Kalman filter transfer function). Actually, the return difference function associated with

point 1 of Figure 1 asymptotically approaches the return difference function associated with point 2 (and similarly for points 4 and 3). For details about this technique, one is referred to Matson's work [51]. However, since the technique has not been extended to the entire class of problems, other robustness approaches have been considered in this research as well as ways to extend the LQG/LTR technique to the entire class of problems.

2.5 Algebra of Transfer Functions

In recent years, much work has been done using a transfer function approach for compensator design [27, 12, 13, 15, 16, 22, 26, 33, 34, 47, 67, 53, 66]. Callier and Desoer developed an algebra of transfer functions and showed how to formulate linear time-invariant DPS in terms of this transfer function algebra [12, 13]. Jacobson [47] later showed that, for the class of systems considered in this research, every exponentially stabilizable and detectable state space representation has its transfer function belonging to this transfer function algebra. He further pointed out that the SDA must be satisfied by exponentially stabilizable and detectable state space systems. Hence, the assumptions of this research allow one to use this transfer function algebra.

Because the problems are in an infinite-dimensional state space, the transfer functions of the systems will be irrational. However, the transfer function algebra allows one to handle irrational transfer functions. The algebra of transfer functions developed by Callier and Desoer is a quotient algebra. A quotient algebra is an algebra in which every element can be written as the quotient of two elements contained in subalgebras of the quotient algebra. The subalgebra from which the denominator elements are taken is the subalgebra containing the algebra inverses. The key elements of the algebra of transfer functions needed to formulate a DPS in terms of this algebra will be defined [12, 53]. In the following definitions, \mathcal{C} denotes the space of complex numbers.

Definition 2.5.1: For $\sigma \in \mathfrak{R}$, a function f is an element of $A(\sigma)$ (the algebra of Laplace transformable functions) if

$$f(t) = \begin{cases} 0 & t < 0 \\ f_0(t) + \sum_{i=0}^{\infty} f_i \delta(t - t_i) & t \geq 0 \end{cases} \quad (43)$$

where

$$f_0 \in L_{1,\sigma}(\mathfrak{R}^+) = \{f|f : \mathfrak{R}^+ \rightarrow \mathcal{C}, \int_0^{\infty} e^{-\sigma t} |f(t)| dt < \infty\}$$

$$0 \leq t_0 \leq t_1 \leq t_2 \leq \dots$$

$$\forall i = 1, 2, \dots \quad f_i \in \mathcal{C}$$

$$\sum_{i=1}^{\infty} |f_i| e^{-\sigma t_i} < \infty$$

and $\delta(t - t_i)$ is the Dirac distribution applied at time $t = t_i$.

Definition 2.5.2: For $\sigma \in \mathfrak{R}$, a function f is an element of $A_-(\sigma)$ (the algebra of stable Laplace transformable functions where σ defines stability) if there exists a $\sigma' < \sigma$ such that $f \in A(\sigma')$.

A better discription of $A(\sigma)$ and $A_-(\sigma)$ will be given in Definition 2.5.3. The thing to note is, by Definition 2.5.1, every element of $A(\sigma)$ or $A_-(\sigma)$ is Laplace transformable, and $\hat{A}(\sigma)$ and $\hat{A}_-(\sigma)$ denote the Laplace transforms of the respective sets (i.e. $\hat{A}(\sigma) = \{\hat{f}|f \in A(\sigma)\}$).

Definition 2.5.3: $\hat{A}_{\infty}(\sigma) = \{\hat{f}|\hat{f} \in \hat{A}_-(\sigma); \hat{f} \text{ is bounded away from zero at infinity in } \mathcal{C}_{\sigma+}\}$ where $\mathcal{C}_{\sigma+} = \{s \in \mathcal{C} \mid \text{Re}(s) \geq \sigma\}$. \hat{f} is said to be bounded away from zero at infinity in $\mathcal{C}_{\sigma+}$ if and only if there exists $\eta > 0; \rho > 0$ such that $\forall s \in D(\sigma, \rho), \quad |\hat{f}(s)| > \eta$, where $D(\sigma, \rho) = \{s \in \mathcal{C}_{\sigma+} : |s - \sigma| \geq \rho\}$ [13]. As pointed out by Callier and Desoer [13], if $\hat{f} \in \hat{A}(\sigma_0)$, then \hat{f} is only guaranteed to be analytic in the interior of \mathcal{C}_{σ_0+} , and not necessarily on the boundary. \hat{f} may have singularities that are dense on the boundary. However, if $\hat{f} \in \hat{A}_-(\sigma_0)$, then \hat{f} is analytic in an

open right half-plane strictly containing $C_{\sigma_0^+}$. Hence, it is analytic at all the zeros of $\hat{f} \in C_{\sigma_0^+}$, and it has a finite number of zeros in any compact subset of $C_{\sigma_0^+}$ [13]. Also, if $\hat{f} \in \hat{A}_-(\sigma_0)$ and \hat{f} is bounded away from zero at infinity in $C_{\sigma_1^+}$ for some $\sigma_1 < \sigma_0$, then \hat{f} has a finite number of zeros in $C_{\sigma_0^+}$ [13].

Before proceeding to the definition of $B(\sigma)$, one needs to understand what is meant by the convolution of two functions [28].

Definition 2.5.4: Given two functions u and h mapping \mathfrak{R}^+ into \mathfrak{R} , one can compute their *convolution* $u * h$ which is defined by the integral

$$(u * h)(t) = \int_0^t h(t - \tau)u(\tau)d\tau \quad t \geq 0$$

The main reasons for using the convolution as a representation for linear time-invariant systems are that under mild conditions (see [61] page 162) any linear time-invariant system can be represented by a convolution kernel which is a distribution, and it allows one to consider lumped-parameter systems and distributed parameter systems under one setting.

A convolution product algebra is one on which addition is defined pointwise, scalar multiplication by real numbers is defined in the normal manner, and the product of two elements is defined to be their convolution [28]. $\hat{B}(\sigma) = [\hat{A}_-(\sigma)] * [\hat{A}_-^\infty(\sigma)]^{-1}$ is a quotient algebra since every element can be written as the quotient of two elements contained in the subalgebras $\hat{A}_-(\sigma)$ and $\hat{A}_-^\infty(\sigma)$.

Definition 2.5.5: $B(\sigma)$ is the convolution product algebra corresponding to the quotient algebra $\hat{B}(\sigma) = [\hat{A}_-(\sigma)] * [\hat{A}_-^\infty(\sigma)]^{-1}$. Elements of $\hat{B}(\sigma)$ can be expressed as the convolution of elements in $\hat{A}_-(\sigma)$ with the reciprocal of elements in $\hat{A}_-^\infty(\sigma)$.

The algebra $B(\sigma)$ is the transfer function algebra developed by Callier and Desoer [12, 13]. Generally speaking, $B(\sigma)$ represents the set of all transfer functions under consideration, and $A_-(\sigma)$ represents the transfer functions defined to be "stable". This leads to the next definition.

Definition 2.5.6: A transfer function $f \in B(\sigma)$ for some $\sigma \in \mathfrak{R}$ is said to be $A_-(\sigma)$ stable, or input-output stable, if $f \in A_-(\sigma)$.

To coincide with the usual notion of stability, σ is taken to be less than or equal to zero. It is pointed out [53] that convolution kernels (i.e. impulse responses) corresponding to physical systems are always real-valued, so that the generality of allowing complex-valued kernels is really not necessary. This algebra is easily extended to multivariable systems by defining $A(\sigma)^{n \times m}$ as a $n \times m$ matrix having elements in $A(\sigma)$. Through the use of left and right coprime factorizations, one is able to factor elements of $B(\sigma)$ in a "minimal" fashion [53, 66]. A factorization of an element in $B(\sigma)$ is minimal if it consists of irreducible quotients. Coprime factorization produces minimum factorizations, and is discussed in detail in [66]. The advantage of working in this algebra is that it places a measure of stability robustness other than the exponential time factor at one's disposal.

Chen and Desoer [16] have extended the singular value concept to this algebra, and have shown that, if a system is $\hat{A}_-(\sigma)$ stable, then, for perturbations which are in $\hat{B}(\sigma)$ and which have singular values bounded by a scalar function for all frequencies ≥ 0 , the multiplicatively perturbed system \tilde{P} ($\tilde{P} = (I + M_p)P$, where M_p is the perturbation and P is the nominal plant) is also $\hat{A}_-(\sigma)$ stable if and only if

$$\sigma_{\max}[PCF(I + PCF)^{-1}(j\omega)] \leq \frac{1}{l_m(\omega)} \quad \forall \omega \in \mathfrak{R}^+ \quad (44)$$

where P, C, F are the transfer functions of the plant, the precompensator, and the feedback compensator respectively (see Figure 2). This condition will be given more

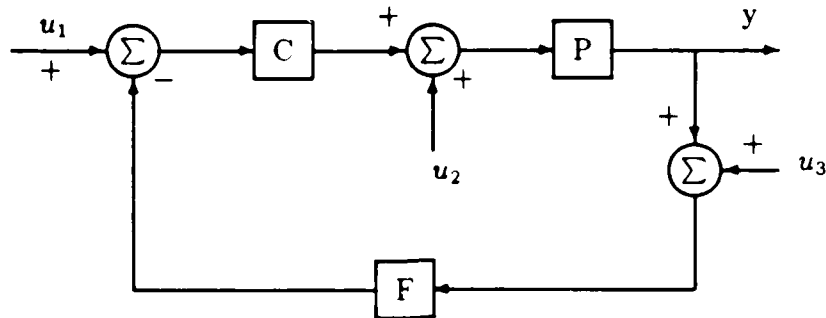


Figure 2. System with Precompensation and Feedback Compensator

completely in Chapter 6, Theorem 6.3.4. The function $l_m(\omega)$ is a scalar function describing the norm of the multiplicative uncertainty of the system as a function of frequency, and σ_{max} is the largest singular value of the operator $PCF(I + PCF)^{-1}$. Construction of $l_m(\omega)$ for multivariable systems is not trivial. The bound assumes a single worst case uncertainty magnitude for all channels of the system. In this sense the uncertainty is unstructured. Typically, uncertainty at higher frequencies is greater than at low frequencies, so that the magnitude of $l_m(\omega)$ increases as $\omega \rightarrow \infty$.

For the case where an LQG compensator is used, one can put the observer based feedback system of Figure 1 into the form of Figure 2 by letting $F = I$ and $C = K_c(sI - A + BK_c + K_fC)^{-1}K_f$. This choice is made in order to correspond to the system configuration used by Jacobson [47], and will be used in Chapter 6. The commanded input is u_1 . u_2 and u_3 can be viewed as disturbances or noise entering the system at the plant input and output respectively (they may also be considered as additional inputs to the system). Thus, the multiplicatively perturbed system is

$\hat{A}_-(\sigma)$ stable if and only if

$$\sigma_{\max}[PC(I + PC)^{-1}(j\omega)] \leq \frac{1}{l_m(\omega)} \quad \forall \omega \in \mathfrak{R}^+ \quad (45)$$

For lumped unity feedback systems, this result is the same as the result stated by Doyle and Stein [30]. Thus, it may be possible to use the singular value as a measure of stability robustness instead of the exponential time constant. This will be used in Chapter 6 when Curtain's work [22] is considered.

2.6 Summary

This chapter has presented background that will be needed in chapters to follow. It is assumed in those chapters that the optimal control theory of Section 2.4 has been applied to yield an LQG controller for the system to be controlled. It is also assumed, except in Chapter 5, that a finite-dimensional controller is to be obtained using Schumacher's direct approach [59]. Chapter 6 will provide the proof of Equation (44), and it will use this result to extend the work of Curtain [22].

The next chapter will show that an LQG compensator satisfies the assumptions needed in Schumacher's approach. This then will prove the existence of finite-dimensional LQG-based controllers for the class of problems described in Section 1.3. It is also proven that the finite-dimensional approximation converges in norm to the infinite-dimensional LQG controller as the dimension of the controller increases toward infinity.

III. Existence of Finite-Dimensional LQG-Based Controllers

3.1 Introduction

This chapter will use the approach developed by Schumacher [58, 59] to prove the existence of finite-dimensional LQG-based controllers for infinite-dimensional systems that satisfy the assumptions outlined in Section 3 of Chapter 1. Throughout the research, it is assumed that one is concerned with improving the stability robustness of a feedback-controlled DPS. It is assumed that:

- 1) A Kalman filter is needed to provide a state estimate as an input to a LQ regulator (i.e. $u = -K_c \hat{x}$).
- 2) The LQG controller will be viewed as a dynamic compensator which has been designed by applying the optimal control theory of Chapter 2.

Thus, it is assumed that an infinite-dimensional controller has been designed, and this chapter will provide the conditions under which a related finite-dimensional controller exists. This controller will be related to the desired infinite-dimensional controller in the sense that the norm difference will be "small". Section 3.3 proves that the finite-dimensional controller converges in norm to the infinite-dimensional controller that is approximated as the dimension of the controller increases toward ∞ . Section 3.4 discusses the limited robustness analysis that can be accomplished after the finite-dimensional controller is designed. Section 3.5 gives an example of Schumacher's approach for a parabolic system.

3.2 Existence Theory

The systems will be described by the infinite-dimensional state space description

$$\dot{x} = Ax + Bu + Gw \quad x(0) \in D(A) \quad (46)$$

$$y = Cx + \eta \quad (47)$$

where $x \in \mathcal{H}$, $u \in \mathfrak{R}^N$ and $y \in \mathfrak{R}^N$, and the assumptions outlined in Chapter 1, Section 3, are satisfied. In this state space description, the state vector x is actually a differentiable \mathcal{H} -valued function of position and time with \dot{x} being the partial derivative of x with respect to time. The system to be controlled will be assumed to have the following mild solution

$$x(t) = T(t)x(0) + \int_0^t T(t-s)Bu(s)ds; \quad x(0) \in \mathcal{H} \quad (48)$$

where \mathcal{H} and U are real Hilbert spaces and $u \in U$. This is not a serious restriction [51].

Consider the "compensator" to be the LQG controller (see Equations (39) - (42)) which has the describing equations

$$\dot{\hat{x}} = A\hat{x} + Bu + K_f(y - C\hat{x}) \quad (49)$$

$$u = -K_c\hat{x} \quad (50)$$

which, when u is substituted into the $\dot{\hat{x}}$ equation gives

$$\dot{\hat{x}} = (A - BK_c - K_fC)\hat{x} + K_fy \quad (51)$$

where the controller and filter gain operators K_c and K_f are given by:

$$K_c = R_c^{-1}B^*P_c; \quad K_c \in \mathcal{L}(\mathcal{H}; \mathfrak{R}^N) \quad (52)$$

$$K_f = P_fC^*R_f^{-1}; \quad K_f \in \mathcal{L}(\mathfrak{R}^N; \mathcal{H}) \quad (53)$$

where P_c and P_f are strictly positive self-adjoint solutions to the controller and filter Riccati equations, Equations (34) and (38), respectively. Note that the number of

inputs equals the number of outputs, namely N . This is assumed since Matson's procedure has only been proven for that case [51] and will be needed in Chapter 4. Also, note that the input u and the output y are finite-dimensional vectors.

It will also be assumed that $(A, Q_c^{1/2})$ is detectable and (A, B) is stabilizable so that the operator K_c will be unique, yield a minimum cost, and produce a stable closed-loop system. Similarly, it will be assumed that (A, C) is detectable and (A, G) is stabilizable in order that the operator K_f be unique, yield a minimum cost associated with the filter, and produce a stable filter [24]. Figure 1 of Chapter 2 depicts the system under consideration.

The closed-loop equations can be written in terms of the extended space $X_E = \mathcal{H} \oplus \hat{X}$, where $\hat{X} \subset \mathcal{H}$, to get

$$\dot{x}_e = \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -BK_c \\ K_f C & A - BK_c - K_f C \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} G & 0 \\ 0 & K_f \end{bmatrix} \begin{bmatrix} w \\ \eta \end{bmatrix} \quad (54)$$

or in terms of the extended space

$$\dot{x}_e = A_e x_e + B_e u_e \quad (55)$$

and B_e is a bounded linear operator. The question to ask now is, under what conditions will the operator A_e generate a stable semigroup when a finite-dimensional compensator is used? The following theorem by Schumacher [59] gives sufficient conditions.

THEOREM 3.2.1: Let the system be described by

$$\dot{x} = Ax + Bu + Gw \quad x(0) \in D(A)$$

$$y = Cx + \eta$$

with the assumptions that

- i) A generates a C_0 -semigroup $T(t)$ on the Hilbert space \mathcal{H}

ii) $B \in \mathcal{L}(\mathbb{R}^N; \mathcal{H})$

iii) $C \in \mathcal{L}(\mathcal{H}; \mathbb{R}^N)$

Also, let the bounded linear operators $F : \mathcal{H} \rightarrow \mathbb{R}^N$ and $K : \mathbb{R}^N \rightarrow \mathcal{H}$ be such that

iv) $(A - BF)$ generates a stable semigroup

v) $(A - KC)$ generates a stable semigroup

If there exists a finite-dimensional subspace $V \subset D(A)$ such that

1) $(A - BF)x \in V \quad \forall x \in V$

2) $Im(-K) \subset V$ where Im denotes the image of an operator

then there exists a stabilizing compensator of order $k = dimV$ (i.e., k is the dimension of V).

Proof: See Schumacher [59] pg 109.

For the case of an LQG compensator, let $K_c = F$ and $K_f = K$. Then the assumptions of Theorem 3.2.1 are satisfied except for the required properties of the subspace V . Theorem 3.2.1 gives a sufficient condition, but it is not clear how to find the required subspace V so that $Im(-K_f) \subset V$ and $(A - BK_c)x \in V \quad \forall x \in V$. One can find K_c and K_f using LQG theory, but V is not as readily obtainable. Therefore, one would like to have some sort of construction procedure to find V . The next theorem from Schumacher does just that.

THEOREM 3.2.2: Let the system be such that A generates a C_0 -semigroup, $B \in \mathcal{L}(\mathfrak{R}^N; \mathcal{H})$; $C \in \mathcal{L}(\mathcal{H}; \mathfrak{R}^N)$, and suppose that the mappings $F \in \mathcal{L}(\mathcal{H}; \mathfrak{R}^N)$ and $K \in \mathcal{L}(\mathfrak{R}^N; \mathcal{H})$ are such that

- 1) $(A - BF)$ generates a stable semigroup
- 2) $(A - BF)$ has a discrete spectrum and its generalized eigenvectors are complete in \mathcal{H} (i.e. $\text{span}\{x \in \mathcal{H} \mid \exists \lambda \in \mathbb{C} \text{ and } \exists n \in \mathbb{N} \ni (\lambda I - A)^n x = 0\}$ is dense in \mathcal{H}).
- 3) There exists a scalar $\alpha > 0$ such that the semigroup generated by $(A - KC + \alpha I)$ is stable (i.e. $(A - KC)$ has an extra margin of stability).

Then there exists a stabilizing finite-dimensional compensator.

Proof: Following Schumacher's proof, let $S(t)$ be the semigroup generated by $(A - KC)$. The growth constant, by assumption (3), is less than or equal to $\omega - \delta$ where $M\alpha = \delta$ and $\omega < 0$. The constant $M > 0$ is the constant such that $\|S(t)\| \leq M e^{(\omega - \delta)t} < M e^{(\omega - \frac{\delta}{2})t}$. Note that since $\omega - \delta < \omega - \frac{\delta}{2}$ then $e^{(\omega - \delta)t} < e^{(\omega - \frac{\delta}{2})t}$. For any operator $\hat{K} : \mathfrak{R}^N \rightarrow \mathcal{H}$ such that $\|K - \hat{K}\| \leq M^{-1} \|C\|^{-1} \frac{\delta}{2}$, then the operator $(A - KC)$ will generate a semigroup $S(t)$ and $(A - \hat{K}C)$ generates $\hat{S}(t)$ with

$$\begin{aligned} \|\hat{S}(t)\| &\leq M e^{(\omega + M\|K - \hat{K}\|\|C\| - \frac{\delta}{2})t} \\ \|\hat{S}(t)\| &\leq M e^{(\omega + MM^{-1}\|C\|\|C\|^{-1} - \frac{\delta}{2})t} \\ \|\hat{S}(t)\| &\leq M e^{\omega t} \end{aligned}$$

so that $\|K - \hat{K}\| \leq \epsilon$ where $\epsilon = M^{-1}\|C\|^{-1}\frac{\delta}{2}$. Then the perturbation of K , \hat{K} , is such that the perturbed operator will still generate a stable semigroup. Now, pick an orthonormal basis for \mathfrak{R}^N , say $\{e_i\}$, and define $g_i = K e_i$ for $i = 1, 2, \dots, n$. Note that $g_i \in \mathcal{H}$. Since the generalized eigenvectors of $(A - BF)$ are complete (i.e.

$\text{span}\{x \in \mathcal{H} \mid \exists \lambda \in \mathcal{C}; \exists n \in \mathbb{N} : (\lambda I - A)^n x = 0\}$ is dense in \mathcal{H}), then for every g_i , there exists a linear combination of finitely many eigenvectors $\{x_{ij}\}, j = 1, 2, \dots, N_i$ (where the double subscript notation is used to emphasize the dependence on g_i) such that $\|g_i - \sum_{j=1}^{N_i} \alpha_{ij} x_{ij}\| \leq \epsilon$ for suitable numbers $\alpha_{ij} (i = 1, 2, \dots, n; j = 1, 2, \dots, N_i)$. Then to every pair of indices (i, j) there exists a $\lambda_{ij} \in \mathcal{C}$ and a $n_{ij} \in \mathbb{N}$ such that $(\lambda_{ij} - (A - BF))^{n_{ij}} x_{ij} = 0$. In other words, every eigenvector x_{ij} has a corresponding eigenvalue λ_{ij} which may have finite multiplicity.

Then define the subspace V as

$$V = \text{span}\{(\lambda_{ij} - (A - BF))^k x_{ij}\}$$

$$i = 1, 2, \dots, n; j = 1, 2, \dots, N_i; k = 0, 1, \dots, n_{ij} - 1$$

Then V is clearly finite-dimensional and $D(A - BF) = D(A)$. Also, since V is spanned by the generalized eigenvectors of $(A - BF)$, then V is invariant under $(A - BF)$ ([59] pg 112, [8] pg 257). Now, write $\hat{g}_i = \sum_{j=1}^{N_i} \alpha_{ij} x_{ij}$ and define the linear transformation $\hat{K} : \mathbb{R}^N \rightarrow \mathcal{H}$ by $\hat{K}e_i = \hat{g}_i$ and note that $\hat{K}(b_1, b_2, \dots, b_n) = \sum_{i=1}^n b_i \hat{g}_i$. Then, $\text{Im}(\hat{K}) \subset V$ and also $\|K - \hat{K}\| \leq \epsilon$ where $\epsilon = (1/2)M^{-1} \|C\|^{-1} \delta$ so that $(A - \hat{K}C)$ generates a stable semigroup. This comes from the fact that

$$\begin{aligned} & \| (K - \hat{K})e_i \| = \| Ke_i - \hat{K}e_i \| \\ & = \| g_i - \hat{g}_i \| = \| g_i - \sum_{j=1}^{N_i} \alpha_{ij} x_{ij} \| \leq \epsilon \end{aligned}$$

and noting that [52]

$$\|K - \hat{K}\| = \sup_{\|e_i\|=1} \| (K - \hat{K})e_i \|$$

Then by Theorem 3.2.1, with this choice of V , K , and \hat{K} a finite-dimensional stabilizing compensator exists of order $= \dim V$. **Q.E.D.**

Notice that this theorem gives us a way to construct V , but there is no upper limit on the order of the compensator. Although Schumacher demonstrates that

the dimension of V may be small for many problems [59], for other problems in which the compensator order is constrained, the dimension of V may have to be too large. Since there is not an upper limit on the dimension of V , one would not know this without trying to solve the problem. The question left to resolve is, are the conditions of Theorem 3.2.2 satisfied using an LQG compensator? $(A - BK_c)$ generates a stable semigroup and with $\omega < 0$ one can find an $\alpha > 0$ such that $(A - K_f C + \alpha I)$ generates a stable semigroup. The requirements on (A, B, C) are satisfied by assumption. Therefore, one needs to ask whether or not the spectrum of $(A - BK_c)$ is discrete with complete eigenvectors. If one can show this to be true, then Theorem 3.2.2 yields that a finite-dimensional LQG-based compensator will exist which will exponentially stabilize the system. First, consider whether the spectrum of $(A - BK_c)$ is discrete. The spectrum of A is discrete and $B : \mathfrak{R}^N \rightarrow \mathcal{H}$ and $K_c : \mathcal{H} \rightarrow \mathfrak{R}^N$. Thus, BK_c has finite rank. Then, as proven in Kato [48] Chapter 4, Theorems 4.6.2 and 4.6.5, the spectrum of $(A - BK_c)$ is discrete.

The final question is, are the eigenvectors of $(A - BK_c)$ complete? Since BK_c has finite rank, it has only a finite number of eigenvalues associated with it, say ℓ . Now by assumption, A satisfies the spectrum decomposition assumption, and there are only a finite number of eigenvalues to the right of any vertical line which separates stable and unstable regions (i.e. the $j\omega$ axis). Also, (A, B) is assumed stabilizable which means that (A_u, B_u) must be controllable, as proven in Schumacher and Curtain [24, 59]. (A_u, B_u) is the restriction of (A, B) to the finite-dimensional subspace \mathcal{H}_u . \mathcal{H}_u is the subspace spanned by the eigenvectors associated with the eigenvalues in the unstable part of the complex plane (i.e. the right half plane). Since K_c stabilizes (A, B) then it must be that (A_u, B_u) is controllable. Thus, when rank BK_c equals $\dim \mathcal{H}_u$, the eigenvalues associated with the subspace \mathcal{H}_u can be arbitrarily perturbed (for proper choice of K_c) so that none of the eigenvalues of $(A_u - B_u K_c)$ coincide with the eigenvalues of A_u , and thus the eigenvectors associated with $(A_u - B_u K_c)$ and A_u are complete, as shown by Sakawa [56]. A_s is the restriction

of A to the subspace \mathcal{H}_s , where $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s$.

Hence, completeness of the eigenvectors of A , the finite rank of BK_c , and the assumption that (A, B) is stabilizable, imply that the eigenvectors of $(A - BK_c)$ can be made complete if K_c is chosen properly. It will be assumed that this has been done during the LQG design. This is not viewed as a real problem since (A_u, B_u) is controllable so that the unstable eigenvalues can be placed at desired locations in the left-half plane. Thus, one can maintain the same number of distinct eigenvalues as the operator A , and for most problems, it is felt that this will yield a set of complete eigenfunctions since A has complete eigenfunctions. Then $(A - BK_c)$ will satisfy assumption 2 in Theorem 3.2.2 so that a finite-dimensional compensator exists which will exponentially stabilize the system for some $\omega < 0$ (not necessarily an arbitrary $\omega < 0$).

