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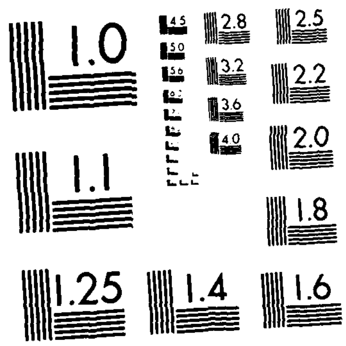
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The main aim of the project was to develop a set of partial differential equations which are the most optimum for the generation of grid lines in arbitrary surfaces. Equations have been developed on the basis of differential-geometric concepts which are elliptic in character and look to be the most optimum among the class of equations which can be used to generate the surface coordinates. The developed equations require a specification of the forcing function which depends on the geometry of the given surface. The surface in which the coordinates are to be introduced is usually given in discrete data form. Methods have been developed which fit a function over the given data to evaluate the forcing function for complicated shapes, e.g., an airplane, the functional fit and the eventual grid generation for the fuselage and wings are done separately and then integrated to obtain the grid lines on the surface.

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Generation of Surface Grids Through Elliptic Partial

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by

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1. INTRODUCTION

The problem of numerical grid generation is of current interest in many branches of engineering particularly in aeronautics, mechanical, and civil engineering. The spatial grids are generated either by algebraic methods using varicus spline and transfinite interpolations or by solving certain partial differential equations. In this regard, a book [1], review articles [2], [3], and conference proceedings [4, 5, 6] may be consulted.

The research accomplishments noted herein are in reference to the problem of grid generation in a given surface by solving a set of elliptic partial differential equations. The mathematical model used in the present research has been developed by the author under this and previous AFOSR grants and has been described in publications [7 - 16]. Reference [16] has been attached as an appendix in which Eqs. (19) and (21) describe the mathematical model.

The developed mathematical model has been used to generate the Cartesian coordinates as functions of the curvilinear coordinates when the surface in which the coordinates are to be generated has been specified either analytically or by discrete data points. In most practical situations the surface is usually specified by discrete data, and therefore, there was a need to develop computer routines to fit a global equation of the form $F(x,y,z) = 0$ to describe the surface. For complicated body shapes e.g. an airplane, a segmentation technique has been used in which the surface is divided into suitable sections and then for each segment the function is separately generated. The need for specifying the function F is due to the fact that the forcing term R in Eqs. (19) or (21) of the appendix is obtained from F. In the period under consideration we have devised vari-

ous routines for the above noted purpose and they are described in Section 2. The developed computer code has been used in many complicated shaped surfaces. Figure 1 shows some current results. Other results are available in the published cited references.

2. NUMERICAL SCHEMES

A computer program for the numerical solution of Eq. (21) has been developed by using point and line SOR. The essential difference between the grid generation in a flat space and in a curved surface is in the appearance of the forcing term on the right of Eqs. (19) and (21). The term R in Eqs. (19) and (21) depends on the first and second partial derivatives of the fitted function F with respect to the Cartesian coordinates. (*Refer to* [15] for the formula of R in terms of F). The following routines have been developed and used in a variety of problems.

1. Multidimensional least square technique.
2. "Overlapping" least square technique.
3. Fourier decomposition of each section of a surface and then blending them to obtain the equation of a surface.

Beside the development of the above techniques, we have also developed the following two separate programs in the solution of Eq. (21).

1. For the acceleration of the iterative process, a "multigrid" technique in the solution of Eq. (21) has been incorporated. From the test cases conducted so far it looks that the computation time with multigrid is definitely much lower than with ordinary SOR. At the time of the progress report in Agusut 1987, these results were not

definite. The application of the multigrid technique in the solution of the surface grid equations seems to be the first attempt of its kind.

2. A routine for the calculation of the optimum acceleration parameter has been developed which works quite well with the SOR technique.

3. CONCLUSIONS

The problem of numerical coordinate generation in arbitrary surfaces has been addressed by first developing a mathematical model as formed of a set of elliptic PDE's and then solving the proposed equations numerically. The proposed mathematical model has deliberately been made to depend on the formulae of Gauss which involve the partial derivatives of the Cartesian coordinates with respect to the curvilinear coordinates. With the formulae of Gauss as the basis for the proposed elliptic equations, it is logical to conclude that on solution the generated Cartesian coordinates will be second order differentiable with respect to the curvilinear coordinates.

The proposed equations have been used to generate the Cartesian coordinates as functions of the curvilienar coordinates in a variety of surfaces. The forcing functions in the PDE's, which distinguish one surface from the other, depend on the equation of the surface in the form $F(x,y,z)=0$. When a surface is given only through a discrete set of data points, the method of least squares and a Fourier method have been developed to fit the required function F.

A computer program has been developed for obtaining the curvilinear coordinates/grids in either a simply- or doubly-connected type of surface. For complicated shapes, a segmentation approach is used in which the whole surface is segmented into simpler shapes and the coordinates for each segment are separately generated. Application of the developed techniques to surfaces having multiple saddle points, airplane surface complete with fuselage and wings, and bodies of revolution, have been made.

