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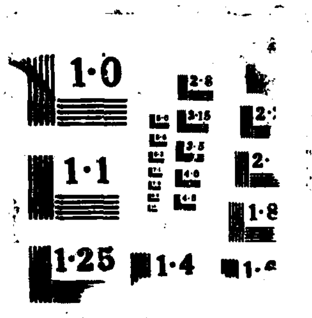
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SOLUTION OF LARGE ORDER FOURIER-BESSEL EQUATIONS

by

S. J. ROBERTS

ABSTRACT

The solution of large order Fourier-Bessel equations, whose derivative is zero at the boundaries of a cylindrical geometry, is found using a series expansion method.

19 pages

4 figures

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1. INTRODUCTION

The solution of the wave equation constrained to a cylindrical geometry gives rise to a radial dependence in terms of Fourier-Bessel functions. This analysis is that appropriate to an annular duct and therefore the eigenfunctions include both Bessel and Neumann contributions. At any radius, the argument of the Fourier-Bessel function is determined by an eigenvalue which depends on both the radial node number (l) and the order of the eigenfunction (n). These eigenvalues are determined from the boundary condition of zero derivative at the inner and outer radii of the duct.

The method used herein seeks to construct solutions of the large order Fourier-Bessel functions from those of the zeroth order by obtaining coefficients to a truncated infinite series expansion. The coefficients are eigenvectors which are associated with modified versions of the large order eigenvalues.

The number of terms necessary for accuracy of the truncated infinite series is investigated and the resulting large order eigenfunctions are plotted.

2.THEORY

2.1 Basic Equations

The solution of *the wave equation* (with or without mean flow)

$$\left(\nabla^2 - \frac{1}{a_0^2}\frac{\partial^2}{\partial t^2} \right)\Psi(\underline{r},t) = 0$$

where $\Psi(\underline{r},t)$ is a wave function and a_0 is the speed of the wave motion, results in a radial dependence given by

$$\frac{d^2R_l^n}{dr^2} + \frac{1}{r}\frac{dR_l^n}{dr} + \left(k_l^{n2} - \frac{n^2}{r^2}\right)R_l^n = 0$$

where $R_l^n(r)$ are a complete set of orthonormal radial eigenfunctions called Fourier-Bessel functions and $R_l^n(r) \equiv R_n(k_l^n r)$, i.e. *n* is the order of the function and l is the radial node number (l=0,1,2,...). Specifically, the $R_l^n(r)$ are a combination of Bessel and Neumann functions. The Fourier-Bessel function differs from the Bessel function because it depends on n which is constrained by the number of circumferential wavelengths fitted around the duct, i.e. n is the circumferential mode number. Hence, because the circumferential dependence is harmonic, the "Bessel" function has a Fourier component and therefore we call it a Fourier-Bessel function. Solutions of this equation are admitted only for certain values of the eigenvalue k_l^n which is determined from the boundary condition

$$\frac{dR_l^n}{dr} = 0$$

at r=1.0,h where h is the ratio of the hub of the duct to the outer radius. This boundary condition arises because the radial gradient of the wave function is zero at the inner and outer duct radii; for example, the radial velocity must vanish at the hub and the tip for non-porous walls.

Normalisation of the eigenfunctions fixes their magnitudes absolutely and is achieved by

$$\int_h^1 r R_l^n(r) R_m^n(r)\, dr = \delta_{lm}$$

2.2 Expansion Of The Higher Order Modes

Following the approach of Namba (ref.1) we expand the n^{th} order eigenfunction in terms of the zeroth mode

$$R_l^n(r) = \sum_{m=0}^{L-1} B_{lm}^n R_m^0(r)$$

where L is as large as we wish. Hence, we have

$$B_{lm}^n R_m^{0\prime\prime} + \frac{1}{r} B_{lm}^n R_m^{0\prime} + \left(k_l^{n2} - \frac{n^2}{r^2}\right) B_{lm}^n R_m^0 = 0$$

and also

$$R_m^{0\prime\prime} + \frac{1}{r} R_m^{0\prime} + k_m^{02} R_m^0 = 0$$

where summation over m is implied for both equations.

