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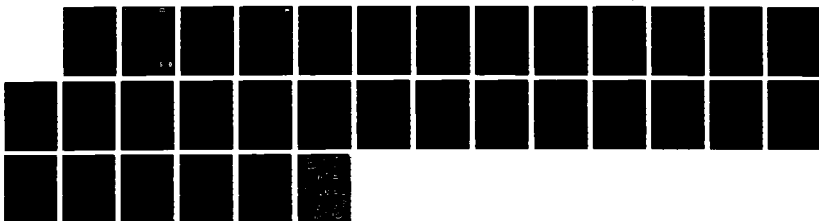
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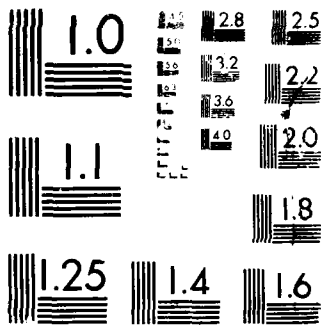
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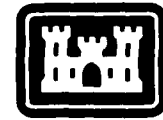


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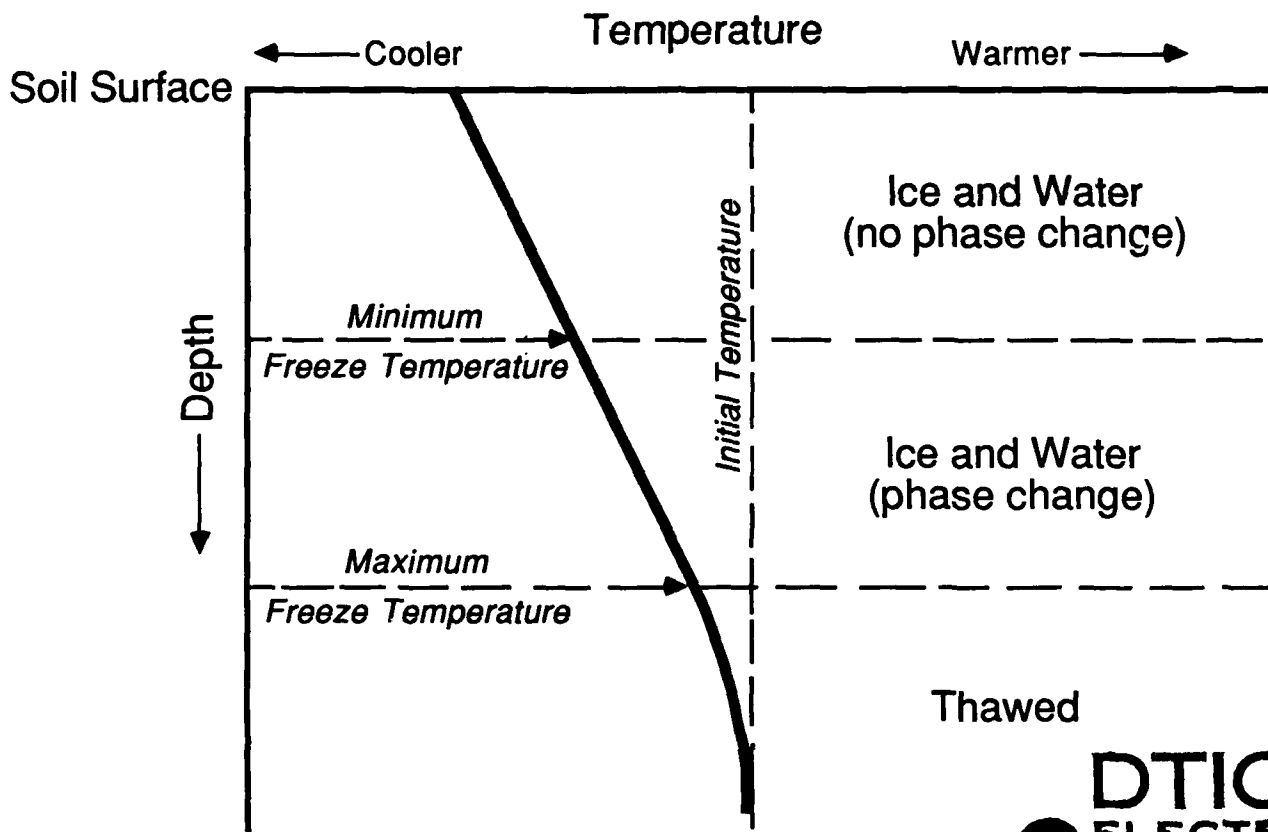


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US Army Corps  
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Cold Regions Research &  
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### *Freezing of soil with an unfrozen water content and variable thermal properties*



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*Cover: Geometry of a semi-infinite soil mass, initially at a temperature above freezing, that freezes due to a constant surface temperature below freezing.*

# CRREL Report 88-2

March 1988



## *Freezing of soil with an unfrozen water content and variable thermal properties*

Virgil J. Lunardini

Prepared for  
OFFICE OF THE CHIEF OF ENGINEERS

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## NOMENCLATURE

- $A = \frac{2\lambda_0}{B}$   
 $B = \frac{\delta - X}{X}$   
 $C$  specific heat  
 $C_{ij} = C_i/C_j$   
 $C_0$  arbitrary value of specific heat for constant-property mushy zone, otherwise  
 $C_0 \equiv C_u$   
 $F$  defined by eq 17  
 $F_1 = \frac{1}{\alpha_0} (1 + \sigma_0 - \beta_{21})$   
 $F_2 = \frac{\beta_{21}}{3\alpha_0\beta_1}$   
 $k$  thermal conductivity  
 $k_{ij} = k_i/k_j$   
 $k_0$  any specified constant-conductivity mushy zone value of  $k$ ; otherwise  $k_0 \equiv k_u$   
 $K = P - A$   
 $N = 1 + 2\beta_1 P$   
 $P = \phi + \frac{\phi^2\beta_1}{2}$   
 $\ell$  latent heat of fusion of water  
 $m$  mass  
 $m_w$  mass of water  
 $q$  heat flux  
 $q_g$  latent heat flux during solidification  
 $R = 1 - \frac{\eta}{\gamma} \sqrt{\alpha_{13}} = 1 - \frac{X_1}{X}$   
 $S_T = \frac{C_s(T_f - T_s)}{\gamma_d \ell \Delta \xi}$ , Stefan number  
 $t$  time  
 $T$  temperature  
 $T_f, T_m$  highest and lowest temperatures for phase change  
 $T_0, T_s$  initial and surface temperatures  
 $x$  Cartesian coordinate  
 $x_t$  volumetric water fraction

$X, X_1$  phase change interface positions for  $T_f, T_m$

$$z = R\lambda/B$$

$\alpha$  thermal diffusivity =  $k/C$

$$\alpha_{ij} = \alpha_i/\alpha_j$$

$$\alpha_0 = k_0/C_0$$

$$\beta_1 = k_{fu} - 1$$

$$\beta_2 = C_{fu} - 1$$

$\delta$  temperature penetration depth

$$\gamma = \frac{X}{2\sqrt{\alpha_3 t}}, \text{ phase change parameter defined by eq 23}$$

$\gamma_d$  dry unit density of soil solids (mass of soil solids per unit volume)

$$\eta = \frac{X_1}{2\sqrt{\alpha_1 t}}, \text{ phase change parameter}$$

$$\theta = \frac{T_f - T}{T_f - T_m}, \text{ dimensionless temperature}$$

$$\lambda = \phi_0 k_{32}$$

$$\lambda_0 = \phi_0 k_{30}$$

$\xi$  ratio of unfrozen water mass to soil solid mass

$\xi_0, \xi_f, \xi_s$  values of  $\xi$  at  $T_f, T_m, T_s$

$$\sigma = \frac{C_{32}\phi}{S_T}$$

$$\sigma_0 = \frac{C_{30}\phi}{S_T}$$

$$\phi = \frac{T_f - T_s}{T_f - T_m}$$

$$\phi_0 = \frac{T_0 - T_f}{T_f - T_m}$$

$\psi$  dimensionless temperature defined by eq 12

#### Subscripts

1,2,3 regions of soil

f, s, u frozen value, surface value, and thawed value



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## **PREFACE**

This report was prepared by Dr. Virgil J. Lunardini, Mechanical Engineer, Applied Research Branch, Experimental Engineering Division, U.S. Army Cold Regions Research and Engineering Laboratory. This study was conducted under ILIR 6XX 71462, *Heat Transfer with Freezing or Thawing*.