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1. "Numerical Grid Generation in Arbitrary Surfaces Through a Second-Order Differential-Geometric Model", by Z.U.A. Warsi, J. Comp. Physics, Vol. 64, (1986), pp. 82-96.
2. "A Synopsis of Elliptic PDE Models for Grid Generation", by Z.U.A. Warsi, Applied Math. and Comp., Vol. 21, (1987), pp. 295-311.
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4. "Numerical Grid Generation Through Second Order Differential-Geometric Models", by Z.U.A. Warsi, and W.N. Tiarn, 11th IMACS World Congress, Oslo, Aug. (1985).
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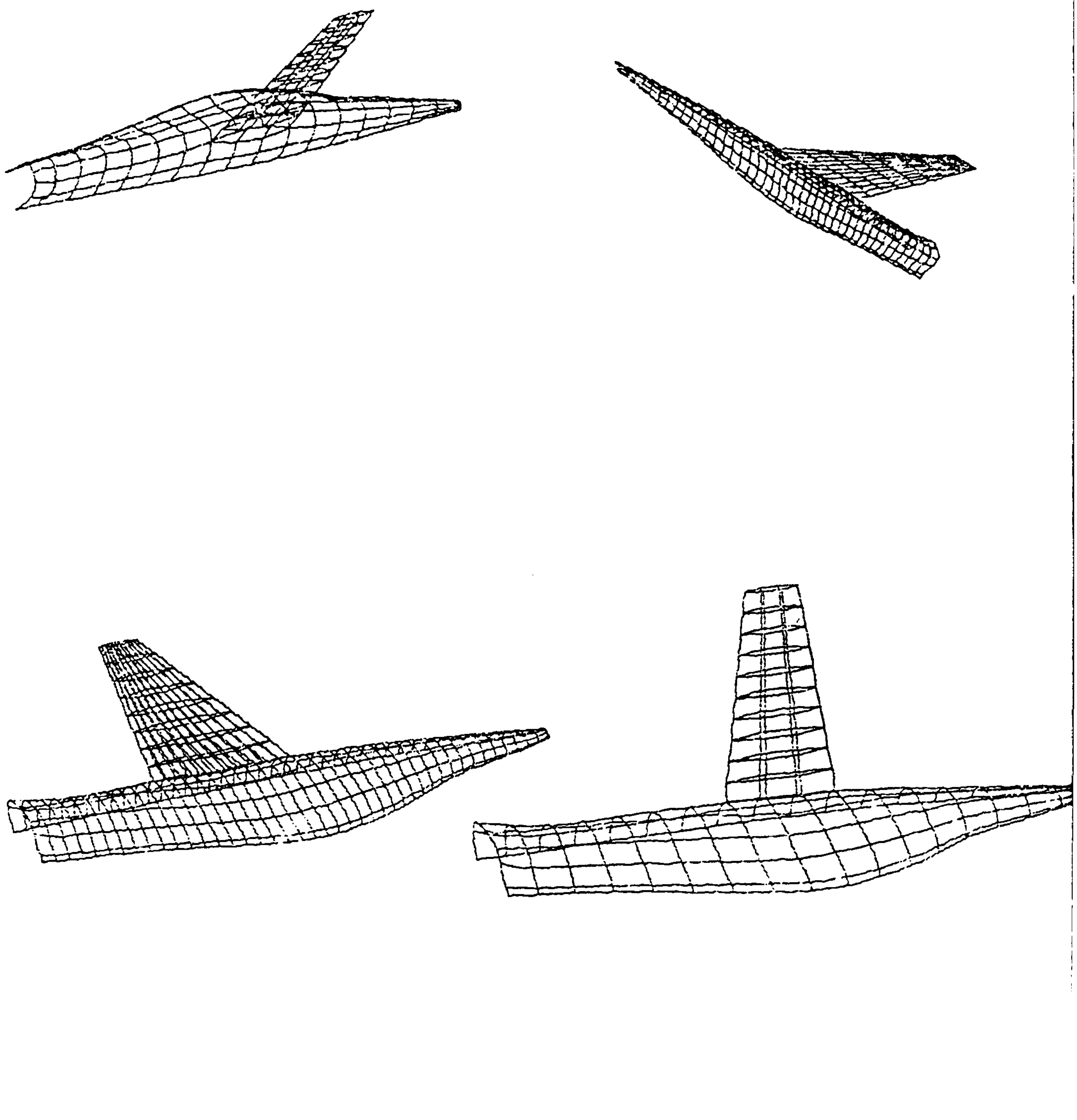

Figure 1. Coordinates On An Airplane With Or Without Swept Wings

APPENDIX

A Synopsis of Elliptic PDE Models for Grid Generation*

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Transmitted by Joe Thompson

ABSTRACT

This paper is devoted to an analytical comparison of the various elliptic partial-differential-equation (PDE) models which are in current use for grid generation. These comparisons, particularly between the equations from the Laplace-Poisson system and the equations from a Gaussian approach, have yielded useful expressions connecting the 3D Laplacians and the surface Beltramians. This effort has specifically been successful when the transverse coordinate leaving the surface is orthogonal to the surface. Equations which are derivable from Cartesian-type Poisson equations and those obtained by using the variational principle in surface coordinates have also been considered.

1. INTRODUCTION

The problem of generating spatial coordinates by numerical methods is of much importance in many branches of engineering mechanics. A review of various methods of coordinate generation in both two- and three-dimensional Euclidean spaces is available in Thompson, Warsi, and Mastin [1, 2].

This paper is exclusively directed to a collection and analytical comparisons of the various elliptic partial-differential-equation (PDE) models which are currently in use for numerical coordinate generation.

The theory of grid generation does not depend on any set of so-called conservation laws, and thus a variety of equations and methods of different characters can be used to obtain the grids. Any consistent method, depending either on the solution of PDEs or any algebraic method, can be used to obtain intersecting trajectories in either 2D or 3D Euclidean space.[1]

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[1] The question of space comes into the picture when it is realized that all the metric coefficients g_{ij} cannot be selected arbitrarily. In fact, these metric coefficients must be so selected that a set of second-order PDEs become satisfied. For details refer to Warsi [3].