$$\therefore \qquad \left(\frac{k_l^{n2}}{n^2} - \frac{1}{r^2} - \frac{k_m^{02}}{n^2}\right) B_{lm}^n R_m^0 = 0 \quad .$$

Putting $K_l^{n2} = k_l^{n2}/n^2$, then multiplying by $rR_j^0(r)$ and integrating

$$\int_h^1 \left(K_l^{n2} - \frac{k_m^{02}}{n^2}\right) B_{lm}^n R_m^0 R_j^0\, r\,dr - \int_h^1 \frac{1}{r} B_{lm}^n R_m^0 R_j^0\, dr = 0$$

$$\therefore \qquad K_l^{n2} B_{lm}^n \delta_{mj} = \left(\bar{R}_{mj} + \frac{k_m^{02}}{n^2}\delta_{mj}\right) B_{lm}^n$$

where $$\bar{R}_{mj} = \int_h^1 \frac{1}{r} R_m^0(r) R_j^0(r)\, dr \qquad (2.2.1)$$

and K_l^{n2} and B_{lm}^n are respectively the eigenvalues and eigenvectors of a matrix

$$\bar{R}_{ij} + \frac{k_i^2}{n}\delta_{ij} \quad . \tag{2.2.2}$$

Note that $\int_h^1 rR_i^n R_p^n \, dr = \int_h^1 B_{im}^n B_{pm}^n R_m R_q r \, dr$

$\therefore \quad \delta_{ip} = B_{im}^n B_{pm}^n \delta_{mq}$

$$\therefore \quad \sum_m B_{im}^{n\,2} = 1 \tag{2.2.3}$$

i.e. the normalisation condition applied to the eigenfunctions means that the sum of the squares of the eigenvectors is one.

2.3 The Boundary Condition

Since $R_l^n(r)$ is a combination of Bessel and Neumann functions, we can represent it by

$$R_l^n(r) = AJ_l^n(r) + BY_l^n(r)$$

where A and B are constants depending on n and l. We have $\frac{dR_l^n}{dr} = 0$ at r=1.0,h

hence, $\quad k_l^n AJ_n'(k_l^n h) + k_l^n BY_n'(k_l^n h) = 0$

and $\quad k_l^n AJ_n'(k_l^n) + k_l^n BY_n'(k_l^n) = 0 \; .$

Assuming $k_l^n \neq 0$, this gives

$$\frac{B}{A} = -\frac{J_n'(k_l^n)}{Y_n'(k_l^n)} = -\frac{J_n'(k_l^n h)}{Y_n'(k_l^n h)} \quad . \tag{2.3.1}$$

Since our solution does not extend to r=0, B≠0 and therefore

$$J_n'(k_l^n)Y_n'(k_l^n h) - J_n'(k_l^n h)Y_n'(k_l^n) = 0 \quad .$$

Abramowitz (ref.2) gives the solution of this as

$$k_l^n h = \beta + \frac{p}{\beta} + \frac{q - p^2}{\beta^3} + \frac{r - 4pq + 2p^3}{\beta^5} + \ldots \tag{2.3.2}$$

where, if $\mu = 4n^2$ and $\lambda = 1.0/h$

$$\beta = \frac{l\pi}{\lambda - 1.0} \qquad p = \frac{\mu + 3}{8\lambda}$$

$$q = \frac{(\mu^2 + 46\mu - 63)(\lambda^3 - 1.0)}{6(4\lambda)^3 (\lambda - 1.0)}$$

$$r = \frac{(\mu^3 + 185\mu^2 - 2053\mu + 1899)(\lambda^5 - 1.0)}{5(4\lambda)^5 (\lambda - 1.0)}$$

