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Washington, DC 20375-5000

AD-A196 230

A Note on Deriving a Result in Decision Theory

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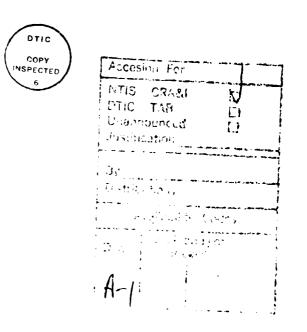*Newark, New Jersey 07103-2757*

June 9, 1988

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REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED			1b RESTRICTIVE MARKINGS			
2a. SECURITY CLASSIFICATION AUTHORITY			3 DISTRIBUTION / AVAILABILITY OF REPORT Approved for public release; distribution unlimited.			
2b. DECLASSIFICATION / DOWNGRADING SCHEDULE						
4. PERFORMING ORGANIZATION REPORT NUMBER(S) NRL Memorandum Report 6177			5. MONITORING ORGANIZATION REPORT NUMBER(S)			
6a. NAME OF PERFORMING ORGANIZATION Naval Research Laboratory		6b. OFFICE SYMBOL (If applicable) Code 2303	7a. NAME OF MONITORING ORGANIZATION			
6c. ADDRESS (City, State, and ZIP Code) Washington, DC 20375-5000			7b ADDRESS (City, State, and ZIP Code)			
8a. NAME OF FUNDING / SPONSORING ORGANIZATION		8b. OFFICE SYMBOL (If applicable)	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER			
8c. ADDRESS (City, State, and ZIP Code)			10 SOURCE OF FUNDING NUMBERS			

			PROGRAM ELEMENT NO	PROJECT NO	TASK NO	WORK UNIT ACCESSION NO

11. TITLE (Include Security Classification)

A Note on Deriving a Result in Decision Theory

12. PERSONAL AUTHOR(S)
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13a. TYPE OF REPORT Final	13b TIME COVERED FROM _____ TO _____	14. DATE OF REPORT (Year, Month, Day) 1988 June 9	15 PAGE COUNT 6

16 SUPPLEMENTARY NOTATION

17	COSATI CODES		18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Decision theory Mathematical statistics

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

This note provides an alternate derivation of a well-known problem in decision theory, which deals with finding the minimum of a certain expectation, i.e., $E|X - c|$. Textbooks generally present this problem as an exercise to verify that the median does, in fact, yield a minimum. The derivation presented here directly finds the minimum, and yields asymptotic properties of the expectation.

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21 ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED	
22a NAME OF RESPONSIBLE INDIVIDUAL Allen R. Miller	22b TELEPHONE (Include Area Code) (20) 767-2215	22c OFFICE SYMBOL Code 2303

DD Form 1473, JUN 86   Previous editions are obsolete.

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A NOTE ON DERIVING A RESULT IN DECISION THEORY

INTRODUCTION

A number of textbooks in mathematical statistics [see, e.g., Ref. 1] include an exercise to verify that if X is a continuous random variable with finite expectation $E(X)$ and density function $f(x)$, then $E|X - c|$ is minimized for $c = m$, where m is the median of f. This result is used in statistical decision theory [see, e.g., Ref. 2].

The usual way authors show this is to first show that

$$E|X - c| = E|X - m| + 2 \int_m^c (c - x) f(x) \, dx \; .$$

Since the term containing the integral in the above equation is nonnegative, one concludes that $E|X - m| \leq E|X - c|$, and thus $E|X - c|$ is minimized at $c = m$.

MINIMIZING THE EXPECTED VALUE

The reader might be interested in knowing how one determines beforehand that the median is a plausible candidate for the global minimum. The exercise above asks to show that the median does, in fact, yield a minimum. However, one may not see readily why it is reasonable to choose the median in the first place. This note addresses this issue. We will show directly that the median is the required value which yields the global minimum for $E|X - c|$. Our derivation will also show the asymptotic nature of this expectation.

Defining $G(c) \equiv E|X - c|$, we compute $G'(c)$ and show that it vanishes at $c = m$. We have,

$$G(c) = \int_{-\infty}^{\infty} |x - c| f(x) dx$$

$$= \int_{-\infty}^{c} (c - x) f(x) dx + \int_c^{\infty} (x - c) f(x) dx$$

$$= c \left[\int_{-\infty}^{c} f(x) dx - \int_c^{\infty} f(x) dx \right]$$

$$+ \int_c^{\infty} x f(x) dx - \int_{-\infty}^{c} x f(x) dx \; .$$

Now differentiating $G(c)$ with respect to c, we arrive at

$$G'(c) = \int_{-\infty}^{c} f(x) dx - \int_c^{\infty} f(x) dx$$

$$= 2F(c) - 1 \; .$$

where $F(x)$ is the cumulative density function. Finally, we see that $G'(c) = 0$ if and only if $c = m$. Thus we have found a possible local minimum at $c = m$.

Now it is easy to show that we have the global minimum at the median. Suppose $c > m$. Then $G'(c) \geq 0$. Further, for $c < m$, we have $G'(c) \leq 0$. Thus, we conclude that $G(c)$ assumes an absolute or global minimum at $c = m$.

ASYMPTOTES FOR THE EXPECTED VALUE

It is interesting to note that as $c \to \pm \infty$, then $G'(c) \to \pm 1$. Also, we note that for $c \to \infty$,

$$G(c) \sim c - E(X) ,$$

and for $c \to -\infty$,

$$G(c) \sim -c + E(X) ,$$

where "$\sim$" means "asymptotically equal to". Equivalently, we can write

$$\lim_{c \to \pm \infty} \frac{E(X) \pm G(c)}{c} = 1 .$$

Further, we remark that $G(c)$ always lies above the asymptotes.

REFERENCES

1. Cramér, Harold (1946), *Mathematical Methods of Statistics*, Princeton University Press, Princeton, N.J.

2. DeGroot, Morris H. (1970), *Optimal Statistical Decisions*, McGraw-Hill, New York.