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# Freezing of Soil with an Unfrozen Water Content and Variable Thermal Properties

VIRGIL J. LUNARDINI

## INTRODUCTION

The mathematical theory of conductive heat transfer with solidification has been largely confined to materials that change phase at a single temperature. The best-known problem of this type is that of Neumann, and its solution has been widely used for the freezing of soils (Neumann 1860, Berggren 1943, Carslaw and Jaeger 1959). However, for media such as soils, the phase change can occur over a range of temperatures (Anderson and Tice 1973, Tice et al. 1978, Lunardini 1981a). In other words, at any temperature below the normal freezing point, there will be an equilibrium state of unfrozen water, ice and soil solids. Figure 1 shows the geometry for a semi-infinite soil mass, initially at a temperature above freezing, that freezes due to a constant surface temperature held below the freezing point. The phase change is assumed to occur within the temperature limits of  $T_m$  and  $T_f$ , representing minimum and maximum phase change temperatures.

Figure 2 shows the unfrozen water  $\xi$  as a function of temperature for a typical soil. At  $T_f$  all of the water is in the liquid form, while at  $T_m$  the free water is all frozen. There may be a residual amount of bound water, denoted by  $\xi_f$ , that will remain unfrozen even at very low temperatures. It will be assumed that for  $T < T_m$ , unfrozen water may exist but no phase change will occur. The region  $T_m \leq T \leq T_f$  is called the zone of phase change, or the mushy zone. In this region, water will solidify to ice, and unfrozen water and ice will coexist. As  $(T_f - T_m) \rightarrow 0$ , the phase change will approach the typical Neumann-type problem, which is

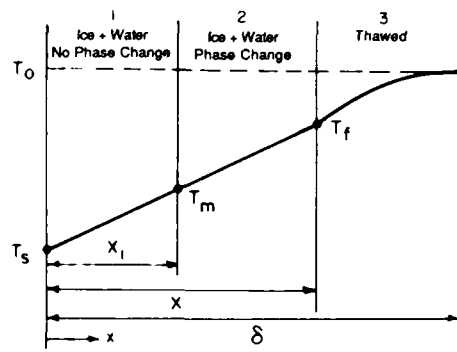


Figure 1. Geometry for solidification with a phase change zone.

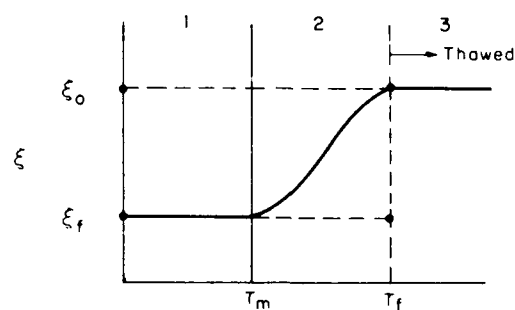


Figure 2. Unfrozen water vs temperature.

particularly valid for coarse materials such as sands and gravels. The form of the  $\xi$  function for soils can be expressed by different functional relations. The simplest relation is linear:

$$\xi = \xi_0 + \frac{\Delta\xi}{\Delta T_m} (T - T_f) \quad (1)$$

where  $\Delta\xi = \xi_0 - \xi_f$  and  $\Delta T_m = T_f - T_m$ . Another functional relation, which can closely model the data and is easy to manipulate analytically, is a quadratic form:

$$\xi = \xi_0 + \frac{2\Delta\xi}{\Delta T_m} (T - T_f) + \frac{\Delta\xi}{\Delta T_m^2} (T - T_f)^2. \quad (2)$$

If  $\xi_0$ ,  $\xi_f$  and  $\Delta T_m$  are the same for these functions, then the mean unfrozen water slope  $d\xi/dT$  will be identical.

The thermal conductivity and the specific heat within the mushy zone are functions of the unfrozen water and may be represented by

$$k = k_u - \frac{(k_f - k_u)}{\Delta\xi} (\xi - \xi_0) \quad (3)$$

$$C = C_u - \frac{(C_f - C_u)}{\Delta\xi} (\xi - \xi_0) \quad (4)$$

where  $k_f, k_u$  = fully frozen and fully thawed thermal conductivities and  $C_f, C_u$  = fully frozen and fully thawed specific heats.

Obviously these properties are functions of the particular form of the unfrozen water function (Frivick 1980). Within the fully frozen region (zone 1) it is assumed that the thermal properties are constant and equal to the frozen values, while for the thawed region (zone 3) the properties are constant and equal to the thawed soil values.

Tien and Geiger (1967) and Ozisik and Uzzell (1979) used an unfrozen liquid content that varied with position within the two-phase zone. They did not deal with soil systems, however. Cho and Sunderland (1974) found a solution for the freezing of a material with the thermal conductivity varying linearly with temperature; however, the material changed phase at a single temperature.

## BASIC EQUATIONS

Consider a small volume of material within the mushy zone. Energy will be conducted in and out of the volume, and latent heat will be released during solidification (Fig. 3). Thus the problem is one of conduction with a distributed energy source. The governing equations were derived by Lunardini (1985, 1988). The net conduction is

$$q_x - q_{x+\Delta x} = \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) \Delta t \Delta x. \quad (5)$$

The latent energy released due to solidification of a mass of water  $\Delta m_w$  is as follows

$$\Delta q_g = -\ell \Delta m_w = -\ell \gamma_d \Delta \xi \Delta x. \quad (6)$$

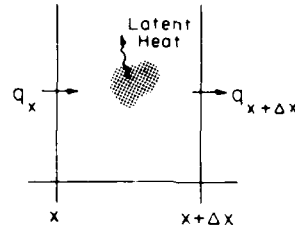


Figure 3. Heat flow in the mushy zone.

The energy equation in the mushy zone is

$$\frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) - \ell \gamma_d \frac{\partial \xi}{\partial t} = C \frac{\partial T}{\partial t}. \quad (7)$$

The most general case will be a problem with three regions as shown in Figure 1. The equations for region 1 are

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\alpha_1} \frac{\partial T_1}{\partial t} \quad (8)$$

$$T_1(0, t) = T_s \quad (8a)$$

$$T_1(X_1, t) = T_m \quad (8b)$$

$$k_1 \frac{\partial T_1(X_1)}{\partial x} = k(X_1) \frac{\partial T_2(X_1)}{\partial x}. \quad (8c)$$

For region 2 (the mushy zone)

$$\frac{\partial}{\partial x} \left( k \frac{\partial T_2}{\partial x} \right) = \left( C + \ell \gamma_d \frac{d\xi}{dT_2} \right) \frac{\partial T_2}{\partial t} \quad (9)$$

$$T_2(X_1, t) = T_m \quad (9a)$$

$$T_2(X, t) = T_f \quad (9b)$$

$$k(X) \frac{\partial T_2(X)}{\partial x} = k_s \frac{\partial T_3(X)}{\partial x}. \quad (9c)$$

The thawed zone is governed by

$$\frac{\partial^2 T_3}{\partial x^2} = \frac{1}{\alpha_3} \frac{\partial T_3}{\partial t} \quad (10)$$

$$\lim_{x \rightarrow \infty} T_3(x, t) = T_o \quad (10a)$$

$$T_3(x, 0) = T_o \quad (10b)$$

$$T_3(X, t) = T_f. \quad (10c)$$

These equations can be partially nondimensionalized by using the following relations:

$$\theta_i = \frac{T_f - T_i}{T_f - T_m} \quad i = 1, 2, 3 \quad \phi = \frac{T_f - T_s}{T_f - T_m} \quad \phi_0 = \frac{T_0 - T_f}{T_f - T_m}$$

$$\sigma = \frac{C_{32}\phi}{S_T} \quad \sigma_0 = \frac{C_{30}\phi}{S_T} \quad \lambda = \phi_0 k_{32} \quad \lambda_0 = \phi_0 k_{30}$$

$$\beta_1 = k_{fu}^{-1} \quad \alpha_0 = \frac{k_0}{C_0} \quad S_T = \frac{C_3(T_f - T_s)}{\gamma_d l \Delta \xi}$$

$$\beta_2 = C_{fu}^{-1}$$

With these relations the thermal conductivity and specific heat in region 2, for the linear  $\xi$  case, are

$$k = k_0(1 + \beta_1 \theta_2) \quad (11a)$$

$$C = C_0(1 + \beta_2 \theta_2). \quad (11b)$$

For the mushy zone with variable thermal properties, eq 11a and b will use  $k_0 = k_u$  and  $C_0 = C_u$ . However, the use of the general notation  $k_0$  and  $C_u$  allows the properties in region 2 to be specified independently of the frozen or thawed values if the thermal properties are constant (Appendix A).