2.4 Coefficients Of The Zeroth Order Eigenfunction

The normalisation condition gives

$$\int_h^1 r R_l^n(r) R_m^n(r)\, dr = \delta_{lm} \qquad (2.4.1)$$

$$\therefore \quad \int_h^1 r R_l^{n^2}(r)\, dr = 1.0 \quad .$$

McLachlan (ref.3) gives

$$\int C_\nu(kz)\, z\,dz = 0.5\, z^2 \left(C_\nu^2(kz)\left(1.0 - \frac{\nu^2}{k^2 z^2}\right) + C_\nu'^2(kz)\right) \qquad (2.4.2)$$

where $C_\nu(kz)$ is a cylinder function (i.e. any combination of Bessel and Neumann functions) of unrestricted order ν.

Therefore, applying this to $R_l^n(r)$ with n=0 and noting that $R_n'(k_l^n) = R_n'(k_l^n h) = 0$, we get

$$\int_h^1 r R_0^2(k_l^0 r)\, dr = 0.5\left(R_0^2(k_l^0) - h^2 R_0^2(k_l^0 h)\right) = 1.0 \quad .$$

Putting $R_0(k_l^0 r) = A J_0(k_l^0 r) + B Y_0(k_l^0 r)$

$$= A\left(J_0(k_l^0 r) + \frac{B}{A} Y_0(k_l^0 r) \right)$$

gives

$$\frac{A^2}{2}\Big(J_0^2(k_l^0) + \left(\frac{B}{A}\right)^2 Y_0^2(k_l^0) + 2\left(\frac{B}{A}\right) J_0(k_l^0) Y_0(k_l^0)$$

$$- h^2 J_0^2(k_l^0 h) - h^2 \left(\frac{B}{A}\right)^2 Y_0^2(k_l^0 h)$$

$$- 2h^2 \left(\frac{B}{A}\right) J_0(k_l^0 h) Y_0(k_l^0 h) \Big) = 1.0$$

For n=0, $\quad \frac{B}{A} = - \frac{J_0'(k_l^0)}{Y_0'(k_l^0)}$

and since $\quad J_0' = -J_1$

$$\frac{B}{A} = - \frac{J_1(k_1^0)}{Y_1(k_1^0)} \quad .$$

Hence, for the zeroth order eigenfunction, both coefficients can easily be determined and the problem now becomes one of evaluating the integral (2.2.2) and extracting the eigenvectors from the resulting matrix.

2.5 Zeroth Node Number Terms

$R_l^n(r)$ is a complete set of orthonormal functions and therefore the l=0 term must be included. Clearly, the boundary condition formula (2.3.2) cannot be used here since l=0 results in a singularity.

Actually, we see that $k_0^0=0$, because n=l=0 means that there are no nodes in the radial and circumferential directions, i.e. the wavelengths in each case are infinite, and therefore the formula cannot be derived. If $k_0^0=0$, the arguments of the Bessel and Neumann functions are zero. However, this results in $Y_0^0(0) \to -\infty$ and $\therefore$ B = 0 for $k_0^0 = 0$. Hence,

$$R_0^0(0) = AJ_0^0(0)$$

for n=0 $\qquad R_0^0(0) = A \qquad$ since $J_0^0(0) = 1.0$

i.e. R_0^0 = constant .

We have $\int_h^1 rR_0^{0\,2}dr = 1.0$

$$\therefore \quad \frac{R_0^{0\,2}}{2}(1-h^2) = 1.0$$

$$\therefore \quad R_0^0 = \left(\frac{2}{1.0 - h^2}\right)^{1/2}$$

Equation (2.2.1) gives

$$\bar{R}_{mj} = \int_h^1 \frac{1}{r} R_m^0(r) R_j^0(r)\, dr$$

$\therefore$ if m = 0

$$\bar{R}_{0j} = \left(\frac{2}{1.0 - h^2}\right)^{1/2} \int_h^1 \frac{1}{r} R_j^0(r)\, dr$$

and if m = j = 0

$$\overline{R}_{\infty} = -\frac{2\log_e h}{1.0-h^2} \quad .$$

We are now able to form all values of the integrand of $\overline{R}_{mj}$ for $m,j = 0,1,2,\ldots,L-1$.

