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ABSTRACT

This work proposes an approach for the control of a general class of discrete-time nonlinear stochastic systems. The system model incorporates a deterministic linear portion together with a nonlinear function of the state and/or control vectors in combination with a white noise vector, where no Gaussian assumption is made.

Under certain conditions imposed on the statistics of the additive nonlinear stochastic term, and assuming perfect state information, the optimal control, which minimizes a quadratic performance index subject to the nonlinear system constraint, is shown to be a linear function of the state vector. This work also shows that for certain infinite horizon problems, an uncertainty threshold can be found such that the designer can, a priori, put an upper bound on the allowable noise covariance to obtain a bounded optimal constant feedback control.

GENERAL RESULTS IN OPTIMAL CONTROL OF DISCRETE-TIME NONLINEAR STOCHASTIC SYSTEMS

A thesis submitted in partial fulfillment
of the requirements for the degree of
Master of Science in Electrical Engineering

By

MARK STEVEN CIANCETTA, B.S.E.E.
University of Arkansas, 1986

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This thesis is approved for
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DEDICATION

This thesis is dedicated to my wife Chris, my daughter Erin, and my son Jesse. I will be eternally grateful for their sacrifices throughout the course of this work.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Optimal control of stochastic systems has far reaching applications, ranging from the control of space satellites [1] to the control of economic systems [2]. It is our aim to show general results applicable to many different systems. Therefore no attempt will be made to specialize our results to any one specific field.

This chapter will present our objectives, as well as some of the work previously done in this area. It will also present an overview of our approach and a brief description of the system model used throughout the sequel.

1.1 Objectives

This work proposes an approach for the control of a general class of discrete-time nonlinear stochastic systems. The system model incorporates a deterministic linear portion together with a nonlinear function of the state and/or control vectors in combination with a white noise vector, where no Gaussian assumption is made.

Under certain conditions imposed on the statistics of the additive nonlinear stochastic term, and assuming perfect state information, the optimal control, which minimizes a quadratic performance index subject to the nonlinear system constraint, is shown to be a linear function of the state vector. This work will also show that for certain infinite horizon problems, an uncertainty threshold can be found such that the designer can, a priori, put an upper bound on the allowable noise covariance to obtain a bounded optimal constant feedback control.

1.2 Previous Work

The overall problem of optimal control of discrete-time nonlinear stochastic systems is quite complex. Therefore, much previous work has been done on specialized cases.

For instance, in the design of control systems, the exact values of system parameters are not often precisely known. These parameter perturbations can be caused by environmental effects, operator error, equipment aging etc. Typically these parameter variations are treated as additive stochastic sequences, but in some cases of large variation this modeling is not accurate. In these instances, one is left with modeling these uncertainties as multiplicative noise. [3] - [8] investigate the optimal control and stability of these linear discrete-time systems with multiplicative noise. These systems are generally referred to as bilinear stochastic systems, systems with multiplicative noise, or stochastic parameter systems. An excellent review of the continuous-time counterpart of this class of systems is given in [9].

[10] - [12] deal with the estimation problems related to stochastic parameter systems. These problems are associated with phenomena such as fading or reflection of a transmitted signal from the ionosphere, and certain situations involving sampling, gating or amplitude modulation.

The more general case of discrete-time nonlinear stochastic systems is investigated in [13], which was used as a basis for much of our work. [14] and [15] suggest several novel control schemes for these systems.

[16] - [18] address the steady state characteristics of linear stochastic parameter systems. [16] derives stability conditions for the matrix Riccati equation arising in the optimal control of linear systems with random gain. [17] and [18] derive an uncertainty threshold for the existence of the infinite-horizon solution to the optimal control of linear discrete-time stochastic parameter systems. This work is generalized to nonlinear stochastic systems in chapters 3 and 4.

1.3 Overview of Our Approach

Our intent is to design the optimal linear feedback controller for a general stochastic system configuration. We will endeavor to keep the system as general as possible so that the resulting controller will be suited for as wide a range of applications as possible. In the remainder of this chapter we describe this general system and then give several examples of both linear and nonlinear systems that it covers. The associated optimal finite-horizon controller for this system will be presented in chapter 2.

The general finite-horizon control solution, although quite interesting academically, is not always the most practical approach. Quite often it is advantageous to utilize a constant feedback controller. This prompts us to investigate the *infinite-horizon* solution. Our concern here is does the infinite-horizon solution exist? And, is there a quantifiable means to predetermine this existence? Through steady-state analysis of a Riccati-like equation in Chapters 3 and 4, threshold conditions are developed for both the scalar and the multivariable cases, such that the designer can, a priori, based on the covariance of the stochastic parameters, guarantee the existence of a steady-state constant feedback controller. Computer simulations follow both developments to verify these threshold conditions.

1.4 General System Description

The optimal control problem is to minimize the following quadratic performance index

$$E\left\{\frac{1}{2}x_N^T S_N x_N + \sum_{k=0}^{N-1}\left(\frac{1}{2}x_k^T Q_k x_k + x_k^T T_k u_k + \frac{1}{2}u_k^T R_k u_k\right)\right\} \tag{1.1}$$

where N is the final time, $E\{.\}$ denotes the expectation operator, S_N and Q_k are positive semi-definite symmetric matrices, R_k is a positive definite symmetric matrix, and

$$\begin{bmatrix} Q_k & T_k \\ T_k^T & R_k \end{bmatrix} \geq 0$$

subject to the system constraint

$$x_{k+1} = A_k x_k + B_k u_k + f_k(x_k, u_k, \alpha_k) \tag{1.2}$$

with x_0 given, and

$$x_k \in R^n,\ u_k \in R^m,\ \alpha_k \in R^q,\ f_k : R^n \times R^m \times R^q \rightarrow R^n$$

The noise sequence α_k is assumed to be independently distributed in time, but not necessarily Gaussian, and $f_k(x_k, u_k, \alpha_k)$ has the following statistical description

$$E\{f_k(x_k,u_k,\alpha_k)\mid x_k u_k\} = 0 \qquad \forall\ x_k \in R^n,\ u_k \in R^m,\ k = 0,\dots,N-1 \tag{1.3}$$

and

$$F_k(x_k,u_k) \triangleq E\{f_k(x_k,u_k,\alpha_k) f_k^T(x_k,u_k,\alpha_k)\mid x_k u_k\} \tag{1.4}$$

is a quadratic function of x_k and u_k having the following form

$$F_k(x_k,u_k) = P_k^0 + \sum_{i=1}^{n'} P_k^i \left(\frac{1}{2} x_k^T W_k^i x_k + x_k^T N_k^i u_k + \frac{1}{2} u_k^T M_k^i u_k\right) \tag{1.5}$$

where

$$P_k^i,\ W_k^i \text{ and } M_k^i \text{ are symmetric and } n' = n\frac{(n+1)}{2}$$

and

$$F_k(x_k,u_k) \ge 0 \qquad \forall\ x_k \in R^n,\ u_k \in R^m$$

Note that the above condition is not at all restrictive as it is necessary in order for equation (1.5) to be a proper covariance representation.