In the mushy zone the following transformation will be used:

$$\psi = \frac{1}{k_0} \int_0^{\theta_2} k(\theta_2) d\theta_2. \quad (12)$$

For the linear  $\xi$  case the function  $\psi$  can be evaluated explicitly:

$$\psi = \theta_2 + \frac{\beta_1}{2} \theta_2^2. \quad (12a)$$

Then

$$k = k_0 \sqrt{1 + 2\beta_1 \psi} \quad (13)$$

$$\theta_2 = \frac{\sqrt{1 + 2\beta_1 \psi} - 1}{\beta_1}. \quad (14)$$

The transformed equations are as follows:

$$\frac{\partial^2 \theta_1}{\partial x^2} = \frac{1}{\alpha_1} \frac{\partial \theta_1}{\partial t} \quad (15)$$

$$\theta_1(0, t) = \phi \quad (15a)$$

$$\theta_1(X_1, t) = 1 \quad (15b)$$

$$k_1 \frac{\partial \theta_1}{\partial x} (X_1, t) = k(X_1) \frac{\partial \theta_2}{\partial x} (X_1, t). \quad (15c)$$

The equations for region 2 are written only for the linear  $\xi$  assumption (the details are given in Appendix A):

$$\sqrt{1 + 2\beta_1\psi} \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial F}{\partial t} \quad (16)$$

$$\psi(X, t) = 0 \quad (16a)$$

$$\psi(X_1, t) = 1 + \beta_1/2 \quad (16b)$$

$$k_0 \frac{\partial \psi}{\partial x} (X, t) = k_3 \frac{\partial \theta_3}{\partial x} (X, t) \quad (16c)$$

$$\alpha_0 F = (1 + \sigma_0 - \beta_{21})\psi + \frac{\beta_{21}}{3\beta_1} (1 + 2\beta_1\psi)^{3/2}. \quad (17)$$

The thawed region has the following relations:

$$\frac{\partial^2 \theta_3}{\partial x^2} = \frac{1}{\alpha_3} \frac{\partial \theta_3}{\partial t} \quad (18)$$

$$\lim_{x \rightarrow -\infty} \theta_3(x, t) = -\phi_0 \quad (18a)$$

$$\theta_3(x, 0) = -\phi_0 \quad (18b)$$

$$\theta_3(X, t) = 0. \quad (18c)$$

Exact solutions for the three-zone problem with variable thermal properties—even with a linear  $\xi$  function—have not been found. It is possible to obtain approximate solutions using the heat balance integral; however, before doing this it is useful to examine the simpler two-zone problem. If the surface temperature  $T_s$  is greater than or equal to the minimum phase change temperature, then a completely frozen zone will not exist. Thus we need only examine regions 2 and 3.

## TWO-ZONE PROBLEMS

The two-zone problem is simpler than the three-zone case and will lead to results that can simplify the need for the full three-zone problem. The linear unfrozen water case will be examined for both variable and constant thermal properties, while the quadratic water content case will be evaluated only for the constant property problem. This will be shown to be adequate for the general problem.

### Linear unfrozen water function

#### *Variable thermal properties*

Equations 16–18 are valid for this case except that the boundary condition in eq 16b becomes

$$\psi(0, t) = \phi + \frac{\beta_1 \phi^2}{2} = P. \quad (18d)$$

An approximation of the solution may be obtained with the heat balance integral method. The heat balance integral method has been adapted to problems of freezing in soils systems by Lunardini (1981b, 1982, 1983) and Lunardini and Varotta (1981). The equations for the heat balance integral method are well known and will not be derived here; the interested reader can consult Lunardini (1981a) for details. Referring to Figure 1 and Appendix B, eq 16 and 18 become

$$\int_0^X \sqrt{1 + 2\beta_2 \psi} \frac{\partial^2 \psi}{\partial x^2} dx = \frac{d}{dt} \left[ \int_0^X F(x, t) dx - F_2 X \right] \quad (19)$$

$$-\alpha_3 \frac{\partial \theta_3}{\partial x} (X, t) = \frac{d}{dt} \left[ \int_X^\delta \theta_3 dx + \phi_0 \delta \right] \quad (20)$$

where

$$F_2 = \frac{\beta_{21}}{3\alpha_0 \beta_1}.$$

For the heat balance integral treatment, eq 18a and b become

$$\theta_3(\delta, t) = -\phi_0 \quad (18e)$$

$$\frac{\partial \theta_3}{\partial x}(\delta, t) = 0. \quad (18f)$$

Quadratic temperature profiles are assumed for  $\psi$  and  $\theta_3$  since experience has shown that they yield good results for the heat balance integral method:

$$\psi = b(X-x) + c(X-x)^2 \quad (21)$$

$$\theta_3 = \phi_0 \left[ \left( \frac{\delta-x}{\delta-X} \right)^2 - 1 \right] \quad (22)$$

where

$$b = \frac{2\lambda_0}{\delta-X}$$

$$cX^2 = P - bX.$$

To simplify the algebra the following parameters are defined:

$$X = 2\gamma\sqrt{\alpha_3 t} \quad (23)$$

$$\delta - X = BX. \quad (24)$$

The use of eq 23 and 24 is not a constraint on the final solution.

The solution of eq 19 and 20 is straightforward but tedious. The algebraic manipulations are given in Appendix B. The unknown parameters  $\gamma$  and  $B$  can be found from the following equations:

$$\gamma^2 = \frac{1}{B \left( \frac{B}{3} + 1 \right)} \quad (25)$$

$$K Q_1 B \left( \frac{B}{3} + 1 \right) = \left[ F_1 \left( \frac{A}{2} + \frac{K}{3} \right) + F_2 (Q_2 - 1) \right] \alpha_3 \quad (26)$$

$$Q_1 = \frac{\sqrt{N}}{2} + \frac{A}{4K} (\sqrt{N} - 1) + \frac{\left( 1 - \frac{A^2 \beta_1}{2K} \right)}{2\sqrt{2\beta_1 K}} \ln Q_3$$

$$Q_2 = \frac{\sqrt{N}}{4} \left[ N + \frac{3}{2} \left( 1 - \frac{A^2 \beta_1}{2K} \right) \right] + \frac{\sqrt{N}}{8} \frac{A(N-1)}{K} + \frac{3 \left( 1 - \frac{A^2 \beta_1}{2K} \right)^2 \ln Q_3}{8\sqrt{2\beta_1 K}}$$

$$Q_3 = \frac{\sqrt{2\beta_1 K} + \beta_1 (2K + A)}{\sqrt{2\beta_1 K} + \beta_1 A}$$

$$N = 1 + 2\beta_1 P$$

$$A = \frac{2\lambda_0}{B}$$

$$K = P - A$$

$$P = \phi + \frac{\beta_1 \phi^2}{2}$$

$$F_1 = \frac{1}{\alpha_0} (1 + \sigma_0 - \beta_{21}).$$

#### *Constant thermal properties*

The constant-thermal-properties solution follows from the preceding case if  $\beta_{21} \rightarrow 0$  and  $\beta_1 \rightarrow 0$ . It can be shown that, for  $\beta_{21} = 0$ ,

$$\lim_{\beta_1 \rightarrow 0} Q_1 = 1 \quad (27a)$$

$$\lim_{\beta_1 \rightarrow 0} Q_2 = 1 \quad (27b)$$

$$\lim_{\beta_1 \rightarrow 0} \psi = \theta_2. \quad (27c)$$

For the constant-property case,  $\alpha_0$  becomes  $\alpha_2$  and  $\sigma_0 = \sigma$ ,  $\lambda_0 = \lambda$ . Then the solution is

$$\left( B - \frac{2\lambda}{\phi} \right) (B + 1) - \left( 1 + \frac{\lambda}{2B} \right) (1 + \sigma) \alpha_{32} = 0. \quad (28)$$

The parameter  $\gamma$  is again found from eq 25.

For this case an exact solution is possible by using a similarity transformation, as was shown by Lunardini (1985). The solution is

$$\frac{\theta_2}{\phi} = 1 - \frac{\operatorname{erf}\left(\frac{x/2}{\sqrt{1+\sigma}}\sqrt{\alpha_2 t}\right)}{\operatorname{erf}\gamma\sqrt{\alpha_{32}(1+\sigma)}} \quad (29)$$

$$\frac{\theta_3}{\phi_0} = \frac{\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha_3 t}}\right) - 1}{\operatorname{erfc}\gamma} \quad (30)$$

$$\lambda \operatorname{erf}\gamma\sqrt{\alpha_{32}(1+\sigma)} = \sqrt{\alpha_{32}(1+\sigma)} \operatorname{erfc}\gamma e^{-\gamma^2[\alpha_{32}(1+\sigma)-1]} \quad (31)$$

where

$$\operatorname{erf}x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz$$

is the familiar error function.

#### Quadratic unfrozen water function

When a quadratic unfrozen water function is used (for example, eq 2), it is not possible to find a closed-form solution for variable thermal properties. Thus a heat balance approximation will be used for constant thermal properties.