2.6 Outline Of The Method Of Solution

Use of a Simpson's rule approximation will now be sufficient to evaluate $\overline{R}_{mj}$ for all values of m and j. Addition of K_m^{o2}/n^2 to the diagonal elements of the matrix thus obtained will provide us with a matrix, the eigenvalues and eigenvectors of which will be K_l^{n2} and B_{lm}^n respectively.

We note in passing that the matrix must be real and symmetric and therefore its eigenvalues will be real which implies that its eigenvectors are real. Therefore, the n^{th} order eigenfunctions must be real. Furthermore, as $n \to \infty$, the influence of the K_m^{o2}/n^2 becomes less and less and hence the matrix becomes constant for given L and h, i.e.

$$R_l^n \sim R_l^{\infty} + O(1/n^2) \ .$$

3. RESULTS

A computer program based on this theory has been written which demonstrates the behaviour of the large order eigenfunctions.

It is useful first to observe from fig.1, which is taken from ref.1, that the eigenvalues K_l^{n2} of the matrix given by (2.2.2) decrease monotonically as n increases, but that the limits are all greater than unity and differ with 1. Additionally, for given 1, K_l^n can be made closer to unity by increasing the number of retained terms L. This behaviour is as expected since as $n \to \infty$, the matrix tends to $\overline{R}_{mj}$,i.e. the n dependence drops off as $1.0/n^2$ and therefore a limit is reached. This limit is not unity because an infinite number of terms are not retained. Namba (ref.1) and McCune (ref.4) discuss this behaviour in more detail.

The eigenvectors B_{lm}^n associated with the K_l^{n2} are calculated subject to the normalisation condition (2.2.3) and these together with the zeroth order Fourier-Bessel functions provide the n^{th} order Fourier-Bessel functions which are plotted in figs. 2, 3, and 4. It is noticed that for given n at 1=0, most of the effect of the radial variation is concentrated near to the outer radius, but that as 1 increases the peak shifts to the inner radius. If we examine the eigenvalues for each R_l^n and take the product $K_l^n r$ then we can choose for the peaks to occur at $K_l^n r_p = 1.0$ (where r_p is the radius at the peak) by judicious choice of L. Hence, as 1 increases, K_l^n increases and therefore the peak occurs at smaller r, i.e. towards the hub. Conversely, as n increases at constant 1, the peak will move to the outer wall.

The reason why the optimum number of terms for the expansion

(L) was chosen such that $K_\ell^n r_p = 1.0$ is explained in the next section. It should be noted that an asymptotic expansion does not necessarily become more accurate as more terms are taken. The value of L which made $K_\ell^n r_p = 1.0$ turned out to be 6 or 7.

4. DISCUSSION OF RESULTS

Abramowitz (ref.2) gives, for large order Bessel and Neumann functions

$$J_n(nz) \sim \left(\frac{4\zeta}{1-z^2}\right)^{1/4} \frac{Ai(n^{2/3}\zeta)}{n^{1/3}} + \ldots$$

$$Y_n(nz) \sim -\left(\frac{4\zeta}{1-z^2}\right)^{1/4} \frac{Bi(n^{2/3}\zeta)}{n^{1/3}} + \ldots$$

where Ai and Bi are the Airy functions given by

$$Ai(n^{2/3}\zeta) \sim \frac{1}{\pi^{1/2}} (n^{2/3}\zeta)^{-1/4} \sin\left(\zeta+\frac{\pi}{4}\right) + \ldots$$

$$Bi(n^{2/3}\zeta) \sim \frac{1}{\pi^{1/2}} (n^{2/3}\zeta)^{-1/4} \cos\left(\zeta+\frac{\pi}{4}\right) + \ldots$$

for $\zeta < 0$ and

$$2/3(-\zeta)^{3/2} = (z^2-1)^{1/2} - \cos^{-1}(1/z) \qquad \text{for } z \geq 1.$$

