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CCS Research Report 591

THE CHANCE-CONSTRAINED CRITICAL PATH
FOR A LARGE CLASS OF DISTRIBUTIONS

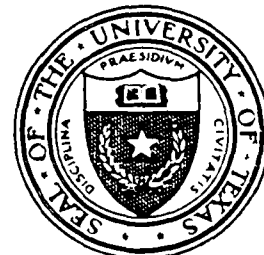
by

A. Charnes
L. Gong

**CENTER FOR
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March, 1988

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ABSTRACT

M. Kress proved for a special class of Location-Scale probability distributions there always exists a probability level for which the Chance Constrained Critical Path (CCCP) remains unchanged for all probabilities greater than or equal to that value. His chance constrained problem has zero-order decision rules and individual chance constraints.

This paper extends his results to most of the common probability distributions.

KEY WORDS

Chance Constrained Programming, (kp)

Chance Constrained Critical Paths

Location-Scale Distributions

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Moshe Kress [1] studied the Chance Constrained Critical Path (CCCP) problem with zero order decision rule and individual chance constraints. He proved that for a class of Location-Scale probability distributions there always exists a probability level for which the CCCP remains unchanged for all probability values greater than or equal to that level. The purpose of this paper is to extend the results of [1] to most of the common probability distributions.

The CCCP problem with zero order decision rule and equal minimal probability level can be formulated as

$$\begin{array}{ll}
 \text{Min } u_m - u_0 & \\
 P(\beta) \quad \text{s.t.} \quad \left\{ \begin{array}{l} P_r \left(\sum_{i=0}^m \epsilon_{ij} u_i \geq t_j \right) \geq \beta \\ j = 1, 2, \dots, n \end{array} \right. &
 \end{array}$$

where u_i 's are the start times of the activities. t_j , $j = 1, 2, \dots, n$, are the durations of activities j 's which are the random variables with marginal distribution function F_j ; ϵ_{ij} is the entry in row i , column j of the node-arc incidence matrix of the activity network.

$$\text{Let } F_j^{-1}(\beta) = \inf \{ t : F_j(t) \geq \beta \}$$

Notice that since the distribution function is right continuous, "inf" can be changed into "min". Now we can write $P(\beta)$ into an equivalent deterministic form.

$$\begin{aligned}
 & \text{Min } u_m - u_0 \\
 P_1(\beta) \quad & \text{s.t.: } \begin{cases} \sum_{i=0}^m \epsilon_{ij} u_i \geq F_j^{-1}(\beta) \\ j = 1, 2, \dots, n \end{cases}
 \end{aligned}$$

The dual problem to $P_1(\beta)$ is

$$\begin{aligned}
 & \text{Max } \sum_{j=1}^n F_j^{-1}(\beta) x_j \\
 D_1(\beta) \quad & \text{s.t.: } \begin{cases} \sum_{j=1}^n \epsilon_{ij} x_j = a_i, \quad i = 0, 1, \dots, m \\ x_j \geq 0, \quad j = 1, 2, \dots, n \end{cases}
 \end{aligned}$$

where $a_0 = -1$, $a_m = 1$, $a_i = 0$, for $i = 1, 2, \dots, m-1$.

Since $D_1(\beta)$ is a pure network problem, for any given $\beta \in (0, 1)$, a basic optimal solution $x^*(\beta) = \{x_1^*(\beta), x_2^*(\beta), \dots, x_n^*(\beta)\}$ of $D_1(\beta)$ has the property $x_j^*(\beta)$ is either 0 or 1, and $\{j : x_j^*(\beta) = 1\}$ forms a critical path for the network of $D_1(\beta)$.