The equations for region 2, using eq 2, are

$$\alpha_2 \frac{\partial^2 \theta_2}{\partial x^2} = \frac{\partial}{\partial t} [(1+2\sigma)\theta_2 - \sigma\theta_2^2] \quad (32)$$

$$\theta_2(X, t) = 0 \quad (32a)$$

$$\theta_2(X_1, t) = 1 \quad (32b)$$

$$k_2 \frac{\partial \theta_2}{\partial x}(X, t) = k_3 \frac{\partial \theta_3}{\partial x}(X, t). \quad (32c)$$

The heat balance integral form of eq 32 is

$$\alpha_2 \left[ \frac{\partial \theta_2}{\partial x}(X, t) - \frac{\partial \theta_2}{\partial x}(0, t) \right] = \frac{d}{dt} \int_0^X [(1+2\sigma)\theta_2 - \sigma\theta_2^2] dx. \quad (33)$$

Equation 20 is still valid for region 3.

The temperature profile for region 3 is again eq 22. The temperature in region 2 is assumed to be

$$\theta_2 = b_1(X-x) + c_1(X-x)^2 \quad (34)$$

where

$$b_1 X = \frac{2\lambda}{B}$$

$$c_1 X^2 = \phi - \frac{2\lambda}{B}$$

The equation for  $B$  is now

$$(\phi B - 2\lambda)(B + 3) = \alpha_{32} \left[ \left( \frac{\lambda}{B} + \phi \right) \left( 1 + 2\sigma - \frac{3\phi\sigma}{5} \right) - \frac{2}{5} \frac{\sigma\lambda^2}{B^2} \right] \quad (35)$$

The parameter  $\gamma$  is again found from eq 25.

The two-zone solutions can be compared by considering some specific cases. For example, consider a typical soil with the following properties suggested by Nakano and Brown (1971):

$$T_0 = 4^\circ\text{C}$$

$$T_s = -4$$

$$T_f = 0$$

$$T_m = -4 \quad (\text{thus } \phi = \phi_0 = 1)$$

$$\xi_0 = 0.20 \frac{\text{gram water}}{\text{gram solids}}$$

$$\xi_f = 0.0782$$

$$\gamma_d = 1.68 \text{ gram solids/cm}^3$$

$$l = 80 \text{ cal/gram water}$$

$$S_T = 0.1539.$$

The soil thermal properties and the results of several cases are summarized in Tables 1 and 2. Cases 1-3 show that the effect of specific heat variation is not important and can safely be neglected. However, case 4 indicates that the thermal conductivity can cause 15-25% variations in the rate of growth of the freezing zone. Case 5 uses average values of  $k$  and  $C$  within

**Table 1. Effect of thermal properties on freeze of soil with average properties and linear  $\xi$ .**

Case	$C_s$ (cal/cm <sup>3</sup> -°C)	$k_s$ (cal/s-cm-°C)	$\beta_1$	$\beta_2$	$\gamma$	Difference from case 1 (%)	Comment
1	0.63	0.0058	0.431	-0.1429	0.3988	—	Base case with variable $k, C$ .
2	0.63	0.0058	0.431	0	0.3996	0.2	Constant specific heat, $C = 0.63$ .
3	0.54	0.0058	0.431	0	0.4016	0.7	Constant specific heat, $C = 0.54$ .
4	0.54	0.0083	0	0	0.4575	14.7	Constant $k, C$ .*
5	0.585	0.0071	0	0	0.4126	3.5	Constant $k, C$ .†
6	0.54	0.0083	0	0	0.4277	7.2	Constant $k, C$ ,* exact solution.

$$* k_s = k_f = 0.0083 \quad C_s = C_f = 0.54 \quad k_i = k_u \quad C_i = C_u.$$

$$\dagger k_s = \frac{k_f + k_u}{2} = 0.0071 \quad C_s = \frac{C_f + C_u}{2} = 0.585.$$

**Table 2. Effect of thermal properties on freeze of soil with extreme property variations.**

Case	$C_u$ (cal/cm <sup>3</sup> -°C)	$k_u$ (cal/s-cm-°C)	$\beta_1$	$\beta_2$	$\gamma$	Difference from case 1 (%)	Comment
1	0.63	0.0058	1	-0.50	0.4626	—	Extreme properties, linear $\xi$ , variable $k$ , $C$ .
2	0.63	0.0058	1	0	0.4607	-0.4	Constant specific heat, $C = 0.63$ , linear $\xi$ .
3	0.315	0.0058	1	0	0.4696	1.5	Constant specific heat, $C = 0.315$ , linear $\xi$ .
4	0.315	0.0116	0	0	0.5671	22.6	Constant $k$ , $C$ ,* linear $\xi$ .
5	0.473	0.0087	0	0	0.4726	2.2	Constant $k$ , $C$ ,† linear $\xi$ .
6	0.315	0.0116	0	0	0.5304	14.7	Constant $k$ , $C$ ,* linear $\xi$ , exact solution.
7	0.315	0.0116	0	0	0.5226	13.0	Constant $k$ , $C$ *, quadratic $\xi$ .

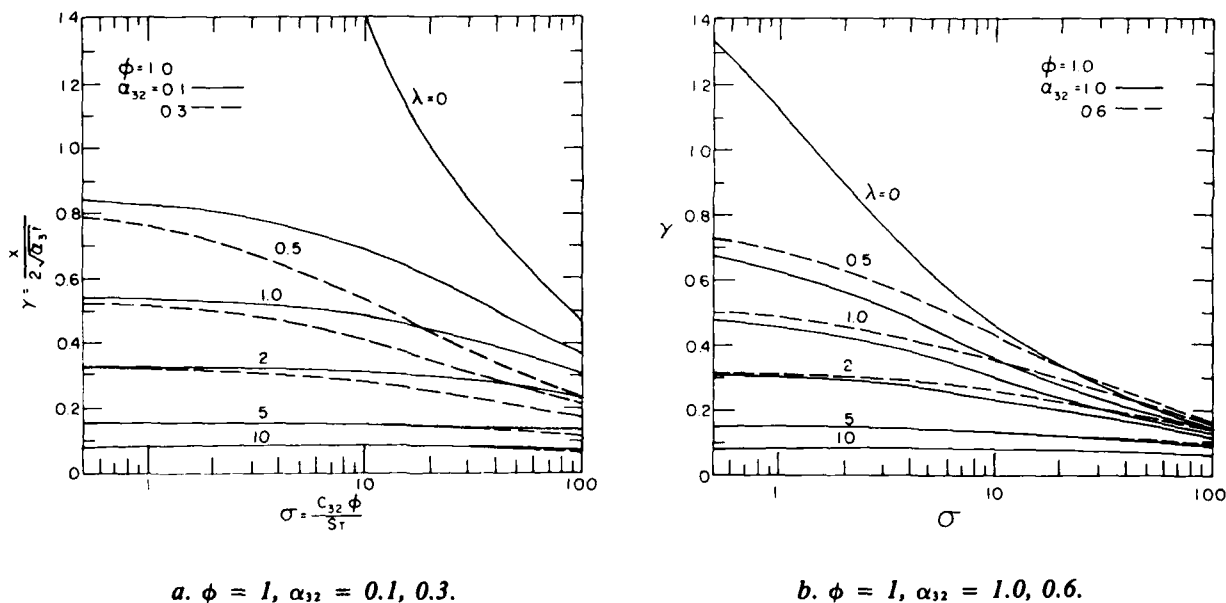
\*  $k_1 = k_f = 0.0116$      $C_1 = C_f = 0.315$ .

†  $k_2 = \frac{k_f + k_u}{2} = 0.0087$      $C_2 = \frac{C_u + C_f}{2} = 0.473$ .

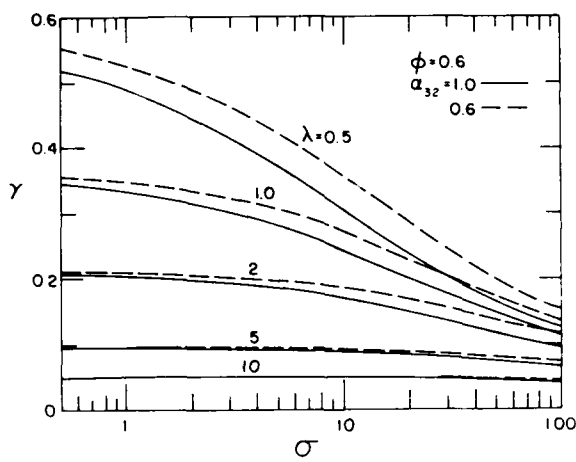
the mushy zone, and the effect of variable properties can be adequately taken into account by using the constant-property solution with the average of the fully frozen and fully thawed thermal properties.

Cases 4 and 6 compare the heat balance approximation to the exact solution (eq 31). The approximate solution is within about 7% of the exact solution. This tends to verify the acceptable accuracy of the heat balance integral method.