If we put $z = K_\ell^n r$, then nz becomes $nK_\ell^n r = K_\ell r$, i.e. the argument of our Fourier-Bessel functions. We can now construct an expression for the large order Fourier-Bessel functions:-

$$R_n(K_\ell^n r) \sim \frac{A}{n^{1/3}} \left(\frac{4\zeta}{1-z^2}\right)^{1/4} \frac{(n^{2/3}\zeta)^{-1/4}}{\pi^{1/2}} \left(\sin(\zeta+\pi/4) + \frac{B}{A}\cos(\zeta+\pi/4)\right)$$

or

$$R_n(K_\ell^n r) \sim \frac{A}{(n\pi)^{1/2}} \left(\frac{4}{1-z^2}\right)^{1/4} \left(\sin(\zeta+\pi/4) + \frac{B}{A}\cos(\zeta+\pi/4)\right) \qquad (4.1)$$

To get quantitative results from this formula we need to find K_ℓ^n. This cannot be found from the boundary condition (2.3.2) despite the fact that $K_\ell^n = K_\ell / n$ because this formula does not hold for large n.

Furthermore, the expression for ζ is only valid for ζ real, i.e. $z \geq 1$. If this is not the case, an alternative expression for ζ is used which leads to exponential terms for the Airy functions and their derivatives. Interested readers are referred to Abramowitz (ref.2). Here, we simply observe from (4.1) that $K_\ell^n r = 1$, i.e. $z = 1$, implies a peak for $R_\ell^n(r)$.

5. CONCLUSION

It is clear from the partial derivation of a formula (4.1) which attempts to evaluate analytically the large order Fourier-Bessel functions that problems are posed such that results can only be extracted with great difficulty. A major hurdle is the evaluation of the eigenvalues at large *n*.

By contrast, an infinite series expansion truncated after an appropriately large number of terms provides the behaviour hinted at (but not described) in the formula (4.1) and utilises a simpler approach. The evaluation of the integral (2.2.2) is central to the method and this presents no difficulty.

Some correlation of the results obtained is provided by reference to the peak value of $R_\ell^n(r)$ and its behaviour with respect to changing n and l. From the examples taken 6 or 7 terms are seen to be necessary for the expansion. Values of $R_\ell^n(r)$ for large *n* show a decreasing dependence on this parameter and in fact, for $n \to \infty$, $R_\ell^n(r) \to R_\ell^\infty(r)$ which depends simply on $\bar{R}_{nj}$ given by (2.2.1).

This work has examined the behaviour of a perhaps recondite function at large orders, but it is one which is essential for correct analysis of the radial dependence of a wave function where the inner boundary of a cylindrical geometry has a finite value.