3.3 Convergence

Although it has been shown that a finite-dimensional stabilizing LQG based controller exists, it has not been shown whether or not this approximation converges to the desired infinite-dimensional LQG controller as the dimension is increased. The following theorem will be needed in proving convergence.

THEOREM 3.3.1: Let $G \in \mathcal{L}(\mathcal{H})$ be the generator of an exponentially stable semigroup $S(t)$. Then, G^{-1} exists and is a bounded operator.

Proof: If one can show that $\lambda = 0$ is in the resolvent set of G , then that will mean G^{-1} exists as a bounded operator in the Hilbert space \mathcal{H} [54].

Recall that a linear operator G is the infinitesimal generator of a C_0 semigroup $S(t)$ satisfying $\|S(t)\| \leq M e^{\omega t}$ for $\omega < 0$ if and only if the resolvent set $\rho(G)$ contains the ray $]\omega, \infty[$ (where $]a, b[$ is the standard notation to denote a ray of the real line

between a and b) and

$$\| R(\lambda; G) \| \leq \frac{M}{(\lambda - \omega)} \quad \forall \lambda > \omega$$

where $R(\lambda; G)$ is the resolvent operator (see Pazy [54] Theorem 5.3). Then since G generates an exponentially stable semigroup, $\omega < 0$ and $\lambda = 0$ is in $\rho(G)$. Hence, G^{-1} will exist as a bounded linear operator. **Q.E.D.**

Now, the finite-dimensional compensator is of the form

$$\dot{z} = Nz + My \tag{56}$$

$$u = Lz \tag{57}$$

where for the case of a LQG compensator described by Equation (42),

$$N = R(A - BK_c - \hat{K}_f C)R^{-1} \tag{58}$$

$$M = R\hat{K}_f \tag{59}$$

$$L = -K_c R^{-1} \tag{60}$$

R is an isomorphism such that $R : V \rightarrow W$, and \hat{K}_f is the approximation of K_f obtained as in Theorem 3.2.2. Since the Theorem 3.2.1 only requires that K_f have its image contained in a finite-dimensional space, there is not a need to approximate K_c . However, the isomorphism R will restrict the domain of K_c to a finite-dimensional space so that $K_c R$ is a finite-dimensional operator. V is the finite-dimensional space such that $(A - BK_c)x \in V \quad \forall x \in V$ and $Im(\hat{K}_f) \subset V$, and W is the finite-dimensional space that the compensator state z is an element of. In terms of LQG notation, Equations (56) and (57) can be written as:

$$\dot{\hat{x}} = R(A - BK_c - \hat{K}_f C)R^{-1}\hat{x} + R\hat{K}_f y \tag{61}$$

$$u = -K_c R^{-1}\hat{x} \tag{62}$$

The describing equation for the finite-dimensional compensator transfer function is given by

$$C_k = FR^{-1}(sI - R(A - BF - \tilde{G}C)R^{-1})^{-1}R\tilde{G} \quad (63)$$

where in the LQG case $F = K_c$ and $G = K_f$ (so $\tilde{G} = \tilde{K}_f$). Let $Z_k = (A - BF - \tilde{G}C)$.

The fact that R is an isomorphism yields

$$C_k = FR^{-1}(sI - RZ_kR^{-1})^{-1}R\tilde{G} \quad (64)$$

$$= F[(sI - RZ_kR^{-1})R]^{-1}R\tilde{G} \quad (65)$$

$$= F[(sR - RZ_k)]^{-1}R\tilde{G} \quad (66)$$

$$= F[R^{-1}(sR - RZ_k)]^{-1}\tilde{G} \quad (67)$$

$$= F(sI - Z_k)^{-1}\tilde{G} \quad (68)$$

This is equivalent to

$$C_k = F(sI - (A - BF - \tilde{G}C))^{-1}\tilde{G} \quad (69)$$

Now, it will be shown that as $k \rightarrow \infty$, then $\lim_{k \rightarrow \infty} C_k = C$, where C is the transfer function of the infinite-dimensional LQG compensator.

THEOREM 3.3.2: The finite-dimensional LQG compensator transfer function C_k converges to the desired infinite-dimensional C compensator as $k \rightarrow \infty$.

Proof: Recall that $G : \mathfrak{R}^N \rightarrow \mathcal{H}$ and $\tilde{G} : \mathfrak{R}^N \rightarrow \mathfrak{R}^k$ where k is the dimension of the finite-dimensional compensator. By construction of \tilde{G} , $\|G - \tilde{G}\| \rightarrow 0$ as $k \rightarrow \infty$ so that the only unresolved question is, will

$$(sI - (A - BF - \tilde{G}C))^{-1} \rightarrow (sI - (A - BF - GC))^{-1}$$

as $k \rightarrow \infty$? One sees that this is true from the following argument. Let $Z_k = (A - BF - \tilde{G}C)$ and let $Z = (A - BF - GC)$. Then since both operators generate a

stable semigroup, by Theorem 3.3.1 their inverses exist and are bounded. Therefore, for complex numbers s , and $s \notin \sigma(Z_k) \subset \sigma(Z) \quad \forall k$,

$$\begin{aligned} \|(sI - Z_k)^{-1} - (sI - Z)^{-1}\| &= \|(sI - Z_k)^{-1}[I - (sI - Z_k)(sI - Z)^{-1}]\| \\ &= \|(sI - Z_k)^{-1}[(sI - Z) - (sI - Z_k)](sI - Z)^{-1}\| \\ &\leq \|(sI - Z_k)^{-1}\| \|Z_k - Z\| \|(sI - Z)^{-1}\| \\ &< d \|Z_k - Z\| \end{aligned}$$

where d is a constant. By definition of Z and Z_k ,

$$(Z_k - Z) = (GC - \hat{G}C)$$

Thus one sees that

$$\begin{aligned} \|Z_k - Z\| &= \|GC - \hat{G}C\| \\ &\leq \|G - \hat{G}\| \|C\| \\ &\leq d \|G - \hat{G}\| \end{aligned}$$

Therefore, by construction of \hat{G} ,

$$\|(sI - Z_k)^{-1} - (sI - Z)^{-1}\| \leq \hat{d} \|G - \hat{G}\|$$

Since $\|G - \hat{G}\| \rightarrow 0$ as $k \rightarrow \infty$, then the transfer function of the finite-dimensional compensator converges to the transfer function of the infinite-dimensional compensator. **Q.E.D.**

Hence, by the linearity of F, G and \hat{G} it has been shown that $\lim_{k \rightarrow \infty} C_k = C$ where in LQG notation

$$C_k = K_c(sI - (A - BK_c - \hat{K}_f C))^{-1} \hat{K}_f$$

$$C = K_c(sI - (A - BK_c - K_f C))^{-1} K_f$$

Recall that only K_f has to be approximated in order to satisfy the conditions needed in Theorem 3.2.1.

Next one may ask whether or not the convergence is monotone or not. If the convergence is monotone, one may be able to select approximating eigenfunctions that reduce the norm difference between K_f and its approximation the most. The following definition will be needed.

Definition 3.3.1: A sequence $\{x_n\} \subset \mathfrak{R}$ is said to be *monotonically increasing* if $x_{n+1} \geq x_n \quad \forall n$, and is said to be *monotonically decreasing* if $x_{n+1} \leq x_n \quad \forall n$.

THEOREM 3.3.3: The finite-dimensional LQG compensator transfer function C_k converges monotonically to the infinite-dimensional transfer function C as $k \rightarrow \infty$ in the sense that $\sum_{j=k+1}^{\infty} \|\alpha_{ij}\|^2$ monotonically decreases as $k \rightarrow \infty$, (where α_{ij} are the coefficients of the eigenfunction expansion of a function in the range of K_f) if and only if the eigenfunctions x_{ij} are orthonormal, and the series $\sum_{j=1}^{\infty} \alpha_{ij} x_{ij}$ converges for all i .

Proof: Let $\{y_i\}$ be a basis for the output space Y . By definition of g_i , $K_f y_i = g_i = \sum_{j=1}^{\infty} \alpha_{ij} x_{ij}$, where x_{ij} are the generalized eigenfunctions of $(A - BK_c)$ which are assumed complete (see discussion in Section 3.2) and normalized. α_{ij} are the Fourier coefficients defined by the inner product $\langle g_i, x_{ij} \rangle$. By construction of the approximation of K_f ,

$$\hat{K}_f y_i = \hat{g}_i = \sum_{j=1}^{n_i} \alpha_{ij} x_{ij}$$

where $n_i \leq k$, the dimension of the compensator. Thus

$$\|K_f y_i - \hat{K}_f y_i\|^2 = \left\| \sum_{j=k+1}^{\infty} \alpha_{ij} x_{ij} \right\|^2$$

From the fact that $x_{i,j}$ are orthonormal, one gets that

$$\|K_f y_i - \hat{K}_f y_i\|^2 \leq \sum_{j=k+1}^{\infty} \|\alpha_{i,j}\|^2$$

Since $\sum_{j=1}^{\infty} \alpha_{i,j} x_{i,j}$ converges for all i if and only if $\sum_{j=1}^{\infty} |\alpha_{i,j}|^2$ converges for all i (see [52] page 308), and $\|\alpha_{i,j}\|$ is a positive number, then this tail of the sequence is monotonically decreasing as $k \rightarrow \infty$. In other words,

$$\lim_{k \rightarrow \infty} \sum_{j=k+1}^{\infty} \|\alpha_{i,j}\|^2 = 0$$

In this sense $\hat{K}_f \rightarrow K_f$ monotonically. **Q.E.D.**

Recall that K_f is approximated by using a finite number of eigenfunctions corresponding to a finite number of eigenvalues. As a minimum, one would want to use the eigenfunctions associated with the unstable subspace \mathcal{H}_u as a first choice to approximate K_f . Otherwise, one would not be able to stabilize the system with the lowest order compensator possible. After that, one could choose eigenfunctions from the subspace \mathcal{H}_s based on where one desires the closed-loop poles to lie.

Figure 3 shows the subspaces \mathcal{H}_u and \mathcal{H}_s for the case where an infinite strip of eigenvalues lies in the left half plane. Obviously, if an infinite vertical strip of eigenvalues is encountered, one cannot improve exponential stability beyond it. The approximating subspaces are finite-dimensional subspaces of the Hilbert space \mathcal{H} , denoted by \mathcal{H}_k , where k is the dimension of the space. The approximating subspaces are ordered so that $\mathcal{H}_k \subset \mathcal{H}_{k+1}$. Note that Theorem 3.3.3 states that the approximation monotonically converges with respect to the tail, regardless of how the approximating eigenfunctions are chosen. The question one may now ask is, can the eigenfunctions be selected so that the sequence is monotone in the sense that

$$\left\| \sum_{j=1}^k \alpha_{i,j} x_{i,j} - \sum_{j=1}^{k+1} \alpha_{i,j} x_{i,j} \right\| \quad (70)$$

decreases monotonically from $k = 1$ on out. This would allow one to choose the eigenfunctions associated with the maximum gain at the start of the design and

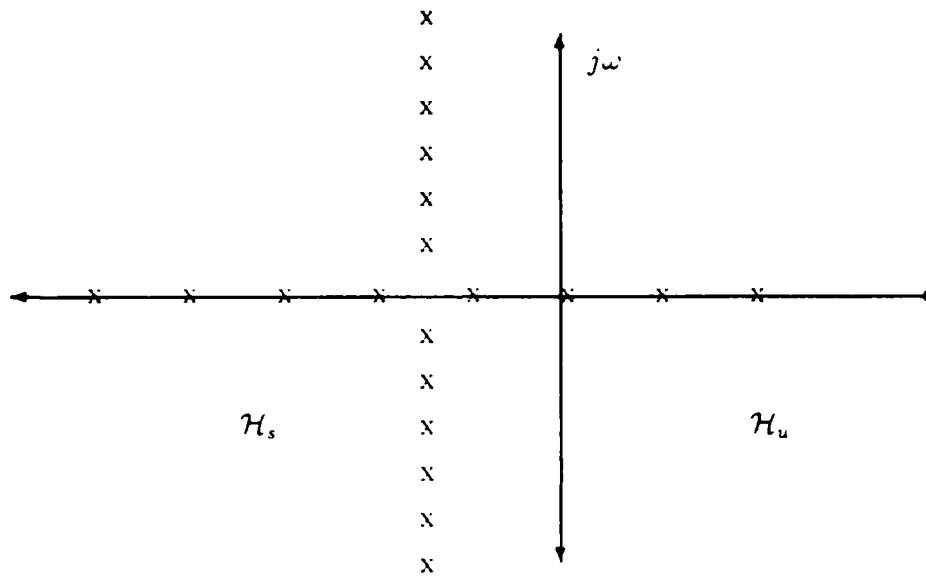


Figure 3. System with Infinite Strip of Eigenvalues in Left Half Plane

possibly get the best controller for any fixed k . Clearly this means that, for $\{x_{ij}\}$ normalized, one must choose the "next" eigenfunction based on the magnitude of $\alpha_i = \langle g_i, x_i \rangle$ for all i , where $g_i = K_f y_i$.

In general, α_i has no relationship to λ_i (the eigenvalue associated with x_i) so that one cannot simply choose the "next" eigenfunction associated with the "next" eigenvalue, no matter how one orders the λ_i 's. What one can do is write the general expression for $\langle g_i, x_i \rangle$ and then choose x_i associated with $\sup_{x_i} \langle g_i, x_i \rangle$. Once this x_i is chosen, one can omit it the next time $\sup_{x_i} \langle g_i, x_i \rangle$ is evaluated. In this way one can "order" the eigenfunction selection to get monotone convergence from $k = 1$ on out (i.e. $\alpha_i \geq \alpha_{i+1} \quad \forall i$). This would yield that Equation (70) is monotone decreasing as $k \rightarrow \infty$.

Of greater importance is where the resulting closed-loop poles lie, and the resulting performance. If the largest value of α_i were associated with a stable eigenvalue, one might not stabilize the system with the lowest possible order for the compensator if one were to try and make Equation (70) monotonically decreasing. Thus, it seems one should be concerned with choosing the eigenfunctions associated

with the unstable subspace \mathcal{H}_u , and make sure that the system is stabilized (i.e., shift at least the unstable eigenvalues to desirable locations). The example problem that follows in Section 3.5 points out this concern.

3.4 Robustness Analysis

Schumacher's approach yields a compensator that stabilizes the nominal system. The question one needs to ask is how robust is that compensator? What type of perturbations can be allowed while maintaining exponential stability?

Following the approach taken by [69], one way to address perturbations is to consider perturbations of the state-space operators. Let the nominal system on which the controller is based be described by the equations

$$\dot{x} = Ax + Bu \quad x_0 \in D(A) \quad (71)$$

$$y = Cx \quad (72)$$

and let the perturbed system be described by

$$\dot{x}_\Delta = (A + \Delta A)x_\Delta + (B + \Delta B)u \quad (73)$$

$$y_\Delta = (C + \Delta C)x_\Delta \quad (74)$$

First, consider the case where perturbations occur only in the dynamics (i.e. ΔB and ΔC are both zero). In this case the perturbed system will be described by

$$\dot{x}_\Delta = (A + \Delta A)x_\Delta + Bu \quad (75)$$

$$y_\Delta = Cx_\Delta \quad (76)$$

For the purpose of discussion, the following definition has been developed by the author of this research.

Definition 3.4.1: The system will be *dynamically robust* if the system remains stable in the presence of perturbations of the dynamics.

For the class of systems considered in this research, the assumptions of detectability and stabilizability yield that, for the nominal system, an LQG controller will provide exponential stability of the nominal closed-loop system. Also, the operators $A - BK_c$ and $A - K_f C$ both generate stable semigroups [24]. If $(A + \Delta A - BK_c)$ and $(A + \Delta A - K_f C)$ also generate stable semigroups, then the system will be dynamically robust for that ΔA .

One possibility is that ΔA is a bounded perturbation. In this case

$$(A + \Delta A - BK_c) = (A - BK_c) + \Delta A \quad (77)$$

$$(A + \Delta A - K_f C) = (A - K_f C) + \Delta A \quad (78)$$

If $\| \Delta A \| < | \omega |$, where ω is the maximum of the exponential time constants associated with the semigroups generated by $(A - BK_c)$ and $(A - K_f C)$, then the perturbed closed-loop system will be exponentially stable. This result is given in the following lemma.

LEMMA 3.4.1: Let the system described by Equations (46) and (47) be such that (A, B) and (A, G) are stabilizable. Also, let (A, C) and $(A, Q_c^{1/2})$ be detectable. Let A generate a C_0 semigroup and have a discrete spectrum. Let B and C be bounded linear operators. Then the LQG regulator will stabilize the nominal closed-loop system and the closed-loop system will be dynamically robust to bounded perturbations if $\| \Delta A \| < | \omega |$ where $\omega = \max[\omega_1, \omega_2]$; ω_1 and ω_2 are the negative time constants associated with the stable semigroups generated by $(A - BK_c)$ and $(A - K_f C)$, respectively.

Proof: The proof will be done if it can be shown that $(A + \Delta A - BK_c)$ and $(A + \Delta A - K_f C)$ both generate stable semigroups. Let $T_c(t)$ and $T_f(t)$ be the semigroups generated by $(A - BK_c)$ and $(A - K_f C)$ respectively. Then the

semigroup $\hat{T}_c(t)$ generated by $(A - BK_c) + \Delta A$ will have the property [54] that

$$\|\hat{T}_c(t)\| \leq M_1 e^{(\omega_1 + \|\Delta A\|)t}$$

where $\omega_1 < 0$. Similarly, the semigroup $\hat{T}_f(t)$ generated by $(A - K_f C) + \Delta A$ will be such that

$$\|\hat{T}_f(t)\| \leq M_2 e^{(\omega_2 + \|\Delta A\|)t}$$

where $\omega_2 < 0$.

Clearly, if $\|\Delta A\| < |\max\{\omega_1, \omega_2\}|$, then both $\hat{T}_c(t)$ and $\hat{T}_f(t)$ will be stable semigroups, and the closed-loop system will be exponentially stable. Thus, the system is dynamically robust. **Q.E.D.**

From Lemma 3.4.1 it becomes clear why the exponential time constant is used as a measure of robustness [19, 21, 23, 58, 59, 60]. However, this is only true for bounded perturbations.

One may choose to consider perturbations to B and C in which case one must determine the stability of the semigroups generated by the operators $[(A + \Delta A) - (B + \Delta B)K_c]$ and $[(A + \Delta A) - K_f(C + \Delta C)]$. Rearranging terms, these operators can be written as $[(A - BK_c) + (\Delta A - \Delta BK_c)]$ and $[(A - K_f C) + (\Delta A - K_f \Delta C)]$. If $(\Delta A - \Delta BK_c)$ and $(\Delta A - K_f \Delta C)$ are both bounded, then one can still use Lemma 3.4.1 as a way of determining if the system will remain stable in the presence of these perturbations.

Another type of perturbation one can consider is a relative bounded perturbation. An operator \mathcal{B} is said to be relatively bounded with respect to an operator A if $D(\mathcal{B}) \supset D(A)$ and if $\|\mathcal{B}x\| \leq b\|x\| + a\|Ax\| \forall x \in D(A)$ (see [8] for further discussion).

LEMMA 3.4.2: Let A be the generator of a stable semigroup $T(t)$ such that $\|T(t)\| \leq M e^{\omega t}$, $\omega < 0$. If \mathcal{B} is a closed linear operator with $D(\mathcal{B}) \supset D(A)$ and

$$\|\mathcal{B}x\| \leq a \|Ax\| \quad \forall x \in D(A)$$

then $A + \mathcal{B}$ generates an exponentially stable semigroup when $a \leq (M + 1)^{-1}$.

Proof: Since $\|T(t)\| \leq M e^{\omega t} < M$, then $T(t)$ is uniformly bounded, and thus analytic by Theorem 2.5.6 of Pazy [54]. Theorem 3.2.1 of Pazy [54] provides the desired result. **Q.E.D.**

Thus, for relatively bounded perturbations, a measure of robustness is the semigroup gain factor. The smaller one can make the constant M associated with $T(t)$, the larger the allowable perturbations can be and retain stability of the closed-loop system. Lemmas 3.4.1 and 3.4.2 are not new ideas, but they point out the limitations of using the exponential time constant as the only measure of robustness. Also, Lemma 3.4.2 demonstrates that the gain factor can be used as a measure of robustness, a fact usually ignored by others.

3.5 Example

This example will point out how to apply Schumacher's approach to finite-dimensional controller design to an evolution equation, and also point out how the coefficient α_i associated with the eigenfunction x_i in the eigenfunction expansion is dependent on the design parameters in the A.R.E. of the LQG controller. This example will demonstrate that it is possible to have the largest coefficient α_i be associated with a stable eigenvalue, and in that case one would not get the "best" controller if one were to choose the eigenfunctions to approximate the filter operator K_f based on the monotone criterion of Section 3.3 instead of closed-loop pole location. In that case one would possibly fail to stabilize the closed-loop system with

the lowest order controller, and be led to believe that a higher order controller is required.

The example problem is a heat equation which describes the temperature distribution on an isolated uniform rod. The equation is a parabolic equation, and the A operator is such that the nominal system is stable. The operators have been chosen so that solutions can be obtained in order to make an important point concerning eigenfunction selection. Thus, the problem is not intended to be an accurate model of a real system, but rather is intended to be illustrative.

Consider the problem given by

$$\frac{\partial}{\partial t}x(z, t) = \left(\frac{1}{\pi^2} \frac{\partial^2}{\partial z^2}\right)x(z, t) + Iu(t) + Iw(t) \quad t \geq 0; 0 \leq z \leq 1 \quad (79)$$

where I is the identity operator, and with Dirichlet boundary conditions

$$x(0, t) = x(1, t) = 0 \quad \forall t \geq 0$$

and initial condition

$$x(z, 0) = 0 \quad \forall z \in [0, 1]$$

Assume a scalar input ($u \in \mathfrak{R}^1$) and a scalar output given by

$$y(t) = \int_0^1 x(z, t) dz + \eta(t) \quad (80)$$

with $\eta(t) \in \mathfrak{R} \quad \forall t$ (i.e. real-valued). The state space will be $\mathcal{H} = L^2(0, 1)$, and choose the strengths of the white Gaussian, zero-mean noise terms w and η to be $Q_o = I$ and $R_f = I^{n \times n}$ respectively. These choices are made so that the reader can follow the mathematics of the problem without confusion, and thereby focus on the application of Schumacher's approach. These choices are also made in order to get numerical results so that an important point can be made.

The domain of the operator A is defined by

$$D(A) = \{x \in \mathcal{H} \mid \frac{\partial^2}{\partial z^2}x \in \mathcal{H}; x(0) = x(1) = 0\} \quad (81)$$

and the operator A is defined by

$$Ax = \left(\frac{1}{\pi^2} \frac{\partial^2}{\partial z^2}\right)x \quad \forall x \in D(A) \quad (82)$$

The output mapping C is given by

$$Cx = \int_0^1 x(z,t) dz \quad \forall x \in \mathcal{H} \quad (83)$$

which can be written in inner product notation as

$$Cx = \langle 1, x \rangle \quad (84)$$

which is simply the inner product of the function which is 1 everywhere with an element of the Hilbert space \mathcal{H} , and is defined by Equation (83). Also, assume that the output vector y is to be controlled so that the operator Q_c will be chosen to be C^*C .

As shown in Schumacher [59], the operator A generates an analytic semigroup for $t > 0$. Furthermore, A has a discrete spectrum with simple eigenvalues at $-i^2$ ($i = 1, 2, \dots$) with corresponding eigenfunctions given by

$$\phi_i = \sqrt{2} \sin i\pi z \quad (85)$$

which is a complete orthonormal set in $L^2(0,1)$. Because of the simple form of B, G , and Q_c , it can be shown that (A, B) and (A, G) are stabilizable and (A, C) and $(A, Q_c^{1/2})$ are detectable [59].

One can design a steady state constant gain LQ controller assuming full state access as shown in Matson [51]. This yields

$$K_c = R_c^{-1} B^* P_c \quad (86)$$

Let $R_c = I^{n \times n}$ (for the same reasons that R_f was chosen to be the identity matrix), so that $K_c = P_c$ where P_c satisfies the Riccati equation

$$\langle Ah, P_c k \rangle + \langle P_c h, Ak \rangle + \langle Q_c h, k \rangle = \langle P_c B R_c^{-1} B^* P_c h, k \rangle \quad \forall h, k \in D(A) \quad (87)$$

As shown in Curtain [24], P_c can be written as

$$P_c h = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p_{ij} \langle h, \phi_i \rangle \phi_j \quad (88)$$

Also, one can solve for the p_{ij} 's by solving the equation (see [24] page 97 for details)

$$(\lambda_i + \lambda_j) \langle Q_c \phi_i, \phi_j \rangle = \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} p_{ik} p_{jr} \langle R_c^{-1} B^* \phi_k, B^* \phi_r \rangle \quad (89)$$

Choose $Q_c = C^* C = I$. Thus, substituting $\lambda_i = 1$, $B = I$, $R_c = I^{n \times n}$, and $Q_c = I$ yields

$$(-i^2 - j^2) p_{ij} + \langle \phi_i, \phi_j \rangle = \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} p_{ik} p_{jr} \langle \phi_k, \phi_r \rangle \quad (90)$$

or

$$(-i^2 - j^2) p_{ij} + 1 = \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} p_{ik} p_{jr} \quad \forall i = j \quad (91)$$

and $p_{ij} = 0$ when $i \neq j$ because of the orthogonality of the $\{\phi_i\}$'s. This gives that [24]

$$(-2i^2) p_{ii} + 1 = p_{ii}^2 \quad (92)$$

Using this fact then yields

$$P_c h = \sum_{i=1}^{\infty} p_{ii} \langle h, \phi_i \rangle \phi_i = K_c h \quad (93)$$

In a similar way one finds that the steady state Kalman filter constant gain operator is given by

$$K_f = P_f C^* R_f^{-1} \quad (94)$$

where P_f satisfies the Riccati equation

$$\langle A^* h, P_f k \rangle + \langle P_f h, A^* k \rangle + \langle Q_f h, k \rangle = \langle P_f C^* R_f^{-1} C P_f h, k \rangle \quad (95)$$

where for this problem $A^* = A$ and Q_f is given by

$$Q_f h = G Q_o G^* h = Q_o h \quad \forall h \in \mathcal{H} \quad (96)$$

For this example $Q_o = I = Q_f$ in order to simplify the mathematics. Also by duality, P_f can be written as

$$P_f h = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} q_{ij} \langle h, \phi_i \rangle \phi_j \quad (97)$$

The q_{ij} 's can be found from the equation

$$(\lambda_i + \lambda_j) q_{ij} + \langle Q_f \phi_i, \phi_j \rangle = \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} q_{ik} q_{jr} \langle R_f^{-1} C \phi_k, C \phi_r \rangle \quad (98)$$

with $Q_f = I$ and $R_f = I^{n \times n}$ for the reasons mentioned earlier. From the fact that $Ch = \langle 1, h \rangle \quad \forall h \in \mathcal{H}$, one can write

$$(\lambda_i + \lambda_j) q_{ij} + \delta_{ij} = \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} q_{ik} q_{jr} \langle C \phi_k, C \phi_r \rangle \quad (99)$$

or

$$(-i^2 - j^2) q_{ij} + 1 = \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} q_{ik} q_{jr} \langle \phi_k, \phi_r \rangle \quad i = j \quad (100)$$

and $q_{ij} = 0$ otherwise. This gives the following equation for q_{ii} :

$$(-2i^2) q_{ii} + 1 = q_{ii}^2 \quad (101)$$

Thus, the operator K_f can be written as

$$K_f h = \sum_{i=1}^{\infty} q_{ii} \langle h, \phi_i \rangle \phi_i \quad (102)$$

The purpose of this example is to illustrate how the approximation of K_f can be affected by the choice of the approximating eigenfunctions. One must determine if the eigenfunctions of $(A - BK_c)$ are complete. Ah can be written as (see [51])

$$Ah = \sum_{i=1}^{\infty} \lambda_i \langle h, \phi_i \rangle \phi_i \quad (103)$$

where λ_i denotes the eigenvalues of the operator A . Using this expression for A yields the following expression for $(A - BK_c)$:

$$(A - BK_c) = (A - K_c) = \sum_{i=1}^{\infty} \lambda_i \langle h, \phi_i \rangle \phi_i - \sum_{i=1}^{\infty} p_{ii} \langle h, \phi_i \rangle \phi_i \quad (104)$$

which shows that the eigenvalues are shifted by $(\lambda_i - p_{ii})$ for all i , and the eigenfunctions will be the same (and therefore complete).