1.5 Examples of Linear Stochastic Systems That Fit Our Description

The above general problem description encompasses many known standard systems. For instance, one of the most common examples is a linear system with an additive noise term. Let

$$f_k(x_k, u_k, \alpha_k) = \alpha_k$$

thus reducing equations (1.2) - (1.5) to the following

$$x_{k+1} = A_k x_k + B_k u_k + \alpha_k \tag{1.6}$$

with

$$F_k(x_k, u_k) = E(\alpha_k \alpha_k^T) = P_k^0 \tag{1.7}$$

Similarly (1.2) - (1.5) can be used to represent linear stochastic parameter systems. The general form is

$$x_{k+1} = A_k(\omega) x_k + B_k(\omega) u_k \tag{1.8}$$

where $A_k(\omega)$ and $B_k(\omega)$ are matrices having elements which are white noises possibly correlated with each other at each time instant k. This modeling evolves naturally in sampled versions of diffusion processes associated with nuclear fission and heat transfer, as well as in the migration of population, and the growth of biological cells, etc. This is also the model typically used to describe the uncertainties in the parameters connected with economic predictions [2]. One can also arrive at (1.8) by:

1. Uniformly sampling a continuous-time stochastic parameter system.

2. Randomly sampling a continuous-time deterministic system.

3. Modeling a deterministic system with an additive noise having random correlation characteristics.

The first two are self explanatory and the third is shown below. Suppose

$$x_{k+1} = A_k x_k + B_k u_k + C_k \alpha_k \tag{1.9}$$

with the noise term α_k possessing the dynamics

$$\alpha_{k+1} = D_k(\omega)\alpha_k + \beta_k \tag{1.10}$$

where $D_k(\omega)$ is a matrix with random elements, and β_k is additive white noise. Then by enlarging the state space, we obtain

$$\begin{bmatrix} x_{k+1} \\ \alpha_{k+1} \end{bmatrix} = \begin{bmatrix} A_k & C_k \\ 0 & D_k(\omega) \end{bmatrix} \begin{bmatrix} x_k \\ \alpha_k \end{bmatrix} + \begin{bmatrix} B_k \\ 0 \end{bmatrix} u_k + \begin{bmatrix} 0 \\ \beta_k \end{bmatrix} \tag{1.11}$$

1.6 Examples of Nonlinear Stochastic Systems

The following are some novel special cases of the general system described by equations (1.2) - (1.5).

1. Norm Dependent Random Vector:

Let

$$f_k(x_k, u_k, \alpha_k) = \alpha_k \sqrt{\frac{1}{2} x_k^T D_1 x_k + x_k^T D_2 u_k + \frac{1}{2} u_k^T D_3 u_k} \qquad (1.12)$$

and

$$E\{\alpha_k \alpha_k^T\} = P_k \quad , \quad E\{\alpha_k\} = 0 \qquad (1.13)$$

Then

$$F_k(x_k, u_k) = P_k \left[\frac{1}{2} x_k^T D_1 x_k + x_k^T D_2 u_k + \frac{1}{2} u_k^T D_3 u_k \right] \qquad (1.14)$$

with

$$D_3 > 0 \text{ and } \left[D_1 - D_2 D_3^{-1} D_2^T \right] \geq 0$$

Note that if $D_1 = I$, $D_2 = D_3 = 0$, then

$$f_k(x_k, u_k, \alpha_k) = \alpha_k \|x_k\| \qquad (1.15)$$

In the following examples only the system will be given, as the covariance matrix can be determined by following the above procedure.

2. Random Vector Dependent Upon the Sign of a Scalar Nonlinear Function of x_k and u_k:

Let

$$f_k(x_k, u_k, \alpha_k) = \operatorname{sgn}[\phi(x_k, u_k)]\alpha_k \qquad (1.16)$$

where $\phi: R^n \times R^m \to R^1$, and the statistics of α_k are as in equation (1.13).

3. Random Vector Dependent Upon the Absolute Value of a Linear Combination of x_k and u_k

Let

$$f_k(x_k, u_k, \alpha_k) = \alpha_k |\beta_k^T x_k + \gamma_k^T u_k| \qquad (1.17)$$

Again the statistics of α_k are given in (1.13).

4. Random Vector Dependent Upon the Norm of x_k and the Absolute Value of Components of x_k

Let

$$f_k(x_k, u_k, \alpha_k) = \alpha_k^0 + \sum_{i=1}^{n} \alpha_k^i |x_k^i| \qquad (1.18)$$

where the n vectors $\alpha_k^0, \alpha_k^1, \ldots, \alpha_k^n$ are uncorrelated and x_k^i is the i^{th} component of x_k.

It is worth noting that the above examples are genuinely nonlinear, and no assumption has been made on the type of probability distribution of a_t.

CHAPTER 2

FINITE-HORIZON OPTIMAL CONTROL

This chapter presents the optimal solution to the general discrete-time nonlinear stochastic system outlined in Chapter 1 equations (1.1) - (1.5). Our approach is to merely state the solution now, and then in section 2.4 use stochastic dynamic programming to prove that this solution is in fact optimal. Rather than stating the solution directly, we will first try to gain insight by examining known simpler cases.

2.1 Deterministic Case

The deterministic case is well documented in most texts on optimal control. [19] is one such reference. Once again the performance index is a quadratic function, but there is no need to take the expected value, due to the deterministic nature of the problem

$$\frac{1}{2}x_N^T S_N x_N + \frac{1}{2}\sum_{k=0}^{N-1}\left(x_k^T Q_k x_k + x_k^T T_k u_k + u_k^T R_k u_k\right) \qquad (2.1)$$

where S_N and Q_k are positive semi-definite symmetric matrices, R_k is a positive definite symmetric matrix, and

$$\begin{bmatrix} Q_k & T_k \\ T_k^T & R_k \end{bmatrix} \geq 0$$

The system model is

$$x_{k+1} = A_k x_k + B_k u_k \tag{2.2}$$

For this system the optimal control is known to be

$$u_k = -K_k x_k \tag{2.3}$$

where K_k is called the Kalman gain, which is given by

$$K_k = \left[B_k^T S_{k+1} B_k + R_k\right]^{-1}\left[B_k^T S_{k+1} A_k + T^T\right] \tag{2.4}$$

S_k is the solution backward in time, from S_N, of the following Riccati equation

$$S_k = A_k^T S_{k+1} A_k - \left[A_k^T S_{k+1} B_k + T_k\right]\left[B_k^T S_{k+1} B_k + R_k\right]^{-1}\left[B_k^T S_{k+1} A_k + T_k^T\right] + Q_k \tag{2.5}$$

And the optimal cost is given by

$$J_i = \frac{1}{2} x_i^T S_i x_i \tag{2.6}$$

2.2 <u>Stochastic</u> <u>Parameter</u> <u>Case</u>

References [4] and [18] make use of the following system description. We find this case interesting because it is very similar to the general form to be proposed in section 2.3. Also, much of the derivation, although not presented here, is the same. We minimize the following performance index

$$E\left\{\frac{1}{2}x_N^T S_N x_N + \sum_{k=0}^{N-1}\left[\frac{1}{2}x_k^T Q_k x_k + x_k^T T_k u_k + \frac{1}{2}u_k^T R_k u_k\right]\right\} \tag{2.7}$$

where S_N and Q_k are positive semi-definite symmetric matrices, R_k is a positive definite symmetric matrix, and

$$\begin{bmatrix} Q_k & T_k \\ T_k^T & R_k \end{bmatrix} \geq 0$$

subject to the following stochastic system model

$$x_{k+1} = \alpha_k A_k x_k + \delta_k B_k u_k \tag{2.8}$$

where α_k and δ_k are scalar Gaussian white random sequences with the following known stationary statistics