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In light of the above discussion, any consistent set of PDEs is sufficient to form a mathematical model. In bounded domains the most natural choice is that of a mathematical model formed of elliptic PDEs. The simplest set of equations, which also yields the smoothest grid, consists of the Laplace equations of the curvilinear coordinates in the Cartesian physical space. As a next logical step, a set of Poisson equations can be selected so as to have a degree of control of the distribution of grid lines. On inverting these equations a set of quasilinear PDEs are obtained [see Equations (2), (4)].

After the development of the problem of grid generation through solving the inverted forms of the Laplace and Poisson equations in 2D Euclidean domains by Allen [4], Winslow [5], Chu [6], and Thompson et al. [7], a logical extension was to use the same equations for 3D domains, as has been done by Thompson and Mastin [8]. Parallel to the above-noted developments, Warsi [3, 9, 10] proposed a Gaussian approach which basically generates surfaces and thus can be used either for generating curvilinear coordinates in a given surface [11] or for generating 3D coordinates by generating a series of surfaces starting from the data on the given surfaces [9, 12]. The Gaussian approach in fact depends on a manipulation of the formulae of Gauss for a surface, and thus the resulting equations have the surface coordinates as the independent variables. This manipulation introduces the Beltramians of the curvilinear coordinates and the sum of the principal curvatures of the surface in a natural way. *Since the formulae of Gauss for a surface hold true for any allowable coordinate system introduced in the surface, the equations proposed by Warsi* [3, 9, 10] *must also have the same properties. Further, because of the use of Gauss formulae, the proposed equations are optimal in the sense that the number of terms and the amount of information in the equations is just sufficient for the generation of either surfaces or coordinates.* In this regard we can justifiably call a mathematical model "optimal" if it can be reduced to the form of the proposed equations [cf. Equation (19a)].

Beside the Laplace-Poisson system and the Gaussian approach, we have also derived the surface generating equations by the use of the variational principle. The resulting equations are near optimal for the Gaussian system.

2. BASIC ELLIPTIC MODELS

Poisson Equations as Grid Generators

Since the publication of the TTM method [7], there has been extensive use of the Poisson equations in the physical r-space to generate both 2D and 3D grids [2]. In practically all cases the main aim is to have those equations in which the computational coordinates appear as the independent variables, and therefore, the Poisson equations have to be inverted by making the physical coordinates $\mathbf{r} = (x, y, z)$ the dependent variables. This inversion can of course be carried out in a nontensorial manner by using the chain rule of partial differentiation. However, it is much simpler to follow either of the following two methods to attain the same result.

Method I: Inner multiplication of Equation (A2)[2] by g^{ij} and the use of Equation (A4) results in

$$g^{ij}\mathbf{r}_{,ij} + (\nabla^2 x^k)\mathbf{r}_{,k} = 0. \tag{1}$$

Introducing the second-order differential operator

$$D = gg^{ij}\frac{\partial^2}{\partial x^i \, \partial x^j}$$

into Equation (1), we get

$$D\mathbf{r} + g(\nabla^2 x^k)\mathbf{r}_{,k} = 0, \tag{2}$$

which is the desired equation in vector form in the computational plane.

Method II: In Equation (A3), writing $\phi = \mathbf{r} = (x, y, z)$ and again using Equation (A4), we obtain the same equation (2).

To form a closed system of equations from Equation (2) one has to specify the Laplacians $\nabla^2 x^k$ in a suitable fashion. As described in Reference [3], a general specification for the Laplacians is to write

$$\nabla^2 x^k = g^{ij}P^k_{ij}, \qquad k = 1,2 \text{ or } k = 1,2,3, \tag{3}$$

and then Equation (1) takes the form

$$g^{ij}\left(\mathbf{r}_{,ij} + P^k_{ij}\mathbf{r}_{,k}\right) = 0. \tag{4}$$

In Equation (3), the $P^k_{ij} = P^k_{ji}$ are intended to be arbitrarily specified control functions. Equating the right-hand sides of Equation (A4) and Equation (3),

[2] Refer to the Appendix.

we have

$$g^{ij}P^k_{ij} = -g^{ij}\Gamma^k_{ij}. \tag{5}$$

Since g^{ij} are not arbitrary, we conclude that

$$P^k_{ij} \neq -\Gamma^k_{ij}$$

for all values of i, j, k.

Recursive Property of P^k_{ij}

We now impose the following requirement on P^k_{ij}: *If the coordinates are such that their Laplacians vanish, i.e., $\nabla^2 x^k = 0$, then the control functions P^k_{ij} vanish individually for all indices i, j, k.*

The importance of this requirement becomes clear by considering Equation (5) in which Γ^k_{ij} are the Christoffel symbols. Thus, when the Laplacians of the coordinates vanish, the right-hand side of Equation (5) vanishes as an inner sum, but the left-hand side vanishes due to the imposed restriction of "individual vanishing" of P^k_{ij} for such coordinates. Denoting the coordinates which satisfy the Laplace equations by $x^i_{(0)}$, we then have

$$P^k_{ij(0)} = 0, \qquad \Gamma^k_{ij(0)} \neq 0, \qquad g^{ij}_{(0)}\Gamma^k_{ij(0)} = 0. \tag{6}$$