S. ROBERTS (S.O.)

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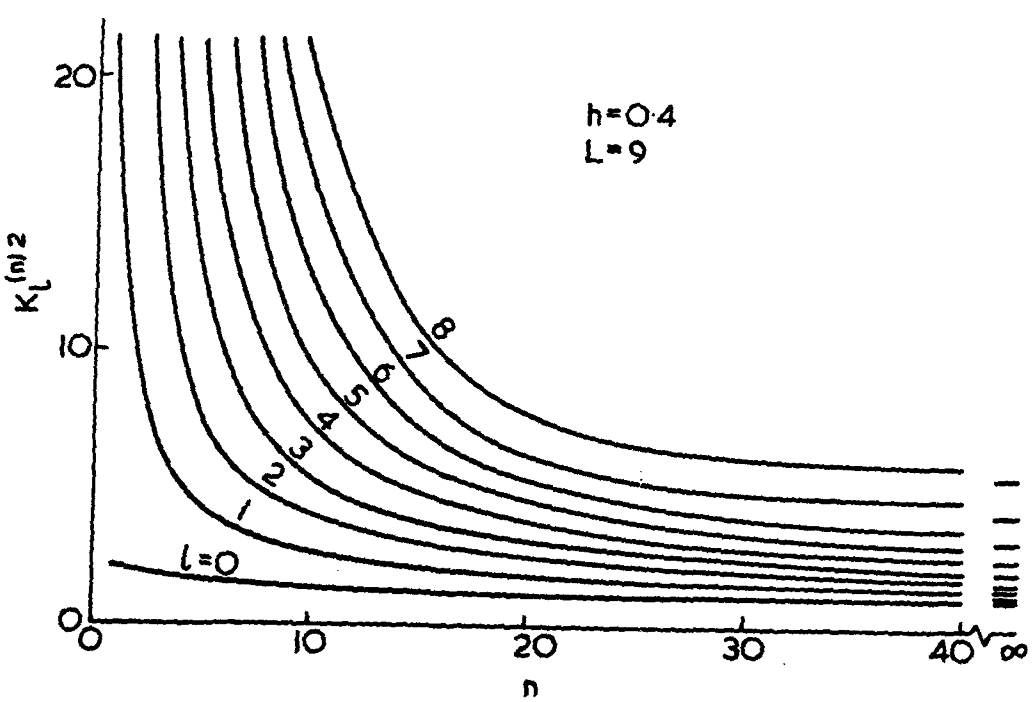
FIG. 1



Dependence of modified radial eigenvalues upon radial mode number l and circumferential mode number n. $h = 0.4$, $L = 9$.