Let $P = \{p_k : k = 1, 2, \dots, K\}$ denote the set of all paths from the source to the sink in the network, and $J_k = \{j : \text{arc } j \text{ is in path } P_k\}$. Then the problem $P(\beta)$ is equivalent to the following problem.

$$\begin{aligned}
 & \text{Max} \sum_{k=1}^K \left(\sum_{j \in J_k} F_j^{-1}(\beta) \right) y_k \\
 \bar{P}(\beta) \quad & \text{s.t.} \quad \begin{cases} \sum_{k=1}^K y_k = 1 \\ y_k \geq 0 \\ k = 1, 2, \dots, K \end{cases}
 \end{aligned}$$

Clearly, if k^* such that

$$\sum_{j \in J_{k^*}} F_j^{-1}(\beta) = \text{Max}_{k \in I} \sum_{j \in J_k} F_j^{-1}(\beta)$$

where $I = \{1, 2, \dots, K\}$,

then $y_{k^*} = 1$, $y_i = 0$ for $i \neq k^*$ is an optimal solution of $\bar{P}(\beta)$, and if we set

$$x_j(\beta) = \begin{cases} 1 & \text{if } j \in J_{k^*} \\ 0 & \text{otherwise} \end{cases}$$

$x^*(\beta) = \{x_1^*(\beta), x_2^*(\beta), x_3^*(\beta), \dots, x_n^*(\beta)\}$ is an optimal solution of problem $D_1(\beta)$.

$$\text{Let } V_k(\beta) = \sum_{j \in J_k} F_j^{-1}(\beta),$$

$$U_{k,l}(\beta) = \sum_{j \in J_k \setminus J_l} F_j^{-1}(\beta),$$

$$H(\beta) = \left\{ k : V_k(\beta) = \text{Max}_{i \in I} V_i(\beta) \right\}.$$

The following theorem gives a sufficient condition under which the CCCP remains unchanged for all probabilities greater than or equal to a level β_0 .

For the distribution function F_j , denote the support set of F_j ($\text{supp}(F_j)$) as the interval

$$[\underline{\gamma}_j, \bar{\gamma}_j]$$

where $\underline{\gamma}_j = \sup \{ \gamma : F_j(\gamma) = 0 \}$

$$\bar{\gamma}_j = \inf \{ \gamma : F_j(\gamma) = 1 \}$$

Theorem: Assume marginal distribution functions $F_j, j = 1, 2, \dots, n$, are continuous and the density functions $f_j > 0$, a.e., in $\text{supp}(F_j)$.

Case 1: $\bar{\gamma}_j < \infty$, for $j = 1, 2, \dots, n$.

If $k^* \in H(1)$, and there exists a $0 \leq \hat{\beta} < 1$ such that

$$U'_{k^*,k}(\beta) \leq U'_{k,k^*}(\beta)$$

for any $\beta \in [\hat{\beta}, 1]$ and $k \in H(1)$,

then there exists a $\beta_0 < 1$ such that for any $\beta \in [\beta_0, 1]$

$$V_{k^*}(\beta) \leq V_k(\beta) \quad \text{for } k \in I$$

Case 2: $\bar{\gamma}_j = +\infty$, for $j = 1, 2, \dots, n$.

(a) If there exists a $0 \leq \hat{\beta} < 1$ such that $k^* \in H(\hat{\beta})$

$$\text{and } U'_{k^*,k}(\beta) \leq U'_{k,k^*}(\beta)$$

for any $\beta \in [\hat{\beta}, 1]$ and $k \in I$.

or,

(b) if k^* is such that

$$\lim_{\beta \rightarrow 1^-} \frac{U'_{k^*,k}(\beta)}{U'_{k,k^*}(\beta)} > 1$$

for $k \in I$

Then, there exists a $\beta_0 < 1$ such that for any $\beta \in [\beta_0, 1]$

$$V_{k^*}(\beta) \geq V_k(\beta) \quad \text{for } k \in I$$

Proof: Case 1.

Since F_j is a continuous monotone distribution function, $f_j = F_j'$ is a Lebesgue measurable function. Since $f_j > 0$ a.e., $1/f_j = (F_j^{-1})'$ is a Lebesgue measurable function. By the condition of theorem defining Case 1, $U'_{k,k^*}(\beta) \geq U'_{k^*,k}(\beta)$ for any $\beta \in [\hat{\beta}, 1]$ and $k \in H(1)$. Therefore, for any $\beta \in [\hat{\beta}, 1]$ and $k \in H(1)$

$$0 \leq \int_{\beta}^1 (U'_{k,k^*}(t) - U'_{k^*,k}(t)) dt$$

$$\begin{aligned}
&= \int_{\beta}^1 \left(\dot{V}_k(t) - \dot{V}_k^*(t) \right) dt \\
&= V_k(1) - V_k^*(1) - V_k(\beta) + V_k^*(\beta)
\end{aligned}$$