We now consider those coordinates which satisfy the Poisson equations. In these coordinates it is of interest to know the relation between the successive P's under successive coordinate transformations. To this end, we consider two *successive* allowable transformations denoted by $x^i_{(m-1)}$ and $x^i_{(m)}$, with $x^i_{(0)}$ as those coordinates which satisfy the Laplace equation. Thus, in the $x^i_{(m-1)}$ coordinates, Equation (4) is

$$g^{ij}_{(m-1)}\left(\frac{\partial^2 \mathbf{r}}{\partial x^i_{(m-1)}\,\partial x^j_{(m-1)}} + P^p_{ij(m-1)}\frac{\partial \mathbf{r}}{\partial x^p_{(m-1)}}\right) = 0, \tag{7}$$

and in the $x^i_{(m)}$ coordinates

$$g^{ij}_{(m)}\left(\frac{\partial^2 \mathbf{r}}{\partial x^i_{(m)}\,\partial x^j_{(m)}} + P^p_{ij(m)}\frac{\partial \mathbf{r}}{\partial x^p_{(m)}}\right) = 0. \tag{8}$$

Using the transformation law (A7) in the form

$$g^{ij}_{(m-1)} = \frac{\partial x^i_{(m-1)}}{\partial x^p_{(m)}} \frac{\partial x^j_{(m-1)}}{\partial x^n_{(m)}} g^{pn}_{(m)}, \tag{9}$$

and the chain rule of partial differentiation of $\mathbf{r}$ in Equation (7), and comparing the resulting equation with Equation (8), we get

$$P^p_{ij(m)} = \left(P^s_{kn(m-1)} \frac{\partial x^p_{(m)}}{\partial x^s_{(m-1)}} + \frac{\partial^2 x^p_{(m)}}{\partial x^k_{(m-1)}\, \partial x^n_{(m-1)}} \right) \frac{\partial x^k_{(m-1)}}{\partial x^i_{(m)}} \frac{\partial x^n_{(m-1)}}{\partial x^j_{(m)}}, \tag{10}$$

establishing a relation between the successive P's. This is how P^k_{ij} transform from one coordinate system to the other. Also, using Equation (A9) in Equation (10), we have

$$P^p_{ij(m)} = -\Gamma^p_{ij(m)} + \left(P^s_{kn(m-1)} + \Gamma^s_{kn(m-1)} \right) \frac{\partial x^p_{(m)}}{\partial x^s_{(m-1)}} \frac{\partial x^k_{(m-1)}}{\partial x^i_{(m)}} \frac{\partial x^n_{(m-1)}}{\partial x^j_{(m)}}. \tag{11}$$

Equation (11) establishes a relation between the P's and Γ's. In particular, for $m = 1$, Equation (10) reduces to Equation (63) of Reference [3].

From the recursive relation (10) it is a straightforward matter to show that the transformation $x^i_{(0)} \to x^i_{(1)} \to x^i_{(2)}$ of P^k_{ij} is the same as the transformation $x^i_{(0)} \to x^i_{(2)}$. The use of the chain rule for

$$\frac{\partial x^i_{(2)}}{\partial x^j_{(1)}}, \qquad \frac{\partial^2 x^i_{(2)}}{\partial x^j_{(1)}\, \partial x^k_{(1)}}$$

and of the formula

$$\frac{\partial^2 x^p_{(0)}}{\partial x^j_{(1)}\, \partial x^k_{(1)}} = -\frac{\partial^2 x^i_{(1)}}{\partial x^r_{(0)}\, \partial x^s_{(0)}} \frac{\partial x^p_{(0)}}{\partial x^i_{(1)}} \frac{\partial x^r_{(0)}}{\partial x^j_{(1)}} \frac{\partial x^s_{(0)}}{\partial x^k_{(1)}}$$

gives the required result.

Other Poisson Systems

It is also possible to have equations in which the dependent variables are non-Cartesian, e.g., cylindrical or spherical coordinates. In a paper by Ghia et al. [13], the computational coordinates are assumed to satisfy a Poisson system in the cylindrical coordinates.

To generalize this concept, let x^s be a coordinate system on which the coordinate system $\bar{x}^s$ (e.g., cylindrical) is to be generated. From Equation (A4),

$$\nabla^2 \bar{x}^s = -\bar{g}^{ij}\bar{\Gamma}^s_{ij} \tag{12}$$

and

$$\begin{aligned}\nabla^2 x^k &= -g^{ij}\Gamma^k_{ij} = g^{ij}P^k_{ij} \\ &= P^k \qquad \text{(say)}.\end{aligned} \tag{13}$$

In Equation (12) both the metric coefficients and the Christoffel symbols are already known in terms of the $\bar{x}^s$ coordinate system. [If $\bar{x}^s$ are the Cartesian coordinates, then Equation (12) is an identity.] From Equation (A3) we then have

$$g^{ij}\frac{\partial^2 \bar{x}^s}{\partial x^i\,\partial x^j} + P^k\frac{\partial \bar{x}^s}{\partial x^k} = -\bar{g}^{ij}\bar{\Gamma}^s_{ij}, \tag{14}$$

which are the transformed equations in the computational space. By using Equation (A10) we can also write Equation (14) as

$$\bar{g}^{mn}C^i_m C^j_n\frac{\partial^2 \bar{x}^s}{\partial x^i\,\partial x^j} + J^2P^k\frac{\partial \bar{x}^s}{\partial x^k} = -(J)^2\bar{g}^{ij}\bar{\Gamma}^s_{ij}, \tag{15}$$

where

$$J = \det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right).$$

Recently Fujii [14] used a Cartesian-type Poisson system between two curvilinear coordinate systems. Let x^s and $\bar{x}^s$ be two curvilinear systems. Consider the Cartesian-type Poisson system