$$E\{\alpha_k\} = \bar{\alpha}, \quad E\{(\alpha_k - \bar{\alpha})^2\} = \Gamma \tag{2.9}$$

$$E\{\delta_k\} = \bar{\delta}, \quad E\{(\delta_k - \bar{\delta})^2\} = \Delta \tag{2.10}$$

$$E\{(\alpha_k - \bar{\alpha})(\delta_k - \bar{\delta})\} = \Lambda \tag{2.11}$$

The form of the optimal control is once again

$$u_k = -K_k x_k \tag{2.12}$$

But K_k is now given by

$$K_k = \left[(\bar{\delta}^2 + \Delta)B_k^T S_{k+1} B_k + R_k\right]^{-1}\left[(\bar{\alpha}\bar{\delta} + \Lambda)B_k^T S_{k+1} A_k + T_k^T\right] \quad (2.13)$$

and S_k is given by

$$S_k = \left(\bar{\alpha}^2 + \Gamma\right)A_k^T S_{k+1} A_k + Q_k +$$
$$-\left[(\bar{\alpha}\bar{\delta} + \Lambda)A_k^T S_{k+1} B_k + T_k\right]\left[(\bar{\delta}^2 + \Delta)B_k^T S_{k+1} B_k + R_k\right]^{-1}\left[(\bar{\alpha}\bar{\delta} + \Lambda)B_k^T S_{k+1} A_k + T_k^T\right]$$
$$(2.14)$$

Notice the similarity between equations (2.4) and (2.13) and between equations (2.5) and (2.14). Also the expression for the optimal cost remains unchanged.

$$J_i = \frac{1}{2} x_i^T S_i x_i \quad (2.15)$$

2.3 General Nonlinear Stochastic Case

The optimal solution to the system described in section 1.4 is as follows. The optimal control has the same feedback form

$$u_k = -K_k x_k \quad (2.16)$$

but K_k is now given by

$$K_k = \left[R_k + B_k^T S_{k+1} B_k + \bar{M}_k\right]^{-1}\left[T_k^T + B_k^T S_{k+1} A_k + \bar{N}_k^T\right] \quad (2.17)$$

S_k is given by

$$S_k = Q_k + A_k^T S_{k+1} A_k + \bar{W}_k - K_k^T\left[R_k + B_k^T S_{k+1} B_k + \bar{M}_k\right]K_k \quad (2.18)$$

and the optimal cost is given by

$$J_i = \frac{1}{2} x_i^T S_i x_i + e_i \tag{2.19}$$

where S_N is given, $e_N = 0$ and

$$e_k = \frac{1}{2} \text{ tr } \left(S_{k+1} P_k^0 \right) + e_{k+1} \tag{2.20}$$

$$\bar{W}_k = \frac{1}{2} \sum_{i=1}^{n'} \left[\text{ tr } \left(S_{k+1} P_k^i \right) W_k^i \right] \tag{2.21}$$

$$\bar{N}_k = \frac{1}{2} \sum_{i=1}^{n'} \left[\text{ tr } \left(S_{k+1} P_k^i \right) N_k^i \right] \tag{2.22}$$

$$\bar{M}_k = \frac{1}{2} \sum_{i=1}^{n'} \left[\text{ tr } \left(S_{k+1} P_k^i \right) M_k^i \right] \tag{2.23}$$

2.4 Proof of the General Result

The proof of the general result can be broken down into three main steps. First, stochastic dynamic programming or Bellman's principle of optimality [19] will be introduced. Then, using this concept, we will generate a form of the Bellman functional equation for the optimal cost to go from time k to finish time N. Finally, induction will be used to verify a proposed solution to this functional equation. The solution for the general nonlinear stochastic case, equations (2.16) through (2.23), is a result of this induction step.

i) Stochastic Dynamic Programming

With dynamic programming, we are not concerned with how we came to be in the present state, but are only concerned with optimizing all future decisions, which is called Bellman's principle of optimality [19]. If this procedure were carried out iteratively, the result would be an overall optimal control sequence. With this thought in mind, let us examine equation (1.1), which is repeated here for convenience

$$E\left\{\frac{1}{2}x_N^T S_N x_N + \sum_{k=0}^{N-1}\left(\frac{1}{2}x_k^T Q_k x_k + x_k^T T_k u_k + \frac{1}{2}u_k^T R_k u_k\right)\right\} \qquad (1.1)$$

Equation (1.1) may be written in two parts (notice the temporary change in subscripts).

$$E\left\{\sum_{i=0}^{k-1}\left(\frac{1}{2}x_i^T Q_i x_i + x_i^T T_i u_i + \frac{1}{2}u_i^T R_i u_i\right)\right\} +$$

$$+ E\left\{\frac{1}{2}x_N^T S_N x_N + \sum_{i=k}^{N-1}\left(\frac{1}{2}x_i^T Q_i x_i + x_i^T T_i u_i + \frac{1}{2}u_i^T R_i u_i\right)\right\} \qquad (2.24)$$

We are trying to minimize equation (2.24) by appropriately choosing the control sequence u_k. The first term of equation (2.24) does not depend on u_k, u_{k+1}, u_{k+2}, ... , u_{N-1}. Therefore, applying Bellman's principle of optimality, it is sufficient to minimize only the second term over u_k, u_{k+1}, u_{k+2}, ... , u_{N-1}.

ii) The Bellman Equation

In analyzing the second part of equation (2.24), the following properties prove useful and can be found in [21]. Here, Y is an arbitrary function of x.

$$E\{Y\} = E\{E[Y \mid x]\}$$

where the right hand side of the above equation is the expected value of the conditional expectation of Y given x. Also

$$\min E\{Y\} = E\{\min Y\}$$

putting these together

$$\min E\{Y\} = E\{\min E[Y \mid x]\}$$

Applying all this to the right hand side of equation (2.25) we have

$$\min E\left\{\frac{1}{2}x_N^T S_N x_N + \sum_{i=k}^{N-1}\left(\frac{1}{2}x_i^T Q_i x_i + x_i^T T_i u_i + \frac{1}{2}u_i^T R_i u_i\right)\right\} = E\{J_k(x_k)\}$$

(2.25)

where

$$J_k(x_k) = \min_{u_k \dots u_{N-1}} E\left\{\frac{1}{2}x_N^T S_N x_N + \sum_{i=k}^{N-1}\left(\frac{1}{2}x_i^T Q_i x_i + x_i^T T_i u_i + \frac{1}{2}u_i^T R_i u_i\right) \mid x_k\right\} \tag{2.26}$$

Representing $J_k(x_k)$ as a summation of minimizations over u_k, u_{k+1}, u_{k+2}, ... , u_{N-1}, we have

$$\begin{aligned} J_k(x_k) = \min_{u_k} E\Big\{&\frac{1}{2}x_k^T Q_k x_k + x_k^T T_k u_k + \frac{1}{2}u_k^T R_k u_k + \\ &+ \min_{u_{k+1}} E\Big[\frac{1}{2}x_{k+1}^T Q_{k+1} x_{k+1} + x_{k+1}^T T_{k+1} u_{k+1} + \frac{1}{2}u_{k+1}^T R_{k+1} u_{k+1} + \\ &+ \min_{u_{k+2}} E\Big[\frac{1}{2}x_{k+2}^T Q_{k+2} x_{k+2} + x_{k+2}^T T_{k+2} u_{k+2} + \frac{1}{2}u_{k+2}^T R_{k+2} u_{k+2} + \\ &+ \min_{u_{k+3}} E \dots \mid x_{k+2}\Big] \mid x_{k+1}\Big] \mid x_k\Big\} \end{aligned} \tag{2.27}$$