Using the quadratic formula, one can solve for p_{ii} and q_{ii} as given by

$$p_{ii} = q_{ii} = \frac{-2i^2}{2} + \frac{\sqrt{(-2i^2)^2 + 4}}{2} \quad (105)$$

which for this example gives

$$q_{11} = .41 = p_{11}$$

$$q_{22} = .12 = p_{22}$$

$$q_{33} = .06 = p_{33}$$

and so forth. The resulting regulator poles will be the spectrum of the operator $(A - BK_c)$ which is given by

$$\sigma(A - BK_c) = \lambda_i - q_{ii} \quad i = 1, 2, \dots \quad (106)$$

and the filter poles will be the spectrum of the operator $(A - K_f C)$ which is given by

$$\sigma(A - K_f C) = \lambda_i - p_{ii} \quad i = 1, 2, \dots \quad (107)$$

The amount of spectrum shift that occurs is a function of the operators K_c and K_f , which are determined by the A.R.E.s and therefore are a function of the operators R_f, R_c, Q_f, Q_c, C, B , and C_1 . Let the order of the approximating subspace be $k = 2$ and select the finite-dimensional subspace associated with $\phi_1 = \sqrt{2} \sin \pi z$. This will give the following approximation for the operator K_f

$$\hat{K}_f = \sum_{i=1}^1 q_{ii} \langle *, \phi_i \rangle \phi_i \quad (108)$$

and as a result of restricting the domain of K_c , it can be expressed as:

$$\hat{K}_c = \sum_{i=1}^1 p_{ii} \langle *, \phi_i \rangle \phi_i \quad (109)$$

The resulting regulator poles will be given by $\sigma(A - BK_c)$ which is

$$\mu_1 = \lambda_1 - q_{11} = -1 - .41 = -1.41$$

$$\mu_i = \lambda_i \quad i = 2, 3, \dots$$

Since for this problem $p_{ii} = q_{ii}$, then the filter will have the same poles.

If the operators R_f and/or Q_f were chosen differently, the values of q_{ii} could be changed so that they are not monotone decreasing. For instance, let Q_f be such that it acts like the identity operator except for when it maps ϕ_1 . For that case let $Q_f : \phi_1 \rightarrow (.1)\phi_1$. Then one finds that q_{11} equals .041 and is less than q_{22} . Then if one chooses ϕ_2 as the approximating eigenfunction because it has the largest coefficient associated with it, the eigenvalue at 0 is not shifted and the system will not be stable. Thus, choosing the eigenfunctions in order to get a monotone sequence from $k = 1$ on out may not be the best thing to do. It is more important to choose the approximating eigenfunctions based on where the resulting closed-loop poles lie.

Since the total closed-loop poles are the poles of the regulator and the poles of the filter, exponential stability may or may not be affected by approximating K_f . If one had chosen the noise strengths differently, the filter and regulator may not have had the same poles. In that case, one might have chosen a different approximation for K_f such that the pole of the regulator closest to the right-half plane was not affected. The system stability would then be determined by the regulator, and unaffected by the approximation of K_f . Matson [51] gave an example for which the system stability is determined by the regulator. The question one now has to be concerned with is, how robust is the finite-dimensional controller?

3.6 Summary

This chapter has shown that the LQG compensator obtained by applying the control theory of Chapter 2, satisfies the conditions established by Schumacher (see Theorem 3.2.2) when it is assumed that the eigenfunctions of $(A - BK_c)$

are complete. As discussed in Section 3.2, this assumption can be satisfied if K_c is chosen properly during the LQG design. Thus, it appears that this is not a restrictive assumption.

Section 3.3 proves that the finite-dimensional controller obtained using Schumacher's approach converges to the desired infinite-dimensional LQG controller as the dimension of the controller increases. This proof was not found in Schumacher's work, and is provided in this research for completeness. Section 3.4 analyzes the robustness of the finite-dimensional controller. It is shown that only a limited analysis is possible after the controller has been designed. As a result, Chapters 5 - 7 develop additional robustness techniques that can either be used in conjunction with Schumacher's approach, or can be used as a basis for a different approach to finite-dimensional controller design.

Section 3.5 provides a simple example to illustrate the application of Schumacher's approach. It is pointed out that one must be concerned with the resulting location of the closed-loop poles, as opposed to selecting eigenfunctions so that monotone convergence occurs.

The next chapter presents a sufficient condition that allows the LQG/LTR technique to be extended to the class of problems considered in this research. The approach taken is different from the one taken by Matson in his work [51]. However, the sufficient condition may not be satisfied by the entire class of problems considered in this research. As an alternative, an approximation of the LQG/LTR technique is presented based on the work by Banks [7]. The LQG/LTR technique allows one to achieve robustness by adjusting the defining A.R.E.s, and therefore could be used in conjunction with Schumacher's approach.

IV. LQG/LTR Extension

4.1 Introduction

This chapter will consider the LQG/LTR technique for robustness enhancement. As developed by Doyle and Stein [30] the technique involves adding a pseudonoise term $\beta^2 B V B^*$ to the operator Q_f in the filter A.R.E., and tuning the equation so as to recover, asymptotically, the loop transfer function of the LQ regulator. The technique was extended to a class of DPS by Matson [51], but not to the entire class of systems considered in this research. Specifically, Matson extended the technique to those problems for which A^* is bounded (which includes finite-dimensional problems), and those problems where $R(B) \cup R(Q_f)$ is contained in a finite-dimensional space spanned by a finite number of eigenfunctions of A^* (see [51] Lemma 4.11).

Consider the class of problems described in Section 3 of Chapter 1, with the added assumption that the number of inputs equals the number of outputs. The steady state Kalman filter constant gain operator K_f is given by

$$K_f = P_f C^* R_f^{-1} \quad (110)$$

where the covariance operator P_f is the unique positive self-adjoint solution to the following A.R.E.:

$$\langle P_f h, A^* k \rangle + \langle A^* h, P_f k \rangle + \langle Q_f h, k \rangle = \langle P_f C^* R_f^{-1} C P_f h, k \rangle \quad (111)$$

for all $h, k \in D(A^*)$, under the standard assumption that (A, G) is stabilizable and (A, C) is detectable. Also, note that $Q_f = G Q_o G^*$, where Q_o is the positive semi-definite operator that describes the strength of the dynamics driving noise w (see Equation (1)).

The LQG/LTR technique modifies the noise term so that $Q_f = Q_{f_o} + \beta^2 B V B^*$ where Q_{f_o} is the nominal value of Q_f that provides the best filter tuning at design

conditions, β is a positive scalar, B is the system input operator, and V is any N by N positive definite matrix. This then yields a family of Kalman filter gain operators $K_{f\beta}$ which are given by

$$K_{f\beta} = P_\beta C^* R_f^{-1} \quad (112)$$

and P_β is the solution of the algebraic Riccati equation

$$\langle P_\beta h, A^* k \rangle + \langle A^* h, P_\beta k \rangle + \langle (Q_f + \beta^2 B V B^* h, k) = \langle P_\beta C^* R_f^{-1} C P_\beta h, k \rangle \quad (113)$$

This A.R.E. can be written in terms of the operator Y_β by setting $Y_\beta = \frac{P_\beta}{\beta^2}$. This yields

$$\langle Y_\beta h, A^* k \rangle + \langle A^* h, Y_\beta k \rangle + \langle (B V B^* + \beta^{-2} Q_f) h, k \rangle = \beta^2 \langle Y_\beta C^* R_f^{-1} C Y_\beta h, k \rangle \quad (114)$$

for all $h, k \in D(A^*)$.

The objective of the LQG/LTR technique is to increase the scalar β (and thus the noise term Q_f) so that the following limit exists:

$$\lim_{\beta \rightarrow \infty} K_{f\beta} [I + C(sI - A)^{-1} K_{f\beta}]^{-1} r = B [C(sI - A)^{-1} B]^{-1} r \quad \forall r \in C^N \quad (115)$$

If this limit exists, then one is able to recover the loop transfer function associated with the LQ regulator at the input to the plant (or by using the dual procedure of Kwakernaak and Sivan [49], one can recover the loop transfer function associated with the Kalman filter at the output of the plant). Matson proved that a sufficient condition for this limit to exist is if

$$\beta^2 Y_\beta C^* R_f^{-1} C Y_\beta \xrightarrow{S} B V B^* \quad \text{as } \beta \rightarrow \infty \quad (116)$$

where \xrightarrow{S} denotes strong convergence. A sequence of bounded linear operators K_β is said to converge strongly to a bounded linear operator K if $\lim_{\beta \rightarrow \infty} \|(K_\beta - K)h\| = 0$ for each h in the normed linear space on which K is defined. Other types of convergence include weak and uniform (see [64] for a discussion of these).

This chapter will look at conditions under which this strong convergence exists. Section 4.2 provides one set of sufficient conditions. Section 4.3 explains why many authors have not worried about strong convergence, and then a second set of sufficient conditions is provided. Since neither of the sufficient conditions developed in this chapter are physically motivated, Section 4 considers how to approximate the LQG/LTR technique using the approach developed by Banks [7].

4.2 Sufficient Condition

The following theorem gives a sufficient condition for the results of Matson [51] to be valid for the class of systems considered in this research.

THEOREM 4.2.1: Let \mathcal{H} be a real Hilbert space and $Y_\beta \in \mathcal{L}(\mathcal{H})$ be the unique, positive, self-adjoint solution to the algebraic Riccati equation

$$\langle Y_\beta h, A^* k \rangle + \langle A^* h, Y_\beta k \rangle + \langle (BVB^* + \beta^{-2}Q_f)h, k \rangle = \beta^2 \langle Y_\beta C^* R_f^{-1} C Y_\beta h, k \rangle$$

for all $h, k \in D(A^*)$ where A, B, Q_f, C , and R_f are as described in Section 3 of Chapter 1. Let V be a positive definite N by N matrix, and let β be a scalar ≥ 0 . Also, assume that the number of inputs equals the number of outputs. Then,

$$\lim_{\beta \rightarrow \infty} \beta^2 \langle Y_\beta C^* R_f^{-1} C Y_\beta h, k \rangle = \langle BVB^* h, k \rangle \quad \forall h, k \in D(A^*)$$

implies that

$$\beta^2 Y_\beta C^* R_f^{-1} C Y_\beta \xrightarrow{S} BVB^* \quad \text{as } \beta \rightarrow \infty$$

if the operator

$$K_\beta = (BVB^* + \beta^{-2}Q_f - \beta^2 Y_\beta C^* R_f^{-1} C Y_\beta)$$

is uniformly bounded (in the operator norm) independent of β , for β sufficiently large, and if K_β is positive semi-definite for β large.

The difficulty with this theorem is that, the conditions on K_β are not physically motivated. There is not a known class of systems which satisfy these conditions by

the nature of their state space operators. To use this theorem requires one to confirm the conditions on a problem by problem basis. However, it is this author's opinion that the conditions will be satisfied for a large class of problems. The boundedness assumption on K_β is reasonable because the operator Y_β is bounded independent of β . Thus, K_β appears to be uniformly bounded as well. The proof of the theorem is now presented.

Proof: Since the system is assumed to be stabilizable and detectable, then as shown in Jacobson [47] and Curtain [19], this implies that the system operator A satisfies the spectrum decomposition assumption and has finitely many unstable eigenvalues of finite multiplicity. Let \mathcal{H}_u be the subspace spanned by the eigenfunctions corresponding to the unstable eigenvalues, and \mathcal{H}_s be the subspace spanned by the eigenfunctions corresponding to the stable eigenvalues. Then as shown in Schumacher [59], one can write $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s$ and as such one can decompose the system operators as follows:

$$A = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix}$$

$$Q_f = \begin{bmatrix} Q_{fs} & 0 \\ 0 & Q_{fu} \end{bmatrix}$$

$$B = \begin{bmatrix} B_s \\ B_u \end{bmatrix}$$

$$C = [C_s \quad C_u]$$

and for any $h \in \mathcal{H}$ one can write $h = h_s \oplus h_u$ where $h_s \in \mathcal{H}_s$ and $h_u \in \mathcal{H}_u$. Now, by this construction one can also decompose Y_β as

$$Y_\beta = \begin{bmatrix} Y_{\beta s} & 0 \\ 0 & Y_{\beta u} \end{bmatrix}$$

In this way one can write the algebraic Riccati equation (A.R.E.) in two parts, corresponding to the stable and unstable subspace restrictions. Note that since \mathcal{H}_u

is finite dimensional, the corresponding A.R.E. is a finite dimensional equation and thus, by Lemma 4.11 in Matson [51],

$$\beta^2 Y_{\beta u} C_u^* R_{f_u}^{-1} C_u Y_{\beta u} \xrightarrow{S} B_u V B_u^*$$

Thus, one needs only to show that this convergence holds for the A.R.E. corresponding to the stable subspace restriction. In other words, one needs to show that

$$\beta^2 Y_{\beta s} C_s^* R_{f_s}^{-1} C_s Y_{\beta s} \xrightarrow{S} B_s V B_s^*$$

Now, as shown in Curtain and Gibson [38, 24], one can write $Y_{\beta} h$ as

$$Y_{\beta} h = \int_0^{\infty} T^*(s) (B V B^* + \beta^{-2} Q_f - \beta^2 Y_{\beta} C^* R_f^{-1} C Y_{\beta}) T(s) h ds \quad \forall h \in \mathcal{H}$$

where $T(t)$ is the stable semigroup generated by A_s . This is a different formulation from the one used by Matson [51] who used the identity

$$\langle x_o, Y_{\beta} x_o \rangle = \min_u \left\{ \int_0^{\infty} [\langle x(s), (B V B^* + \beta^{-2} Q_f) x(s) \rangle + \beta^{-2} \langle u(s), R_f u(s) \rangle] ds \right\}$$

Also, Matson used the semigroup generated by the operator A instead of the semigroup generated by the operator A_s . The expression for Y_{β} used in this theorem yields the following inner product equation

$$\langle Y_{\beta} h, k \rangle = \left\langle \int_0^{\infty} T^*(s) K_{\beta} T(s) h ds, k \right\rangle \quad \forall h, k \in \mathcal{H}$$

where $K_{\beta} = (B V B^* + \beta^{-2} Q_f - \beta^2 Y_{\beta} C^* R_f^{-1} C Y_{\beta})$.

By continuity of the inner product, Curtain (see [24] page 93) shows that one can write

$$\begin{aligned} \langle Y_{\beta} h, k \rangle &= \left\langle \int_0^{\infty} T^*(s) K_{\beta} T(s) h ds, k \right\rangle \\ &= \int_0^{\infty} \langle K_{\beta} T(s) h, T(s) k \rangle ds \end{aligned}$$

Now Matson [51] has shown that, for the case where the number of inputs equals the number of outputs, and under the assumptions of this development,

$$\lim_{\beta \rightarrow \infty} \langle Y_{\beta} h, k \rangle = 0$$

so that, from the previous equality,

$$\lim_{\beta \rightarrow \infty} \left\langle \int_0^\infty T^*(s) K_\beta T(s) h, k \right\rangle ds = 0 \quad \forall h, k \in \mathcal{H}$$

Next, it will be shown that for K_β positive semi-definite,

$$\int_0^\infty \|K_\beta T(s)h\|^2 ds \leq \left\{ \int_0^\infty \langle K_\beta T(s)h, T(s)h \rangle ds \right\} \left\{ \int_0^\infty \langle K_\beta^2 T(s)h, K_\beta T(s)h \rangle \right\}$$

This will be needed in order to show that $\lim_{\beta \rightarrow \infty} \int_0^\infty \|K_\beta T(s)h\|^2 ds = 0$. Let $f(s) = T(s)h$, and define a pseudonorm on $f(s)$ as:

$$\| \| f \| \| = [f, f]^{1/2}$$

where $[f, y] \equiv \int_0^\infty \langle K_\beta f(s), y(s) \rangle ds$. A pseudonorm differs from a norm only in that $\| \| f \| \|$ can be zero for $f \neq 0$. Since K_β is positive semi-definite, and not strictly positive, this can occur. Note that since $K_\beta \geq 0$, then $[f, f] \geq 0 \quad \forall f$. Thus, for any $\delta, \epsilon \in \mathfrak{R}$, and f and z in \mathcal{H} ,

$$0 \leq \| \| \delta f + \epsilon z \| \|^2$$

which implies

$$0 \leq \delta^2 \| \| f \| \|^2 + 2\delta\epsilon [f, z] + \epsilon^2 \| \| z \| \|^2$$

For $[f, z] \neq 0$, let $\epsilon = \frac{[f, z]}{\| \| f \| \|^2}$ to yield

$$0 \leq \delta^2 \| \| f \| \|^2 + 2\delta | [f, z] | + \| \| z \| \|^2$$

Now let $\delta = \frac{-| [f, z] |}{\| \| f \| \|^2}$. This then implies that:

$$0 \leq \frac{[f, z]^2}{\| \| f \| \|^2} - \frac{2[f, z]^2}{\| \| f \| \|^2} + \| \| z \| \|^2$$

This implies that

$$| [f, z] | \leq \| \| f \| \| \| \| z \| \|$$

Letting $f(s) = T(s)h$ and $z(s) = K_\beta T(s)h$, from the last inequality one gets that

$$\int_0^\infty \|K_\beta T(s)h\|^2 ds \leq \left\{ \int_0^\infty \langle K_\beta T(s)h, T(s)h \rangle ds \right\} \left\{ \int_0^\infty \langle K_\beta^2 T(s)h, K_\beta T(s)h \rangle \right\}$$

Now, it can be shown that if $K_\beta \geq 0$ and if

$$\lim_{\beta \rightarrow \infty} \left\langle \int_0^\infty T^*(s) K_\beta T(s) h, k \right\rangle ds = 0$$

then, when K_β is uniformly bounded independent of β , one gets that:

$$\lim_{\beta \rightarrow \infty} \int_0^\infty \|K_\beta T(s) h\|^2 ds = 0$$

It was shown that

$$\int_0^\infty \|K_\beta T(s) h\|^2 ds \leq \left\{ \int_0^\infty \langle K_\beta T(s) h, T(s) h \rangle ds \right\} \left\{ \int_0^\infty \langle K_\beta^2 T(s) h, K_\beta T(s) h \rangle \right\}$$

However,

$$\int_0^\infty \langle K_\beta^2 T(s) h, K_\beta T(s) h \rangle ds \leq \int_0^\infty \|K_\beta\|^3 \|T(s)\|^2 \|h\|^2 ds$$

which yields

$$\int_0^\infty \langle K_\beta^2 T(s) h, K_\beta T(s) h \rangle ds \leq k^3 \|h\|^2 \int_0^\infty \|T(s)\|^2 ds \equiv D < \infty$$

where $\|K_\beta\| \leq k$. Thus, the integral is bounded independent of β . Since

$$\lim_{\beta \rightarrow \infty} \left\langle \int_0^\infty T^*(s) K_\beta T(s) h, k \right\rangle ds = 0$$

Then

$$\int_0^\infty \|K_\beta T(s) h\|^2 ds \leq \left\{ \int_0^\infty \langle K_\beta T(s) h, T(s) h \rangle ds \right\} \left\{ \int_0^\infty \langle K_\beta^2 T(s) h, K_\beta T(s) h \rangle \right\}$$

yields the result that

$$\lim_{\beta \rightarrow \infty} \int_0^\infty \|K_\beta T(s) h\|^2 ds = 0$$

With this last result, and using the fact that K_β is uniformly bounded independent of β (which is not integer valued) for β sufficiently large, it can now be shown that:

$$\lim_{\beta \rightarrow \infty} \|K_\beta h\| = 0$$

for all $h \in \mathcal{H}$, which is the desired convergence result. For any $t > 0$, one gets that

$$\begin{aligned}
 | \|K_\beta T(t)h\|^2 - \|K_\beta h\|^2 | &= | \int_0^t \frac{d}{ds} \|K_\beta T(s)h\|^2 ds | \\
 &= 2 | \int_0^t \langle K_\beta Ah, K_\beta T(s)h \rangle ds | \\
 &\leq 2 [\int_0^t \|K_\beta Ah\|^2 ds]^{1/2} [\int_0^t \|K_\beta T(s)h\|^2 ds]^{1/2} \\
 &= 2t^{1/2} \|K_\beta Ah\| [\int_0^t \|K_\beta T(s)h\|^2 ds]^{1/2} \\
 &\leq 2t^{1/2} M_h [\int_0^t \|K_\beta T(s)h\|^2 ds]^{1/2} \\
 &\leq 2t^{1/2} M_h [\int_0^\infty \|K_\beta T(s)h\|^2 ds]^{1/2}
 \end{aligned}$$

Since K_β is uniformly bounded, then $\|K_\beta Ah\| \leq M_h$. Now, choose $t = t_\beta$ where

$$t_\beta = [\int_0^\infty \|K_\beta T(s)h\|^2 ds]^{-1/2}$$

Notice that $t_\beta \rightarrow \infty$ as $\beta \rightarrow \infty$. Also, substituting this choice of $t = t_\beta$ into the inequality

$$| \|K_\beta T(t)h\|^2 - \|K_\beta h\|^2 | \leq 2t^{1/2} M_h [\int_0^\infty \|K_\beta T(s)h\|^2 ds]^{1/2}$$

yields

$$| \|K_\beta T(t_\beta)h\|^2 - \|K_\beta h\|^2 | \leq 2M_h [\int_0^\infty \|K_\beta T(s)h\|^2 ds]^{1/4}$$

This last inequality implies that

$$\|K_\beta h\|^2 \leq \|K_\beta T(t_\beta)h\|^2 + 2M_h [\int_0^\infty \|K_\beta T(s)h\|^2 ds]^{1/4}$$

But since

$$\begin{aligned}
 \|K_\beta T(t_\beta)h\| &\leq \|K_\beta\| \|T(t_\beta)\| \|h\| \\
 &\leq \|K_\beta\| M e^{\omega t_\beta} \|h\| \\
 &\leq D \|h\| e^{\omega t_\beta}
 \end{aligned}$$

where D is a constant depending on the bound of K_β and the semigroup constant M . Since $\omega < 0$ ($T(t)$ is a stable semigroup generated by A_s) and $t_\beta \rightarrow \infty$ as $\beta \rightarrow \infty$, then

$$D\|h\|e^{\omega t_\beta} \rightarrow 0 \quad \text{as } \beta \rightarrow \infty$$

This then yields from the inequality for $\|K_\beta h\|^2$ that

$$\|K_\beta h\| \rightarrow 0 \quad \text{as } \beta \rightarrow \infty \quad \forall h \in \mathcal{H}$$

since it was established that $\int_0^\infty \|K_\beta T(s)h\|^2 ds$ as $\beta \rightarrow \infty$. Thus, the desired convergence has been proven for K_β being uniformly bounded independent of β , and for K_β positive semi-definite. **Q.E.D.**

Since strong convergence is a sufficient condition for the LQG/LTR technique to be valid, then Theorem 4.2.1 gives a sufficient condition for the technique in terms of the operator K_β . Since Y_β is bounded independent of β , it is reasonable to assume that K_β is also. Therefore, the only restrictive assumption would appear to be the assumption that K_β is positive semi-definite. Theorem 4.2.1 took a different approach than that found in Matson [51], and therefore may yield insights not before available that may allow the LQG/LTR to be extended to the entire class of problems considered in this research (see Chapter 1, Section 3). The next section discusses why many authors have not been concerned with strong convergence (and thus have ignored the issue), and also provides another sufficient condition for strong convergence.

4.3 Conditions for Strong Convergence

This section will consider why it is difficult to prove strong convergence exists, and why many authors have avoided the issue when dealing with algebraic Riccati equations on infinite-dimensional spaces. The results of Lukes and Russell [50] will be applied to demonstrate that the A.R.E. of Theorem 4.2.1 has a bounded extension to the entire Hilbert space \mathcal{H} , which is why most authors have considered

the A.R.E. to be defined on the entire Hilbert space. A bounded extension \mathcal{B}_ε is a bounded linear operator that equals an operator Ξ , the operator to be extended, on $D(\Xi)$, and which is defined on the entire Hilbert space \mathcal{H} containing $D(\Xi)$.

The difficulty in proving that strong convergence exists, is the fact that the algebraic Riccati equation (A.R.E.) is only defined on $D(A^*)$. The equation has been considered by many to be defined on the entire Hilbert space \mathcal{H} because of the following lemma from Lukes and Russell [50].

LEMMA 4.3.1: The operator $(Y_\beta A^* + AY_\beta)$ associated with the A.R.E. given by

$$\begin{aligned} & \langle (Y_\beta A^* + AY_\beta)h, k \rangle + \langle BVB^*h, k \rangle + \langle \beta^{-2}Q_f h, k \rangle \\ & = \beta^2 \langle Y_\beta C^* R_f^{-1} C Y_\beta h, k \rangle \quad \forall h, k \in D(A^*) \end{aligned}$$

has a bounded linear extension to all of \mathcal{H} for each β .