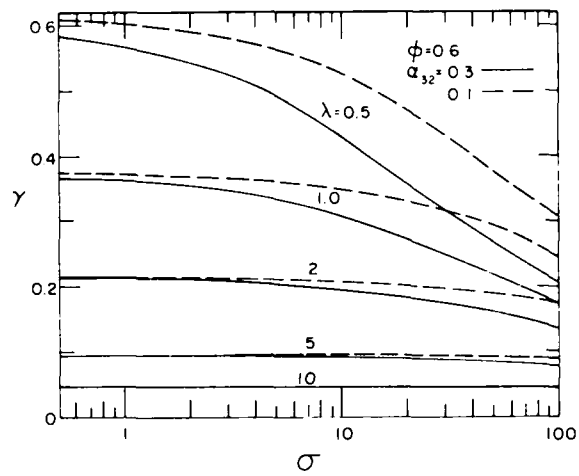
The effect of the different unfrozen water content functions can be deduced from cases 4 and 7 of Table 2. The growth rate for the quadratic water function lags behind that of the linear water function by about 9%. This was also noted by Lunardini (1985). Since the quadratic unfrozen water function will be a more accurate representation of an actual soil, the quadratic solution is presented in Figure 4 for the two-zone problem.



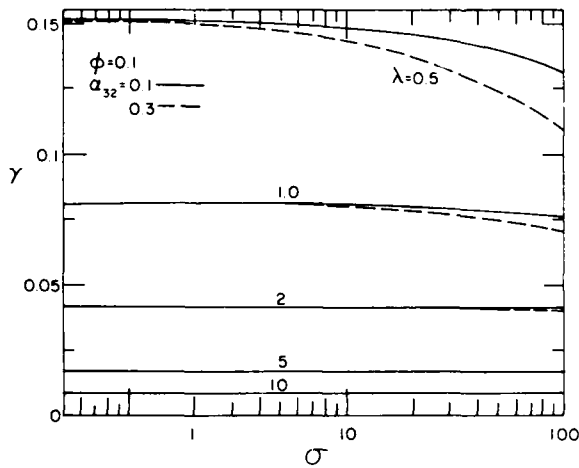
**Figure 4. Quadratic solution for the two-zone problem.**



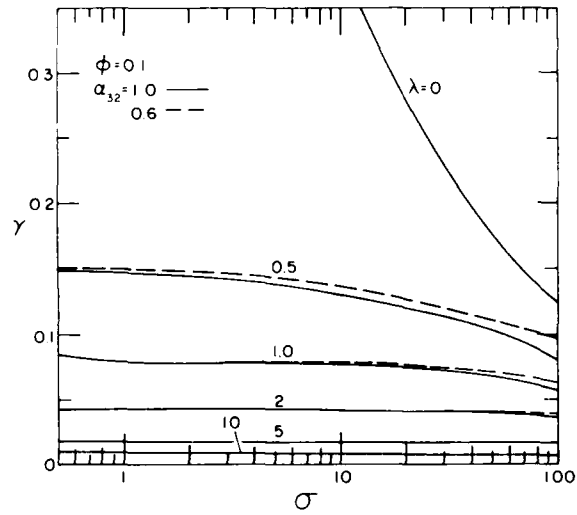
c.  $\phi = 0.6, \alpha_{32} = 1.0, 0.6.$



d.  $\phi = 0.6, \alpha_{32} = 0.3, 0.1.$



e.  $\phi = 0.1, \alpha_{32} = 0.1, 0.3.$



f.  $\phi = 0.1, \alpha_{32} = 1.0, 0.6.$

Figure 4 (cont'd).

### THREE-ZONE PROBLEMS

Since the variable-property case can be adequately handled by an appropriate constant-property solution, only the constant-property problem will be examined.

#### Linear unfrozen water function

##### Exact solution

Equations 15 and 18 are the governing equations for regions 1 and 3, while the equations for region 2 are

$$\alpha_2 \frac{\partial^2 \theta_2}{\partial x^2} = (1 + \sigma) \frac{\partial \theta_2}{\partial t} \quad (36)$$

$$\theta_2(X, t) = 0 \quad (36a)$$

$$\theta_2(X_1, t) = 1 \quad (36b)$$

$$k_2 \frac{\partial \theta_2}{\partial x}(X, t) = k_2 \frac{\partial \theta_3}{\partial x}(X, t). \quad (36c)$$

The similarity method was used by Lunardini (1985) to obtain an exact solution to this problem. The solution is as follows:

$$\theta_1 = (1 - \phi) \left[ 1 + \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_1 t}}\right)}{\operatorname{erf} \eta} \right] \quad (37)$$

$$\theta_2 = \frac{\operatorname{erf} \gamma \sqrt{\alpha_{32}(1 + \sigma)} - \operatorname{erf}\left(\frac{x}{2\sqrt{\frac{\alpha_2 t}{1 + \sigma}}}\right)}{\operatorname{erf} \gamma \sqrt{\alpha_{32}(1 + \sigma)} - \operatorname{erf} \eta \sqrt{\alpha_{12}(1 + \sigma)}} \quad (38)$$

$$\theta_3 = \left[ \frac{\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha_3 t}}\right)}{\operatorname{erfc} \gamma} - 1 \right] \phi_0 \quad (39)$$

$$X_1 = 2\eta\sqrt{\alpha_1 t} \quad (40)$$

$$X = 2\gamma\sqrt{\alpha_3 t}. \quad (41)$$

The parameters  $\eta$  and  $\gamma$  are found from the simultaneous solution of the two following equations:

$$\sqrt{\alpha_{32}(1 + \sigma)} (\operatorname{erfc} \gamma) e^{-\gamma^2[\alpha_{32}(1 + \sigma) - 1]} = \lambda [\operatorname{erf} \gamma \sqrt{\alpha_{32}(1 + \sigma)} - \operatorname{erf} \eta \sqrt{\alpha_{12}(1 + \sigma)}] \quad (42)$$

$$(\phi - 1) e^{-\eta^2[1 - \alpha_{12}(1 + \sigma)]} [\operatorname{erf} \gamma \sqrt{\alpha_{32}(1 + \sigma)} - \operatorname{erf} \eta \sqrt{\alpha_{12}(1 + \sigma)}] = k_{21} \sqrt{\alpha_{12}(1 + \sigma)} \operatorname{erf} \eta. \quad (43)$$

Lunardini (1985) showed that this solution approached the Neumann solution as the phase change zone decreased (Table 3). It is clear from Table 3 that the solution does converge to the Neumann solution as  $(T_f - T_m) \rightarrow 0$ . The thaw/freeze interface greatly exceeds the value for the Neumann solution if phase change occurs over a 4 °C zone as in the example calculation.

#### Heat balance integral solution

The heat balance integral equations for the three-zone problem are as follows:

$$\alpha_1 \left[ \frac{\partial \theta_1}{\partial x}(X_1, t) - \frac{\partial \theta_1}{\partial x}(0, t) \right] = \frac{d}{dt} \int_0^{X_1} (\theta_1 - 1) dx \quad (44)$$

$$\theta_1(0, t) = \phi \quad (44a)$$

$$\theta_1(X_1, t) = 1 \quad (44b)$$

$$\frac{\partial \theta_1}{\partial X}(X_1, t) = k_{21} \frac{\partial \theta_2}{\partial X}(X_1, t) \quad (44c)$$

$$\alpha_1 \left[ \frac{\partial \theta_2}{\partial X}(X_1, t) - \frac{\partial \theta_2}{\partial X}(X_1, t) \right] = (1 + \sigma) \frac{d}{dt} \left[ \int_{X_1}^X \theta_2 dx + X \right] \quad (45)$$

$$\theta_2(X, t) = 0 \quad (45a)$$

$$\theta_2(X_1, t) = 1 \quad (45b)$$

$$\frac{\partial \theta_2}{\partial X}(X, t) = k_{32} \frac{\partial \theta_3}{\partial X}(X, t) \quad (45c)$$

$$-\alpha_3 \frac{\partial \theta_3}{\partial X}(X, t) = \frac{d}{dt} \left[ \int_X^\delta \theta_3 dx + \phi_0 \delta \right] \quad (46)$$

$$\theta_3(\delta, t) = \phi_0 \quad (46a)$$

$$\theta_3(X, t) = 0 \quad (46b)$$

$$\frac{\partial \theta_3}{\partial X}(\delta, t) = 0. \quad (46c)$$

Quadratic temperature profiles for the three regions lead to

$$\theta_3 = \phi_0 \left[ \left( \frac{\delta - x}{\delta - X} \right)^2 - 1 \right] \quad (47)$$

$$\theta_1 = 1 + b_2(X_1 - x) + c_2(X_1 - x)^2 \quad (48)$$

$$\theta_2 = e_2(X - x) + f_2(X - x)^2 \quad (49)$$

where

$$b_2 X = 2k_{21} \left( \frac{1}{R} - \frac{\lambda}{B} \right)$$

$$c_2 X^2 (1 - R)^2 = \phi - 1 - 2k_{21} (1 - R) \left( \frac{1}{R} - \frac{\lambda}{B} \right)$$

$$e_2 = \frac{2\lambda}{BX}$$

$$f_2 X^2 R^2 = 1 - \frac{2\lambda R}{B}$$

$$R = 1 - \frac{\eta}{\gamma} \sqrt{\alpha_{13}}.$$

**Table 3. Effect of phase change temperature. (After Lunardini 1985.)**

Case	$T_m$ (°C)	$\eta$	$\gamma$	$X^*$ (cm)	$X_f^*$ (cm)	$\Delta X = X - X_i$
1	-4	0.0617	0.3029	33.33	8.13	25.2
2	-2	0.1135	0.2576	28.34	14.95	13.39
3	-1	0.1376	0.2272	25.0	18.12	6.88
4	-0.5	0.1492	0.2106	23.17	19.65	3.52
5	-0.1	0.1571	0.1946	21.41	20.69	0.72
Neumann	0	0.1606	—	21.15	21.15	0