One can easily see that, excluding the first term, equation (2.27) is simply $J_{k+1}(x_{k+1})$ by our definition of $J_k(x_k)$. Substituting $J_{k+1}(x_{k+1})$ into (2.27) results in

$$J_k(x_k) = \min_{u_k}\left[E\left\{\frac{1}{2}x_k^T Q_k x_k + x_k^T T_k u_k + \frac{1}{2}u_k^T R_k u_k \mid x_k\right\} + E\left\{J_{k+1}(x_{k+1}) \mid x_k\right\}\right] \tag{2.28}$$

As u_k is a function of x_k, all the quantities in the first expected value of equation (2.28) are known constants, for a given k. This results in the following form of the Bellman functional equation

$$J_k(x_k) = \min_{u_k}\left[\frac{1}{2}x_k^T Q_k x_k + x_k^T T_k u_k + \frac{1}{2}u_k^T R_k u_k + E\{J_{k+1}(x_{k+1}) \mid x_k\}\right] \tag{2.29}$$

iii) Solution of the Functional Equation

The solution to equation (2.29) will be shown to be

$$J_k(x_k) = \frac{1}{2}x_k^T S_k x_k + \theta_k \quad S_N \text{ given, } \theta_N = 0 \tag{2.30}$$

First, by definition

$$J_N(x_N) = \frac{1}{2}x_N^T S_N x_N \tag{2.31}$$

Next, assume

$$J_{k+1}(x_{k+1}) = \frac{1}{2}x_{k+1}^T S_{k+1} x_{k+1} + \theta_{k+1} \tag{2.32}$$

Finally, to show that equation (2.32) implies equation (2.30), we will start by inserting equation (2.32) into equation (2.29)

$$J_k(x_k)=\min_{u_k}\left[\frac{1}{2}x_k^TQ_kx_k+x_k^TT_ku_k+\frac{1}{2}u_k^TR_ku_k+E\left\{\frac{1}{2}x_{k+1}^TS_{k+1}x_{k+1}+e_{k+1}\mid x_k\right\}\right]$$

(2.33)

Inserting equation (1.2) for x_{k+1}

$$J_k(x_k)=\min_{u_k}\left[\frac{1}{2}x_k^TQ_kx_k+x_k^TT_ku_k+\frac{1}{2}u_k^TR_ku_k+\right.$$

$$\left.+E\left\{\frac{1}{2}[A_kx_k+B_ku_k+f_k]^TS_{k+1}[A_kx_k+B_ku_k+f_k]+e_{k+1}\mid x_k\right\}\right]$$

(2.34)

Expanding equation (2.34)

$$J_k(x_k)=\min_{u_k}\left[\frac{1}{2}x_k^TQ_kx_k+x_k^TT_ku_k+\frac{1}{2}u_k^TR_ku_k+\right.$$

$$+E\left\{\frac{1}{2}x_k^TA_k^TS_{k+1}A_kx_k+\frac{1}{2}u_k^TB_k^TS_{k+1}B_ku_k+\frac{1}{2}f_k^TS_{k+1}f_k+\right.$$

$$+\frac{1}{2}x_k^TA_k^TS_{k+1}B_ku_k+\frac{1}{2}u_k^TB_k^TS_{k+1}A_kx_k+\frac{1}{2}x_k^TA_k^TS_{k+1}f_k+$$

$$\left.\left.+\frac{1}{2}u_k^TB_k^TS_{k+1}f_k+f_k^TS_{k+1}A_kx_k+f_k^TS_{k+1}B_ku_k+e_{k+1}\mid x_k\right\}\right] \quad (2.35)$$

Now, let us consider equation (2.35) in depth. First, the expected value can be distributed over the sum of terms. Second, because x_k is known (assuming complete state information), all of the terms, with the exception of terms containing f_k, can be considered constants for a given k. Furthermore, the expected value of any term containing a single f_k term is zero, as the expected value of f_k is defined to be zero. Also, each term is a scalar (1x1). Therefore, each term is equal to its transpose, allowing us to further reduce the equation

$$J_k(x_k) = \min_{u_k}\Big[\frac{1}{2}x_k^T Q_k x_k + x_k^T T_k u_k + \frac{1}{2}u_k^T R_k u_k + \frac{1}{2}x_k^T A_k^T S_{k+1} A_k x_k +$$

$$+\frac{1}{2}u_k^T B_k^T S_{k+1} B_k u_k + u_k^T B_k^T S_{k+1} A_k x_k + E\left\{\frac{1}{2}f_k^T S_{k+1} f_k \mid x_k\right\} + e_{k+1}\Big] \tag{2.36}$$

Now we need to examine the remaining expected value term in equation (2.36). As was stated earlier, we are working with a scalar term. This allows us to take the trace of the inside term, as the trace of a scalar is equal to itself. Also, the trace and expected value may be interchanged, and the expected value of $S_{k+1} = S_{k+1}$. Using these facts and the property

$$\text{tr}\,[ABC] = \text{tr}\,[CAB] \tag{2.37}$$

yields the following equations

$$E\left\{\frac{1}{2}f_k^T S_{k+1} f_k \mid x_k\right\} = E\left\{\frac{1}{2} \text{ tr } \left[f_k^T S_{k+1} f_k\right] \mid x_k\right\} = E\left\{\frac{1}{2} \text{ tr } \left[f_k f_k^T S_{k+1}\right] \mid x_k\right\} =$$

$$= E\left\{\frac{1}{2} \text{ tr } \left[S_{k+1} f_k f_k^T\right] \mid x_k\right\} = \frac{1}{2} \text{ tr } \left[E\left\{S_{k+1} f_k f_k^T \mid x_k\right\}\right] = \frac{1}{2} \text{ tr } \left[S_{k+1} E\left\{f_k f_k^T \mid x_k\right\}\right] \tag{2.38}$$

Looking at the rightmost equation of (2.38), one can see that the expected value is simply F_k from equations (1.4) and (1.5). Therefore, substituting equations (1.5), and (2.38) into equation (2.36) yields

$$J_k(x_k) = \min_{u_k}\left[\frac{1}{2}x_k^T Q_k x_k + x_k^T T_k u_k + \frac{1}{2}u_k^T R_k u_k + \frac{1}{2}x_k^T A_k^T S_{k+1} A_k x_k + \right.$$

$$+ \frac{1}{2}u_k^T B_k^T S_{k+1} B_k u_k + u_k^T B_k^T S_{k+1} A_k x_k + e_{k+1} +$$

$$+ \frac{1}{2} \text{ tr } \left[S_{k+1} P_k^0\right] + \frac{1}{4} \text{ tr } \left[S_{k+1} \sum_{i=1}^{n'} P_k^i x_k^T W_k^i x_k\right] +$$

$$\left. + \frac{1}{2} \text{ tr } \left[S_{k+1} \sum_{i=1}^{n'} P_k^i x_k^T N_k^i u_k\right] + \frac{1}{4} \text{ tr } \left[S_{k+1} \sum_{i=1}^{n'} P_k^i u_k^T M_k^i u_k\right]\right] \tag{2.39}$$

We need to investigate the last three terms further, and, as they are basically the same, we will consider just the first of the three. The trace and S_{k+1} can be taken inside the summation, as shown in equation (2.40).