$$\sum_p \frac{\partial^2 x^r}{\partial \bar{x}^p\,\partial \bar{x}^p} = P^r, \tag{16}$$

where x^r is the curvilinear system on which the $\bar{x}^p$-system (e.g., spherical) is to be generated. To obtain the inversion of Equation (16), we take the inner product of Equation (A11) with $\partial\bar{x}^t/\partial x^r$ to obtain

$$\frac{\partial^2 \bar{x}^t}{\partial x^i\,\partial x^j}\frac{\partial x^i}{\partial \bar{x}^k}\frac{\partial x^j}{\partial \bar{x}^n} = -\frac{\partial^2 x^r}{\partial \bar{x}^k\,\partial \bar{x}^n}\frac{\partial \bar{x}^t}{\partial x^r}. \tag{17}$$

Setting $k = n = 1$, $k = n = 2$, $k = n = 3$ and adding, we get

$$C^{ij}\frac{\partial^2 \bar{x}^t}{\partial x^i\,\partial x^j} = -\frac{\partial \bar{x}^t}{\partial x^r}\sum_p \frac{\partial^2 x^r}{\partial \bar{x}^p\,\partial \bar{x}^p},$$

so that

$$C^{ij}\frac{\partial^2 \bar{x}^t}{\partial x^i\,\partial x^j} = -\frac{\partial \bar{x}^t}{\partial x^r}P^r, \tag{18}$$

where

$$C^{ij} = \sum_p \frac{\partial x^i}{\partial \bar{x}^p}\frac{\partial x^j}{\partial \bar{x}^p} = \frac{1}{J^2}\sum_p C^i_p C^j_p.$$

Equation (18) is the inverted form of Equation (16). Similarly to the case of actual Poisson system, the functions P^r are again the arbitrary specified functions. Alternatively, we can also write

$$P^r = C^{ij}P^r_{ij},$$

and then P^r_{ij} are the arbitrary control functions.

Beltramians as Grid Generators in Curved Surfaces

The problem of generating curvilinear coordinates in a surface has important applications in many branches of engineering. In this paper we are interested only in one differential model which generates coordinates in surfaces embedded in R_3 and which reduces to Equation (2) in a natural way when the surface degenerates into a plane. To achieve this aim, we consider the formulae of Gauss [Equation (A12)] for a surface $\nu = \text{const}$. Inner multiplication of Equation (A12) by $G_\nu g^{\alpha\beta}$ yields

$$D\mathbf{r} + G_\nu\left(\Delta_2^{(\nu)}x^\delta\right)\mathbf{r}_{,\delta} = \mathbf{n}^{(\nu)}R, \tag{19a}$$

where

$$D = G_\nu g^{\alpha\beta}\partial_{\alpha\beta}, \qquad \mathbf{n}^{(\nu)} = \text{unit surface normal},$$

$$G_\nu = g_{\alpha\alpha}g_{\beta\beta} - (g_{\alpha\beta})^2, \qquad \nu, \alpha, \beta \text{ cyclic}, \tag{19b}$$

$$R = G_\nu g^{\alpha\beta}b_{\alpha\beta} = \left(k_{\mathrm{I}}^{(\nu)} + k_{\mathrm{II}}^{(\nu)}\right)G_\nu,$$

$b_{\alpha\beta}$ are the coefficients of the second fundamental form, and $k_{\mathrm{I}}^{(\nu)}$, $k_{\mathrm{II}}^{(\nu)}$ are the principal curvatures at a point of the surface $\nu = \text{const}$. The Beltramians $\Delta_2^{(\nu)}x^\delta$ are defined by Equation (A14). It is readily seen that when the surface $\nu = \text{const}$ degenerates into a plane, then $R = 0$ and Δ_2 becomes the Laplace operator.

Equation (19a) can be used either for the generation of coordinates in a *given* surface or for the generation of 3D spatial coordinates between two given surfaces. In the latter case it has been shown [10] that

$$R = G_\nu g^{\alpha\beta}\Gamma_{\alpha\beta}^{\nu}\lambda^{(\nu)}, \qquad \lambda^{(\nu)} = \mathbf{n}^{(\nu)}\cdot\mathbf{r}_{,\nu}, \tag{19c}$$

where $\Gamma_{\alpha\beta}^{\nu}$ are the 3-space Christoffel symbols and x^ν is the transverse coordinate.

Surface Coordinates

The surface-oriented generating system of equations, with the option of arbitrary coordinate control, is now obtained by putting suitable restrictions on the Beltramians appearing in Equation (19a). Similar to Equation (3), a general specification of the Beltramians is

$$\Delta_2^{(\nu)}x^\delta = g^{\alpha\beta}P_{\alpha\beta}^{\delta}, \tag{20}$$

where $P_{\alpha\beta}^{\delta}$ satisfy all the properties stated earlier, including Equations (10) and (11) with Γ replaced by T.

To be specific, we take the surface $x^3 = \text{const}$ as the given surface and $x^1 = \xi$, $x^2 = \eta$. Then Equation (19a) becomes

$$L\mathbf{r} = \mathbf{n}^{(3)}R, \tag{21a}$$

where

$$L = g_{22}\partial_{\xi\xi} - 2g_{12}\partial_{\xi\eta} + g_{11}\partial_{\eta\eta} + \bar{P}\partial_\xi + \bar{Q}\partial_\eta,$$

$$\bar{P} = g_{22}P_{11}^1 - 2g_{12}P_{12}^1 + g_{11}P_{22}^1, \tag{21b}$$

$$\bar{Q} = g_{22}P_{11}^2 - 2g_{12}P_{12}^2 + g_{11}P_{22}^2.$$

3. SURFACE-GRID EQUATIONS FROM THE LAPLACIANS

In this section an attempt is made to establish a relation between the surface equations derived by using the formulae of Gauss [Equation (19a)] and the surface equations as derived from the inversions of the 3D Laplacians.