\* For  $t = 24$  hours.

$$T_s = 4^\circ\text{C} \quad T_i = -6 \quad T_f = 0^\circ\text{C} \quad k_1 = 0.00828 \text{ cal/s-cm-}^\circ\text{C}$$

$$k_2 = 0.00703 \quad k_3 = 0.00578 \quad C_1 = C_2 = C_3 = 0.165 \text{ cal/cm}^3\text{-}^\circ\text{C}$$

$$S_T = 0.0605.$$

The solutions to eq 44-46 are straightforward; the details will be omitted. The results are given below:

$$\gamma^2 = \frac{1}{B\left(\frac{B}{3} + 1\right)} \quad (50)$$

$$\eta^2 = (1 - R)^2 \gamma^2 \alpha_{31} \quad (51)$$

$$(\phi - 1)(3 - \eta^2) - (6 + \eta^2) \left[ (1 - R)k_{21} \left( \frac{1}{R} - \frac{\lambda}{B} \right) \right] = 0 \quad (52)$$

$$1 - \frac{2\lambda R}{B} - \frac{(1 + \sigma)}{3} R \alpha_{32} \gamma^2 \left( 3 - 2R + \frac{R^2 \lambda}{B} \right) = 0. \quad (53)$$

These four equations can be easily solved for  $\eta$  and  $\gamma$ .

Table 4 shows that the approximate solution is in error by less than 6% when compared to the exact solution.

**Table 4. Comparison of exact and heat balance integral solutions with linear  $\xi$  and constant  $k$  and  $C$ .**

Case	$T_m$ (°C)	Exact solution		Heat balance integral solution		Variation (%)	
		$\eta$	$\gamma$	$\eta$	$\gamma$	$\eta$	$\gamma$
1	-4	0.0617	0.3029	0.0622	0.2938	0.8	-3.1
2	-2	0.1135	0.2576	0.1150	0.2411	1.3	-6.4
3	-1	0.1376	0.2272	0.1390	0.2168	1.0	-4.6
4	-0.5	0.1492	0.2106	0.1505	0.2050	0.9	-2.6
5	-0.1	0.1571	0.1946	0.1595	0.1958	1.5	0.6

$$T_s = 4^\circ\text{C} \quad T_i = -6 \quad T_f = 0^\circ\text{C} \quad k_1 = 0.00828 \text{ cal/s-cm-}^\circ\text{C}$$

$$k_2 = 0.00703 \quad k_3 = 0.00578 \quad C_1 = C_2 = C_3 = 0.165 \text{ cal/cm}^3\text{-}^\circ\text{C}$$

$$S_T = 0.0605.$$

### Quadratic unfrozen water function

The equations for the quadratic unfrozen water relation are identical to eq 44-46 except that the energy equation for the mush region (eq 45) is

$$\alpha_2 \left[ \frac{\partial \theta_2}{\partial x} (X, t) - \frac{\partial \theta_2}{\partial x} (X_1, t) \right] = \frac{d}{dt} \int_{X_1}^X [(1 + 2\sigma)\theta_2 - \sigma\theta_2^2] dx + (1 + \sigma)X_1 \quad (54)$$

The quadratic temperature approximations (eq 47-49) are used again. The solution is again given by eq 50-52, with the mushy zone equation given by

$$1 - 2z - R^2 \alpha_{32} \gamma^2 \left[ \frac{z-2}{3} + \frac{1+\sigma}{R} - \frac{\sigma}{15} (2z^2 - 7z + 18) \right] = 0 \quad (55)$$

where

$$z = \frac{R\lambda}{B}$$

Case 1 of Table 4 was evaluated for the quadratic  $\xi$ , and it was found that  $\eta = 0.0572$  and  $\gamma = 0.2730$ . These values differ from the linear  $\xi$  approximation by about 8%.

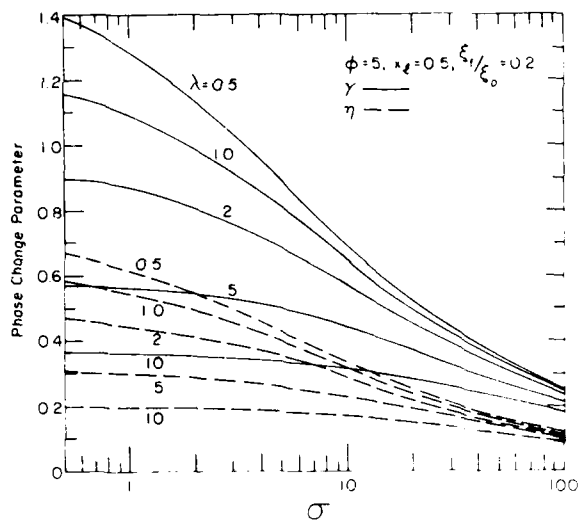
The quadratic  $\xi$ , three-zone problem is a function of  $\lambda$ ,  $\sigma$ ,  $\phi$ ,  $\alpha_{31}$ ,  $\alpha_{32}$  and  $k_{21}$ . Graphical solutions are shown in Figure 5 for typical soil parameters.

### CONCLUSIONS

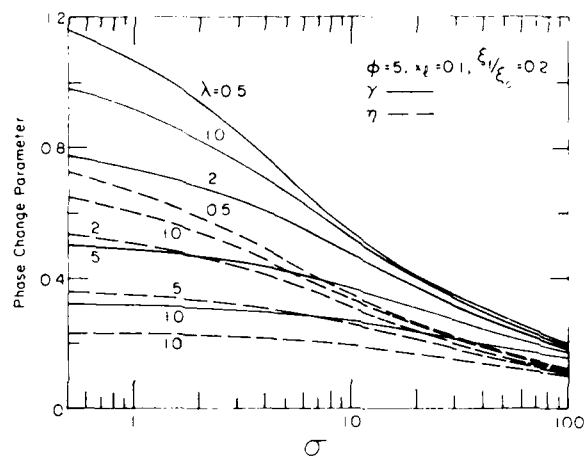
The mathematical model used assumed the latent heat to be a source of energy distribution throughout the volume of a soil with phase-change temperature limits of  $T_f$  and  $T_m$ . This contrast with the treatment of the latent heat is totally released at the upper phase-change temperature  $T_f$ . A comparison of the exact solution for the former case showed that it converged exactly to the Neumann solution as  $T_f - T_m$  approached zero. Thus, the mathematical model is based on sound physical principles.

The existence of a mushy zone can have a significant effect on the mechanical strength of freezing or thawing soils. For many predictions of the strength of a freezing soil, the assumption is made that the soil temperature is that calculated from the Neumann solution. A soil with a mushy zone would have a completely frozen layer of soil that is thinner than that for the Neumann case and therefore would have less bearing capacity. In addition a thick zone of frozen soil would exist that has a variable unfrozen water content. Again the presence of this liquid water will decrease the mechanical strength of the soil. The thawing case would also tend to yield a bearing strength of soil that is less than that predicted with the Neumann solution temperatures.

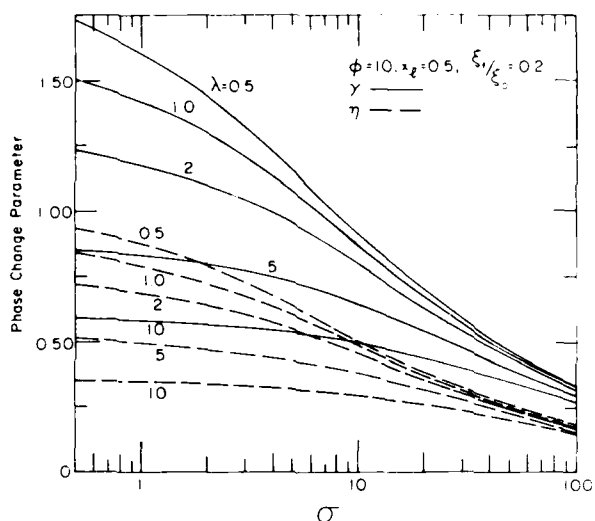
The effect of variable specific heat on the rate of phase change is negligible for the cases examined; thus, it is acceptable to use an average specific heat value in the mushy zone. The variation of the thermal conductivity with water content is much more significant and can cause a 15% change in the rate of freezing of the soil. However, it was shown that the constant-property solution, with average values of the thermal properties in the mushy zone, gives a solution that is quite close to that for the variable-property solution. It is acceptable to use the much simpler, constant-property solution with average thermal properties to compensate for the actual variable thermal conductivity.