Proof: (see Lukes and Russell [50]). From Theorem 4.3.1, for $h, k \in D(A^*)$, one has that

$$\langle A^*h, Y_\beta k \rangle + \langle Y_\beta h, A^*k \rangle = \langle \mathcal{B}_\varepsilon h, k \rangle$$

where \mathcal{B}_ε is a bounded linear operator defined on all of \mathcal{H} , and $\mathcal{B}_\varepsilon = \beta^2 Y_\beta C^* R_f^{-1} C Y_\beta - (BVB^* + Q_f/\beta^2)$. Since \mathcal{B}_ε is defined on all of \mathcal{H} , then it is true that $(Y_\beta A^* + AY_\beta)$ has a bounded extension equal to \mathcal{B}_ε on all of \mathcal{H} for each β . One should note that \mathcal{B}_ε is dependent on β , so that there is not necessarily a \mathcal{B}_ε that satisfies the A.R.E. of Theorem 4.3.1 for all β . **Q.E.D.**

It is not known whether or not one can show that \mathcal{B}_ε of Lemma 4.3.1 is uniformly bounded for all β . If one could show that to be true, then K_β of Theorem 4.2.1 would be uniformly bounded. Another useful insight comes from Gibson [38], who demonstrates that Y_β maps $D(A^*)$ onto $D(A)$. This fact allows one to write

$\langle Y_\beta h, A^* k \rangle$ as $\langle AY_\beta h, k \rangle$ so that the A.R.E. can be written as

$$\begin{aligned} & \langle Y_\beta A^* h, k \rangle + \langle AY_\beta h, k \rangle + \langle BVB^* h, k \rangle + \langle Q_f / \beta^2 h, k \rangle \\ & = \beta^2 \langle Y_\beta C^* R_f^{-1} CY_\beta h, k \rangle \quad \forall h, k \in D(A^*) \end{aligned} \quad (117)$$

and the strong convergence of Y_β to 0 [51], for the class of systems in Theorem 4.2.1, yields

$$\lim_{\beta \rightarrow \infty} \beta^2 \langle Y_\beta C^* R_f^{-1} CY_\beta h, k \rangle = \langle BVB^* h, k \rangle \quad \forall h, k \in D(A^*) \quad (118)$$

Therefore, the question one needs to ask is, when does $\beta^2 Y_\beta C^* R_f^{-1} CY_\beta \rightarrow BVB^*$ strongly as $\beta \rightarrow \infty$? The following lemma gives a sufficient condition for this to happen.

LEMMA 4.3.2: For the assumptions of Theorem 4.2.1, if $\langle \beta^2 Y_\beta C^* R_f^{-1} CY_\beta h, h \rangle$ is a monotone (either increasing or decreasing) sequence for increasing β (see Definition 3.3.1), then

$$\beta^2 Y_\beta C^* R_f^{-1} CY_\beta \xrightarrow{s} BVB^*$$

where convergence is strong.

Proof: Only the case of $\beta^2 Y_\beta C^* R_f^{-1} CY_\beta$ being monotonically increasing will be considered. If $\beta^2 Y_\beta C^* R_f^{-1} CY_\beta$ is monotonically increasing, then, since the limit of Equation (118) exists, it must be true that $[BVB^* - \beta^2 Y_\beta C^* R_f^{-1} CY_\beta]$ is positive $\forall \beta$ and $\forall h \in D(A^*)$. Then Theorem 3.6 in [48], page 453, yields that $\beta^2 Y_\beta C^* R_f^{-1} CY_\beta \rightarrow BVB^*$, with convergence being strong. A similar result applies if $\beta^2 Y_\beta C^* R_f^{-1} CY_\beta$ is monotonically decreasing. In that case the operator $[\beta^2 Y_\beta C^* R_f^{-1} CY_\beta - BVB^*]$ is positive, and again [48] yields that strong convergence exists. **Q.E.D.**

It is not known whether or not this sufficient condition can be demonstrated for the entire class of systems described in Section 3 of Chapter 1. However, it is

believed that it will occur in many problems. As a designer, one must evaluate $\beta^2 Y_\beta C^* R_f^{-1} C Y_\beta$ on a problem by problem basis to determine if it is monotone, to see if the LQG/LTR technique can be used confidently. This may not be easy to do. Even if the operator is not monotone, the LQG/LTR technique may still work since this is only a sufficient condition. The next lemma gives one sufficient design condition for $\beta^2 Y_\beta C^* R_f^{-1} C Y_\beta$ to be monotone. In the lemma, V and R_f will be chosen to be identity operators. These choices are made to simplify the details of the proof, which only requires that V and R_f be as described in Section 4.2. Other assumptions needed in the next lemma are that the inputs u be elements of L_2 space, and that the state variable x be bounded away from zero. The assumption on u is not very restrictive for real problems. The assumption on x implies that the system has stochastic controllability so that x can be bounded away from zero by adding white noise. Two additional assumptions involve the operators B^* and C . These assumptions do not have a physical significance, but rather are chosen in order to make the proof work.

LEMMA 4.3.3: Make the assumptions made in Theorem 4.2.1 (except for the ones for K_β). Let Q_f be a bounded positive linear operator with a bounded square root $Q_f^{1/2}$, and let the positive definite matrix $V = I$, let $R_f = I$, and let

$$\int_0^\infty \|u(s)\|^2 ds \leq M \quad \forall u \in U$$

$$\int_0^\infty \|x(s)\|^2 ds \geq \epsilon > 0 \quad \forall x \in \mathcal{H}$$

and let B^* be such that its inverse exists (not necessarily typical of physical problems) so that

$$\int_0^\infty \|B^* x(s)\|^2 ds \geq k \int_0^\infty \|x(s)\|^2 ds \quad , k > 0$$

Finally, let C be a unitary operator. Then under these conditions,

$$\beta^2 Y_\beta C^* R_f^{-1} C Y_\beta$$

is monotone increasing.

Proof: Since Y_β satisfies the A.R.E., then it can be expressed [24] as

$$\langle x_o, Y_\beta x_o \rangle = \min_u \int_0^\infty [\langle x(s), \theta x(s) \rangle + \beta^{-2} \langle u(s), R_f u(s) \rangle] ds$$

where $x_o \in \mathcal{H}$ and $\theta = BB^* + Q_f/\beta^2$. This then yields

$$\langle x_o, \beta Y_\beta x_o \rangle = \min_u \int_0^\infty [\langle x(s), \beta \theta x(s) \rangle + \beta^{-1} \langle u(s), R_f u(s) \rangle] ds$$

As demonstrated by [24], the minimum is attained with $u = -K_f x$. Let u^* and x^* be the solutions obtained by performing the minimization. Then one gets

$$\langle x_o, \beta Y_\beta x_o \rangle = \int_0^\infty [\langle x^*, \beta \theta x^* \rangle + \beta^{-1} \langle u^*, R_f u^* \rangle] ds$$

Substituting for θ then gives

$$\begin{aligned} \langle x_o, \beta Y_\beta x_o \rangle = \int_0^\infty [\langle x^*, \beta BB^* x^* \rangle + 1/\beta \langle x^*, Q_f x^* \rangle + \\ 1/\beta \langle u^*, R_f u^* \rangle] ds \end{aligned}$$

which is equivalent to

$$\begin{aligned} \langle x_o, \beta Y_\beta x_o \rangle = \int_0^\infty \beta \| B^* x^* \|^2 ds + 1/\beta \int_0^\infty \| Q_f^{1/2} x^* \|^2 ds + \\ 1/\beta \int_0^\infty \| R_f^{1/2} x^* \|^2 ds \end{aligned}$$

Define this inner product as $H(\beta) = \langle x_o, \beta Y_\beta x_o \rangle$. This inner product will be an increasing function of β if its derivative $H'(\beta)$ is positive, which occurs if

$$H'(\beta) = \int_0^\infty \| B^* x^* \|^2 ds - 1/\beta^2 \int_0^\infty \| Q_f^{1/2} x^* \|^2 ds - 1/\beta^2 \int_0^\infty \| R_f^{1/2} x^* \|^2 ds$$

is positive. The fact that $Q_f^{1/2}$ is a bounded operator and the assumptions of the lemma yield

$$H'(\beta) \geq K \int_0^\infty \| x^* \|^2 ds - \frac{\tilde{K}}{\beta^2} \int_0^\infty \| x^* \|^2 ds - \frac{M}{\beta^2} \int_0^\infty \| x^* \|^2 ds$$

$$H'(\beta) > K\varepsilon - \frac{\tilde{K}\varepsilon}{\beta^2} - \frac{M}{\beta^2}$$

$$H'(\beta) > K\varepsilon - \frac{\tilde{K}\varepsilon + M}{\beta^2}$$

which, for β sufficiently large, is positive $\forall x_o \in \mathcal{H}$. Now, since βY_β is self-adjoint, then it is also normal (i.e. $\beta Y_\beta \beta Y_\beta^* = \beta Y_\beta^* \beta Y_\beta$). This yields that the square root is normal [41]. Then by Theorem 5.23.15 in [52],

$$\| \beta^{1/2} Y_\beta^{1/2} x_o \|^2 = \| \beta Y_\beta x_o \|^2 = \| (\beta^{1/2} Y_\beta^{1/2})^2 x_o \|^2$$

Thus, if $\langle x_o, \beta Y_\beta x_o \rangle$ increases as β increases (which is proven in [51]), then $\| \beta^{1/2} Y_\beta^{1/2} x_o \|^2$ increases, as does $\| \beta Y_\beta x_o \|^2$. This gives that $\langle \beta Y_\beta x_o, \beta Y_\beta x_o \rangle$ is a monotonically increasing function for β sufficiently large.

Now, let C be a unitary operator. Then for $R_f = I$,

$$\begin{aligned} \langle \beta^2 Y_\beta C^* R_f^{-1} C Y_\beta x_o, x_o \rangle &= \langle R_f^{-1/2} C \beta Y_\beta x_o, R_f^{-1/2} C \beta Y_\beta x_o \rangle \\ &= \langle C \beta Y_\beta x_o, C \beta Y_\beta x_o \rangle \\ &= \langle C^* C \beta Y_\beta x_o, \beta Y_\beta x_o \rangle \\ &= \langle \beta Y_\beta x_o, \beta Y_\beta x_o \rangle \end{aligned}$$

so that for β sufficiently large, $\beta^2 Y_\beta C^* R_f^{-1} C Y_\beta$ is a monotonically increasing operator. **Q.E.D.**

The problem with Lemma 4.3.3 is that the conditions are not physically motivated. In the case where the input space is finite-dimensional, $\| R_f^{-1/2} u(s) \|^2$ will be bounded and stochastic controllability would imply that the states can be bounded away from zero by adding white noise. However, one cannot usually expect to satisfy the condition

$$\int_0^\infty \| B^* x(s) \|^2 ds \geq k \int_0^\infty \| x(s) \|^2 ds \quad (119)$$

or to have the operator C be a unitary operator. The lemma does give some added insight into how to approach showing that $\beta^2 Y_\beta C^* R_f^{-1} C Y_\beta$ is a monotonically increasing operator for other classes of problems. For example, it may be possible to show that, for certain problems,

$$\frac{\int_0^\infty \| Q_f^{1/2} x^* \|^2 ds + \int_0^\infty \| R_f^{1/2} x^* \|^2 ds}{\int_0^\infty \| B^* x(s) \|^2 ds} \leq K \quad (120)$$

i.e., it is uniformly bounded $\forall x$ and u . This would result in $H'(\beta)$ being positive, so that (using a proof similar to that of Lemma 4.3.3) one can show there exists a β such that $\langle \beta Y_\beta x_o, \beta Y_\beta x_o \rangle$ is monotone. Also, it may be possible to prove that $\langle C \beta Y_\beta x_o, C \beta Y_\beta x_o \rangle$ is monotone for arbitrary $C \in \mathcal{L}(\mathcal{H}; \mathfrak{R}^N)$, given that $\langle \beta Y_\beta x_o, \beta Y_\beta x_o \rangle$ is monotone.

However, if $\beta^2 Y_\beta C^* R_f^{-1} C Y_\beta$ is not monotone, then nothing can be said about whether or not the LQG/LTR technique is valid, because Lemma 4.3.2 only provides a sufficient condition, as opposed to necessary and sufficient. Since a set of physically meaningful conditions was not obtained to ensure the desired strong convergence, one may try to apply the procedure in an ad hoc fashion, or one may try to use another approach to achieve robustness. One alternative is to try to approximate the LQG/LTR procedure. The next section will look at using the approximation approach of Banks [7] as a way to approximate the LQG/LTR procedure.

4.4 LQG/LTR Approximation

With Schumacher's approach (see Chapter 3), one first must have an infinite-dimensional compensator that he wants to approximate. For the LQG design, this means having to solve an infinite-dimensional algebraic Riccati equation (A.R.E.), which may be extremely difficult to do. However, assuming one can perform this solution, one really has only two known parameters to adjust for robustness. These are the exponential time constant for the case of bounded perturbations, and the semigroup gain constant for relative bounded perturbations, as discussed

in Section 4 of Chapter 3. Therefore, when using Schumacher's approach, it would seem that one would want to make the exponential time constant as negative as possible and the gain constant as small as possible in order to improve robustness.

Another approach to robust LQG controller design would be to make the robustness recovery a part of the A.R.E. This is the idea behind the LQG/LTR technique. In this way one "tunes" the filter in order to regain the guaranteed robustness of the LQ regulator [30], or one can use Kwakernaak and Sivan's [49] dual procedure, depending on the point in the loop where one chooses to enhance robustness. If one chooses to enhance robustness at the input to the plant (point 4 of Figure 1 in Chapter 2), then one can use Doyle and Stein's approach [62] to recover the loop transfer function of the regulator at the plant input. If one enhances robustness at the plant output (point 1 of Figure 1 in Chapter 2), one can use the dual procedure of Kwakernaak and Sivan [49] to recover the loop transfer function associated with the Kalman filter at the plant output. However, if one were able to apply the LQG/LTR technique validly for our class of systems, one would still have to solve an infinite-dimensional A.R.E., and then use some technique to approximate the desired "robust" compensator. The problem is, that it is not known if the LQG/LTR technique can be validly applied to the entire class of distributed systems considered in this research.

One way to approach designing a robust compensator might be to approximate the infinite-dimensional A.R.E. with a sequence of finite-dimensional A.R.E.s such that the sequence of solutions converges to the infinite-dimensional A.R.E. solution, as done in [7]. In this way, one could approximate the desired solution, and perform robustness enhancement on the approximation. Thus, one could possibly approximate the LQG/LTR technique for the entire class of systems.

In order to make direct use of the results of [7], the dual robustness recovery procedure of Kwakernaak and Sivan will be considered. By duality, the type of results obtained are applicable to the robustness recovery procedure of Doyle and

Stein. For that approach, one merely has to consider the dual A.R.E. and the operators associated with it. The development that follows involves the A.R.E. associated with the LQ regulator gain operator K_c (see Equation (34)). The operator Π used by Banks is the same as the operator P_c that solves Equation (34).

Following the presentation by Banks [7], and using his notation, $\Pi \in \mathcal{L}(\mathcal{H})$ is a solution of the A.R.E. given by

$$A^*\Pi + \Pi A + Q_c - \Pi B R_c^{-1} B^* \Pi = 0 \quad (121)$$

if Π maps $D(A)$ into $D(A^*)$ and if Π satisfies the A.R.E. on $D(A)$.

One thing to note is that, this A.R.E. is actually an inner product equation. The inner product notation has been dropped in order to simplify the writing of the equations. Kwakernaak and Sivan's approach to robustness recovery involves modifying the state cost weighting operator Q_c , by adding the additional cost term $q^2 C^* V C$ to Q_c . Then, as $q \rightarrow \infty$, one is able to asymptotically recover the loop transfer function associated with the Kalman filter, at the plant output. The problem with this LQG/LTR technique is that, in order to converge to the guaranteed stability robustness of the filter, one must show that (see Chapter 3 for the dual discussion)

$$\lim_{q \rightarrow \infty} q^2 \langle \Pi B R_c^{-1} B^* \Pi h, k \rangle = \langle C^* V C h, k \rangle \quad \forall h, k \in D(A) \quad (122)$$

implies strong convergence [51]. Matson [51] has proven the dual case (and thus this case by duality) when A^* is a finite-dimensional (i.e. bounded) operator, and also when $R(B) \cup R(Q_f)$ is contained in a finite-dimensional space spanned by a finite number of eigenfunctions of A^* .

Following the development of Banks [7], the infinite-dimensional state space problem given by

$$\dot{x} = Ax + Bu \quad x_0 \in D(A) \quad (123)$$

$$y = Cx \quad (124)$$

where $x \in \mathcal{H}$, can be redefined by projecting it onto a finite-dimensional subspace of \mathcal{H} (denoted \mathcal{H}_k) by means of an orthogonal projection P^k such that $P^k : \mathcal{H} \rightarrow \mathcal{H}_k$. A projection P^k is said to be orthogonal if its range and null space are orthogonal, if $P^k P^k = P^k$, and if each $h \in \mathcal{H}$ can be written uniquely as $h = r + n$ where r is in the range of P^k , and n is in the null space of P^k . Also, an orthogonal projection is continuous.

This projection is used to define the operators A^k, B^k, C^k and Q_c^k as follows:

$$P^k A = A^k \in \mathcal{L}(\mathcal{H}_k) \quad (125)$$

$$P^k B = B^k \in \mathcal{L}(\mathbb{R}^N; \mathcal{H}_k) \quad (126)$$

$$P^k C = C^k \in \mathcal{L}(\mathcal{H}_k; \mathbb{R}^N) \quad (127)$$

$$P^k Q_c = Q_c^k \in \mathcal{L}(\mathcal{H}_k) \quad (128)$$

Using these operators, the family of finite-dimensional regulator problems described by

$$\dot{x} = A^k x + B^k u; \quad x(0) = P^k x_0 \quad (129)$$

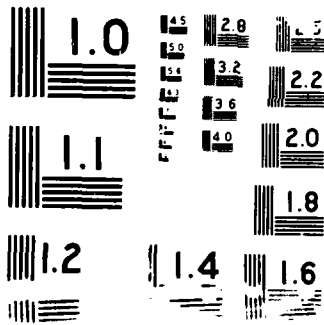
$$y = C^k x \quad (130)$$

with associated cost functional given by

$$J^k = \int_0^\infty [\langle Q_c^k x, x \rangle + \langle R_c u, u \rangle] dt \quad (131)$$

can be considered. It will also be necessary to make the following assumptions on each finite-dimensional problem:

1. For each $x_0 \in \mathcal{H}_k$, there exists an admissible control $u \in L^2(0, \infty; U)$ such that the cost functional J^k is finite
2. (a) Q_c is positive semi-definite on \mathcal{H} , as is $P^k Q_c$ on \mathcal{H}_k for all k
(i.e. $\langle Q_c x, x \rangle \geq 0 \quad \forall x \in \mathcal{H}$) and $\langle Q_c^k x, x \rangle \geq 0 \quad \forall x \in \mathcal{H}_k$



- (b) R_c is positive on the input space U (i.e. $\langle R_c u, u \rangle > 0 \quad \forall u \in U$)
3. (a) $P^k \rightarrow I$ as $k \rightarrow \infty$ in the sense that, for any $x \in \mathcal{H}$, $P^k x \rightarrow x$ as $k \rightarrow \infty$
- (b) $T^k(t)P^k x \rightarrow T(t)x$ as $k \rightarrow \infty$, where $T(t)$ is the semigroup generated by the operator A
- (c) $T^{*k}(t)P^k x \rightarrow T^*(t)x$ as $k \rightarrow \infty$, where $T^*(t)$ is the dual semigroup generated by A^*
- (d) $B^k u \rightarrow Bu$ and $B^{*k} x \rightarrow B^* x$ as $k \rightarrow \infty$
- (e) $Q_c^k P^k \rightarrow Q_c x$ as $k \rightarrow \infty$

Assumption 1 is a standard LQG assumption and is satisfied by our assumptions of stabilizability and detectability made in Chapter 1 and in Chapter 3. Specifically, it is assumed that (A, B) and (A, G) are stabilizable, and (A, C) and $(A, Q_c^{1/2})$ are detectable. Assumption 2 is made in order to use the results of [51] and it is not very restrictive in applications. If assumption 1 holds, then the optimal control for each finite-dimensional problem is given by

$$u^k = -R_c^{-1} B^{*k} \Pi^k x \quad (132)$$

where $\Pi^k \in \mathcal{L}(\mathcal{H}_k)$ is the unique nonnegative self-adjoint solution of the A.R.E. on \mathcal{H}_k given by

$$A^{*k} \Pi^k + \Pi^k A^k + Q_c^k - \Pi^k B^k R_c^{-1} B^{*k} \Pi^k = 0 \quad (133)$$

Assumptions 3 a-f can be satisfied by many approximation schemes, for example modal approximation [7], so that these assumptions are not overly restrictive. In fact, 3(a) implies 3(d) since B is a bounded linear operator. The assumption that (A, B) is exponentially stabilizable along with the assumption that $(A, Q_c^{1/2})$ is exponentially detectable (as is done in this research) allows the hypothesis of Theorem 2.2 of [7] to be satisfied for a large class of distributed parameter systems using several approximation schemes (i.e. modal, spline, Ritz, etc. [36]). The following theorem is taken from [7].

THEOREM 4.4.1 : Suppose that the previous Assumptions 1 thru 3(e) hold. Let $Q_c \geq 0$, $R_c > 0$, and let Π^k denote the unique nonnegative self-adjoint Riccati solution on \mathcal{H}_k . Also assume that a unique nonnegative self-adjoint Riccati operator Π exists for the problem defined on \mathcal{H} . Let $S^k(t)$ and $S(t)$ be the semigroups generated by $(A^k - B^k R_c^{-1} B^{*k} \Pi^k)$ and $(A - B R_c^{-1} B^* \Pi)$ on \mathcal{H}_k (a finite-dimensional subspace of the Hilbert space \mathcal{H}) and \mathcal{H} respectively, and let $\|S(t)x\| \rightarrow 0$ as $t \rightarrow \infty \quad \forall x \in \mathcal{H}$. If there are two positive constants (M_1, M_2) and $\omega < 0$ that are all independent of both k and t such that

$$\text{i) } \|S^k(t)\| \leq M_1 e^{\omega t} \quad \forall t \geq 0 \text{ and } k = 1, 2, \dots,$$

$$\text{ii) } \|\Pi^k\| \leq M_2$$

then, as $k \rightarrow \infty$,

$$\text{a) } \Pi^k P^k x \rightarrow \Pi x \quad \forall x \in \mathcal{H}$$

$$\text{b) } S^k(t) P^k x \rightarrow S(t)x \quad \forall x \in \mathcal{H}$$

where the convergence is uniform in t on bounded subsets of $[0, \infty)$, and

$$\|S(t)\| \leq M_1 e^{\omega t} \quad \forall t \geq 0, \quad \omega < 0$$

Proof: See Banks [7] page 695.

Note that, since the LQ regulator generates a stable semigroup (see Section 4 of Chapter 2), then $\|S(t)x\| \rightarrow 0$ as $t \rightarrow \infty$. Thus, using this approach, one can approximate the solution of the infinite-dimensional A.R.E. with a sequence of

finite-dimensional A.R.E.'s. Banks [7] demonstrates that the hypothesis of Theorem 4.4.1 can be satisfied by many popular approximation schemes (such as modal, spline, etc). Once the solution is approximated closely enough, then one could apply the LQG/LTR technique using the finite-dimensional A.R.E. The drawback to this approach is that guaranteed stability margins to be approached asymptotically are only with respect to the finite-dimensional model, not the infinite-dimensional model. However, models are always an approximation of reality. The idea is to choose a model that matches the real world well enough so that the compensator's robustness can tolerate the model uncertainty. One also wants robustness to deal with model perturbations for the case when a system changes during operation. Even DPS models are approximations, but they are often used because they are better approximations for some systems than finite-dimensional models. Since Banks' approach involves approximating the DPS solution by approximating the associated infinite-dimensional A.R.E. with a sequence of finite-dimensional A.R.E.'s., the development of this section can be viewed as an approximation of the LQG/LTR technique.

If perturbations are modeled as finite-dimensional, then this approach can be used to get a desirable compensator. If not, then this approach provides added robustness with respect to a finite-dimensional model which can be made arbitrarily close to the DPS under consideration. In this way one can say he is approximating the LQG/LTR technique on the infinite-dimensional system. However, for DPS for which the LQG/LTR is not known to be valid, it is not clear what type of robustness is achieved in the limit as the order k of the approximation goes to ∞ . It is believed that the approximation technique will yield desirable results. However, the LQG/LTR technique is just one way to achieve robustness. Other approaches are available, as will be discussed in Chapters 5 through 7.

4.5 Summary

This chapter presents sufficient conditions that allow the LQG/LTR technique to be extended to a class of problems of interest. In particular, Theorem 4.2.1 gives conditions based on the operator $K_\beta = (BV B^* + \beta^{-2} Q_f - \beta^{-2} Y_\beta C^* R_f^{-1} C Y_\beta)$. It is shown that, if K_β is uniformly bounded independent of β , and if K_β is positive semi-definite, then the LQG/LTR technique is valid for the class of problems that satisfy the assumptions of Section 3 in Chapter 1. Theorem 4.2.1 is based on a different development than that found in Matson's work [51], and may provide insight as to how to extend the LQG/LTR technique to the entire class of problems without any conditions on K_β .

Section 4.3 discussed the reason many people have not considered strong convergence an issue. Since the A.R.E. has a bounded extension to the entire Hilbert space \mathcal{H} , some have simply assumed strong convergence. It is shown in Section 4.3, that weak convergence does exist for the class of problems considered in this research. Section 4.3 also has a second set of sufficient conditions that allow the LQG/LTR technique to be extended. The conditions are not physically meaningful, but Lemma 4.3.3 provides yet another approach to solving the problem of extending the LQG/LTR technique.

Since the technique has not been extended to the entire class of problems considered in this research, an approximation of the technique is developed, based on the work of Banks [7]. Section 4.4 shows how to approximate the LQG/LTR technique by approximating the infinite-dimensional A.R.E. with a sequence of finite-dimensional A.R.E.s. Since the LQG/LTR technique is valid for finite-dimensional systems, then one can use the finite-dimensional A.R.E.'s. to approximate the technique.

The LQG/LTR technique is just one way to achieve robustness. Another technique being currently taken is the optimal projection equation (O.P.E.) approach

developed by Bernstein and others. The next chapter extends the O.P.E. approach based on the work by Bernstein [10].

V. Optimal Projection Equations

5.1 Introduction

Another approach being currently taken to address the problem of robust reduced order controllers [10, 11, 46, 9, 45, 39, 40] is to fix the order of the compensator based on physical constraints, and determine the optimal robust controller using the optimal projection equation approach [11].

Ignoring the issue of robustness for the moment, Bernstein [10] gives a set of necessary conditions for a reduced order controller to be the "optimum controller". A controller will be considered "optimum" if it stabilizes the system at design conditions, and if it produces a feedback control law that minimizes a desired cost functional which characterizes the system's steady state performance. The cost functional to be minimized will be of the form

$$J = \lim_{t \rightarrow \infty} \mathcal{E}[\langle Q_c x(t), x(t) \rangle + \langle R_c u(t), u(t) \rangle]$$

where \mathcal{E} denotes the expectation operator. This is discussed in more detail in Section 3.