Let ξ, η, ζ be a general curvilinear coordinate system in R^3 such that ξ, η form the coordinates in the surface $\zeta = \text{const}$, with ζ as the transverse coordinate. Starting from Equation (B.3) of Reference [1] and using Equation (A2), we get (refer to [11] for more details)

$$\Omega \mathbf{r} = \frac{G_3\left(k_{\mathrm{I}}^{(3)} + k_{\mathrm{II}}^{(3)}\right)\mathbf{r}_\zeta}{\lambda}, \tag{22a}$$

where the differential operator Ω is

$$\Omega = D + T^\alpha \frac{\partial}{\partial x^\alpha} \tag{22b}$$

$$= D + \left(S^\alpha + G_3 \Delta_2 x^\alpha\right) \frac{\partial}{\partial x^\alpha}. \tag{22c}$$

In (22b, c) the differential operator D defined in (19b) is

$$D = g_{22}\partial_{\xi\xi} - 2g_{12}\partial_{\xi\eta} + g_{11}\partial_{\eta\eta}, \tag{22d}$$

and

$$T^\alpha = 2g_{12}\Gamma_{12}^\alpha - g_{22}\Gamma_{11}^\alpha - g_{11}\Gamma_{22}^\alpha, \tag{22e}$$

$$S^\alpha = 2g_{12}\left(\Gamma_{12}^\alpha - \Upsilon_{12}^\alpha\right) - g_{22}\left(\Gamma_{11}^\alpha - \Upsilon_{11}^\alpha\right) - g_{11}\left(\Gamma_{22}^\alpha - \Upsilon_{22}^\alpha\right), \tag{22f}$$

where $\alpha = 1, 2$. Further

$$\lambda = \mathbf{n} \cdot \mathbf{r}_\zeta, \tag{22g}$$

$\mathbf{n}$ being the unit surface normal vector on $\zeta = \text{const}$.

It must be emphasized that Γ_{jk}^i and $\Upsilon_{\alpha\beta}^\delta$ are the 3-space and 2-space Christoffel symbols of the second kind respectively, and in general they are not equal to each other at $\zeta = \text{const}$. Thus, a comparison of Equations (19a) and (22a) shows that the two sets of surface generators are entirely different. However, it has been shown below that Equations (19a) and (22a) become exactly the same equations under the following conditions:

(i) When the surface degenerates into a plane, in which case ξ, η are the curvilinear coordinates (of any nature) in a plane.

(ii) When ξ, η are any general coordinates in a surface, *but* ζ (the transverse coordinate) *is orthogonal to the surface*.

Case (i) is patently straightforward. For case (ii), noting that ζ is orthogonal to the surface, we have

$$g_{13} = g_{23} = 0, \tag{23a}$$

and

$$\mathbf{n} = \frac{\mathbf{r}_\zeta}{\sqrt{g_{33}}}, \qquad \lambda = \sqrt{g_{33}}. \tag{23b}$$

Under the conditions (23a) it is easy to show that

$$\Gamma^{\alpha}_{\beta\gamma} = \Upsilon^{\alpha}_{\beta\gamma},$$

where all Greek indices assume only two values (here 1,2). Thus $S^{\alpha} = 0$, and using (23b) we find that Equations (19a) and (23a) become the same equations under the condition of orthogonality of the ζ-coordinate to the surface $\zeta = \text{const}$. Further, under this condition the Beltramians and the Laplacians of ξ and η are related through the following equations:

$$\nabla^2 \xi = \Delta_2 \xi - \frac{\Gamma^1_{33}}{g_{33}}, \tag{24a}$$

$$\nabla^2 \eta = \Delta_2 \eta - \frac{\Gamma^2_{33}}{g_{33}}. \tag{24b}$$

Further,

$$\nabla^2 \zeta = -\frac{\Gamma^3_{33}}{g_{33}} - \frac{k^{(3)}_{\mathrm{I}} + k^{(3)}_{\mathrm{II}}}{\sqrt{g_{33}}}, \tag{24c}$$

where

$$k^{(3)}_{\mathrm{I}} + k^{(3)}_{\mathrm{II}} = -\frac{1}{2\sqrt{g_{33}}\,G_3}\frac{\partial G_3}{\partial \zeta}, \tag{24d}$$

and [referring to (19b)]

$$G_3 = g_{11}g_{22} - (g_{12})^2.$$

4 SURFACE GENERATING EQUATIONS BASED ON A VARIATIONAL PRINCIPLE

The use of a variational principle in the generation of both 2D and 3D grids has already been considered by Brackbill and Saltzman [15, 16] and Thompson and Warsi [17]. In this section we shall consider only the surface grid generation problem based on the use of a variational principle. In essence the following analysis is a unified approach to both the plane 2D and surface 2D cases from a variational viewpoint.