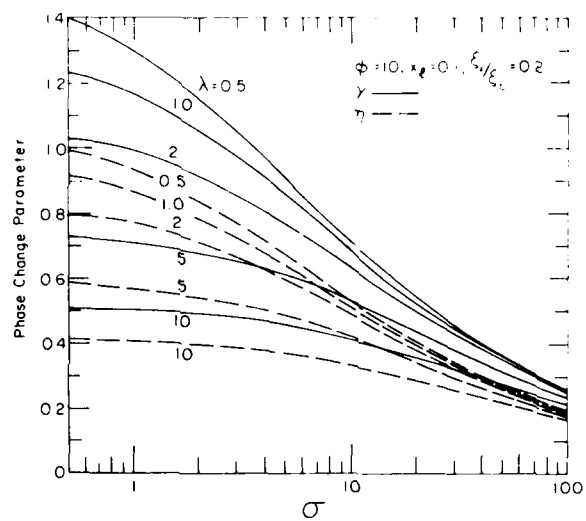
a.  $\phi = 5, x_f = 0.5, \xi_f/\xi_0 = 0.2.$



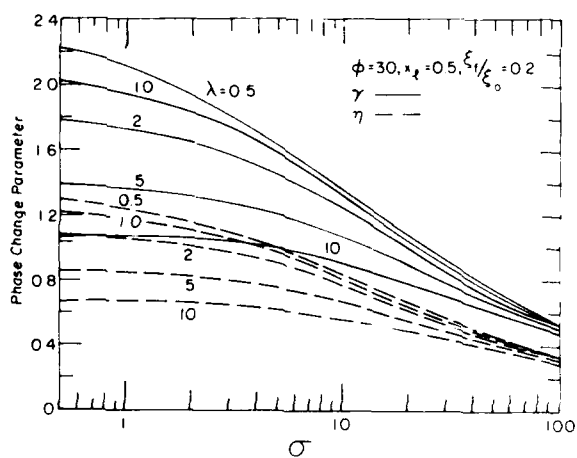
b.  $\phi = 5, x_f = 0.1, \xi_f/\xi_0 = 0.2.$



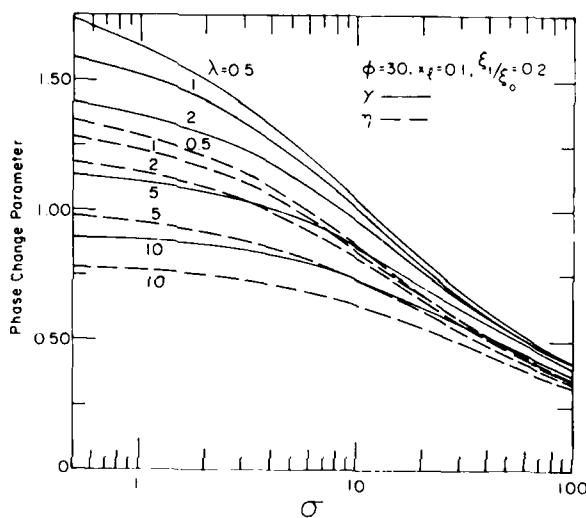
c.  $\phi = 10, x_f = 0.5, \xi_f/\xi_0 = 0.2.$



d.  $\phi = 10, x_f = 0.1, \xi_f/\xi_0 = 0.2.$



e.  $\phi = 30, x_f = 0.5, \xi_f/\xi_0 = 0.2.$



f.  $\phi = 30, x_f = 0.1, \xi_f/\xi_0 = 0.2.$

Figure 5. Quadratic solution for the three-zone problem.

A series of graphs are presented of the constant-property, three-zone problem for typical ranges of soil parameters. These graphs allow rapid predictions to be made for the freezing of soils with an unfrozen water content that is a function of temperature and variable thermal properties.

#### LITERATURE CITED

- Anderson, D.M. and A. Tice** (1973) The unfrozen interfacial phase in frozen soil water systems. In *Analysis and Synthesis. Ecological Studies*, vol. 4 (A. Nodos et al., Ed.). New York: Springer-Verlag, pp. 107-124.
- Berggren, W.P.** (1943) Prediction of temperature distribution in frozen soils. *Transactions, American Geophysical Union*, **24**(3): 71-77.
- Carslaw, H.S. and J.C. Jaeger** (1959) *Conduction of Heat in Solids*. Oxford: Clarendon Press.
- Cho, S.H. and J.E. Sunderland** (1974) Phase change problems with temperature-dependent thermal conductivity. *Journal of Heat Transfer*, **96**(2): 214-217.
- Frivik, P.E.** (1980) State-of-the-art report, Ground freezing: Thermal properties, modeling of processes and thermal design. In *Proceedings, International Symposium on Ground Freezing*, pp. 354-373.
- Lunardini, V.J.** (1981a) *Heat Transfer in Cold Climates*. New York: Van Nostrand Reinhold Company.
- Lunardini, V.J.** (1981b) Phase change around a circular cylinder. *Journal of Heat Transfer*, **103**(3): 598-600.
- Lunardini, V.J.** (1982) Freezing of soil with surface convection. In *Proceedings of Third International Symposium on Ground Freezing, Hanover, N.H.*, pp. 205-212.
- Lunardini, V.J.** (1985) Freezing of soil with phase change occurring over a finite temperature zone. In *Proceedings, 4th International Offshore Mechanics and Arctic Engineering Symposium, II*: 38-46. American Society of Mechanical Engineers.
- Lunardini, V.J.** (1988) Heat conduction with freezing or thawing. USA Cold Regions Research and Engineering Laboratory, CRREL Monograph 88-1.
- Nakano, Y. and J. Brown** (1971) Effect of a freezing zone of finite width on the thermal regime of soils. *Journal of Water Resources Research*, **7**(5): 1226-1233.
- Neumann, F.** (ca. 1860) Lectures given in 1860s. (See Riemann-Weber. Die partiellen Differentialgleichungen. *Physik* (5th ed., 1912), **2**: 121.)
- Ozisik, M.N. and J.C. Uzzell** (1979) Exact solution for freezing in cylindrical symmetry with extended freezing temperature range. *Journal of Heat Transfer*, **101**: 331-334.
- Tice, A.R., C.M. Burrows and D.M. Anderson** (1978) Phase composition measurements on soils at very high water content by the pulsed nuclear magnetic resonance technique. In *Moisture and Frost-Related Soil Properties*. Washington, D.C.: Transportation Research Board, National Academy of Sciences, pp. 11-14.
- Tien, R.H. and G.I. Geiger** (1967) A heat transfer analysis of the solidification of a binary eutectic system. *Journal of Heat Transfer, ASME, Ser. C*, **89**: 230-243.

## APPENDIX A: DERIVATION OF THE MUSHY ZONE EQUATION

The energy equation in the mushy zone (region 2) is

$$\frac{\partial}{\partial x} \left( k \frac{\partial T_2}{\partial x} \right) = \left( C + \ell \gamma_d \frac{d\xi}{dT_2} \right) \frac{\partial T_2}{\partial t} . \quad (\text{A1})$$

The dimensionless temperature in region 2 is defined as

$$\theta_2 = \frac{T_f - T_2}{T_f - T_m} . \quad (\text{A2})$$

Equation 1 transforms immediately to

$$\frac{\partial}{\partial x} \left( k \frac{\partial \theta_2}{\partial x} \right) = C \frac{\partial \theta_2}{\partial t} - \frac{\ell \gamma_d}{(T_f - T_m)} \frac{d\xi}{d\theta_2} \frac{\partial \theta_2}{\partial t} . \quad (\text{A3})$$

With the parameters defined in the text, it is also clear that, for the linear  $\xi$  case,

$$k = k_u(1 + \beta_1 \theta_2) \quad (\text{A4a})$$

$$C = C_u(1 + \beta_2 \theta_2). \quad (\text{A4b})$$

Equations A4a and b are written explicitly for the definitions of  $k$  and  $C$  discussed earlier. However, it will be advantageous to allow for the case where the mushy zone will have constant thermal properties different than those of the thawed state. Thus, we will define  $k_o$  and  $C_o$  to be any constant values of the thermal conductivity and specific heat desired. Then

$$k = k_o(1 + \beta_1 \theta_2) \quad (\text{A5a})$$

$$C = C_o(1 + \beta_2 \theta_2). \quad (\text{A5b})$$

Clearly, for the mushy zone with variable thermal properties,  $k_o \equiv k_u$  and  $C_o \equiv C_u$ . However, if we wish to examine the constant thermal property case, then we can simply set  $k_o$ ,  $C_o$  to any convenient values. For example we could let the properties of the mushy zone be the mean values of the frozen and thawed states and define  $k_o$  and  $C_o$  appropriately.