$$\frac{1}{4} \text{ tr } \left[S_{k+1} \sum_{i=1}^{n'} P_k^i x_k^T W_k^i x_k\right] = \frac{1}{4} \sum_{i=1}^{n'} \text{ tr } \left[S_{k+1} P_k^i x_k^T W_k^i x_k\right] \tag{2.40}$$

Using the fact that $x_k^T W_k^i x_k$ is a scalar, we can remove it from the trace leaving

$$\frac{1}{4}\sum_{i=1}^{n'} \mathrm{tr}\ [S_{k+1} P_k^i] x_k^T W_k^i x_k \tag{2.41}$$

Now we can insert the trace between x_k^T and W_k^i, as the trace is simply another scalar. Also x_k does not depend on i and can be removed from the summation.

$$\frac{1}{4} x_k^T \left\{ \sum_{i=1}^{n'} \mathrm{tr}\ [S_{k+1} P_k^i] W_k^i \right\} x_k = \frac{1}{2} x_k^T \overline{W}_k x_k \tag{2.42}$$

where

$$\overline{W}_k = \frac{1}{2}\sum_{i=1}^{n'} \mathrm{tr}\ [S_{k+1} P_k^i] W_k^i \tag{2.43}$$

Similar procedures can be followed to simplify the last two terms in equation (2.39), generating the following two definitions

$$\overline{N}_k = \frac{1}{2}\sum_{i=1}^{n'} \mathrm{tr}\ [S_{k+1} P_k^i] N_k^i \tag{2.44}$$

$$\overline{M}_k = \frac{1}{2}\sum_{i=1}^{n'} \mathrm{tr}\ [S_{k+1} P_k^i] M_k^i \tag{2.45}$$

Substituting all of this into equation (2.39), and grouping like terms results in the following equation

$$J_k(x_k) = \min_{u_k}\left[\frac{1}{2}x_k^T(Q_k + A_k^T S_{k+1} A_k + \overline{W}_k)x_k + x_k^T T_k u_k + \frac{1}{2}u_k^T R_k u_k + \right.$$

$$+\frac{1}{2}u_k^T B_k^T S_{k+1} B_k u_k + u_k^T B_k^T S_{k+1} A_k x_k + \theta_{k+1} +$$

$$\left. +\frac{1}{2}\ \mathrm{tr}\ \left[S_{k+1} P_k^0\right] + x_k^T \overline{N}_k u_k + \frac{1}{2}u_k^T \overline{M}_k u_k\right] \tag{2.46}$$

To find the minimum, we can take the partial derivative of equation (2.46) with respect to u_k, and set this equal to zero

$$\frac{\partial}{\partial u_k} J_k(x_k) = T_k^T x_k + R_k u_k + B_k^T S_{k+1} B_k u_k + B_k^T S_{k+1} A_k x_k + \overline{N}_k^T x_k + \overline{M}_k u_k = 0 \tag{2.47}$$

grouping like terms

$$\left[R_k + B_k^T S_{k+1} B_k + \overline{M}_k\right] u_k = -\left[T_k^T x_k + B_k^T S_{k+1} A_k x_k + \overline{N}_k^T x_k\right] \tag{2.48}$$

$\overline{M}_k$ and S_{k+1} can be shown to be positive semi-definite and positive definite respectively, and, since R_k is positive definite the following inverse exists

$$\left[R_k + B_k^T S_{k+1} B_k + \overline{M}_k\right]^{-1}$$

Therefore, solving for u_k

$$u_k = -\left[R_k + B_k^T S_{k+1} B_k + \overline{M}_k\right]^{-1}\left[T_k^T + B_k^T S_{k+1} A_k + \overline{N}_k^T\right] x_k \tag{2.49}$$

As can be seen in equation (2.49), we have finally arrived at the optimal controller proposed in equations (2.16) and (2.17). We have also shown the derivation of equations (2.21) - (2.23), but it remains yet to complete the induction on J_k, and verify equations (2.18) - (2.20). This can be done by substituting equation (2.49) into equation (2.46). After appropriate algebraic manipulations this results in

$$J_k(x_k) = \frac{1}{2} x_k^T \{ Q_k + A_k^T S_{k+1} A_k + \bar{W}_k - K_k^T [R_k + B_k^T S_{k+1} B_k + \bar{M}_k] K_k \} x_k +$$

$$+ \frac{1}{2} \operatorname{tr} \left[S_{k+1} P_k^0 \right] + \theta_{k+1} \tag{2.50}$$

We began the induction step with J_{k+1} and needed to show that this implied

$$J_k(x_k) = \frac{1}{2} x_k^T S_k x_k + \theta_k \tag{2.51}$$

Comparing equation (2.51) with equation (2.50) we can see that this is true if we assume the following

$$S_k = Q_k + A_k^T S_{k+1} A_k + \bar{W}_k - K_k^T [R_k + B_k^T S_{k+1} B_k + \bar{M}_k] K_k \tag{2.52}$$

$$\theta_k = \frac{1}{2} \operatorname{tr} \left[S_{k+1} P_k^0 \right] + \theta_{k+1} \tag{2.53}$$

Equations (2.52) and (2.53) not only complete the induction, but they also verify equations (2.18) - (2.20), thus completing the proof.

CHAPTER 3

SCALAR INFINITE-HORIZON CASE

Examining equation (2.17) one observes that the gain K_k is dependent, at stage k, on several parameters. In the case of a stationary system, A, B, R, M, N, W, T, Q, and P constant matrices, K_k is dependent on only S_{k+1}. Therefore, if the evolution of S_k comes to a steady-state, the evolution of K_k will subsequently come to a steady-state. In most well-behaved systems this is precisely the case, and the steady-state value of K_k, although not optimal for the finite-horizon case, quite often is an excellent and cost effective simple feedback control. This chapter derives a threshold condition for the scalar case, based on the statistics of the system, under which one can guarantee the existence of a steady-state solution for S_k.

3.1 Scalar System

The performance index is

$$E\left\{\frac{1}{2}S_N x_N^2 + \sum_{k=0}^{N-1}\frac{1}{2}Qx_k^2 + Tx_k u_k + \frac{1}{2}Ru_k^2\right\} \qquad (3.1)$$

and the system model is

$$x_{k+1} = ax_k + bu_k + f_k(x_k, u_k, \alpha_k) \qquad (3.2)$$

Before proceeding, we need to carefully examine equation (1.5), which is repeated below.

$$F_k(x_k,u_k)=P_k^0+\sum_{i=1}^{n'}P_k^i\left(\frac{1}{2}x_k^TW_k^ix_k+x_k^TN_k^iu_k+\frac{1}{2}u_k^TM_k^iu_k\right) \quad (1.5)$$