Let x^1, x^2 be the coordinates in a surface. Consider the surface functional

$$I = \int \sqrt{G_3}\, \phi \, dx^1 \, dx^2, \tag{25}$$

where $G_3 = g_{11}g_{22} - (g_{12})^2$ and ϕ is a specified function. The condition $\delta I = 0$ then leads one to the Euler-Lagrange equations (using the summation convention)

$$\frac{\partial}{\partial x_r}\left(\sqrt{G_3}\,\phi\right) - \frac{\partial}{\partial x^\beta}\frac{\partial\left(\sqrt{G_3}\,\phi\right)}{\partial x_{r,\beta}} = 0, \tag{26}$$

where x_r $(r = 1,2,3)$ are the rectangular Cartesian coordinates, x^β are the curvilinear surface coordinates, and

$$x_{r,\beta} = \frac{\partial x_r}{\partial x^\beta}, \qquad x_{r,\alpha\beta} = \frac{\partial^2 x_r}{\partial x^\alpha \, \partial x^\beta}.$$

From Equation (26), it is a purely algebraic problem to show that

$$Lx_r = \frac{\sqrt{G_3}}{2}\frac{\partial}{\partial x^\gamma}\left(\frac{1}{\sqrt{G_3}}\frac{\partial G_3}{\partial x_{r,\gamma}}\right), \tag{27a}$$

where

$$L = G_3\left[g^{\alpha\beta}\frac{\partial^2}{\partial x^\alpha \, \partial x^\beta} + (\Delta_2 x^\alpha)\frac{\partial}{\partial x^\alpha}\right]. \tag{27b}$$

Let now ϕ be a function of $x_{r,\beta}$. Then expanding Equation (26) and using Equation (27), we get

$$Lx_r = -M, \tag{28a}$$

where

$$M = \frac{\phi}{2}\left(\frac{\partial G_3}{\partial x_{r,\beta}}\frac{\partial \phi}{\partial x^\beta} + \frac{\partial \phi}{\partial x_{r,\beta}}\frac{\partial G_3}{\partial x^\beta}\right) + \frac{G_3}{\phi}\frac{\partial}{\partial x^\beta}\left(\frac{\partial \phi}{\partial x_{r,\beta}}\right). \tag{28b}$$

On the other hand, if ϕ is taken as

$$\phi = F/G_3, \tag{29a}$$

where F is still a function of $x_{r,\beta}$, then in place of (28b), we have

$$\begin{aligned} M = {} & \frac{1}{2F}\left(\frac{\partial G_3}{\partial x_{r,\beta}}\frac{\partial F}{\partial x^\beta} + \frac{\partial G_3}{\partial x^\beta}\frac{\partial F}{\partial x_{r,\beta}}\right) \\ & - \frac{1}{2G_3}\frac{\partial G_3}{\partial x_{r,\beta}}\frac{\partial G_3}{\partial x^\beta} - \frac{G_3}{F}\frac{\partial}{\partial x^\beta}\left(\frac{\partial F}{\partial x_{r,\beta}}\right). \end{aligned} \tag{29b}$$

The generating system (28a) with L defined in (27b) is similar to Equation (19a). However, the selection of the form of the function ϕ or F which yields the right-hand side of Equation (19a) seems to be a difficult task. One simple case is when $\phi = 1$. In this case the minimization of I implies [from (28a)]

$$Lx_r = 0,$$

and these are the equations for a minimal surface. The form (29a) is of interest because the choice

$$F = g_{11} + g_{22}, \tag{30a}$$

or

$$\phi = g^{11} + g^{22}, \tag{30b}$$

is equivalent to the "smoothness" problem in 2D plane coordinates, as has been shown in Reference [15]. It must, however, be stated that "smoothness" of coordinates in a 2D plane problem is due to the satisfaction of the Laplace equation. No such criterion is obvious on using either of the equations (30) in (28a).

5. ANALYTICAL COMPARISONS AND CONCLUSIONS

Based on the foregoing analysis, we conclude as follows:

(a) The Laplace-Poisson system for 2D regions is optimal, since its inversion coincides exactly with the Gaussian equations in a plane. [Note: In a plane the right-hand side of Equation (19a) is zero and the Beltramians become the Laplacians.]

(b) The inversion of the Laplace-Poisson system for 3D regions for a constant coordinate value, viz., for a surface, does *not* reduce to the Gaussian equations (cf. Section 3) except when the transverse coordinate is taken as orthogonal to the surface [cf. Equation (22a)]. This implies that the Laplace-Poisson system in 3D regions with three nonorthogonal coordinates is not optimal, though it is a valid system. The extra terms (22f) should somehow be managed, and in practice, they are taken as part of the arbitrary specified control functions. This means that the generated coordinates will assume a distribution which may not be to one's desire. It must, however, be again emphasized that the terms (22f) vanish when the transverse coordinate is orthogonal to the surface. (In Reference [18], the author had to make other assumptions besides orthogonality.)

(c) For the elliptic system described by Equation (15) the conclusions discussed in (b) hold good.

(d) The generating system described by Equation (16) is a Cartesian-type Poisson system. It looks difficult to assess its optimality in relation to the Gaussian equations.

(e) The equations derived from the variational principle, viz., (28), are nearly optimal, though it looks difficult to find the appropriate function ϕ which makes the right hand side of (28a) the same as that of Equation (19a).

APPENDIX

The following formulae have been used in the main text of the paper and can be found in any standard text on tensors, e.g., [19], [20].