If the thermal conductivity varies with temperature, the Kirchoff transformation can be used to define a new temperature variable:

$$\psi = \frac{1}{k_o} \int_0^{\theta_2} k(y) dy \quad (\text{A6})$$

where  $y$  is a dummy variable.

From eq A6

$$\frac{\partial \theta_2}{\partial t} = \frac{k_o}{k} \frac{\partial \psi}{\partial t} \quad (\text{A7})$$

$$k \frac{\partial \theta_2}{\partial x} = k_o \frac{\partial \psi}{\partial x} . \quad (\text{A8})$$

The derivation will proceed for the case of a linear  $\xi$  function. The quadratic assumption can also be used, of course, but the equations will be more complicated.

From eq A4 and A6 an explicit relation between  $\psi$  and  $\theta_2$  can be found:

$$\psi = \theta_2 + \frac{\beta_1}{2} \theta_2^2. \quad (\text{A9})$$

It follows that

$$k = k_0 \sqrt{1 + 2\beta_1 \psi} \quad (\text{A10})$$

$$C = C_0(1 - \beta_{21} + \beta_{21} \sqrt{1 + 2\beta_1 \psi}). \quad (\text{A11})$$

Equation A3 then becomes

$$\frac{k_0}{C_0} \sqrt{1 + 2\beta_1 \psi} \frac{\partial^2 \psi}{\partial x^2} = \left(1 - \beta_{21} + \frac{\ell \gamma_d \Delta \xi}{C_0(T_f - T_m)}\right) \frac{\partial \psi}{\partial t} + \beta_{21} \sqrt{1 + 2\beta_1 \psi} \frac{\partial \psi}{\partial t}. \quad (\text{A12})$$

Equation A12 can be put into a more convenient form by noting that

$$\sqrt{1 + 2\beta_1 \psi} \frac{\partial \psi}{\partial t} = \frac{1}{3\beta_1} \frac{\partial}{\partial t} [(1 + 2\beta_1 \psi)^{3/2}]. \quad (\text{A13})$$

Finally the energy equation for the mushy zone with a linear unfrozen water content function is

$$\alpha_0 \sqrt{1 + 2\beta_1 \psi} \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial t} \left[ (1 - \beta_{21} + \sigma_0) \psi + \frac{\beta_{21}}{3\beta_1} (1 + 2\beta_1 \psi)^{3/2} \right]. \quad (\text{A14})$$

**APPENDIX B: SOLUTION OF THE TWO-ZONE PROBLEM WITH  
A LINEAR  $\xi$  AND VARIABLE THERMAL PROPERTIES**

The equations for zones 2 and 3 are

$$\sqrt{1+2\beta_1\psi} \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial F}{\partial t} \quad (\text{B1})$$

$$\psi(0, t) = \phi + \frac{\beta_1 \phi^2}{2} = P \quad (\text{B1a})$$

$$\psi(X, t) = 0 \quad (\text{B1b})$$

$$k_0 \frac{\partial \psi}{\partial x}(X, t) = k_3 \frac{\partial \theta_3}{\partial x}(X, t) \quad (\text{B1c})$$

$$\frac{\partial^2 \theta_3}{\partial x^2} = \frac{1}{\alpha_3} \frac{\partial \theta_3}{\partial t} \quad (\text{B2})$$

$$\theta_3(\delta, t) = -\phi_0 \quad (\text{B2a})$$

$$\frac{\partial \theta_3}{\partial x}(\delta, t) = 0 \quad (\text{B2b})$$

where

$$F = F_1 \psi + F_2 (1 + 2\beta_1 \psi)^{3/2} \quad (\text{B3a})$$

$$F_1 = \frac{1}{\alpha_0} (1 + \sigma_0 - \beta_{21}) \quad (\text{B3b})$$

$$F_2 = \frac{\beta_{21}}{3\alpha_0 \beta_1} \quad (\text{B3c})$$

The heat balance integral method (HBI) integrates the energy equation over the volume of interest. For region 2

$$\int_0^X \sqrt{1+2\beta_1\psi} \frac{\partial^2 \psi}{\partial x^2} dx = \int_0^X \frac{\partial F}{\partial t} dx \quad (\text{B4})$$

Using Leibniz's rule this is

$$\int_0^X \sqrt{1+2\beta_1\psi} \frac{\partial^2 \psi}{\partial x^2} dx = \frac{d}{dt} \int_0^X F(x, t) dx - F(X, t) \frac{dX}{dt} \quad (\text{B5})$$

Now

$$F(X, t) = F_2 \quad (\text{B6})$$

and the HBI equation for region 2 is

$$\int_0^X \sqrt{1+2\beta_1\psi} \frac{\partial^2 \psi}{\partial x^2} dx = \frac{d}{dt} \left[ \int_0^X F(x,t) dx - F_2 X \right]. \quad (\text{B7})$$

The HBI equation for region 3 is derived in the same manner and is

$$-\alpha_3 \frac{\partial \theta_3}{\partial x} (X,t) = \frac{d}{dt} \left[ \int_X^\delta \theta_3 dX + \phi_0 \delta \right]. \quad (\text{B8})$$

Quadratic temperature profiles for regions 2 and 3 that satisfy the boundary conditions are

$$\psi = b(X-x) + c(X-x)^2 \quad (\text{B9})$$

$$\theta_3 = \phi_0 \left[ \left( \frac{\delta-x}{\delta-X} \right)^2 - 1 \right] \quad (\text{B10})$$

where

$$b = \frac{2\phi_0 k_{30}}{\delta-X}$$

$$cX^2 = P - bX.$$

The moving interfaces  $X$  and  $\delta$  are assumed to have the forms

$$X = 2\gamma\sqrt{\alpha_3 t} \quad (\text{B11})$$

$$\delta = (B+1)X \quad (\text{B12})$$

where  $B$  is a constant. Using eq B10 with eq B8 leads to the following equation:

$$\frac{6\alpha_3}{\delta-X} = \frac{d}{dt} (\delta + 2X). \quad (\text{B13})$$

The solution to eq B13 follows easily with the aid of eq 11 and 12:

$$\gamma^2 = \frac{1}{B\left(\frac{B}{3} + 1\right)}. \quad (\text{B14})$$

Equation B7 can be solved to yield an algebraic equation for the unknown quantity  $B$ . From eq B9 we note

$$\frac{\partial^2 \psi}{\partial x^2} = 2c = \frac{2(P - bx)}{X^2} \quad (\text{B15})$$

where  $c$  is only a function of time. Then eq B7 is

$$2c \int_0^X \sqrt{1+2\beta_1\psi} dx = \frac{d}{dt} \left[ \int_0^X F(x,t) dx - F_2 X \right]. \quad (\text{B16})$$

The solution of eq B16 is straightforward but rather tedious. The results are

$$\int_0^X \sqrt{1+2\beta_1\psi} dx = Q_1 X \quad (\text{B17})$$

$$\int_0^X F dx = F_1 \left( \frac{A}{2} + \frac{K}{3} \right) X + F_2 \int_0^X (1+2\beta_1\psi)^{3/2} dx \quad (\text{B18})$$

$$\int_0^X (1+2\beta_1\psi)^{3/2} dx = Q_2 X \quad (\text{B19})$$

where

$$Q_1 = \frac{\sqrt{N}}{2} + \frac{A}{4K} (\sqrt{N} - 1) + \left( \frac{1 - \frac{A^2\beta_1}{2K}}{2\sqrt{2\beta_1 K}} \right) \ln Q_3$$

$$Q_2 = \frac{\sqrt{N}}{4} \left[ N + \frac{3}{2} \left( 1 - \frac{A^2\beta_1}{2K} \right) \right] + \frac{\sqrt{N}}{8} \frac{A}{K} (N-1) + \frac{3 \left( 1 - \frac{A^2\beta_1}{2K} \right)^2}{8\sqrt{2\beta_1 K}} \ln Q_3$$

$$Q_3 = \frac{\sqrt{2\beta_1 K} + \beta_1(2K+A)}{\sqrt{2\beta_1 K} + \beta_1 A}$$

$$A = \frac{2\phi_0 k_{30}}{B}$$

$$K = P - A.$$

The equation for  $B$  is then

$$K Q_1 B \left( \frac{B}{3} + 1 \right) = \left[ F_1 \left( \frac{A}{2} + \frac{K}{3} \right) + F_2 (Q_2 - 1) \right] \alpha_3. \quad (\text{B20})$$

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v, 30 p., illus.; 28 cm. (CRREL Report 88-2.)

Bibliography: p. 17.

1. Frozen soils 2. Phase change 3. Soils I. United States Army. Corps of Engineers. II. Cold Regions Research and Engineering Laboratory. III. Series: CRREL Report 88-2.

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