By hypothesis $V_k(1) = V_k^*(1)$, hence

$$V_k^*(\beta) \geq V_k(\beta) \quad \text{for any } k \in H(1)$$

If $k \notin H(1)$, then $V_k^*(1) > V_k(1)$

and since $f_j > 0$ a.e. in $\text{supp}(F_j)$, F_j^{-1} is a strictly increasing continuous function and so is the sum of these which is $V_k(\beta)$. Therefore there exists a $\beta_k < 1$ such that for any $\beta \in [\beta_k, 1]$

$$V_k^*(\beta) \geq V_k(\beta).$$

Let $\beta_0 = \max \{ \hat{\beta}, \beta_k \text{ for } k \in I \setminus H(1) \} < 1$

Then for $\beta \in [\beta_0, 1]$, $k \in I$

$$V_k^*(\beta) \geq V_k(\beta)$$

Case 2.a.

For any $\beta \in [\hat{\beta}, 1]$ and $k \in I$

$$\begin{aligned}
0 &\leq \int_{\beta}^{\hat{\beta}} \left(\dot{U}_{k^*,k}(t) - \dot{U}_{k,k^*}(t) \right) dt \\
&= \int_{\beta}^{\hat{\beta}} \left(\dot{V}_{k^*}(t) - \dot{V}_k(t) \right) dt \\
&= V_{k^*}(\hat{\beta}) - V_k(\hat{\beta}) - V_{k^*}(\beta) + V_k(\beta)
\end{aligned}$$

By hypothesis $k^* \in H(\hat{\beta})$, so we have

$$V_{k^*}(\beta) \geq V_k(\beta)$$

Therefore we can choose β_0 at least equal to $\hat{\beta}$.

Case 2.b.

$$\text{Since } \lim_{\beta \rightarrow 1} \frac{U'_{k^*,k}(\beta)}{U'_{k,k^*}(\beta)} > 1$$

there exists a $\hat{\beta}$ such that for any $\beta \in [\hat{\beta}, 1]$

$$\frac{\dot{U}_{k^*,k}(\beta)}{\dot{U}_{k,k^*}(\beta)} \geq 1 + \epsilon$$

that is

$$\dot{U}_{k^*,k}(\beta) - \dot{U}_{k,k^*}(\beta) \geq \epsilon \dot{U}_{k,k^*}(\beta)$$

Hence

$$\begin{aligned} \epsilon \int_{\hat{\beta}}^{\beta} \dot{U}_{k,k^*}(t) dt &\leq \int_{\hat{\beta}}^{\beta} (\dot{U}_{k^*,k}(t) - \dot{U}_{k,k^*}(t)) dt = \int_{\hat{\beta}}^{\beta} (V_{k^*}(t) - V_k(t)) dt \\ &\leq V_{k^*}(\beta) - V_k(\beta) - (V_{k^*}(\hat{\beta}) - V_k(\hat{\beta})) \end{aligned}$$

Notice that if $\bar{\gamma}_j = \infty$ and $f_j > 0$ a.e. in $\text{supp}(f_j)$ there exists a $\beta_0 < 1$ such that

$$\int_{\hat{\beta}}^{\beta_0} \dot{U}_{k,k^*}(t) dt \geq (V_k(\hat{\beta}) - V_{k^*}(\hat{\beta})) / \epsilon$$

So, for any $\beta > \beta_0$,

$$V_{k^*}(\beta) - V_k(\beta) \geq 0$$

Hence we have obtained a β_0 which satisfies our requirement.

Remark 1: From the argument of our proof, it is clear that the theorem is also true when the F_j 's satisfy the assumed condition only in a interval of $[\alpha, \bar{\gamma}]$, where α is a certain real value less than $\bar{\gamma}$.