In an Euclidean space R^n (though here we are concerned only with R^2 or R^3), endowed with the curvilinear coordinates x^i, the first partial derivatives of the covariant basis vectors $\mathbf{a}_i$ are expressible as linear functions of $\mathbf{a}_i$:

$$\frac{\partial \mathbf{a}_i}{\partial x^j} = \Gamma_{ij}^k \mathbf{a}_k, \tag{A1}$$

where Γ_{ij}^k are the Christoffel symbols of the second kind. Here, and in what follows, a repeated lower and upper index will always imply summation over the range of index values. Equation (A1) can alternatively be written as

$$\mathbf{r}_{,ij} = \Gamma_{ij}^k \mathbf{r}_{,k}, \tag{A2}$$

where a comma preceding an index implies a partial derivative, and $\mathbf{r}$ is the position vector, viz.,

$$\mathbf{r} = (x, y, z) \text{ or } (x_1, x_2, x_3).$$

The Laplacian of a scalar ϕ is given by

$$\nabla^2 \phi = g^{ij}\left(\phi_{,ij} - \Gamma_{ij}^k \phi_{,k}\right). \tag{A3}$$

From (A3), we have

$$\nabla^2 x^k = -g^{ij}\Gamma_{ij}^k \tag{A4}$$

The quantities g^{ij} and g_{ij} are respectively the contravariant and covariant components of the metric tensor, and the two are related as

$$g^{ij} = \frac{g_{pn}g_{rs} - g_{ps}g_{nr}}{g},$$

where (i, p, r) and (j, n, s) are to be taken in the cyclic permutations of $(1, 2, 3)$, and

$$g = \det(g_{ij}).$$

Let x^i and $\bar{x}^i$ be two allowable coordinate systems in a Euclidean space such that each of the functions $\bar{x}^i = \phi^i(x^j)$ and $x^i = \psi^i(\bar{x}^j)$ define a transformation with

$$J = \det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) \neq 0, \qquad \bar{J} = \det\left(\frac{\partial x^i}{\partial \bar{x}^j}\right) \neq 0.$$

On transformation of coordinates from x^i to $\bar{x}^i$, we have

$$\bar{g} = \frac{g}{J^2}, \qquad g = \frac{\bar{g}}{\bar{J}^2}, \tag{A5}$$

$$\bar{g}_{pn} = \frac{\partial x^i}{\partial \bar{x}^p}\frac{\partial x^j}{\partial \bar{x}^n} g_{ij}, \tag{A6}$$

$$\bar{g}^{pn} = \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial \bar{x}^n}{\partial x^j} g^{ij}, \tag{A7}$$

$$\bar{\Gamma}^p_{kn} = \Gamma^s_{ij}\frac{\partial \bar{x}^p}{\partial x^s}\frac{\partial x^i}{\partial \bar{x}^k}\frac{\partial x^j}{\partial \bar{x}^n} + \frac{\partial^2 x^j}{\partial \bar{x}^k \partial \bar{x}^n}\frac{\partial \bar{x}^p}{\partial x^j}, \tag{A8}$$

$$\frac{\partial^2 \bar{x}^p}{\partial x^k \partial x^n} = \Gamma^s_{kn}\frac{\partial \bar{x}^p}{\partial x^s} - \bar{\Gamma}^p_{rt}\frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial \bar{x}^t}{\partial x^n}. \tag{A9}$$

The first partial derivatives of x^i with respect to $\bar{x}^i$ are given by

$$\frac{\partial x^i}{\partial \bar{x}^j} = \frac{C^i_j}{J},$$

where

$$C^i_j = \frac{\partial \bar{x}^r}{\partial x^s}\frac{\partial \bar{x}^k}{\partial x^n} - \frac{\partial \bar{x}^r}{\partial x^n}\frac{\partial \bar{x}^k}{\partial x^s}, \tag{A10}$$

and (i, s, n) and (j, r, k) are cyclic permutations of $(1, 2, 3)$.

The second partial derivatives of one set of coordinates are related with those of the other set as

$$\frac{\partial^2 x^r}{\partial \bar{x}^k \partial \bar{x}^n} = -\frac{\partial^2 \bar{x}^s}{\partial x^i \partial x^j}\frac{\partial x^r}{\partial \bar{x}^s}\frac{\partial x^i}{\partial \bar{x}^k}\frac{\partial x^j}{\partial \bar{x}^n}. \tag{A11}$$

We now consider a surface embedded in R^3. All tensor indices associated with a surface will be denoted by Greek letters (except the letter ν). In contrast to Equation (A2), the formula of Gauss is

$$\mathbf{r}_{,\alpha\beta} = \Gamma^\delta_{\alpha\beta}\mathbf{r}_{,\delta} + \mathbf{n}^{(\nu)} b_{\alpha\beta}, \tag{A12}$$

where $\Upsilon^{\delta}_{\alpha\beta}$ are the surface Christoffel symbols of the second kind, $b_{\alpha\beta}$ are the coefficients of the second fundamental form, and $n^{(\nu)}$ is the unit surface normal on the surface $\nu = \text{const}$. The values of ν and other Greek indices follow the following scheme:

$\nu = 1$: Greek indices α, β, etc. assume values 2 and 3.
$\nu = 2$: Greek indices α, β, etc. assume values 3 and 1.
$\nu = 3$: Greek indices α, β, etc. assume values 1 and 2.

The Beltramian of a scalar ϕ is given by

$$\Delta_2^{(\nu)}\phi = g^{\alpha\beta}\left(\phi_{,\alpha\beta} - \Upsilon^{\delta}_{\alpha\beta}\phi_{,\delta}\right). \tag{A13}$$

From (A13), we have

$$\Delta_2^{(\nu)}x^{\delta} = -g^{\alpha\beta}\Upsilon^{\delta}_{\alpha\beta}. \tag{A14}$$

For a surface the formulae (A5)–(A9) are equally applicable with proper choice of indices and replacing Γ by Υ.

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