Let Λ and Γ be a bounded linear operators mapping $\mathcal{H} \rightarrow \mathfrak{R}^k$ (where k is the dimension of the finite-dimensional controller) such that $\Gamma\Lambda^* = I_k$. I_k is the identity operator on \mathfrak{R}^k . The conditions developed by [10] are a pair of modified Riccati equations and a pair of coupled Lyapunov equations. The two modified A.R.E.s are given by

$$AQ + QA^* + Q_f - QC^*R_f^{-1}CQ + \tau_{\perp}QC^*R_f^{-1}CQ\tau_{\perp}^* = 0 \quad (134)$$

$$A^*P + PA + Q_c - PBR_c^{-1}B^*P + \tau_{\perp}^*PBR_c^{-1}B^*P\tau_{\perp} = 0 \quad (135)$$

and the two Lyapunov equations are given by

$$(A - BR_c^{-1}B^*P)\hat{Q} + \hat{Q}(A - BR_c^{-1}B^*P)^* + QC^*R_f^{-1}CQ - \tau_{\perp}QC^*R_f^{-1}CQ\tau_{\perp}^* = 0 \quad (136)$$

$$(A - QC^*R_f^{-1}C)^* \hat{P} + \hat{P}(A - QC^*R_f^{-1}C) + PBR_c^{-1}B^*P - \tau_{\perp}^*PBR_c^{-1}B^*P\tau_{\perp} = 0 \quad (137)$$

where the operator τ is defined by the operators Λ and Γ ($\tau = \Lambda^*\Gamma$) which determine the projection of the full order compensator to a fixed order compensator.

The projection operator τ_{\perp} satisfies the relation

$$\tau_{\perp} = I - \tau = I - \Lambda^*\Gamma \quad (138)$$

and using the operators Λ and Γ , the compensator is defined by the equations

$$\dot{x}_c = A_c x_c + B_c y \quad (139)$$

$$u = C_c x_c \quad (140)$$

where A_c , B_c and C_c are given by

$$A_c = \Gamma(A - QC^*R_f^{-1}C - BR_c^{-1}B^*P)\Lambda^* \quad (141)$$

$$B_c = \Gamma QC^*R_f^{-1} \quad (142)$$

$$C_c = -R_c^{-1}B^*P\Lambda^* \quad (143)$$

Notice that the Lyapunov equations are coupled to the A.R.E.s by the projection operator τ_{\perp} . If the order of the system equals the order of the compensator, then $\tau_{\perp} = 0$, and one gets the standard LQG A.R.E.s, and the Lyapunov equations become decoupled from the A.R.E.s. In that case, the Lyapunov equations can be satisfied separately from the A.R.E.s, and the standard LQG results are obtained. For the case when the order of the compensator is less than the system order, one might wonder if the optimum projection equations (O.P.E.) can be modified not only to stabilize the system and minimize the associated cost functional J , but also to provide robustness as is done using the LQG/LTR technique. The next section will consider whether or not loop transfer functions can be recovered when the order of the compensator is less than or equal to the order of the system.

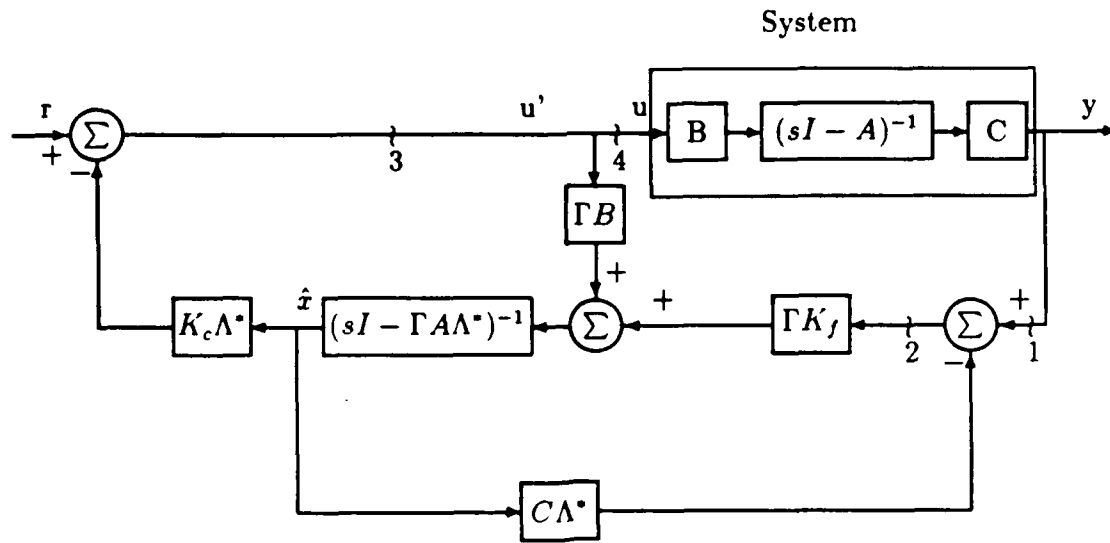


Figure 4. System with Finite-Dimensional LQG Based Controller

5.2 Loop Recovery with Reduced Order Controllers

Let Λ and Γ be the bounded linear operators that define the projection of the full order compensator. Figure 4 represents the infinite-dimensional system using the finite-dimensional controller.

The infinite-dimensional system is described by the equations

$$\dot{x} = Ax + Bu \quad (144)$$

$$y = Cx \quad (145)$$

and the compensator is described by Equations (139)-(143). With A_c , B_c and C_c given as in Equations (141)-(143), the compensator state equations can be written as

$$\dot{x}_c = (\Gamma A \Lambda^* - \Gamma K_f C \Lambda^* - \Gamma B K_c \Lambda^*) x_c + \Gamma K_f y \quad (146)$$

or:

$$\dot{x}_c = \Gamma A \Lambda^* x_c + \Gamma B u + \Gamma K_f (y - C \Lambda^* x_c) \quad (147)$$

where $u = -K_c \Lambda^* x_c$.

The system input appears at point 4 in Figure 4. This point is physically important since it is one point where the compensator interfaces with the system

being controlled (the other point that is important is point 1 where the system output interfaces with the compensator). The transfer function at point 4 is given by

$$G_4(s) = K(s)P(s) = C_c(sI - A_c)^{-1}B_cC(sI - A)^{-1}B \quad (148)$$

which can be written as

$$G_4(s) = K_c\Lambda^*(\Phi_c^{-1} + \Gamma K_f C\Lambda^* + \Gamma B K_c\Lambda^*)^{-1}\Gamma K_f C\Phi_p B \quad (149)$$

where the controller state transition operator is $\Phi_c = (sI - \Gamma A\Lambda^*)^{-1}$ and the plant state transition operator is $\Phi_p = (sI - A)^{-1}$. Since A generates a C_0 semigroup, $(sI - A)^{-1}$ will exist. $(sI - \Gamma A\Lambda^*)^{-1}$ will exist since $\Gamma A\Lambda^*$ is a finite-dimensional operator. Note that when the order of the compensator is less than order of the system, Φ_c does not equal Φ_p . For infinite-dimensional systems, the state transition operator is the semigroup $T(t)$ which is generated by the operator A .

In a similar fashion one can write the transfer function at point 3 in the system. This is the transfer function that one would try to recover at point 4 using the LQG/LTR technique for robustness enhancement. This point is internal to the compensator, and has guaranteed robustness properties as discussed in Section 4 of Chapter 4. Using the equations

$$u = -K_c\Lambda^*x_c \quad (150)$$

and

$$y = C\Phi_p B u' \quad (151)$$

where u' is the input at point 3, the state x_c can be expressed as

$$x_c = (sI - \Gamma A\Lambda^*)^{-1}\Gamma B u' + (sI - \Gamma A\Lambda^*)^{-1}\Gamma K_f y - (sI - \Gamma A\Lambda^*)^{-1}\Gamma K_f C\Lambda^* x_c \quad (152)$$

or equivalently

$$x_c = (I + \Phi_c \Gamma K_f C\Lambda^*)^{-1}[\Phi_c \Gamma B u' + \Phi_c \Gamma K_f y] \quad (153)$$

Substituting for y via Equation (151), and rearranging terms, the state can be written as

$$x_c = (I + \Phi_c \Gamma K_f C \Lambda^*)^{-1} (\Phi_c \Gamma) (I + K_f C \Phi_p) B u' \quad (154)$$

Substituting this into the expression for u , Equation (150), the loop transfer function at point 3 is given as

$$G_3(s) = -K_c \Lambda^* (I + \Phi_c \Gamma K_f C \Lambda^*)^{-1} (\Phi_c \Gamma) (I + K_f C \Phi_p) B \quad (155)$$

If the order of the compensator equals the order of the system, then τ is an identity operator (and τ_{\perp} is a zero operator), and from Equation (155) it is clear that the loop transfer function at point 3 can be expressed as

$$G_3(s) = -K_c \Phi_p B \quad (156)$$

Since Φ_c does not equal Φ_p when a reduced order controller is used, then one cannot recover a desired transfer function that involves an infinite-dimensional design model. Note that, since any model is an approximation of the true system, then the controller order will always be less than the true system order. The issue considered here is the case when the controller order is less than the design model order. The design model is the mathematical model one chooses to describe the physical system to be controlled, and as such, it is an approximation of the true system. In practice, one assumes that the design model is the true system so that a result can be synthesized. Robustness is needed due to the fact that the design model does not equal the true system being controlled. Thus, loop transfer recovery using a finite-dimensional controller can only be accomplished using a reduced order model.

5.3 O.P.E. Robustness

The O.P.E. approach allows one to achieve robustness by modifying the infinite-dimensional A.R.E., and will give a new interpretation of LQG/LTR when

the controller is finite-dimensional but the design model is infinite-dimensional. The O.P.E. approach provides a way to achieve robustness to uncertainty, and to minimize the cost functional J , at conditions other than the nominal design condition [40].

For a fixed order compensator where $k < \dim \mathcal{H} = \infty$, one wants to determine the operators (A_c, B_c, C_c) such that the closed-loop system consisting of the controlled system

$$\dot{x} = (A + \Delta A)x + (B + \Delta B)u \quad (157)$$

where $x \in \mathcal{H}$, along with measurements

$$y = (C + \Delta C)x \quad (158)$$

with $y \in \mathfrak{R}^N$, and a finite-dimensional compensator described by

$$\dot{x}_c = A_c x_c + B_c y \quad (159)$$

$$u = C_c x_c \quad (160)$$

is exponentially stable for all perturbations $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$ where \mathcal{U} is the set of admissible operator triplets describing the perturbations to the operators A, B , and C one wishes to consider. If, for instance, one only allows bounded perturbations, then \mathcal{U} is the set of bounded linear operators which are bounded by some constant, say D . Through \mathcal{U} , one describes the robustness desired.

However, in addition to being exponentially stable, one also would like to minimize the cost functional associated with the optimal control problem at other than design conditions. This will be made clearer later, in Equation (165). The cost functional to be considered is denoted by J and will be defined as

$$J(A_c, B_c, C_c) = \lim_{t \rightarrow \infty} \mathcal{E}[\langle Q_c x(t), x(t) \rangle + \langle R_c u(t), u(t) \rangle] \quad (161)$$

where \mathcal{E} is the expectation operator and is defined as [29]

$$\mathcal{E}(x) = \int_{\Psi} x(\omega) dP \quad (162)$$

where Ψ is the space of all possible ω that the random variable $x(t)$ maps to some Borel space over which a probability measure P is defined. This cost functional is chosen instead of the one given by Equation (33) since the objective of the O.P.E. approach is to achieve optimum steady state performance, and not necessarily optimum performance over the entire time interval.

For a reduced order compensator where $k < \dim \mathcal{H}$, one wants to determine (A_c, B_c, C_c) such that when the closed-loop system consisting of

$$\dot{x} = (A + \Delta A)x + (B + \Delta B)u + Gw \quad (163)$$

with noisy measurements given by

$$y = (C + \Delta C)x + \eta \quad (164)$$

is coupled with the compensator, the steady-state performance cost functional

$$J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C) \in \mathcal{U}} \lim_{t \rightarrow \infty} \mathcal{E} \{ \langle Q_c x(t), x(t) \rangle + \langle R_c u(t), u(t) \rangle \} \quad (165)$$

is minimized. In other words, one wants to design a controller so that the largest value that the cost functional can take on for all possible perturbations is minimized. In the development that follows, G is assumed to be the identity operator (without loss of generality) in order to correspond to the development of Bernstein [9].

A control will be considered admissible only if it forces the cost functional J to take on a finite value. To help simplify the notation, the closed-loop system can be written in terms of the augmented state-space $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathfrak{R}^k$. It is assumed that the noise terms w and η are independent. In terms of the augmented state vector

$$\tilde{x} = [x \quad x_c]^T \quad (166)$$

this yields

$$\dot{\tilde{x}} = (\tilde{A} + \Delta \tilde{A})\tilde{x} + \tilde{G}\zeta \quad (167)$$

where

$$\tilde{G} = \begin{bmatrix} I & 0 \\ 0 & B_c \end{bmatrix}$$

$$\zeta = \begin{bmatrix} w \\ \eta \end{bmatrix}$$

$$\mathcal{E}[\zeta(t)\zeta^T(t+\tau)] = \begin{bmatrix} Q_o & 0 \\ 0 & R_f \end{bmatrix} \delta(\tau)$$

$$\tilde{A} = \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix}$$

$$\Delta\tilde{A} = \begin{bmatrix} \Delta A & \Delta BC_c \\ B_c \Delta C & 0 \end{bmatrix}$$

and

$$\tilde{V} = \tilde{G} \begin{bmatrix} Q_o & 0 \\ 0 & R_f \end{bmatrix} \tilde{G}^*$$

or

$$\tilde{V} = \begin{bmatrix} Q_o & 0 \\ 0 & B_c R_f B_c^* \end{bmatrix}$$

Also, in terms of \tilde{x} , the performance cost functional can be written as

$$J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C) \in \mathcal{U}} \lim_{t \rightarrow \infty} \mathcal{E}[\langle \hat{R}\tilde{x}, \tilde{x} \rangle] \quad (168)$$

where

$$\hat{R} = \begin{bmatrix} Q_c & 0 \\ 0 & C_c^T R_c C_c \end{bmatrix}$$

The following lemma allows one to express the cost functional in terms of the second moment of $\tilde{x}(t)$. This will be needed so that an upper bound of the cost functional can be established in a theorem to follow.

LEMMA 5.3.1: For (A_c, B_c, C_c) and $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$, the performance cost functional can be expressed in terms of the covariance of $\tilde{x}(t)$, defined as:

$$\tilde{Q}_\Delta(t) = \mathcal{E}[(\tilde{x}(t) - \mathcal{E}\tilde{x}(t))(\tilde{x}(t) - \mathcal{E}\tilde{x}(t))^*]$$

Furthermore, if the system is stable for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$, then the performance cost functional can be expressed as

$$J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C) \in \mathcal{U}} \text{tr}[\tilde{Q}_\Delta \tilde{R}]$$

where tr denotes the trace operator, and where \tilde{Q}_Δ satisfies the equation

$$(\tilde{A} + \Delta \tilde{A})\tilde{Q}_\Delta + \tilde{Q}_\Delta(\tilde{A} + \Delta \tilde{A})^* + \tilde{V} = 0$$

Proof: Balakrishnan [1], page 317, defines the covariance of $\tilde{x}(t)$ as the nonnegative-definite operator $\tilde{Q}_\Delta(t)$ given above. Bernstein and Hyland [10], Lemma 4.1, page 137, prove that $J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C) \in \mathcal{U}} \text{tr}[\tilde{Q}_\Delta \tilde{R}]$ and Lemma 4.4, page 139 [10] proves that \tilde{Q}_Δ satisfies the last equation of Lemma 5.3.1. **Q.E.D.**

In the development to follow, the operator \tilde{V} will be modified by a nonnegative operator Υ . A result that will be needed is, if $(\tilde{V}^{1/2}, \tilde{A} + \Delta \tilde{A})$ is detectable, then so is $([\tilde{V} + \Upsilon]^{1/2}, \tilde{A} + \Delta \tilde{A})$ under certain conditions. The following theorem from Wonham [68] gives conditions under which the property of detectability is preserved.

THEOREM 5.3.2: Let \mathcal{H} be a real Hilbert space, and let M_m be a bounded linear operator mapping $\mathcal{H} \rightarrow \mathcal{H}$. If M_m is a nonnegative operator and if $(M_m^{1/2}, A)$ is detectable, then for all nonnegative operators N , the pair $([M_m + N]^{1/2}, A)$ is detectable.

Proof: See Wonham's book [68] page 79, and let the operators of Theorem 3.6, Q and B , be such that $Q = B = 0$. **Q.E.D.**

The next theorem is the main theorem of this section, and it provides sufficient conditions for robust stability and optimum performance. In the theorem to follow, the operator Ω is a positive self-adjoint operator that "bounds" the uncertainty described by the operator $\Delta\tilde{A}$. The operator Ω is part of a Lyapunov condition involving the nominal system operators. The operator \underline{Q} is the bounded positive-semidefinite self-adjoint operator solution for the Lyapunov condition, and is the only unknown in the Lyapunov equation. It is assumed that Ω and \underline{Q} both exist. Satisfying the Lyapunov equation will ensure stability in the presence of perturbations described by the operator $\Delta\tilde{A}$. The next theorem demonstrates that the operator Ω is also a function of the operator \underline{Q} .

THEOREM 5.3.3: Let $\Omega : \mathcal{H} \rightarrow \mathcal{H}$ be a self-adjoint positive operator such that

$$i) \quad \langle \Delta\tilde{A}^*x, \underline{Q}x \rangle + \langle \underline{Q}x, \Delta\tilde{A}^*x \rangle \leq \langle \Omega x, x \rangle \quad \forall x \in \mathcal{H}$$

for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$.

Also, for a given (A_c, B_c, C_c) , assume that there exists a $\underline{Q} \in \mathcal{S}$ (where \mathcal{S} is the class of bounded, positive-semidefinite, self adjoint operators) such that \underline{Q} satisfies the equation

$$ii) \quad \tilde{A}\underline{Q} + \underline{Q}\tilde{A}^* + \Omega + \tilde{V} = 0$$

on $D(\tilde{A})$.

In addition, assume that $(\tilde{V}^{1/2}, \tilde{A} + \Delta\tilde{A})$ (where $\Delta\tilde{A}$ is defined by Equation (167)) is detectable for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$. Then, $\tilde{A} + \Delta\tilde{A}$ is exponentially stable for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$. Also,

$$\tilde{Q}_\Delta \leq \underline{Q}$$

where \tilde{Q}_Δ satisfies Equation (ii) and

$$J(A_c, B_c, C_c) \leq \text{tr}[Q\tilde{R}]$$

Proof: Following Bernstein [9, 40], and recalling Lemma 5.3.1, for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$, Equation (ii) is equivalent to

$$(\tilde{A} + \Delta\tilde{A})\underline{Q} + \underline{Q}(\tilde{A}^* + \Delta\tilde{A}^*) + \tilde{V} + \Upsilon(\underline{Q}, B_c, C_c, \Delta\tilde{A}) = 0$$

where $\Upsilon = \Omega - (\Delta\tilde{A}\underline{Q} + \underline{Q}\Delta\tilde{A}^*)$.

Notice that, by Equation (i), Υ is nonnegative for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$. Since $(\tilde{V}^{1/2}, \tilde{A} + \Delta\tilde{A})$ is assumed detectable, then by Theorem 5.3.2, it follows that $([\tilde{V} + \Upsilon]^{1/2}, \tilde{A} + \Delta\tilde{A})$ is detectable for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$. Bernstein [10] Lemma 4.1 gives that this detectability condition, along with the assumption that \underline{Q} is bounded, implies that $(\tilde{A} + \Delta\tilde{A})$ must be stable.

Next, subtracting

$$(\tilde{A} + \Delta\tilde{A})\tilde{Q}_\Delta + \tilde{Q}_\Delta(\tilde{A} + \Delta\tilde{A})^* + \tilde{V} = 0$$

(which is a result from Lemma 5.3.1) from the first equation in this proof,

$$(\tilde{A} + \Delta\tilde{A})\underline{Q} + \underline{Q}(\tilde{A}^* + \Delta\tilde{A}^*) + \tilde{V} + \Upsilon = 0$$

yields that

$$(\tilde{A} + \Delta\tilde{A})(\underline{Q} - \tilde{Q}_\Delta) + (\underline{Q} - \tilde{Q}_\Delta)(\tilde{A}^* + \Delta\tilde{A}^*) + \Upsilon = 0$$

Since Bernstein [10] Lemma 4.1 yields that $(\tilde{A} + \Delta\tilde{A})$ is stable, then Lemmas 4.4 and 4.1 of [10] allow one to write

$$\underline{Q} - \tilde{Q}_\Delta = \int_0^\infty T(t)\Upsilon T^*(t)dt \geq 0$$

where $T(t)$ is the semigroup generated by $(\bar{A} + \Delta\bar{A})$. Thus, $\underline{Q} \geq \tilde{Q}_\Delta$ and as a result

$$\text{tr}[\underline{Q}\hat{R}] \geq \text{tr}[\tilde{Q}_\Delta\hat{R}]$$

for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$, so that

$$J(A_c, B_c, C_c) \leq \text{tr}[\underline{Q}\hat{R}]$$

Note that since \underline{Q} is bounded, then this last inequality places an upper bound on the $\sup \text{tr}[\tilde{Q}_\Delta\hat{R}]$. **Q.E.D.**

Theorem 5.3.3 provides a sufficient condition that ensures robustness, and provides an upper bound on the value of the cost functional J . By choosing an operator Ω so that conditions (i) and (ii) of Theorem 5.3.3 are satisfied, one can achieve robustness. The difficulty with the theorem is the ability to find an operator Ω that satisfies the conditions. One would like to choose Ω so that it reflects uncertainty in a meaningful way, and this may be difficult to do. Also, the existence of a bounded operator \underline{Q} is assumed.

5.4 Insights

Choosing Ω is based on the type of perturbations one wishes to consider, and depends on how one chooses to model uncertainty. Bernstein [9] gives one choice of Ω that works for finite-dimensional problems in which uncertainty is described in terms of stability radius. The only constraint on Ω is that it be chosen so that the hypotheses of Theorem 5.3.3 are satisfied. Assuming that is done, then using the operator Ω allows one to accomplish a procedure like that of Bernstein [10, 9], so that by setting the Frechet derivatives of the cost functional

$$L(\underline{Q}, A_c, B_c, C_c) = \text{tr}[\lambda \underline{Q}\hat{R} + (\bar{A}\underline{Q} + \underline{Q}\bar{A} + \Omega + \hat{V})p]$$

(where λ and p are nonzero scalar Lagrange multipliers) to zero, one could derive conditions similar in form to those of Theorem 8.1 of [9] with the obvious differ-

ences due to operators being considered instead of matrices. The exact form of the resulting equations depends on the form chosen for Ω .

The advantage of this approach is that one can possibly address perturbations other than bounded ones. The disadvantage is that an algorithm to solve the complex coupled equations is not readily available. However, the approach does allow a new interpretation of the LQG/LTR technique.

Let $\Delta B = 0$ and $\Delta C = 0$. Then one obtains that $\Delta \tilde{A} = \Delta A$. Following the development of Bernstein's Theorem 8.1 [9], one finds that one of the necessary A.R.E.s that must be satisfied is

$$AQ + QA^* + Q_o + \Omega - QC^*R_f^{-1}CQ + \tau_{\perp}QC^*R_f^{-1}CQ\tau_{\perp}^* = 0 \quad (169)$$

Note that if $\Delta \tilde{A}$ and $\Delta \tilde{A}^*$ are bounded, then one can choose $\Omega = \beta^2 BV B^*$, and as $\beta \rightarrow \infty$ one gets that

$$\Omega \geq \Delta \tilde{A} \underline{Q} + \underline{Q} \Delta \tilde{A}^* \quad (170)$$

Substituting $\Omega = \beta^2 BV B^*$ into this equation yields

$$BV B^* \geq \Delta \tilde{A} \frac{Q}{\beta^2} + \frac{Q}{\beta^2} \Delta \tilde{A}^* \quad (171)$$

and the work of Matson [51] demonstrates that the operator $\frac{Q}{\beta^2}$ monotonically decreases as $\beta \rightarrow \infty$ since \underline{Q} is bounded. Therefore, one can achieve robustness to bounded perturbations by using a "tuning" procedure like LQG/LTR. Note that, by doing this, one does not asymptotically recover a loop transfer function with guaranteed margins, as was pointed out before. Also, this form for Ω will result in the other A.R.E. given by

$$A^*P + PA + R_1 - PBR_2^{-1}B^*P + \tau_{\perp}^*PBR_2^{-1}B^*P\tau_{\perp} = 0 \quad (172)$$

which does not involve Ω . The only assumption is that Equation (ii) of Theorem 5.3.3 be satisfied in order to guarantee stability. In this way \underline{Q} will be admissible

only if Equation (ii) is satisfied, which will also ensure that the cost functional J is finite since \underline{Q} is bounded.

Thus, LQG/LTR can be viewed as a way to achieve robustness even under the constraint of a reduced order controller, even though one is not necessarily recovering a desired transfer function asymptotically. Also, note that if $\Delta\tilde{A}$ and $\Delta\hat{A}^*$ are unbounded operators, then one cannot find a β large enough to satisfy Theorem 5.3.3 when Ω is chosen to be $\beta^2 BVB^*$. This is similar to the problem in Theorem 4.2.1 where one needs to find a β sufficiently large so that K_β is uniformly bounded.

The O.P.E. approach gives an expanded view of the LQG/LTR technique when the order of the controller is intentionally less than the order of the system design model. Also, the O.P.E. approach allows one to choose other forms for Ω which may give more flexibility as to how one models the system perturbations.

5.5 Summary

This chapter has extended the optimal projection equation robustness conditions of Bernstein [9] to the class of systems described in Section 3 of Chapter 1. The O.P.E. approach for reduced order controller design was extended to infinite dimensional systems by Bernstein in 1986 [10]. Using that work, Bernstein's conditions for robustness were extended to infinite-dimensional systems by following his development in [9]. Thus, Theorem 5.3.3 is a result of his earlier work.