First, as we are dealing with a scalar system, n' is equal to one. Therefore, we no longer have a summation. Second, as we have stated, P, W, N, and M are constants, so we can drop their respective subscripts. Next, let us consider the term P^0. This term is equal to zero unless there is a purely additive noise term in $f_k(x_k,u_k,a_k)$. If P^0 is nonzero, then e_k, equation (2.20), grows without bound, and subsequently, the optimal cost, equation (2.19), grows without bound. Therefore, for our investigation of the infinite horizon case, we will require $f_k(x_k,u_k,a_k)$ to not contain a purely additive noise term, making P^0, and e_k both equal to zero. To further simplify notation we will let $P^1W=W$, $P^1N=N$, and $P^1M=M$. All this reduces equation (1.5) to

$$F_k(x_k,u_k)=\frac{1}{2}Wx_k^2+Nx_ku_k+\frac{1}{2}Mu_k^2 \quad (3.3)$$

and the optimal solution to

$$u_k = -\frac{T + S_{k+1}\left(ab + \frac{N}{2}\right)}{R + S_{k+1}\left(b^2 + \frac{M}{2}\right)} x_k \tag{3.4}$$

$$S_k = Q + S_{k+1}\left(a^2 + \frac{W}{2}\right) - \frac{\left[T + S_{k+1}\left(ab + \frac{N}{2}\right)\right]^2}{R + S_{k+1}\left(b^2 + \frac{M}{2}\right)} \tag{3.5}$$

with the optimal cost to go from time t equal to

$$J_t = \frac{1}{2} x_t^2 S_t \tag{3.6}$$

3.2 <u>Threshold</u> <u>Condition</u>

Consider equation (3.5) and suppose S_{k+1} gets very large. Then we can consider Q, T, and R, added to a very large S_{k+1}, as negligible, thus reducing equation (3.5) to

$$S_k \approx S_{k+1}\left(a^2 + \frac{W}{2}\right) - \frac{S_{k+1}\left(ab + \frac{N}{2}\right)^2}{b^2 + \frac{M}{2}} \tag{3.7}$$

As we are only interested in the steady-state value of S_k, forward or backward evolution in time does not matter, therefore, we will interchange S_k and S_{k+1}.

$$S_{k+1} \approx \left[\left(a^2 + \frac{W}{2}\right) - \frac{\left(ab \quad \frac{N}{2}\right)^2}{b^2 + \frac{M}{2}}\right] S_k \tag{3.8}$$

or

$$S_{k+1} \approx g\,S_k$$

where

$$g = \left(a^2 + \frac{W}{2}\right) - \frac{\left(ab + \frac{N}{2}\right)^2}{b^2 + \frac{M}{2}} \tag{3.9}$$

Equation (3.9) is the threshold condition. It is obvious that if $g < 1$ then the evolution of S_k stays bounded and S_k, and subsequently K_k, has a steady-state solution.

There is an alternate approach to finding the threshold condition g that does not utilize the above approximations. First suppose S_k does reach a steady-state, say S, then equation (3.5) takes on a somewhat simpler form.

$$S = Q + S\left(a^2 + \frac{W}{2}\right) - \frac{\left[T + S\left(ab + \frac{N}{2}\right)\right]^2}{R + S\left(b^2 + \frac{M}{2}\right)} \tag{3.10}$$

multiplying both sides by the denominator and rearranging terms yields

$$\left[T + S\left(ab + \frac{N}{2}\right)\right]^2 = \left[R + S\left(b^2 + \frac{M}{2}\right)\right]\left[Q + S\left(a^2 + \frac{W}{2}\right) - S\right] \tag{3.11}$$

Next, squaring the left side and multiplying out the right side yields

$$T^2 + 2TS\left(ab + \frac{N}{2}\right) + S^2\left(ab + \frac{N}{2}\right)^2 - RQ - RS\left(a^2 + \frac{W}{2}\right) + RS =$$

$$= QS\left(b^2 + \frac{M}{2}\right) + S^2\left(b^2 + \frac{M}{2}\right)\left(a^2 + \frac{W}{2}\right) - S^2\left(b^2 + \frac{M}{2}\right) \qquad (3.12)$$

finally combining the like powers of S

$$S^2\left[\left(b^2 + \frac{M}{2}\right)\left(a^2 + \frac{W}{2}\right) - \left(ab + \frac{N}{2}\right)^2 - \left(b^2 + \frac{M}{2}\right)\right] +$$

$$+ S\left[R\left(a^2 + \frac{W}{2}\right) - 2T\left(ab + \frac{N}{2}\right) + Q\left(b^2 + \frac{M}{2}\right) - R\right] + RQ - T^2 = 0$$

$$(3.13)$$

Upon examining equation (3.13) one can see that it is just a quadratic equation in S, of the general form shown below.

$$AS^2 + BS + C = 0 \qquad (3.14)$$

Thinking in two dimensions, if $A<0$, then we have a parabola concave downward. If at the same time $C>0$, then the Y intercept is positive and we have a unique positive definite solution for S. Now, let's investigate these conditions with respect to equation (3.13). $C>0$ implies $RQ-T^2>0$. This is a natural restriction as T is the cross weighting term from x_k and u_k and can not be weighted larger than Q or R, which are the weighting terms associated with x_k^2 and u_k^2 respectively. As a result $RQ-T^2>0$ is automatically true.

Therefore all that is necessary for S to have a unique positive definite solution is the following

$$\left[\left(b^2+\frac{M}{2}\right)\left(a^2+\frac{W}{2}\right)-\left(ab+\frac{N}{2}\right)^2-\left(b^2+\frac{M}{2}\right)\right]<0 \qquad (3.15)$$

Simplifying equation (3.15) we arrive at the threshold condition

$$g=\left(a^2+\frac{W}{2}\right)-\frac{\left(ab+\frac{N}{2}\right)^2}{b^2+\frac{M}{2}}<1 \qquad (3.16)$$

It is worth noting that, as the above derivations show, the threshold condition g is both a necessary and sufficient condition. If g is less than one, then the steady-state solution exists, but, if g is greater than one, then S_k increases without bound.

3.3 Simulations

Figures 3.1 through 3.3 plot the evolution of S_k given by equation (3.5), forward in time, vs k for fifty values of k. Each graph contains four plots of S_k for various values of W, N, and M. These values were specifically chosen to demonstrate the threshold condition. It should be noted that, as the threshold condition is approached, the values of S_k become very large. Therefore the base ten logarithm of S_k vs k is what actually appears in the graphs.

It should also be noted that, as we are plotting S_k forward in time, the equation for the optimal cost to go from time 0 to time k is now

$$J_k = \frac{1}{2} x_0^2 S_k \tag{3.17}$$

As shown in equation (3.17), the optimal cost, for a given x_0, is merely a constant times S_k. Therefore, figures 3.1 through 3.3 can also be considered the evolution of the optimal cost J_k.