FIG. 2

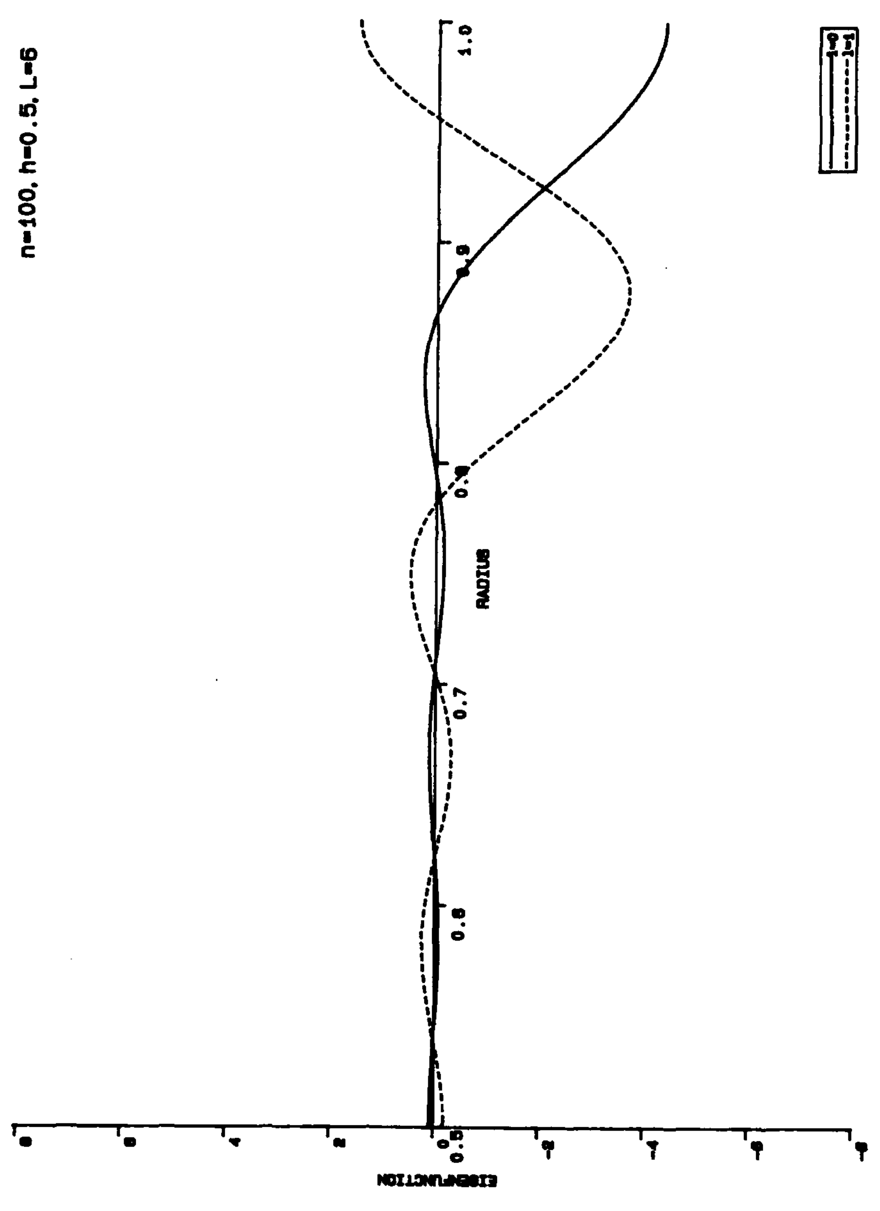


FIG.3

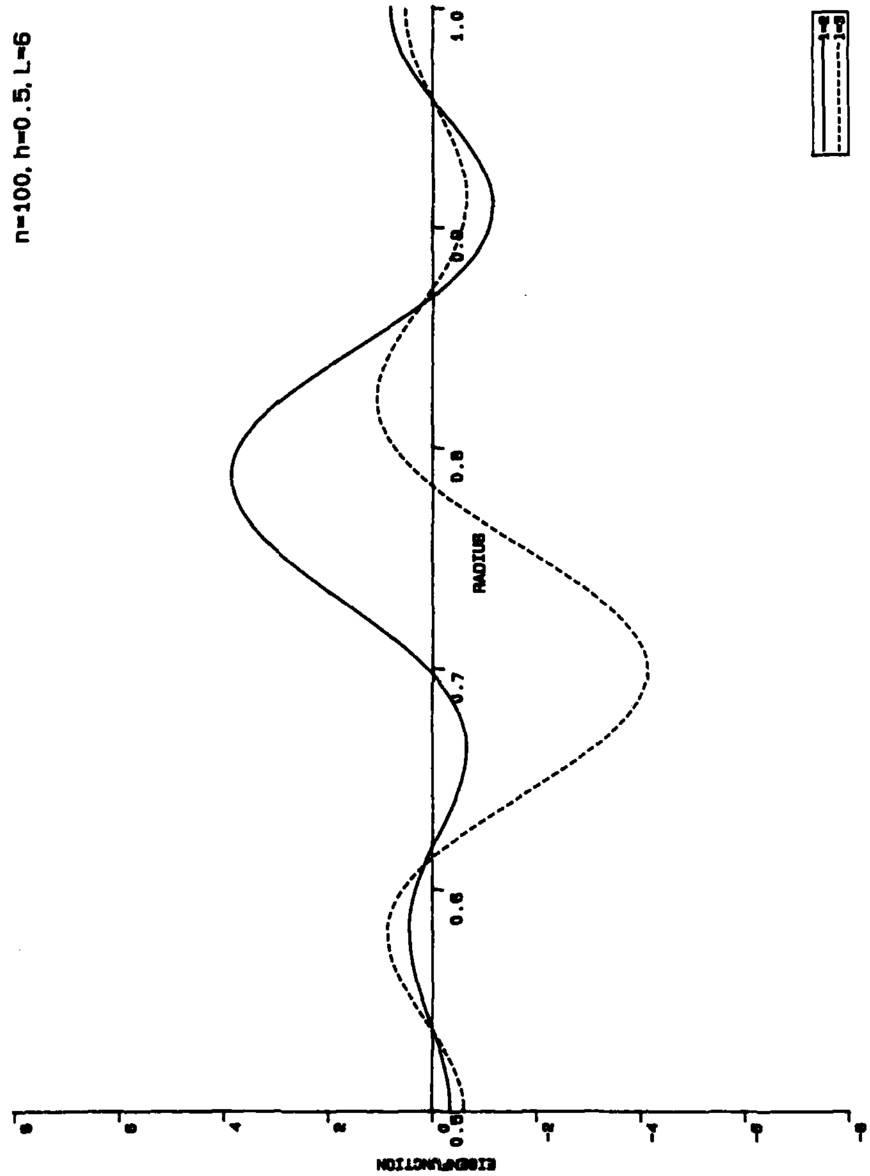