Section 5.2 demonstrated that one cannot recover a loop transfer function when the order of the controller is less than the order of the design model. This points out that one cannot perform loop transfer function recovery using a finite-dimensional controller and an infinite-dimensional design model. However, as a result of the O.P.E. robustness conditions of Theorem 5.3.3, a new interpretation of the LQG/LTR technique was given in Section 5.4. It was demonstrated that, although loop transfer function recovery does not occur when a finite-dimensional

controller is used along with an infinite-dimensional design model, one can achieve robustness to bounded perturbations by using the LQG/LTR technique. This is an insight not before mentioned in the literature since the LQG/LTR technique has been used to recover loop transfer functions, and thus achieve robustness. The technique now has an expanded interpretation. It was mentioned that the solution to the O.P.E. approach will, in general, be different from the one obtained by projecting the LQG compensator to finite-dimensional space. However, the lack of an algorithm to solve the coupled equations associated with the O.P.E. approach makes it impossible at this time to evaluate differences.

The next chapter will develop a sufficient condition for robustness when the transfer function is perturbed, instead of when the state space operators are perturbed, as has been the case so far. The development is an extension of Curtain's [22] sufficient condition which considered only additive perturbations of the system transfer function.

VI. Transfer Function Approach

6.1 Introduction

Another approach taken in recent literature [16, 22, 33, 34, 30, 47] has been to consider perturbations of the nominal plant transfer function instead of the state-space operators. Let the nominal plant transfer function be given by

$$G_{nom}(s) = C_{nom}T(t)B_{nom} \quad (173)$$

where $T(t)$ is the C_0 semigroup generated by A_{nom} . This has the corresponding state equations

$$\dot{x}_{nom} = A_{nom}x_{nom} + B_{nom}u \quad (174)$$

$$y_{nom} = C_{nom}x_{nom} \quad (175)$$

Similarly, denote the perturbation of the plant transfer function by

$$G_{\delta}(s) = C_{\delta}T_{\delta}(t)B_{\delta} \quad (176)$$

where $T_{\delta}(t)$ is a C_0 semigroup generated by an operator A_{δ} . The corresponding state equations are

$$\dot{x}_{\delta} = A_{\delta}x_{\delta} + B_{\delta}u \quad (177)$$

$$y_{\delta} = C_{\delta}x_{\delta} \quad (178)$$

For additive perturbations of the transfer function, the perturbed plant has the transfer function given by

$$G(s) = G_{nom}(s) + G_{\delta}(s) \quad (179)$$

which can be written equivalently as

$$G(s) = [C \quad C_{\delta}] \begin{bmatrix} T(t) & 0 \\ 0 & T_{\delta}(t) \end{bmatrix} \begin{bmatrix} B \\ B_{\delta} \end{bmatrix} \quad (180)$$

One thing to note is that, A_δ that generates $T_\delta(t)$ is not equal to $A_{nom} + \Delta A$, where ΔA denotes a perturbation of the system A operator (as was the case in Chapter 5). This chapter deals with perturbations of the transfer function, and not perturbations of the state space operators. Additive perturbations of the plant transfer function are considered first in order to apply the robustness analysis developed by Jacobson [47]. In Section 3, multiplicative perturbations will be considered. Multiplicative perturbations are preferable over additive since compensated transfer functions (i.e. $GK(s)$) have the same uncertainty as the uncompensated plant, and multiplicative perturbations correspond more closely to the classical ideas that lead to the definition of gain margin and phase margin. Section 3 will apply the sufficient condition for robustness developed by Chen and Desoer [16] which is applicable to multiplicative perturbations. Section 4 will then extend the results of Curtain [22] so that a sufficient condition for robust finite-dimensional controllers for infinite-dimensional plants can be obtained. The approach will rely on H_∞ techniques, and will be applicable to multiplicative perturbations.

6.2 LQG Robustness

The LQG design will yield a pair of operators K_f and K_c such that the nominal system can be stabilized under the assumptions that (A, B) and (A, G) are stabilizable, and (A, C) and $(A, Q_c^{1/2})$ are detectable. The closed-loop system with the compensator included can be described by [47]

$$G_{cl} = C_{cl}T_{cl}(t)B_{cl} \quad (181)$$

where

$$C_{cl} = \begin{bmatrix} 0 & -C \\ K_c & 0 \end{bmatrix}$$

$$B_{cl} = \begin{bmatrix} K_f & 0 \\ 0 & B \end{bmatrix}$$

and $T_{cl}(t)$ is the semigroup generated by the operator

$$A_{cl} = \begin{bmatrix} A - BK_c - K_f C & -K_f C \\ BK_c & A \end{bmatrix}$$

and $T_d(t)$ is a stable semigroup with growth constant $\omega = \max[\omega_1, \omega_2]$, where ω_1 and ω_2 are the growth constants associated with the stable semigroups generated by $(A - BK_c)$ and $(A - K_f C)$.

The closed-loop system will remain stable in the presence of additive perturbations if $T_\delta(t)$ is a stable semigroup (i.e. if K_c and K_f stabilize the transfer function of the perturbed plant as well as the nominal plant transfer function).

The transfer functions G_{nom} and G_δ correspond to state-space realizations $(A_{nom}, B_{nom}, C_{nom})$ and $(A_\delta, B_\delta, C_\delta)$ respectively. The triplet describing G_δ is not unique, and A_δ does not equal ΔA corresponding to a perturbation of the state-space operator A (and similarly for B_δ and C_δ). The corresponding state-space equations are given by

$$\dot{x}_{nom} = A_{nom}x_{nom} + B_{nom}u \quad (182)$$

$$y_{nom} = C_{nom}x_{nom} \quad (183)$$

and

$$\dot{x}_\delta = A_\delta x_\delta + B_\delta u \quad (184)$$

$$y_\delta = C_\delta x_\delta \quad (185)$$

which can be put into augmented state-space form as

$$\begin{bmatrix} \dot{x}_{nom} \\ \dot{x}_\delta \end{bmatrix} = \begin{bmatrix} A_{nom} & 0 \\ 0 & A_\delta \end{bmatrix} \begin{bmatrix} x_{nom} \\ x_\delta \end{bmatrix} + \begin{bmatrix} B_{nom} \\ B_\delta \end{bmatrix} u \quad (186)$$

$$\begin{bmatrix} y_{nom} \\ y_\delta \end{bmatrix} = \begin{bmatrix} C_{nom} & 0 \\ 0 & C_\delta \end{bmatrix} \begin{bmatrix} x_{nom} \\ x_\delta \end{bmatrix} \quad (187)$$

Recall that the compensator is based on the nominal plant and has the state-space equations given by

$$\dot{x}_c = (A_{nom} - B_{nom}K_c - K_f C_{nom})x_c + K_f y_p \quad (188)$$

$$u = -K_c x_c \quad (189)$$

Using these state-space equations, the total closed-loop system with perturbations can be described by

$$\begin{bmatrix} \dot{x}_c \\ \dot{x}_{nom} \\ \dot{x}_\delta \end{bmatrix} = \begin{bmatrix} (A_{nom} - B_{nom}K_c - K_f C_{nom}) & -K_f C_{nom} & -K_f C_\delta \\ B_{nom}K_c & A_{nom} & 0 \\ B_\delta K_c & 0 & A_\delta \end{bmatrix} \begin{bmatrix} x_c \\ x_{nom} \\ x_\delta \end{bmatrix} \quad (190)$$

or $\dot{x} = A_e x$ and

$$y_p = y_\delta + y_{nom} = \begin{bmatrix} 0 & C_{nom} & C_\delta \end{bmatrix} \begin{bmatrix} x_c \\ x_{nom} \\ x_\delta \end{bmatrix} \quad (191)$$

Jacobson [47] shows that the system will be input-output stable (and thus exponentially stable) if A_e generates a stable semigroup. If A_δ is the generator of a stable semigroup, then A_e will generate a stable semigroup whose growth constant is determined by $\|K_f C_\delta\|$. This can be seen by writing A_e as

$$A_e = \begin{bmatrix} (A_{nom} - B_{nom}K_c - K_f C_{nom}) & -K_f C_{nom} & | & 0 \\ B_{nom}K_c & A_{nom} & | & 0 \\ \hline B_\delta K_c & 0 & | & A_\delta \end{bmatrix} + \begin{bmatrix} 0 & 0 & | & -K_f C_\delta \\ 0 & 0 & | & 0 \\ \hline 0 & 0 & | & 0 \end{bmatrix} \quad (192)$$

Proposition 4.7 of Schumacher [59] provides a sufficient condition for A_e to generate a stable semigroup. The following proposition is Schumacher's Proposition 4.7.

PROPOSITION 6.2.1: Suppose that A_{11} and A_{22} are generators of semigroups $T_1(t)$ and $T_2(t)$ on the Banach space X_1 and X_2 , with growth constants ω_1

and ω_2 respectively. Suppose also that $A_{21} : X_1 \rightarrow X_2$ is a bounded linear operator. Then the operator

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

is a generator of a C_0 semigroup on $X_1 \oplus X_2$. The semigroup has a growth constant equal to $\max\{\omega_1, \omega_2\}$.

Proof: See Schumacher [59] page 86.

Since the upper left block generates a stable semigroup, if A_δ generates a stable semigroup, the preceding proposition states that the first matrix term in the expression for A_ϵ generates a stable semigroup $S(t)$. A_ϵ will generate a stable semigroup if $\|K_f C_\delta\| < |\omega|$ where ω is the growth constant of $S(t)$. Recall that the triplet describing G_δ is not unique, so that it may be possible to satisfy these conditions for some state-space realizations and not for others.

If A_δ does not generate a stable semigroup, it is not apparent how one would determine the stability of the semigroup generated by A_ϵ . Thus, if one wants to express perturbations in terms of transfer functions, it is not clear how to relate the semigroup factors ω and M to robustness unless the transfer function perturbations have stable generators, which is a restrictive class. Because of this, one may wish to consider another approach. One alternative is to use the transfer function algebra discussed in Chapter 2. Chen and Desoer [16] have developed a set of necessary and sufficient conditions for robustness for not only additive perturbations, but multiplicative perturbations as well (a preferred model for perturbations). The next section will present the result for multiplicative perturbations. This result will be the basis for extending the work of Curtain [22] which demonstrates how to apply H_∞ techniques to the problem, and derive a sufficient condition for robustness of finite-dimensional controllers in the presence of multiplicative perturbations. Curtain [22]

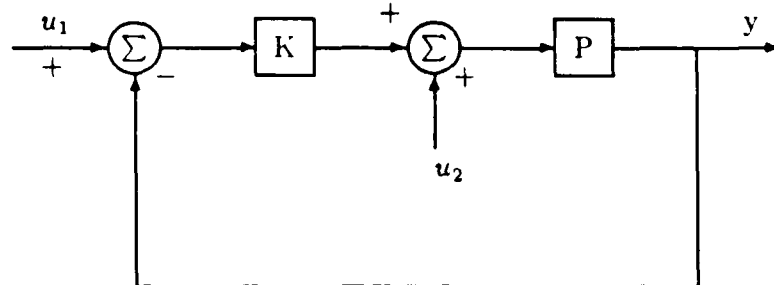


Figure 5. Unity Feedback Control System

has developed sufficient conditions for robustness for additive perturbations using an H_∞ approach. This research will extend that work in order to consider not only additive perturbations, but multiplicative perturbations of the plant transfer function as well.

6.3 Robustness Sufficient Condition

Consider the configuration shown in Figure 5. This is the configuration used by Jacobson [47] in his work. In Figure 5, P is the plant to be controlled, and K is the compensator. Ignoring disturbances, let $u_2 = 0$. Jacobson has proven that, for systems satisfying the assumptions of Chapter 1, the transfer function $P(s)$ belongs to the algebra $\hat{B}(\sigma)$ described in Definition 2.5.5 of Chapter 2. Specifically, the following theorem applies:

THEOREM 6.3.1: If the system described by

$$\dot{x} = Ax + Bu \quad x(0) \in D(A)$$

$$y = Cx$$

satisfies the assumptions in Section 3 of Chapter 1 on (A, B, C) , and if the system is both exponentially stabilizable and detectable, then the transfer function $P(s) = C(sI - A)^{-1}B$ has a representation in $\hat{B}(\sigma)^{n \times n}$ for some $\sigma < 0$.

Proof: See Jacobson [47] page 14.

The fact that $P(s)$ has a representation in $\hat{B}(\sigma)$ will be important to the development of the necessary and sufficient conditions for robustness. Note that the stochastic system of equations is given by

$$\dot{x} = Ax + Bu + Gu \tag{193}$$

$$y = Cx + \eta \tag{194}$$

However, the Gu and η noise terms do not alter the transfer function mapping $P : u \rightarrow y$. Thus, the plant input-output transfer function for the stochastic equations is the same as for the equations in Theorem 6.3.1. Therefore, by Theorem 6.3.1, the plant transfer function $P(s) = C(sI - A)^{-1}B$ is in the algebra $\hat{B}(\sigma)^{n \times n}$ for some $\sigma < 0$. Jacobson's proof of Theorem 6.3.1 also shows that the compensator transfer function must lie in $\hat{B}(\sigma)^{n \times n}$ for some $\sigma < 0$. This idea is also contained in a theorem by Nett [53].

THEOREM 6.3.2: If $P \in \hat{B}(\sigma)^{n \times n}$, and if K is a Laplace transformable distribution which $A_-(\sigma)$ stabilizes (input-output stable, see Definitions 2.5.2 and 2.5.6) the closed-loop system, and if either PK or KP is proper, then both P and K have entries in $\hat{B}(\sigma)^{n \times n}$.

A distribution is a linear continuous functional defined on the space of continuous functions which have continuous derivatives of all orders, and which vanish outside some closed and bounded region (see [37] for details).

Proof: See Nett [53] pg 56.

Thus, since P can be assumed to be strictly proper (valid assumption since the frequency response of any physical system is bandlimited) then one has that $K \in \hat{B}(\sigma)^{n \times n}$. Also, if $K = \tilde{K}$ (the finite-dimensional stabilizing controller), then clearly $\tilde{K} \in \hat{B}(\sigma)^{n \times n}$.

At this point the only measures of stability robustness have been the magnitude of the exponential time constant, the magnitude of the semigroup bound constant, or the operator Ω for the O.P.E. approach. However, it is hard to relate model uncertainties to the exponential factor ω . A measure of robustness that has been used in the design of finite-dimensional control systems [30, 32, 35] is the singular value.

Let $A \in \mathcal{C}^{n \times n}$ (where $\mathcal{C}^{n \times n}$ is the ring of square matrices with elements in \mathcal{C}). A ring is a set on which the operations addition and multiplication are defined, and multiplication is distributive over addition. The largest singular value of A is defined to be [63]

$$\sigma_{\max}[A] = \sup_{\|z\|=1} \|Az\| \quad (195)$$

which is just the L_2 norm of A . The algebraic properties of $\hat{B}(\sigma)$ places the loop gain operator $PK(I + PK)^{-1}$ in this ring, so that one can define its largest singular value. Since for each $\omega \in \mathfrak{R}^+$

$$[PK(I + PK)^{-1}(j\omega)] \in \mathcal{C}^{n \times n}$$

then $\sigma_{\max}\{[PK(I + PK)^{-1}(j\omega)]\}$ is defined. This will allow the singular value to be used as measure of robustness for the systems considered in this research.

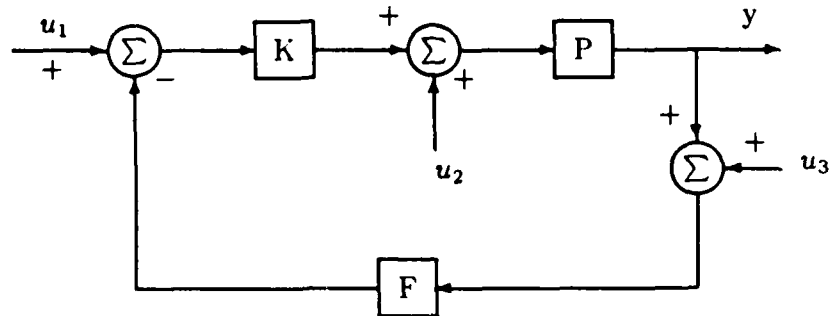


Figure 6. Feedback Control System

One can put the LQG problem into the setup of Figure 6 by defining

$$P = C(sI - A)^{-1}B \quad (196)$$

$$K = K_c(sI - A + BK_c + K_fC)^{-1}K_f \quad (197)$$

$$F = I \quad (198)$$

$$u_2 = Gw \quad u_3 = \eta \quad (199)$$

Unity feedback is used in order to correspond to the development in Chapter 2. Since $F = I$, then $F \in \hat{B}(\sigma)^{n \times n}$. This will be needed in development of the robustness conditions. u_1 is the command input to the system. One may also consider u_2 as an added input to the plant as well as a disturbance. This system will be denoted as $S(P, K, F)$.

Now, consider multiplicative perturbations since the compensated plant has the same uncertainty associated with it as the nominal plant [30], and since they correspond more closely to the classical ideas of gain and phase margins. Hence, the perturbed plant \hat{P} will be defined as $\hat{P} = (I + M_p)P$, where M_p is an element

of a class of perturbations. The properties of the perturbations will be outlined in the last theorem of this section. The perturbed closed-loop system is referred to as $S(\tilde{P}, K, F)$. Recall the definition of $A_-(\sigma)$ stability found in Definition 2.5.6. $A_-(\sigma)$ stability is the same as input-output stability. Jacobson's work [47] provides the following theorem:

THEOREM 6.3.3: If the system is exponentially stable, then it is $A_-(\sigma)$ stable for some $\sigma < 0$.

Proof: See Jacobson [47] pg 43.

Thus, since the compensator K exponentially stabilizes the closed-loop system, the nominal closed-loop system is $A_-(\sigma)$ stable for some $\sigma < 0$. The question is, under what conditions will the closed-loop system remain $A_-(\sigma)$ stable in the presence of multiplicative perturbations? Chen and Desoer [16] give necessary and sufficient conditions for the robust stability of $S(\tilde{P}, K, F)$. In the theorem to follow, l_m is a scalar function that describes the uncertainty of the system as a function of frequency, and n_{P+} denotes the number of poles of the plant P in the right half plane. The backslash notation used to define the perturbations M_p (i.e. $A_-(\sigma) \setminus \{0\}$) means exclusion of the zero vector.

THEOREM 6.3.4: Consider the multiplicatively perturbed system $S(\tilde{P}, K, F)$ with $\tilde{P} = (I + M_p)P$. Let $P \in \hat{B}(\sigma)^{n \times n}$, $K \in \hat{B}(\sigma)^{n \times n}$, and $F \in \hat{B}(\sigma)^{n \times n}$ for some $\sigma < 0$. Also, let $M_p \in \mathcal{M}$ where

$$\mathcal{M} = \{M_p : M_p \in [A_-(\sigma)] * [A_-(\sigma) \setminus \{0\}]^{-1};$$

$$M_p P \in \hat{B}(\sigma)^{n \times n};$$

$$n_{\tilde{P}+} = n_{P+}$$

$$\sigma_{\max}[M_p(j\omega)] < l_m(\omega) \quad \forall \omega \in \mathfrak{R}^+$$

Under these conditions, if $S(P, K, F)$ is $A_-(\sigma)$ stable, then $\forall M_p \in \mathcal{M}$, $S(\tilde{P}, K, F)$ is $A_-(\sigma)$ stable if and only if

$$\sigma_{\max}[PKF(I + PKF)^{-1}(j\omega)] < l_m(\omega)^{-1} \quad \forall \omega \in \mathfrak{R}^+$$

Proof: See Chen and Desoer [16] pg 264.

The size restriction imposed by the "tolerance" function l_m on M_p implies that the perturbation has no poles on the $j\omega$ -axis. This, with the condition on the number of poles of the perturbed system, implies that the perturbations do not introduce additional right half plane poles. The function l_m in essence defines the size of the "ball of uncertainty" around the nominal plant P , such that stability is retained, by defining how large M_p (and therefore $M_p P$) can be. For a discussion of $l_m(\omega)$, the reader is referred to [30].

Note that:

i) The description of \mathcal{M} implies that $M_p \in \mathcal{M}$ may not be proper, but both P and \tilde{P} must be strictly proper in order for $M_p P \in \hat{B}(\sigma)$, which is assumed. As pointed out earlier, this is not felt to be restrictive since the frequency response of any physical system is bandlimited.

ii) The description of \mathcal{M} tells one that all elements M must belong to the field of fractions of the commutative domain $A_-(\sigma)$ (the algebra $A_-(\sigma)$ forms the domain of the field, and multiplication on this algebra commutes). This is needed in order to make the proof work.

iii) $n_{\tilde{P}+}$ is the number of right-half plane poles of \tilde{P} . Thus, it is assumed that the number of unstable poles of \tilde{P} is the same as for P (i.e. the unstable poles of

\tilde{P} are obtained by moving around the unstable poles of P). This is needed in the proof so that stability of the nominal system will imply stability of the perturbed system.

Theorem 6.3.4 gives a singular value measure of stability robustness for distributed parameter systems, and it will have the same type of conservatism as discussed in [30]. Since the singular value is not structured, it assumes a worst case bound for the perturbation for all channels in a MIMO system. This leads to conservatism in the design. This procedure gives a way to check sufficient conditions for robustness of the closed-loop system.

Based on the work of Curtain [22], this measure of robustness can be used in conjunction with H_∞ techniques (the norm used to define the largest singular value is the H_∞ norm) to give a design tool by which a finite-dimensional controller can be evaluated for robustness. The next section's results are a direct consequence of Theorem 6.3.4 and Curtain's results [22] for additive perturbations. However, since Curtain only considered additive perturbations, the material in Section 4 is presented in order to extend her work to multiplicative perturbations, which are preferred.

6.4 H_∞ Approach

Extending Curtain's results [22] to consider multiplicative perturbations follows directly from her work and that of [16]. However, it is very useful to accomplish since multiplicative perturbations more accurately reflect simultaneous gain and phase change in a system.

The desire is to stabilize a family of plants described by the transfer function

$$\tilde{G}(s) = (I + M_p)G_{nom}(s) \quad (200)$$

using a finite-dimensional LQG based compensator which has the transfer function

$$K(s) = K_c(sI - A + BK_c + K_fC)^{-1}K_f \quad (201)$$

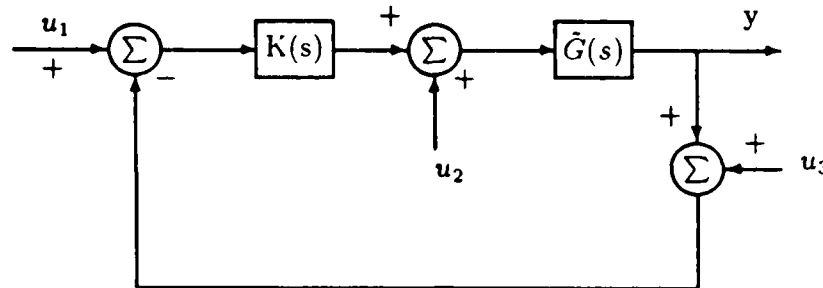


Figure 7. System with Multiplicative Perturbations

The problem corresponds to the block diagram of Figure 7 which is equivalent to Figure 6 with $F = I$ and $P = \tilde{G}$.

To correspond with the definition of exponential stability, σ of the algebra $\hat{B}(\sigma)$ will be chosen to be zero. The following assumptions will be needed in order to use the results of [22] and [16].

1. $M_p G(s) \in \hat{B}(0)^{n \times n}$ where $\hat{B}(0)^{n \times n}$ is the Banach algebra developed by Callier and Desoer [27, 12, 13, 14, 15, 26], and defined in Section 5 of Chapter 2.
2. $G(s)$ and $\tilde{G}(s)$ have the same number of poles in C^+ (the right half of the complex plane).
3. $G(s) \in \hat{B}(0)^{n \times n}$ with no poles on the $j\omega$ axis.
4. $\|M_p(\omega)\|_\infty < f_m(\omega) \quad \forall \omega \in \mathfrak{R}^+$ where $f_m(\omega)$ is a scalar function of ω .
5. $K(s) \in \hat{B}(0)^{n \times n}$

These assumptions are needed in order to apply the results of Chen and Desoer [16], and are the same ones made by them in Theorem 6.3.4 [16]. The discussion of the assumptions following Theorem 6.3.4 applies here also. Assumption 3 and 4 allow one to use results found in [12, 13] to express the perturbed transfer function as the sum of a rational transfer function and a stable irrational transfer function. This will be stated more precisely later, and is essential to the development. Assumptions 1 and 5 are made in order to apply the algebraic properties of the Banach algebra $\hat{B}(\sigma)^{n \times n}$. Assumption 2 says simply that the perturbations do not introduce additional right half plane poles, and allows one to conclude stability of the perturbed system when the nominal system is stabilized. Notice that assumption 4 can be equivalently expressed as

$$\|f_m^{-1}M_p\|_\infty < 1 \quad (202)$$

where the ∞ norm is defined as

$$\|*\|_\infty = \sup \sigma_{\max}\{*(s)\} = \text{ess sup}_\omega \sigma_{\max}\{*(j\omega)\} \quad (203)$$

The small gain theorem as found in [71] gives a sufficient condition input-output stability (i.e. a bounded input produces a bounded output). The open loop gain is the operator which maps the input to the compensator K , to the output of the feedback, and is obtained by opening the feedback loop in Figure 7 just to the left of the block labeled $K(s)$.

THEOREM 6.4.1: Let the feedback system be described by the block diagram of Figure 7. The system will be input-output stable if the open loop gain is less than unity.

Proof: See Zames [71] page 232.

For the system of Figure 7, one can perform the block diagram transformations as shown in [33, 34] and express the small gain sufficient conditions as

$$\| M_p GK(I - GK)^{-1} \|_{\infty} < 1 \quad (204)$$

In view of Equation (202), a sufficient condition for Equation (204) to be satisfied is

$$\| f_m GK(I - GK)^{-1} \|_{\infty} \leq 1 \quad (205)$$

which can be equivalently expressed as

$$\| GK(I - GK)^{-1} \|_{\infty} \leq f_m^{-1} = \| W_1 \|_{\infty} \quad (206)$$

Francis and Curtain [34, 22] refer to W_1 as a sensitivity function. W_1 is any stable rational transfer function for which $\|W_1\|_{\infty} = f_m^{-1}$, and for which W_1^{-1} is also a stable rational transfer function. Notice that assumption 4 can be expressed in terms of W_1 as

$$\| W_1 M_p \|_{\infty} < 1 \quad (207)$$

Curtain's development [22] is formulated in terms of two sensitivity functions, W_1 and W_2 , where W_2 is also a stable rational transfer function. The assumption she imposes is that $\|W_1 M_p W_2\|_{\infty} < \epsilon$ where ϵ is a positive scalar. Following her notation and development, ϵ has been set equal to 1, and $W_2 = W_2^{-1} = I$.

The nominal feedback system is obtained by setting $M_p = 0$. It is assumed that a compensator has been chosen that stabilizes the nominal system so that the nominal system is $\hat{A}_-(0)$ stable as defined in Chapter 2, Section 2.5. Thus, as shown in [16], the operators $(I - GK)^{-1}$, $K(I - GK)^{-1}$, $(I - GK)^{-1}G$, and $[I - K(I - GK)^{-1}G]$ are all elements of the algebra of transfer functions $\hat{A}_-(0)$. These facts allow one to manipulate these operators algebraically, and apply standard block diagram type arguments.