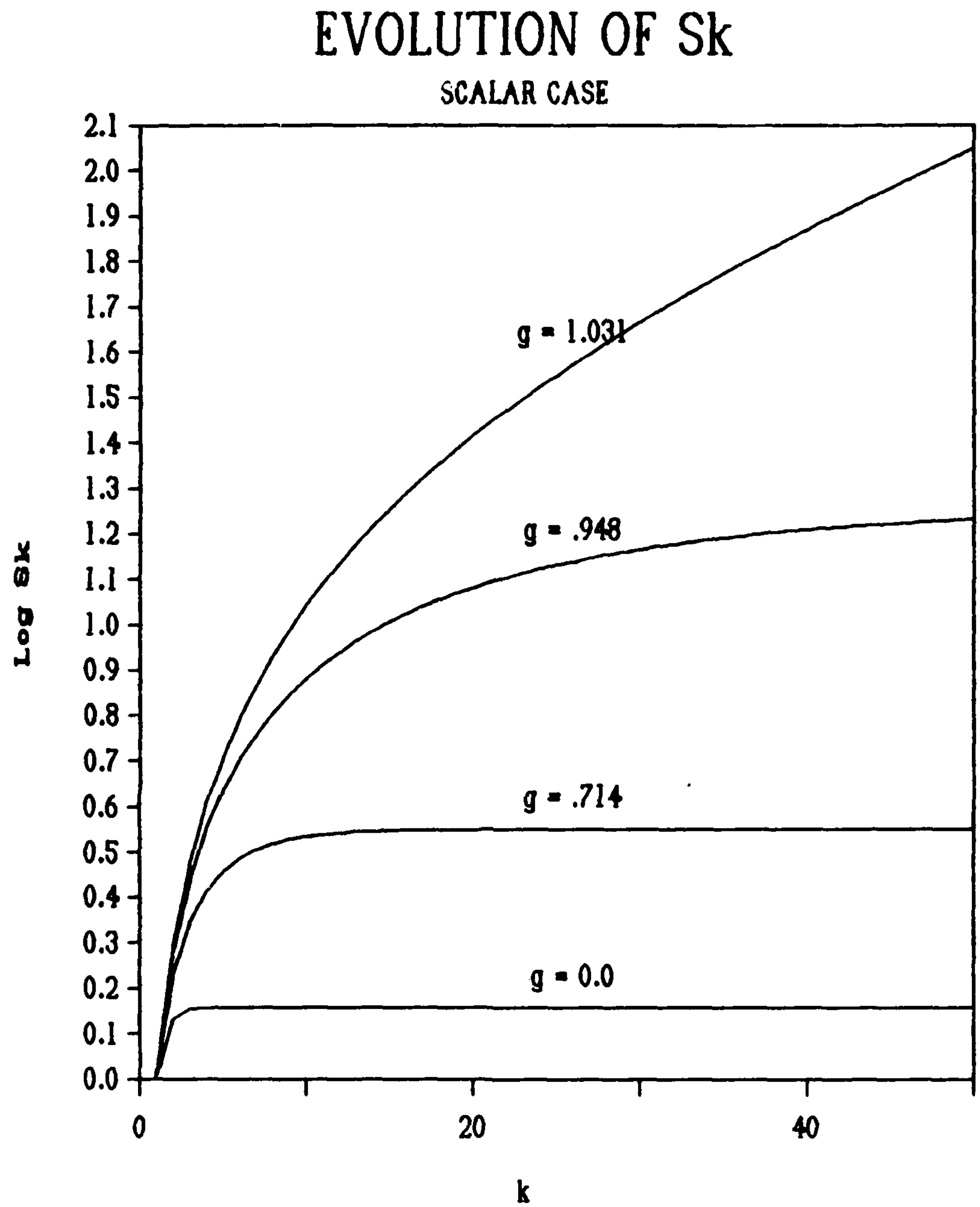

Figure 3.1 Evolution of s_k with $s_0=0$, $T=0.2$, $Q=R=1.0$, $a=1.1$, $b=1.0$, $V=N=0$, and $M=0$, 2.88, 7.2, and 11.52

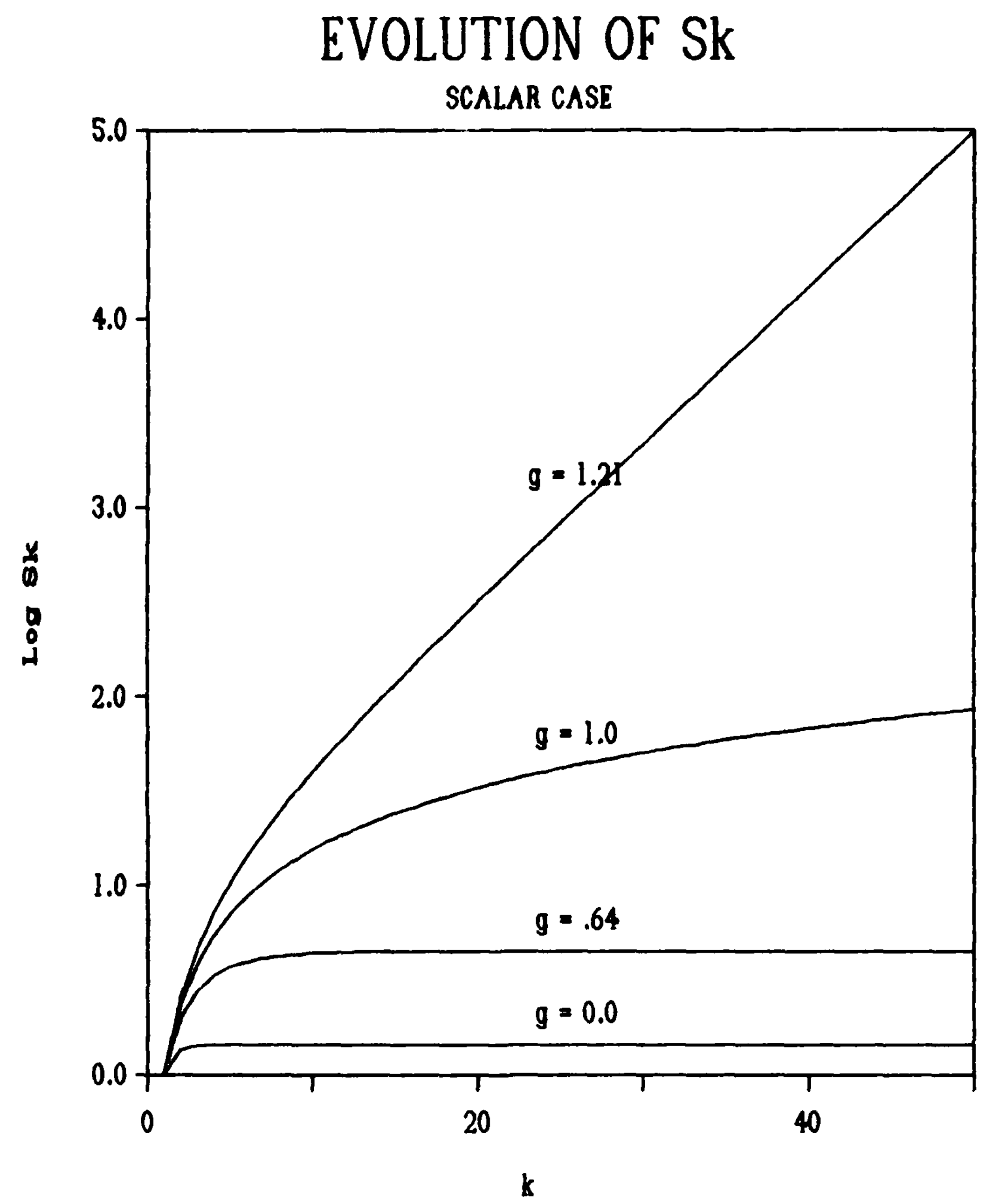

Figure 3.2 Evolution of s_k with $s_0=0$, $r=0.2$, $Q=R=1.0$, $a=1.1$, $b=1.0$, $M=N=0$, and $V=0$, 1.28, 2.0, and 2.42