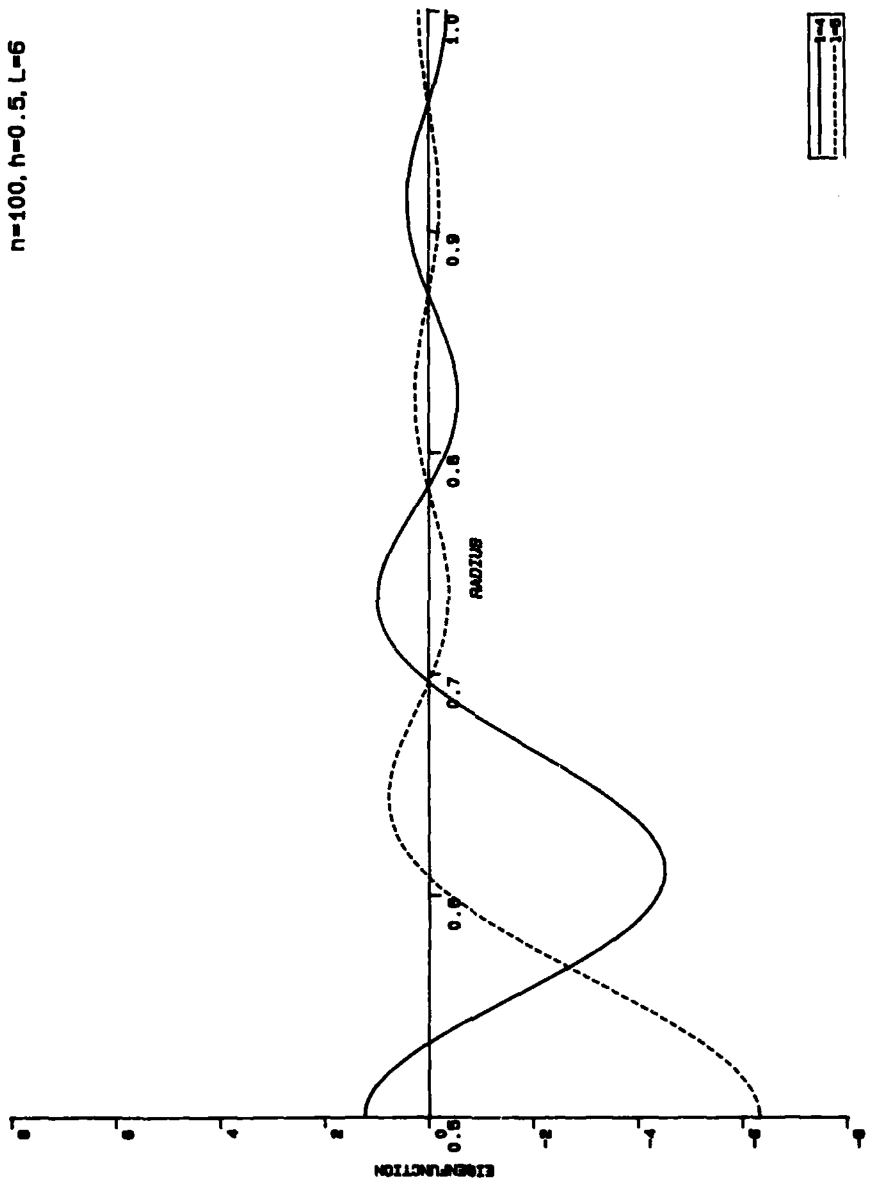


FIG.4



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