Remark 2: When the set of random variables t_i , $i = 1, 2, \dots, n$ contains both finite support and infinite support distribution functions, then we can treat it as we did in the infinite support case.

We are now going to show that the location-scale distribution family which Kress [1] studied satisfies the hypotheses of our theorem.

The F_j , $j = 1, 2, \dots, n$, for the Kress family are location-scale distributions with the same generating distribution $\emptyset$, that is,

$$F_j(x) = \emptyset\left(\frac{x - \mu_j}{\sigma_j}\right) \text{ for } j = 1, 2, \dots, n. \text{ where } \mu_j, \sigma_j \text{ are the location and scale parameters of } F_j$$

respectively.

The inverse of a location-scale distribution of a non-negative random variable is

$$F_j^{-1}(\beta) = \sigma_j \emptyset^{-1}(\beta) + \mu_j$$

where $\emptyset^{-1}(\beta) \geq 0$ is the β th fractile of the generating distribution $\emptyset$.

$$\text{Let } S_k = \sum_{j \in J_k} \sigma_j \geq 0, \quad M_k = \sum_{j \in J_k} \mu_j \geq 0.$$

$$\text{Then } V_k(\beta) = S_k \emptyset^{-1}(\beta) + M_k$$

and the following corollary is obtained.

Corollary: Suppose F_j , $j = 1, 2, \dots, n$, are such location-scale distributions, $\emptyset$ a non-negative continuous distribution.

Case 1. Let $\bar{\gamma} = \inf\{\gamma: \emptyset(\gamma) = 1\} < \infty$

If $k^* \in I$ such that $k^* \in H(1)$ and

$$M_{k^*} = \max_{j \in H(1)} M_j$$

Then, for any $\beta \in [0, 1]$, $k \in H(1)$

$$U_{k^*,k}(\beta) \geq U_{k,k^*}(\beta)$$

Case 2. Let $\bar{\gamma} = +\infty$

If $k^* \in I$ such that

$$S_{k^*} \geq S_k, \text{ for all } k \in I \text{ and } M_{k^*} \geq M_k, \text{ for all } k \in H, \text{ where } H = \left\{ i: S_i = \max_{j \in I} S_j \right\},$$

then there exists a $\hat{\beta}$ such that for $\beta \in [\hat{\beta}, 1]$, $k \in H(\hat{\beta})$ and $U_{k^*,k}(\beta) \geq U_{k,k^*}(\beta)$ for $k \in I$.

Proof: Case 1. Let $k \in H(1)$, then since

$$M_{k^*} \geq M_k \text{ and } V_{k^*}(1) = S_{k^*} \emptyset^{-1}(1) + M_{k^*} = S_k \emptyset^{-1}(1) + M_k = V_k(1),$$

we have $S_{k^*} \leq S_k$.

Therefore $V_{k^*}(\beta) = S_{k^*}(\varnothing^{-1}(\beta)) \leq S_k(\varnothing^{-1}(\beta)) = V_k(\beta)$

That is $U_{k^*,k}(\beta) \leq U_{k,k^*}(\beta)$ for $k \in H(1)$

Case 2: By hypothesis, we know if $k \in H$

$$V_{k^*}(\beta) = S_{k^*}\varnothing^{-1}(\beta) + M_{k^*} \geq S_k\varnothing^{-1}(\beta) + M_k = V_k(\beta)$$

that is, $U_{k^*,k}(\beta) \geq U_{k,k^*}(\beta)$ (*)

If $k \in \bar{H}$, then $S_{k^*} > S_k$. Let $\mu = S_{k^*} - S_k > 0$.