The following definition is taken from Curtain [22]. In the definition, the triplet notation is used to denote the feedback system of Figure 7 with perturbations described by the sensitivity function W_1 .

Definition 6.4.1: The feedback system denoted by the triplet (G, W_1, K) will be called *robustly stable* if and only if $(G + M_p G, K)$ is input-output stable for all perturbations $M_p G \in \hat{B}(0)^{n \times n}$ satisfying assumption 2 and $\|W_1 M_p\|_\infty \leq 1$.

As a result of this definition, a system is *robustly stabilizable* if there exists a compensator $K \in \hat{B}(0)^{n \times n}$ such that (G, W_1, K) is robustly stable. Since $W_1 \in \mathcal{RH}^\infty$ (the class of stable rational transfer functions), then the fact that $G \in \hat{B}(0)^{n \times n}$ with no poles on the imaginary axis along with the condition of Equation (207) yields [12] that

$$W_1 G = G_1 + G_2 \quad (208)$$

Note that G_1 is the finite-dimensional unstable part of the system, and G_1^* denotes the complex conjugate transpose of the transfer function matrix G_1 . In other words, for the rational transfer function $G_1(s)$, $G_1^*(s) = [G_1(\overline{-s})]^*$. Also, $G_2 \in \hat{A}_-(0)$ (stable infinite-dimensional part) and $G_1^* \in \mathcal{RH}^\infty$ (the class of rational stable transfer functions) with $G_1(\infty) = 0$ (rational transfer function with poles in \mathcal{C}^+).

In Curtain's development [22], she shows that the system is robustly stable to additive perturbations if the finite-dimensional transfer function G_1 can be robustly stabilized. Similar results will be obtained for the case of multiplicative perturbations by following the lemmas in her work. As in her work, one can write a version of Chen and Desoer's corollary (see [16] page 264) to yield

LEMMA 6.4.2: Under assumptions 1 thru 5 on page 108, (G, W_1, K) is robustly stable if and only if (G, K) is stable under nominal conditions and

$$\|GK(I - GK)^{-1}W_1^{-1}\|_\infty \leq 1$$

Proof: See Chen and Desoer [16] page 264.

This then leads to the next Lemma.

LEMMA 6.4.3: Under assumptions 1 thru 5, there exists a compensator $K \in \hat{B}(0)^{n \times n}$ such that (G, W_1, K) is robustly stable if and only if

$$K = (I + K_1 G_2)^{-1} K_1 W_1$$

for some $K_1 \in \hat{B}(0)^{n \times n}$ such that (G_1, W_1, K_1) is robustly stable.

Proof: Following the proof in Curtain [22], since W_1 and $W_1^{-1} \in \mathcal{RH}^\infty$, a block diagram type of argument corresponding to Figure 7 will suffice. (G, W_1, K) is robustly stable if

$$\| GK(I - GK)^{-1} W_1^{-1} \|_\infty \leq 1$$

But, as shown by Curtain, this is true if and only if

$$\| W_1 G K W_1^{-1} (I - W_1 G K W_1^{-1})^{-1} W_1^{-1} \|_\infty \leq 1$$

or equivalently $(W_1 G, W_1, K W_1^{-1})$ is robustly stable.

Now, one can write $W_1 G = G_1 + G_2$ so that $(G_1 + G_2, W_1, K W_1^{-1})$ is robustly stable. Let $K_2 = K W_1^{-1}$ so that $(G_1 + G_2, W_1, K_2)$ is robustly stable. As shown in Doyle and Francis [34], this equivalent to $(G_1, W_1, K_2(I - G_2 K_2)^{-1})$ being robustly stable.

Let $K_1 = K_2(I - G_2 K_2)^{-1}$ so that (G_1, W_1, K_1) is robustly stable. Note that

$$K_2 = K_1(I - G K_2) = K_1 - K_1 G K_2$$

or equivalently

$$K_1 = K_2 + K_1 G_2 K_2 = (I + K_1 G_2) K_2$$

so that $K_2 = (I + K_1 G_2)^{-1} K_1$ or

$$K = (I + K_1 G_2)^{-1} K_1 W_1$$

The fact that $K \in \hat{B}(0)$ if and only if $K_1 \in \hat{B}(0)$ follows from this last equation since $G_2 \in \hat{A}_-(0)$. **Q.E.D.**

Lemma 6.4.3 says that the total system is robust if and only if the finite-dimensional unstable part of the transfer function is robustly stable with K_1 and if K has the specified form. One can use this to yield the following corollary.

COROLLARY 6.4.4: (G, W_1, K) is robustly stable if and only if

$$\inf_{K_1} \| G_1 K_1 (I - G_1 K_1)^{-1} W_1^{-1} \|_{\infty} \leq 1$$

Proof: This is an immediate consequence of combining Lemma 6.4.2 and Lemma 6.4.3. **Q.E.D.**

At this point, when multiplicative perturbations are considered, a way to determine robustness of the system using a finite-dimensional compensator is available. Let \tilde{K} be the approximation of K obtained, say, using Schumacher's technique. Then for a sufficiently high order approximation, the nominal system is input-output stable. The question one has to ask is, how robust is the controlled system using this choice of \tilde{K} ? Corollary 6.4.4 gives a sufficient condition for \tilde{K} to satisfy that will ensure stability of the perturbed system.

Let G_1 be the part of the plant transfer function corresponding to the finite-dimensional unstable space. Then (G_1, W_1) is robustly stabilizable if and only if

$$\| G_1 K_1 (I - G_1 K_1)^{-1} W_1^{-1} \|_{\infty} \leq 1 \quad (209)$$

and, as given in Lemma 6.4.3,

$$K_1 = \hat{K}W_1^{-1}(I - G_2\hat{K}W_1^{-1})^{-1} \quad (210)$$

with $G_2 = G \setminus G_1$, where the backslash notation is used to denote the exclusion of the operator G_1 .

Thus, given an approximation \hat{K} , a sensitivity function W_1 , and a nominal plant transfer function G (and thus G_1 and G_2), one can solve for K_1 and then determine if G_1 is robustly stabilized with respect to W_1 . If this occurs, then since

$$\hat{K} = K_1(I - G_2\hat{K}W_1^{-1})W_1 \quad (211)$$

which equates to

$$\hat{K} = (I + K_1G_2)^{-1}K_1W_1 \quad (212)$$

Lemma 6.4.3 yields that the total system will be robustly stable using this choice of \hat{K} . Hence, one can reduce the robustness issue to a singular value test by applying Lemma 6.4.3.

One thing to note is that \hat{K} 's robustness properties will be directly related to the properties of the compensator K it is approximating. In a way just like that described for \hat{K} , the robustness of K should be evaluated. One should not expect to get good robustness with \hat{K} if K does not have good robustness. Assuming K has desirable robustness properties, it is not clear whether one should improve K 's robustness, or the approximation \hat{K} , when \hat{K} is not robust enough.

Also, this approach allows one to leave the sensitivity function W_1 undefined, and solve for it, given G and K . In this way one may be able to evaluate the robustness of one design versus another, and therefore get some idea as to when an approximation is good enough. An approximation \hat{K} can be considered acceptable when it satisfies the condition of Equation (209) for a desired W_1 .

In a way like that used by Sandell [57], one can also develop sufficient conditions which may have more physical meaning. Using the following sufficient con-

dition allows one to leave the sensitivity function W_1 undefined, and determine a conservative estimate of the system's robustness for a given G and K . Thus, if one does not have a good idea of what type of perturbations to expect, the following condition yields a conservative measure of the perturbations a system can accept and remain stable.

LEMMA 6.4.5: Let W_1 be a sensitivity function describing the uncertainty of a system. A sufficient condition for the system to be robustly stable is

$$\|W_1^{-1}\|_{\infty} < \frac{1}{\| [I - (I + GK)^{-1}] \|_{\infty}}$$

Proof: This condition implies that

$$\| [I - (I + GK)^{-1}] \|_{\infty} \|W_1^{-1}\|_{\infty} < 1$$

which implies that

$$\| [I - (I + GK)^{-1}]W_1^{-1} \|_{\infty} < 1$$

The identity [57]:

$$I - (I + GK)^{-1} = GK(I + GK)^{-1}$$

can be substituted into the second inequality to yield

$$\| GK(I + GK)^{-1}W_1^{-1} \|_{\infty} < 1$$

and Lemma 6.4.3 gives the desired result. **Q.E.D.**

In this development, the sensitivity function W_2 of Curtain's development [22] has been chosen to be the identity operator and W_1 was the only sensitivity function considered. This seems reasonable since many examples [22, 33] choose $W_2 = I$. However, as shown in Francis [33, 34], one may wish to let W_2 be a second

sensitivity function in order to describe the structure of the perturbations more accurately. For example, one may want to model uncertainty at two points in the system. Using two sensitivity functions instead of one may allow one to do that. This research will not consider this since many of the examples in the literature have not, and also, to do so is a simple extension of this work.

The disadvantage of using H_∞ methods is the difficulty to solve the associated equations for MIMO systems other than delay equations. However, it is the only method developed in this research that allows one to consider perturbations of the transfer function directly. The other methods involve modeling perturbations as changes to the state space operators. It may be difficult to do that if one only has input-output responses to describe a system. It is difficult to relate perturbations of a transfer function to a triplet of operator perturbations $(\Delta A, \Delta B, \Delta C)$.

6.5 Summary

This chapter began by analyzing the robustness of a system controlled by a LQG controller, assuming additive perturbations to the plant transfer function. Only for the case of stable perturbations could stability of the closed-loop system be determined using direct analysis methods. Section 6.3 used the algebra of transfer functions discussed in Chapter 2, and presented a necessary and sufficient condition for robustness for the case of multiplicative perturbations of the plant transfer function. Since multiplicative perturbations are usually preferred, this result is quite useful.

Using this result, along with the work by Curtain [22], Section 6.4 developed a sufficient condition for robustness using H_∞ techniques. Theorem 6.4.2 provides a condition for robustness of the infinite-dimensional LQG controller when multiplicative perturbations are considered. From this, Corollary 6.4.4 is developed which yields a sufficient condition for robustness based on the finite-dimensional controller and an infinite-dimensional plant. An alternative sufficient condition is presented

in Lemma 6.4.5. These sufficient conditions can be used as a measure of the finite-dimensional controller's robustness by leaving the sensitivity function W_1 undefined, or they can be used to determine when a controller is acceptable when W_1 is defined.

The next chapter is an application of the techniques developed in this research. Some of the advantages and disadvantages of the approaches are discussed.

VII. Application Example

7.1 Introduction

This chapter contains a simple example that will demonstrate the application of the techniques developed in this research. The equation considered is a parabolic partial differential equation that describes the temperature distribution on an isolated uniform rod (i.e., it is a heat equation). The problem is not intended to be realistic, but rather is chosen so that solutions can be obtained and the techniques applied without losing readers in the mathematical details. Since a good design depends on many engineering factors, no attempt is made to get a best design. This problem is the same type as the one in Chapter 3 with the exception that the A operator has been changed in order to have two unstable eigenvalues.

7.2 Problem

Let the problem be given by

$$\frac{\partial}{\partial t}x(z, t) = \left(\frac{1}{\pi^2} \frac{\partial^2}{\partial z^2} + 4\right)x(z, t) + Iu(t) + Iw(t) \quad t \geq 0; 0 \leq z \leq 1 \quad (213)$$

with Dirichlet boundary conditions

$$x(0, t) = x(1, t) = 0 \quad \forall t \geq 0$$

and initial condition

$$x(z, 0) = 0 \quad \forall z \in [0, 1]$$

Assume a scalar input ($u \in \mathfrak{R}^1$) and a scalar output given by

$$y(t) = \int_0^1 x(z, t) dz + \eta(t) \quad (214)$$

with $\eta(t) \in \mathfrak{R} \quad \forall t$ (i.e. real valued). The state space will be $\mathcal{H} = L^2(0, 1)$ and choose the strengths of the white Gaussian, zero-mean noise terms w and η to be

$Q_o = I$ and $R_f = I^{n \times n}$ respectively. As before, these choices are made to simplify the solution so that numerical results can be obtained.

The domain of the operator A is defined by

$$D(A) = \{x \in \mathcal{H} \mid \frac{\partial^2}{\partial z^2} x \in \mathcal{H}; x(0) = x(1) = 0\} \quad (215)$$

where A has been changed from the example in Chapter 3 in order to have two unstable eigenvalues, and the operator A satisfies

$$Ax = \left(\frac{1}{\pi^2} \frac{\partial^2}{\partial z^2} + 4\right)x \quad \forall x \in D(A) \quad (216)$$

The output mapping C is given by

$$Cx = \int_0^1 x(z,t) dz \quad \forall x \in \mathcal{H} \quad (217)$$

which can be written in inner product notation as

$$Cx = \langle 1, x \rangle \quad (218)$$

which is simply the inner product of the function which is 1 everywhere with elements in the Hilbert space \mathcal{H} as defined by Equation (217). It is assumed that the output vector y is to be controlled, and Q_c is then chosen to be C^*C .

Schumacher [59] shows that the operator A generates an analytic semigroup for $t > 0$, and A has a discrete spectrum with simple eigenvalues at $4 - i^2$ ($i = 1, 2, \dots$) with corresponding eigenfunctions given by

$$\phi_i = \sqrt{2} \sin i\pi z \quad (219)$$

which is a complete orthonormal set in $L^2(0, 1)$. Because of the simple form of B, G , and Q_o , Schumacher shows that (A, B) and (A, G) are stabilizable, and (A, C) and (A, Q_o) are detectable [59].

Following the same development found in Chapter 3, one can design a steady state constant gain LQ controller assuming full state access as shown in Matson

[51]. This yields for the regulator gain operator

$$K_c = R_c^{-1} B^* P_c \quad (220)$$

Let $R_c = I^{n \times n}$ (again to aid in obtaining a solution), so that $K_c = P_c$ where P_c satisfies the Riccati equation

$$\langle Ah, P_c k \rangle + \langle P_c h, Ak \rangle + \langle Q_c h, k \rangle = \langle P_c B R_c^{-1} B^* P_c h, k \rangle \quad \forall h, k \in D(A) \quad (221)$$

As shown in Curtain (see [24]), P_c can be written as

$$P_c h = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p_{ij} \langle h, \phi_i \rangle \phi_j \quad (222)$$

and one can solve for the p_{ij} 's by solving the equation [24]

$$(\lambda_i + \lambda_j) \langle Q_c \phi_i, \phi_j \rangle = \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} p_{ik} p_{jr} \langle R_c^{-1} B^* \phi_k, B^* \phi_r \rangle \quad (223)$$

As shown in Chapter 3, this yields

$$(-2i^2 + 8)p_{ii} + 1 = p_{ii}^2 \quad (224)$$

Using this fact then yields

$$P_c h = \sum_{i=1}^{\infty} p_{ii} \langle h, \phi_i \rangle \phi_i = K_c h \quad (225)$$

In a similar way one finds that the steady state Kalman filter constant gain operator is given by

$$K_f = P_f C^* R_f^{-1} \quad (226)$$

where P_f satisfies the Riccati equation

$$\langle A^* h, P_f k \rangle + \langle P_f h, A^* k \rangle + \langle Q_f h, k \rangle = \langle P_f C^* R_f^{-1} C P_f h, k \rangle \quad \forall h, k \in D(A^*) \quad (227)$$

where for this problem $A^* = A$ and Q_f is given by

$$Q_f h = G Q_o G^* h = Q_o h \quad \forall h \in \mathcal{H} \quad (228)$$

For this example $Q_o = I = Q_f$ in order to make obtaining a solution easier. Also by duality, P_f can be written as

$$P_f h = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} q_{ij} \langle h, \phi_i \rangle \phi_j \quad (229)$$

With $Q_f = I$ and $R_f = I^{n \times n}$ (so that a solution can be obtained), the q_{ij} 's can be found from the equation

$$(-2i^2 + 8)q_{ii} + 1 = q_{ii}^2 \quad (230)$$

and the operator K_f can be written as

$$K_f h = \sum_{i=1}^{\infty} q_{ii} \langle h, \phi_i \rangle \phi_i \quad (231)$$

In order to apply Schumacher's approximation technique, one must determine if the eigenfunctions of $(A - BK_c)$ are complete. As was demonstrated in Chapter 3, this is the case. This problem differs from the one in Chapter 3 in that there are two unstable eigenvalues associated with this problem. Both of these eigenvalues must be shifted to the left half plane in order to stabilize the system.

Using the quadratic formula, one can solve for p_{ii} and q_{ii} as given by

$$p_{ii} = q_{ii} = \frac{-2i^2 + 8}{2} + \frac{\sqrt{(-2i^2 + 8)^2 + 4}}{2} \quad (232)$$

which for this example gives

$$q_{11} = 6.16 = p_{11}$$

$$q_{22} = 1 = p_{22}$$

$$q_{33} = .1 = p_{33}$$

and so forth. The resulting regulator poles will be the spectrum of the operator $(A - BK_c)$ which is given by

$$\sigma(A - BK_c) = \lambda_i - q_{ii} \quad i = 1, 2, \dots \quad (233)$$

and the filter poles will be the spectrum of the operator $(A - K_f C)$ which is given by

$$\sigma(A - K_f C) = \lambda_i - p_{ii} \quad i = 1, 2, \dots \quad (234)$$

The amount of spectrum shift that occurs is a function of the operators K_c and K_f , which are determined by the A.R.E.s and therefore are a function of the operators $R_f, R_c, Q_f, Q_c, C, B,$ and C_1 .

7.3 Schumacher's Approach

The purpose of this example is to illustrate the application of the techniques developed and explored in this research effort. The first technique to be considered will be that of Schumacher [58, 59, 60]. One can choose the isomorphism R of Theorem 3.3.1 to be a projection operator that maps the Hilbert space \mathcal{H} to a finite-dimensional subspace \mathcal{H}_k so that

$$\tilde{K}_f = RK_f = P^k K_f : \mathfrak{R}^N \rightarrow \mathcal{H}_k \quad (235)$$

$$\tilde{K}_c = K_c R^{-1} = K_c (P^k)^{-1} : \mathcal{H}_k \rightarrow \mathfrak{R}^N \quad (236)$$

Let the order of the subspace be $k = 2$ and select the finite-dimensional subspace so that $\mathcal{H}_k = \mathcal{H}_u$ where \mathcal{H}_u is the unstable subspace spanned by the unstable eigenfunctions associated with the eigenvalues

$$\lambda_1 = 3 \quad \lambda_2 = 0$$

so that

$$\phi_1 = \sqrt{2} \sin \pi z \quad \phi_2 = \sqrt{2} \sin 2\pi z$$

This will give the following approximations for the operators K_c and K_f

$$\tilde{K}_f = \sum_{i=1}^2 q_{ii} \langle *, \phi_i \rangle \phi_i \quad (237)$$

$$\tilde{K}_c = \sum_{i=1}^2 p_{ii} \langle *, \phi_i \rangle \phi_i \quad (238)$$

The resulting regulator poles will coincide with $\sigma(A - BK_c)$, which are

$$\mu_1 = \lambda_1 - q_{11} = 3 - 6.16 = -3.16$$

$$\mu_2 = \lambda_2 - q_{22} = 0 - 1 = -1$$

$$\mu_i = \lambda_i \quad i = 3, 4, \dots$$

Since for this problem $p_n = q_{ii}$, the filter will have the same poles.

This approximation will provide a second order controller that stabilizes the system with an exponential growth constant of -1. Therefore, this controller will retain stability for bounded perturbations whose norm is less than 1. If one were to solve for the semigroup generated by $(A - BK_c)$ and $(A - K_f C)$ and find a constant M such that

$$\|T(t)\| \leq M e^{-t}$$

then one could use Lemma 3.4.2 and determine that the system would remain stable in the presence of relatively bounded perturbations that satisfy

$$\|\Delta\| \leq a \|A\|$$

where $a \leq (M + 1)^{-1}$. Other than these two types of perturbations, it is not readily apparent as to how robust the resulting closed-loop system will be. Also, for more complicated problems, it may not be so easy to calculate resulting closed-loop poles to determine the growth constant of the closed-loop system [17].

7.4 Banks Approximation

Another approach one can take is that of Banks [7] as discussed in Section 4 of Chapter 4. With this approach, one selects an approximation scheme and uses it to approximate the state space operators. Then, using these approximations, one can approximate the solution of the A.R.E. and base the finite-dimensional controller on that solution. One possible approximation scheme is to use modal

approximation as was mentioned in Section 4.4. Using this technique, one can write A^k as

$$A^k = P^k A = \sum_{i=1}^k \lambda_i \langle *, \phi_i \rangle \phi_i \quad (239)$$

and the semigroup generated by A^k can be written as [51]:

$$T^k(t) = \sum_{i=1}^k e^{\lambda_i t} \langle *, \phi_i \rangle \phi_i \quad (240)$$

Also, the operators $B, C,$ and Q_c can be approximated as

$$B^k = P^k I = P^k$$

$$P^k C = P^k I = P^k$$

$$P^k Q_c = P^k I = P^k$$

Using these modal approximations results in the Riccati equation

$$(\lambda_i + \lambda_j) p_{ii} + \langle P^k \phi_i, \phi_j \rangle = \sum_{k=1}^k \sum_{r=1}^k p_{ik} p_{jr} \langle R_c^{-1} B^{k*} \phi_k, B^{r*} \phi_r \rangle \quad (241)$$

which yields the following equation for p_{ii} :

$$(-2i^2 + 8)p_{ii} + 1 = p_{ii}^2 \quad (242)$$

This will result in the same controller that was obtained using Schumacher's approach. A different approximation scheme would likely result in a different controller form, just as a different choice of the isomorphism R would likely produce a different result using Schumacher's approach. The robustness is analyzed in the manner demonstrated the previous section.

7.5 Optimal Projection Equations

Because of the limited robustness analysis of the two methods used so far, a different approach would be useful if it gave more insight into the robustness of the resulting controller. Ideally it would be desirable to place the robustness

constraint into the design equations. One such technique is the LQG/LTR technique [51]. However, that technique has not been extended to the class of problems where A^* is unbounded and $R(B) \cup R(Q_f)$ is not contained in a finite-dimensional space.

One could approximate the LQG/LTR technique as was discussed in Section 4.4, or one could use the O.P.E. approach [10] discussed in Chapter 5. The optimum projection equations (O.P.E.) have been extended to the entire class of problems considered in this research. Using this approach, one must select operators Λ and Γ which define the projection of the compensator. The resulting compensator operators (see Equations (141) - (143)) are defined by:

$$A_c = \Gamma(A - QC^*R_f^{-1}C - BR_c^{-1}B^*P)\Lambda^* \quad (243)$$

$$B_c = \Gamma QC^*R_f^{-1} = \Gamma K_f \quad (244)$$

$$C_c = -R_c^{-1}B^*P\Lambda^* = -K_c\Lambda^* \quad (245)$$

where Q and P satisfies the four coupled equations (see Equations (134) - (137)):

$$AQ + QA^* + Q_f - QC^*R_f^{-1}CQ + \tau_{\perp}QC^*R_f^{-1}CQ\tau_{\perp}^* = 0 \quad (246)$$

$$A^*P + PA + Q_c - PBR_c^{-1}B^*P + \tau_{\perp}^*PBR_c^{-1}B^*P\tau_{\perp} = 0 \quad (247)$$

$$(A - BR_c^{-1}B^*P)\hat{Q} + \hat{Q}(A - BR_c^{-1}B^*P)^* + QC^*R_f^{-1}CQ + \tau_{\perp}QC^*R_f^{-1}CQ\tau_{\perp}^* = 0 \quad (248)$$

$$(A - QC^*R_f^{-1}C)^*\hat{P} + \hat{P}(A - QC^*R_f^{-1}C) + PBR_c^{-1}B^*P + \tau_{\perp}^*PBR_c^{-1}B^*P\tau_{\perp} = 0 \quad (249)$$

These four equations are coupled through the projection operator $\tau = \Lambda^*\Gamma$, and the solution involves the simultaneous solution of all four equations. This will in general produce a different solution than that obtained by projecting the standard uncoupled A.R.E. solutions.

One possible choice of Γ is the projection operator P^k , and then Λ^* will be the inverse of P^k . At first it would appear that this would give the same result that was obtained using Schumacher's approach. However, the answers will differ

in that, the approximation of K_f , \tilde{K}_f , appears in the defining equation for A_c when using Schumacher's approach, but the O.P.E. approach has the operator K_f in the defining equation for A_c (and likewise for K_c). This is due to the coupling that is present in the A.R.E. when using the O.P.E.'s. One thing to note is that the O.P.E. approach will include the Schumacher approach solutions by setting the noise term Q_f used in Schumacher's approach equal to the term $Q_f + \tau_{\perp} Q C^* R_f^{-1} C Q \tau_{\perp}^*$ used in the O.P.E. approach (and similarly for Q_c).

The O.P.E. approach will allow one to select an operator Ω (which bounds the system uncertainty) that satisfies Theorem 5.3.3, and thus achieve robustness to perturbations by including Ω in the A.R.E.'s, as was discussed in Chapter 5. The drawback to this approach is the difficulty in solving the coupled equations and the lack of an available algorithm. Algorithm development is being pursued by Hyland [46] who has presented limited results for finite-dimensional systems. The algorithm used by Hyland has not been published, and is not readily available.

7.6 H_{∞} Approach

The fourth approach to robustness that was discussed in this research was the use of H_{∞} techniques of Section 4 of Chapter 6. When using this approach, one needs to solve for the plant transfer function and the compensator transfer function. For the problem at hand, the plant transfer function is given by

$$G(s) = C(sI - A)^{-1}B = (sI - A)^{-1} \quad (250)$$

Matson [51] shows that one can write $(sI - A)^{-1}$ as:

$$(sI - A)^{-1} = \sum_{i=1}^{\infty} \frac{1}{s - \lambda_i} \langle *, \phi_i \rangle \phi_i \quad (251)$$

Using this fact, and recalling the definitions of B and C for this problem, the operator $G(s)$ can be written as

$$G(s) = \sum_{i=1}^{\infty} \frac{1}{s - \lambda_i} \langle *, \phi_i \rangle \phi_i \quad (252)$$

From Equation (197) of Chapter 6, the compensator has the transfer function

$$K(s) = K_c(sI - A + BK_c + K_f C)^{-1} K_f \quad (253)$$

which, for a second order approximation (which is chosen since only two unstable eigenvalues exist), can be written as

$$\tilde{K}(s) = \left(\sum_{i=1}^2 p_{ii} \langle *, \phi_i \rangle \phi_i \right) (sI - A + \tilde{K}_c + \tilde{K}_f)^{-1} \left(\sum_{i=1}^2 q_{ii} \langle *, \phi_i \rangle \phi_i \right) \quad (254)$$

Note that the operator $(A - \tilde{K}_c - \tilde{K}_f)$ can be expressed as

$$(A - \tilde{K}_c - \tilde{K}_f) = \sum_{i=1}^{\infty} \lambda_i \langle *, \phi_i \rangle \phi_i - \sum_{i=1}^2 p_{ii} \langle *, \phi_i \rangle \phi_i - \sum_{i=1}^2 q_{ii} \langle *, \phi_i \rangle \phi_i \quad (255)$$

which for this problem is equivalent to

$$(A - \tilde{K}_c - \tilde{K}_f) = \sum_{i=1}^{\infty} (\lambda_i - 2p_{ii}) \langle *, \phi_i \rangle \phi_i \quad (256)$$

Using this equation, one can write the operator $(sI - A + \tilde{K}_c + \tilde{K}_f)^{-1}$ as

$$(sI - A + \tilde{K}_c + \tilde{K}_f)^{-1} = \sum_{i=1}^{\infty} \langle *, \phi_i \rangle \frac{1}{s - \mu_i} \phi_i \quad (257)$$

where $\mu_i = (\lambda_i - 2p_{ii})$.

Substituting this into Equation (254) gives

$$\tilde{K}(s) = \frac{p_{11}^2}{s - \mu_1} + \frac{p_{22}^2}{s - \mu_2} \quad (258)$$

which, upon substituting for p_{11} , p_{22} , μ_1 , and μ_2 , gives

$$\tilde{K}(s) = \frac{38.95(s + 2.19)}{(s + 9.32)(s + 2)} \quad (259)$$

Using this expression for $\tilde{K}(s)$ then yields that $G(s)\tilde{K}(s)$ is equal to

$$G(s)\tilde{K}(s) = \left(\frac{1}{(s - 3)} + \frac{1}{s} \right) (\tilde{K}(s)) \quad (260)$$

$$G(s)\tilde{K}(s) = \frac{77.9(s - 1.5)(s + 2.19)(s - 1.5)}{s(s - 3)(s + 9.32)(s + 2)} \quad (261)$$

$G(s)\hat{K}(s)$ can be expressed as in Equation (260) because $p_{ii} = 0$ for $i > 2$. $G(s)$ is really an infinite sum as shown in Equation (252).

At this point one can use the H_∞ sufficient conditions to evaluate the robustness of this approximation with respect to some sensitivity function W_1 as in Section 4 of Chapter 6. The system will be robustly stable with respect to W_1 if

$$\| GK(I - GK)^{-1}W_1^{-1} \|_\infty \leq 1 \quad (262)$$

or, using the sufficient condition of Lemma 6.4.5, the system will be robustly stable using this compensator if

$$\| W_1^{-1} \|_\infty \leq \frac{1}{\| \{I - (I + GK)^{-1}\} \|_\infty} \quad (263)$$

At this point one could substitute for $G(s)K(s)$ and plot the right hand side of Equation (263) to get a bound on the norm of the sensitivity function as a function of frequency. If the norm bound is too small in a frequency range of concern, then one would have to iterate the design of the compensator. However, as pointed out by others [30], increased robustness at one frequency range will mean giving up performance somewhere. This tradeoff has to be made based on the goals to be achieved.

The advantage of the H_∞ approach is that it allows one to model uncertainty as either a multiplicative or additive perturbation, and the perturbations occur in the transfer function instead of the state space operators. The other approaches do not allow for this since they are based on perturbations to the state space operators. Since a transfer function description of a system and its perturbations is often easier to obtain than a state space model, in many cases, the H_∞ approach may be more advantageous. However, the disadvantage of the H_∞ approach is that it is very difficult to solve the problem for MIMO distributed parameter systems. Even for finite-dimensional problems, MIMO systems are quite often difficult to solve using this approach, although techniques to do so exist [33].

VIII. Conclusions and Recommendations

8.1 Conclusions

This research has considered finite-dimensional LQG-based control of infinite-dimensional systems with robustness as a prime issue. In Chapter 3, Schumacher's direct approach [59] to finite-dimensional compensator design was used to prove the existence of LQG-based finite-dimensional controllers for the class of systems considered in this research. Schumacher had proven the existence of finite-dimensional controller, but he did not try to force them to be of any particular type; this research did. Also, Schumacher never gave a proof that the finite-dimensional controller that results from his approach converges, as the controller dimension is increased toward ∞ , to the infinite-dimensional controller upon which the approximation is based. This research provides that proof. Finally, the limited ability to analyze robustness of the finite-dimensional controller obtained via Schumacher's approach was pointed out. It was shown how one can evaluate robustness for the case of bounded and relatively bounded perturbations.

Chapter 4 reviewed the LQG/LTR technique that Matson [51] extended in his work. A sufficient condition was developed that, if satisfied, allows the technique to be applied to a specific class of systems. First, it was shown that if the operator $K_\beta = (BV'B^* + \beta^{-2}Q_f - \beta^{-2}Y_\beta C^* R_f^{-1} C Y_\beta)$ is uniformly bounded independent of β , and if K_β is positive semi-definite, then the LQG/LTR technique is valid for the class of systems described in Section 3 of Chapter 1. Second, it was shown that the LQG/LTR technique is valid for the class of problems for which the inputs are uniformly bounded (using the L^2 norm), the states are bounded away from zero, the operator C is unitary, the adjoint of the B operator is bounded below, and $Q_f^{1/2}$ is a bounded linear operator (see Lemma 4.3.3). The class of systems is not physically meaningful, but both approaches taken to extend the LQG/LTR

technique are different from that taken by Matson. It is hoped that these new approaches will shed new light as to how to extend the LQG/LTR technique to the entire class of systems considered in this research. As an alternative, it was shown how to use the results of Banks [7] to approximate the LQG/LTR technique. In the same way that he approximates the solution of the infinite-dimensional algebraic Riccati equation, one can approximate the LQG/LTR procedure.

Chapter 5 considered the approach being taken by Bernstein and others [10, 11, 46, 9, 45, 39, 40] to achieve finite-dimensional control, robustness, and optimum performance at perturbed conditions, all in one design procedure. Using the optimal projection equation approach, the recent results of Bernstein [40] were extended to the infinite-dimensional systems considered in this research. The extension is a result of the theory developed by Bernstein in [10], but the important result obtained was a new interpretation of the LQG/LTR technique when the compensator order is fixed to be less than the model order describing the system to be controlled. It was shown that, even though one cannot recover a desired loop transfer function when the compensator order is less than the "truth" model order, one can achieve robustness to a class of perturbations that are a function of the modifying noise term $\beta^2 B V B^*$. As shown in Theorem 5.3.3, and discussed in Section 4 of Chapter 5, the class of perturbations are those for which $\Delta A Q + Q \Delta A^* \leq \Omega \quad \forall x \in \mathcal{H}$, where one can choose $\Omega = \beta^2 B V B^*$ when the uncertainty is restricted to the A operator, and ΔA is a bounded linear operator. This is an insight not before demonstrated in the literature.

Chapter 6 considered perturbations that occur in the transfer function. By using the algebraic theory developed by Callier and Desoer [12, 13], coupled with the work of Curtain [22] and Chen and Desoer [16], an H_∞ sufficient condition was established that can be used as a design tool to evaluate the robustness of a controller to multiplicative perturbations. This is different from the results of Curtain [22] which are for the case of additive perturbations. In this way her results have been

extended.

Finally, Chapter 7 gives an example problem. The approach of Schumacher [59] was used to obtain a finite-dimensional LQG-based controller, and its robustness was analyzed using the techniques of Chapter 3. Next, Banks' approximation approach was used, and it was shown that, for the problem considered, it resulted in the same controller that was obtained using Schumacher's approach. The advantage of using Banks's approach is that it involves approximating the algebraic Riccati equations (resulting in a finite-dimensional algebraic Riccati equation), and so can be used as an approximation to the LQG/LTR technique. Schumacher's technique involves approximating the solution of the infinite-dimensional A.R.E. for which LQG/LTR tuning has not been extended.

The optimal projection equation approach is applied to the problem, but it is not solved since an algorithm for solving the four coupled equations is not currently available. However, by looking at the structure of the defining equations, one can see that the result will not be the same as was obtained using Schumacher's or Banks' approaches. Finally, the transfer function of the the system is established, and it is shown how to use the H_∞ sufficient condition for robustness to evaluate the controller design. In the example, a sensitivity function W_1 (which is inherent to the methodology) was not assumed. Rather, the problem was set up so one could solve for the W_1 that satisfies the robustness sufficient condition. In this way one can establish a sufficient condition that, if satisfied, will guarantee robustness of a stabilizing controller.

8.2 Proposed Design Procedure

Based on the developments in this dissertation, a proposed design procedure is presented. This procedure is just one possible approach to the problem. A better procedure may be possible by changing the order of the steps, or by performing any of the steps in a different fashion.

1. Determine the operators K_f and K_c , using the optimal control theory of Chapter 2, that provide an acceptable degree of exponential stability of the closed-loop system.
2. Adjust the operator K_c if necessary to ensure that the eigenfunctions of $(A - BK_c)$ are complete.
3. Determine the exponential growth constant associated with the operator $(A - K_f C)$. Adjust K_f if necessary so that the growth constant of $(A - K_f C)$ is sufficiently less than zero. Increasing this growth constant will likely lower the order of the finite-dimensional controller since the magnitude of the growth constant determines how closely K_f has to be approximated (see Theorem 3.2.2).
4. Use Shumacher's approach to design a finite-dimensional approximation of the infinite-dimensional LQG controller.
5. Use a robustness enhancement technique, and determine the robustness of the system with respect to the type of perturbations one wishes to consider.
6. Determine the closed-loop pole locations and analyze the system's robustness and performance.

Clearly, there are alternatives for some of the steps. For example, if the LQG/LTR technique is valid, then one could include robustness recovery as part of the design in step 1. However, if one is approximating the LQG/LTR technique, then the robustness recovery will take place when the finite-dimensional controller is designed. Also, since none of the approaches can consider every type of perturbation, the type of perturbation will influence the steps in the design. Finally, iterations at one step may impact the ability to perform a later step. Therefore, the procedure may not be as sequential as indicated. It may be better to do some steps at the same time. Development of a good design procedure using the results of this dissertation

is a good topic for future research. The procedure given here is simply a starting point.

8.3 Recommendations

There are several recommendations for future research. It would be very helpful to develop an algorithm for solving the optimal projection equations. It may be possible to use an approach similar to that of Banks and approximate the solution. However, this would mean solving the optimal projection equations in a finite-dimensional form, and the algorithm for that is just now being developed [44].

Obviously, it would be worthwhile if one could extend the LQG/LTR technique to a larger class of systems than that for which it is currently valid. It is hoped that the work found in Chapter 4 will give some insight as to how to do that.

Also, it would be interesting to see if Schumacher's approach could be extended to incorporate robustness. If the LQG/LTR technique is ever extended to the entire class of problems, this could be used as the basis of the approximation. Also, by making a connection between the optimal projection equation approach and Schumacher's, it may be possible to gain insight into this.

Another area for research is considering other types of controllers. This research considered regulators. It may be useful to consider proportional plus integral controllers as was done by Pohjolainen [55]. Trackers, disturbance rejection controllers, and other forms of controllers also warrant attention. One also needs to consider the issue of performance which was ignored in this research. Increasing robustness is normally accomplished at the expense of performance at design conditions. A tradeoff occurs in practice, and it would be worthwhile to express this trade-off in mathematical terms for infinite-dimensional systems.

As was mentioned earlier, it would be very useful to refine the design procedure of the previous section. It is unknown exactly how the design steps interact.

However, it is believed that the LQG/LTR is valid for the entire class of systems considered in Chapter 1. This is expressed in the following conjecture.

Conjecture: For the class of systems described in Section 3 of Chapter 1, the operator K_β of Theorem 4.2.1 is uniformly bounded independent of β , and is either positive semi-definite, or negative semi-definite.

The conjecture allows K_β to be either positive or negative semi-definite since Theorem 4.2.1 can be proven for either case (just reverse the signs in the proof). Though neither condition has been proven, it is believed that one or the other will be true.

Finally, it would be worthwhile to extend any of the results in this research to the case of unbounded B and C operators. This would make the results applicable to a larger class of problems.

Bibliography

1. A. V. Balakrishnan. *Applied Functional Analysis*. Springer-Verlag, New York, 1981.
2. M. J. Balas. Adaptive Parameter Estimation of Large-Scale Systems by Reduced-Order Modeling. In *Proceedings of the Automatic Controls Conference*, pages 575-579, San Francisco, Ca., 1983.
3. M. J. Balas. Feedback Control of Flexible Structures. *IEEE Transactions on Automatic Control*, 23(4):673-679, 1978.
4. M. J. Balas. Linear Distributed Parameter Systems: Closed-loop Exponential Stability with a Finite-Dimensional Controller. *Automatica*, 20(3):371-377, 1984.
5. M. J. Balas. Modal Control of Certain Flexible Dynamic Systems. *SIAM Journal of Control and Optimization*, 16(3):450-462, 1978.
6. M. J. Balas. Trends in Large Space Structure Control Theory: Fondest Hopes. Wildest Dreams. *IEEE Transactions on Automatic Control*, AC-27(3):522-535, 1982.
7. H. T. Banks and K. Kunisch. The Linear Regulator Problem for Parabolic Systems. *SIAM Journal of Control and Optimization*, 22(5):684-698, September 1984.
8. A. Belleni-Morante. *Applied Semigroups and Evolution Equations*. Oxford University Press, Walton St., Oxford, 1979.
9. D. S. Bernstein and W. M. Haddad. The Optimal Projection Equations with Peterson-Hollot Bounds: Robust Controller Synthesis with Guaranteed Structured Stability Radius. In *Proceedings of the 26th Conference on Decision and Control*, pages 1308-1318, Los Angeles, Ca, December 1987.
10. D. S. Bernstein and D. C. Hyland. The Optimal Projection Equations for Finite-Dimensional Fixed-Order Dynamic Compensation of Infinite-Dimensional Systems. *SIAM Journal of Control and Optimization*, 24(1):122-151, January 1986.
11. D. S. Bernstein and D. C. Hyland. The Optimal Projection Equations for Fixed-Order Dynamic Compensation of Distributed-Parameter Systems. In *Proceedings of AIAA Dynamics Specialists Conference*, pages 396-400, Palm Springs, Ca, May 1984.
12. F. M. Callier and C. A. Desoer. An Algebra of Transfer Functions for Distributed Linear Time-Invariant Systems. *IEEE Transactions on Circuits and Systems*, CAS-25(9):651-662, 1978.

13. F. M. Callier and C. A. Desoer. Simplifications and Clarifications on the Paper "An Algebra of Transfer Functions for Distributed Linear Time-Invariant Systems". *IEEE Transactions on Circuits and Systems*, CAS-27(4):320-323, 1980.
14. F. M. Callier and C. A. Desoer. Stabilization, Tracking, and Disturbance Rejection in Multivariable Convolution Systems. *Annales de la Societe Scientifique de Bruxelles*, T. 94(1):7-51, 1980.
15. F. M. Callier and J. Winkin. Distributed System Transfer Functions of Exponential Order. *International Journal of Control*, 43(5):1353-1373, 1986.
16. M. J. Chen and C. A. Desoer. Necessary and Sufficient Conditions for Robust Stability of Linear Distributed Feedback Systems. *International Journal of Control*, 35(2):255-267, 1982.
17. R. F. Curtain. Pole Assignment for Distributed Systems by Finite-Dimensional Control. *Automatica*, 57-67, 1985.
18. R. F. Curtain. A Semigroup Approach to the LQG Problem for Infinite-Dimensional Systems. *IEEE Transactions on Circuits and Systems*, CAS-25(9):713-721, September 1978.
19. R. F. Curtain. *Equivalence of Input-Output Stability and Exponential Stability for Infinite Dimensional Systems*. Technical Report TW-276, Mathematics Institute Rijksuniversiteit Groningen, P.O. Box 800, 9700 AV Groningen. The Netherlands, August 1987.
20. R. F. Curtain. Finite-Dimensional Compensator Design for Parabolic Distributed Systems with Point Sensors and Boundary Input. *IEEE Transactions on Automatic Control*, AC-27(1):98-104, 1982.
21. R. F. Curtain. Finite-Dimensional Compensators for Parabolic Distributed Systems with Unbounded Control and Observation. *SIAM Journal of Control and Optimization*, 22(2):255-276, 1984.
22. R. F. Curtain and K. Glover. Robust Stabilization of Infinite Dimensional Systems by Finite Dimensional Controllers. *Systems and Control Letters*, 7:41-47, February 1986.
23. R. F. Curtain and A. J. Pritchard. Compensators for Infinite Dimensional Linear Systems. *Journal of the Franklin Institute*, 315(5/6):331-346, 1983.
24. R. F. Curtain and A. J. Pritchard. *Infinite Dimensional Linear System Theory*. Springer-Verlag, New York, 1978.
25. J. J. D'Azzo and C. Houpis. *Linear Control System Analysis*. McGraw-Hill, New York, 1981.
26. C. A. Desoer et al. Feedback System Design: The Fractional Representation Approach to Analysis and Synthesis. *IEEE Transactions on Automatic Control*, AC-25(3):399-412, 1980.

27. C. A. Desoer and C. L. Gustafson. Algebraic Theory of Linear Multivariable Feedback Systems. *IEEE Transactions on Automatic Control*, AC-29(10):909-916, 1984.
28. C. A. Desoer and M. Vidyasagar. *Feedback Systems: Input-Output Properties*. Academic Press, New York, 1975.
29. J. L. Doob. *Stochastic Processes*. John Wiley and Sons, Inc., New York, 1953.
30. J. C. Doyle and G. Stein. Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis. *IEEE Transactions on Automatic Control*, AC-26(1):4-16, 1981.
31. J. C. Doyle and M. Vidyasagar. Guaranteed Margins for LQG Regulators. *IEEE Transactions on Automatic Control*, AC-23(4):756-757, 1978.
32. J. C. Doyle, J. E. Wall, and G. Stein. Performance and Robustness Analysis for Structured Uncertainty. In *Proceedings of the 1982 IEEE Conference on Decision and Control*, pages 629-636, Orlando, Florida, 1982.
33. B. A. Francis. *A Course in H-Infinity Control Theory. Lecture Notes in Control and Information Sciences*, Springer-Verlag, New York, 1987.
34. B. A. Francis and J. C. Doyle. Linear Control Theory with an H-infinity Optimality Criterion. *SIAM Journal of Control and Optimization*, 25(4):815-844, July 1987.
35. J. S. Friedenberg, D. P. Looze, and J. B. Cruz. Robustness Analysis Using Singular Value Sensitivities. *International Journal of Control*, 35(1):95-116, 1982.
36. I. M. Gelfand and S. V. Fomin. *Calculus of Variations*. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963.
37. I. M. Gelfand and G. E. Shilov. *Generalized Functions Vol 1*. Academic Press, New York, 1964.
38. J. S. Gibson. The Riccati Integral Equations for Optimal Control Problems on Hilbert Space. *SIAM Journal of Control and Optimization*, 17(4):537-565, January 1979.
39. S. W. Greeley, D. C. Hyland, and S. Richter. Reduced-Order Compensation: LQG Reduction Versus Optimal Projection Using a Homotopic Continuation Method. In *Proceedings of the 26th Conference on Decision and Control*, pages 742-747, Los Angeles, Ca, December 1987.
40. W. M. Haddad and D. S. Bernstein. The Unified Optimal Projection Equations for Simultaneous Reduced-Order, Robust Modeling, Estimation and Control. In *Proceedings of the 26th Conference on Decision and Control*, pages 449-454, Los Angeles, Ca, December 1987.

41. P. R. Halmos. *A Hilbert Space Problem Book*. Princeton: D. Van Nostrand Co., Inc., 1967.
42. E. Hille and R. S. Phillips. *Functional Analysis and Semigroups*. Volume 31, American Mathematical Society Colloquium Publications, 1957.
43. I. Horowitz. *Synthesis of Feedback Systems*. Academic Press, New York, 1963.
44. D. C. Hyland. The Optimal Projection Approach to Fixed-Order Compensation: Numerical Methods and Illustrative Results. In *AIAA 21st Aerospace Sciences Mtg.*, pages 83-0303, Reno, Nv., 1983.
45. D. C. Hyland and D. S. Bernstein. Explicit Optimality Conditions for Fixed-Order Dynamic Compensation. In *Proceeding of Conference on Decision and Control*, pages 161-165, San Antonio, Tx, December 1983.
46. D. C. Hyland and D. S. Bernstein. The Optimal Projection Equations for Fixed-Order Dynamic Compensation. *IEEE Transactions on Automatic Control*, AC-29(11):1034-1037, November 1984.
47. C. A. Jacobson. *Some Aspects of the Structure and Stability of a Class of Linear Distributed Systems*. Master's thesis, Rensselaer Polytechnic Institute, Troy, New York, May 1984.
48. T. Kato. *Perturbation Theory for Linear Operators*. Springer-Verlag, New York, 1966.
49. H. Kwakernaak and R. Sivan. *Linear Optimal Control Systems*. Wiley-Interscience, New York, 1972.
50. D. L. Lukes and D. L. Russell. The Quadratic Criterion for Distributed Systems. *SIAM Journal of Control and Optimization*, 7:101-121, 1969.
51. C. L. Matson. *The Linear Quadratic Gaussian / Loop Transfer Recovery Technique for a Class of Distributed Parameter Systems*. PhD thesis, Air Force Institute of Technology, Wright-Patterson AFB, Ohio, December 1986.
52. A. W. Naylor and G. R. Sell. *Linear Operator Theory in Engineering and Science*. Springer-Verlag, New York, 1982.
53. C. N. Nett. *The Fractional Representation Approach to Robust Linear Feedback Design: A Self-Contained Exposition*. Master's thesis, Rensselaer Polytechnic Institute, Troy, New York, 1984.
54. A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag, New York, 1983.
55. S. A. Pohjolainen. Robust Multivariable PI-Controller for Infinite-Dimensional Systems. *IEEE Transactions on Automatic Control*, AC-27(1):17-30, 1982.
56. M. Sakawa. Feedback Stabilization of a Class of DPS and Construction of a State Estimator. *IEEE Transactions on Automatic Control*, AC-6:748-753, 1975.

57. N. R. Sandell. *Robust Stability of Linear Dynamic Systems with Application to Singular Perturbation Theory, Recent Developments in the Robustness Theory of Multivariable Systems*. Technical Report LIDS-R-954, MIT Laboratory for Information Decision Systems, August 1979.
58. J. M. Schumacher. A Direct Approach to Compensator Design for Distributed Systems. *SIAM Journal of Control and Optimization*, 21:823-836, 1983.
59. J. M. Schumacher. *Dynamic Feedback in Finite- and Infinite-Dimensional Linear Systems*. PhD thesis, Mathematisch Centrum, M. C. Tracts No. 143, Amsterdam, 1982.
60. J. M. Schumacher. Finite-Dimensional Regulators for a Class of Infinite-Dimensional Systems. *Systems and Control Letters*, 3:7-12, 1983.
61. L. Schwartz. *Theorie des Distributions*. Hermann, Paris, 1966.
62. G. Stein and J. C. Doyle. Robustness with Observers. *IEEE Transactions on Automatic Control*, AC-24(4):607-611, August 1979.
63. G. W. Stewart. *Introduction to Matrix Computations*. Academic Press, New York, 1973.
64. A. E. Taylor and D. C. Lay. *Introduction to Functional Analysis*. Wiley and Sons, second edition, 1980.
65. R. Triggiani. On the Stabilizability Problem in Banach Space. *Journal of Mathematical Analysis and Applications*, 52:383-403, 1975.
66. M. Vidyasagar. *Control System Synthesis: A Factorization Approach*. MIT Press, Cambridge, Mass, 1985.
67. M. Vidyasagar, H. Schneider, and B. A. Francis. Algebraic and Topological Aspects of Feedback Stabilization. *IEEE Transactions on Automatic Control*, AC-27(4):880-894, 1982.
68. W. M. Wonham. *Linear Multivariable Control: A Geometric Approach*. Springer-Verlag, New York, 1979.
69. R. K. Yedavalli and Z. Liang. Reduced Conservatism in Stability Robustness Bounds by State Transformation. *IEEE Transactions on Automatic Control*, AC-31(9):863-866, 1986.
70. K. Yosida. *Functional Analysis*. Springer-Verlag, Berlin, sixth edition, 1980.
71. G. Zames. On The Input-Output Stability of Nonlinear Time-Varying Feedback Systems, Part I. *IEEE Transactions on Automatic Control*, AC-11:228-238, 1966.

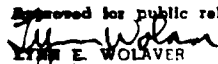
Vita

Randall Norris Paschall was born on 23 June 1958 in Memphis, Tennessee. He graduated from Raleigh-Egypt High School in 1976, and attended Oklahoma University on a Navy ROTC scholarship during his freshman year of college. He left Oklahoma and returned home where he attended Christian Brothers College in Memphis. He graduated in May 1980 as a distinguished graduate of the Air Force ROTC program at Memphis State University with a Bachelor of Science in Electrical Engineering from Christian Brothers College. Upon graduation, he received a commission as a Second Lieutenant, and entered the School of Engineering at the Air Force Institute of Technology in June 1980. He entered the graduate program in guidance and control, and graduated in December 1981 with a Master of Science in Electrical Engineering. Upon graduation he went to Eglin AFB, Florida where he was an engineer for the range instrumentation program office and a controls engineer for the low level laser guided bomb program office. In June 1985 he entered the Ph.D. program at the Air Force Institute of Technology.

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Block 19

This dissertation considers the problem of robustly stabilizing infinite-dimensional systems using finite-dimensional controllers. The controllers are assumed to be linear quadratic Gaussian (LQG) based controllers. This research uses a direct approach to demonstrate the existence of finite-dimensional LQG-based controllers that stabilize the nominal system. Once existence is proven, the research focuses on ways to analyze the robustness of the controller. It is pointed out that the exponential growth constant of the semigroup generated by the system A operator is not the only measure of robustness.

Several types of perturbations are considered, including bounded, relatively bounded, additive, and multiplicative. ~~As a result,~~ several approaches to analyzing robustness are developed. Direct analysis using results from functional analysis is accomplished, followed by an approach called the optimal projection equation approach, and then H -infinity techniques are used to develop a sufficient condition for robustness in the presence of multiplicative perturbations of the plant transfer function. It is pointed out that each approach can be used to account for different types of perturbations.

A development in this research is a new interpretation of the linear quadratic Gaussian / loop transfer recovery technique (LQG/LTR) for the case of reduced order controllers. The technique can be interpreted as modeling system uncertainty through the added noise term rather than tuning to recover a desired loop transfer function.

Also contained in this research is a sufficient condition for which the LQG/LTR technique can be extended. The development of the sufficient condition is different than approaches taken by others, and may provide the needed insight to extend the LQG/LTR technique to the class of problems considered without any added assumption being required. A way to approximate the LQG/LTR technique is also given.

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