Since $\bar{\gamma} = \infty$, there is a $\hat{\beta}$ such that for any $\beta \in [\hat{\beta}, 1]$, $\varnothing^{-1}(\beta) \geq \frac{M_k - M_{k^*}}{\mu}$

$$\begin{aligned} \text{Hence, } V_{k^*}(\beta) &= S_{k^*}\varnothing^{-1}(\beta) + M_{k^*} \\ &= S_k\varnothing^{-1}(\beta) + \mu\varnothing^{-1}(\beta) + M_{k^*} \\ &\geq S_k\varnothing^{-1}(\beta) + M_k = V_k(\beta) \end{aligned} \quad (**)$$

Combining (*) and (**), for $\beta \in [\hat{\beta}, 1]$

$$U_{k^*,k}(\beta) \geq U_{k,k^*}(\beta)$$

On the other hand, because $S_{k^*} \geq S_k$ for $k \in I$, we can directly conclude

$$V_{k^*}(\beta) = S_{k^*}(\varnothing^{-1}(\beta)) \geq S_k(\varnothing^{-1}(\beta)) = V_k(\beta)$$

which is equivalent to

$$U_{k^*,k}(\beta) \geq U_{k,k^*}(\beta)$$

Q.E.D.

Example 1:

Consider the project network depicted in Figure 1.

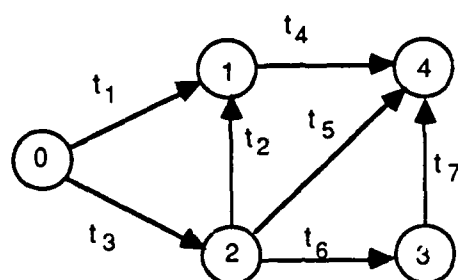


Figure 1

Let $t_1 = t_2$ have distribution $U(0, 1)$

$$\begin{array}{ll}
 t_3 \sim \text{density function} & f_3(x) = \begin{cases} \frac{1}{2}(2-x) & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases} \\
 t_4 = t_5 \sim \text{density function} & f_4(x) = f_5(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \\
 t_6 = t_7 \sim \text{density function} & f_6(x) = f_7(x) = \begin{cases} 3x^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}
 \end{array}$$

Then $\bar{\gamma}_j \leq 2 < \infty$

By solving the pure network problem $D_1(1)$, we get the optimal critical paths $J_1 = \{0, 2, 1, 4\}$,

$J_2 = \{0, 2, 3, 4\}$. See Figure 2.

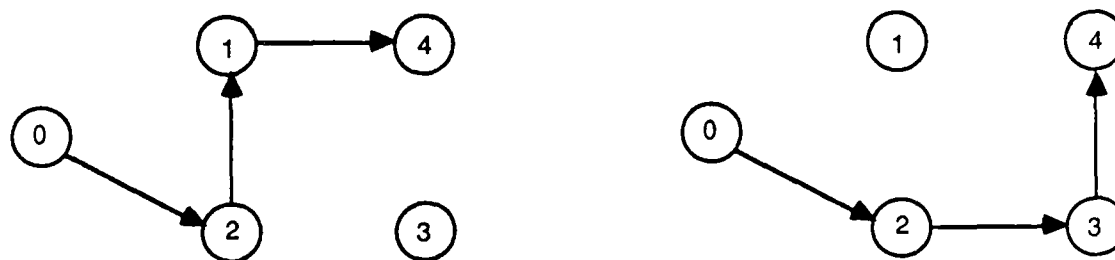


Figure 2

Since $U'_{1,2}(1) = \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$
 and $U'_{2,1}(1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

the critical path J_2 is what we are seeking. By use of the optimal solution corresponding to J_2 in $D_1(\beta)$ and computing its reduced cost, we can get the minimal probability level $\beta_0 = 0$, for which CCCP remains unchanged for all probability values greater than or equal to it.

Example 2:

Consider the project network depicted in Figure 1 but let $t_i \sim (1 - \alpha_i) F(x) + \alpha_i G(x)$ where $F(x)$, $G(x)$ are the distribution functions of $N(0,1)$ and $\text{Exp}(1)$ respectively and $\alpha_i = \frac{1}{i+1}$, $i = 1, 2, \dots, 7$.

Since the possible critical paths are $J_1 = \{0, 2, 1, 4\}$, $J_2 = \{0, 2, 3, 4\}$, $J_3 = \{0, 2, 4\}$, $J_4 = \{0, 1, 4\}$, we have

$$\lim_{\beta \rightarrow 1^-} \frac{U_{2,k}(\beta)}{U_{k,2}(\beta)} > 1 \quad \text{for } k = 1, 3, 5$$

So J_2 is the desired critical path.

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