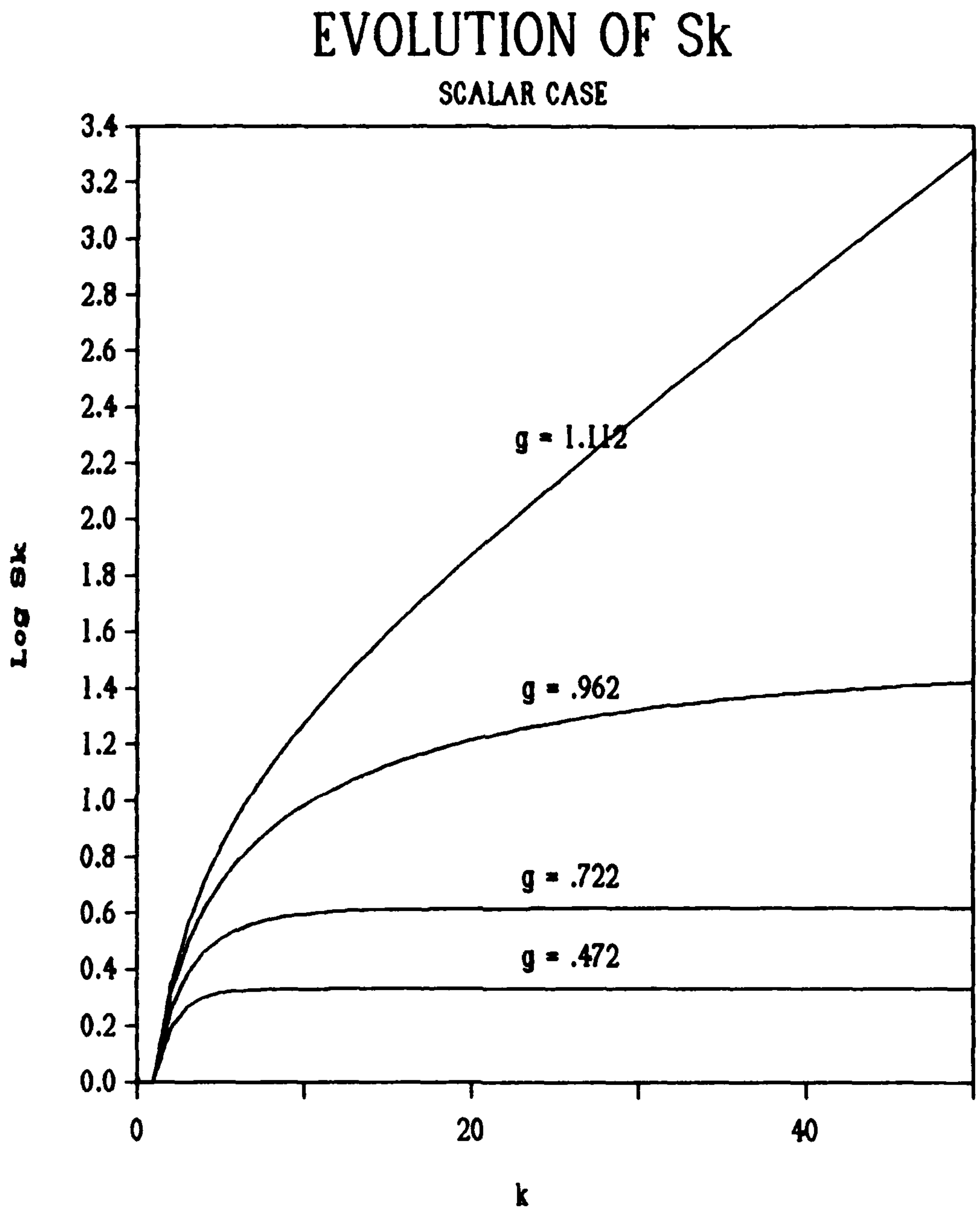

Figure 3.3 Evolution of S_k with $S_0=0$, $T=0.2$, $Q=R=1.0$, $a=1.1$, $b=1.0$, $N=0$, $M=1.28$, and $V=0$, 0.5, 0.98, and 1.28

CHAPTER 4

MULTIVARIABLE INFINITE-HORIZON CASE

Unlike the scalar threshold condition, equation (3.16), an absolute threshold for the multivariable case is much more difficult to define. Therefore, a sufficient condition is found that guarantees the existence of the infinite-horizon solution. If the threshold is violated, the steady-state solution may or may not exist. In spite of this, the relative magnitude that the threshold condition is exceeded by is still an excellent indicator of the steady-state characteristics of S_k.

4.1 Threshold Condition

As before, we are only interested in the steady-state behavior of S_k, and we can choose to evaluate this forward in time rather than backward. Therefore S_k, which is equation (2.18), will be rewritten forward in time. As we are interested in the steady-state analysis, all matrices except S_k will be assumed constant. Also, to make the problem more tractable, without a great loss of generality, we will drop the cross terms T_k and N_k and assume Q is positive definite. Therefore we will begin with the following equation

$$S_{k+1} = Q + A^T S_k A + \overline{W}_k - A^T S_k B\left[R + B^T S_k B + \overline{M}_k\right]^{-1} B^T S_k A$$

(4.1)

where $\overline{W}_k$ and $\overline{M}_k$ now represent the following values

$$\overline{W}_k = \frac{1}{2}\sum_{i=1}^{n'}\left[\ \mathrm{tr}\ \left(S_k P^i\right) W^i\right] \tag{4.2}$$

$$\overline{M}_k = \frac{1}{2}\sum_{i=1}^{n'}\left[\ \mathrm{tr}\ \left(S_k P^i\right) M^i\right] \tag{4.3}$$

Inserting equations (4.2) and (4.3) into equation (4.1)

$$S_{k+1} = Q + A^T S_k A + \frac{1}{2}\sum_{i=1}^{n'}\ \mathrm{tr}\ \left(S_k P^i\right) W^i +$$

$$-A^T S_k B\left[R + B^T S_k B + \frac{1}{2}\sum_{i=1}^{n'}\ \mathrm{tr}\ \left(S_k P^i\right) M^i\right]^{-1} B^T S_k A \tag{4.4}$$

We will now show three properties of the evolution of S_k that guarantee the existence of a steady-state solution. First, we will show that S_k is always positive definite. Next we will show that S_k is a monotonically increasing matrix sequence. And finally, if we also show that S_k remains bounded above, then it must come to a steady-state value [22]. Therefore, we will prove the first two properties, and then find a threshold condition such that the third is true.

First, we will show that, starting from $S_0=0$, all subsequent S_k are positive definite. Evaluating equation (4.4) with $k=0$ yields $S_1=Q$, and, as Q is positive definite, S_1 must be positive definite. Now we may apply the well known matrix inversion lemma [19]

$$\left[A_{11}^{-1}+A_{12}A_{22}A_{21}\right]^{-1}=A_{11}-A_{11}A_{12}\left[A_{21}A_{11}A_{12}+A_{22}^{-1}\right]^{-1}A_{21}A_{11}$$

where A_{11} and A_{22} are square and non-singular, to equation (4.4) to obtain

$$S_{k+1}=A^T\left[S_k^{-1}+B\left[R+\frac{1}{2}\sum_{i=1}^{n'}\text{tr}\,(S_kP^i)M^i\right]^{-1}B^T\right]^{-1}A+Q+\frac{1}{2}\sum_{i=1}^{n'}\text{tr}\,(S_kP^i)W^i \tag{4.5}$$

Substituting $S_1=Q$, which is positive definite, into equation (4.5), one can see that the two summation terms are at least positive semi-definite, and, as R is positive definite, the inner inverse exits and is positive definite. Now, using these facts and the property that any matrix quadratic form ZXZ^T is at least positive semi-definite if X is at least positive semi-definite, we can show that

$$B\left[R+\frac{1}{2}\sum_{i=1}^{n'}\text{tr}\,(S_kP^i)M^i\right]^{-1}B^T\geq 0$$

We have shown S_1 is positive definite, therefore S_1^{-1} exists, and, moreover, S_1^{-1} is positive definite which means the outer inverse exists and

$$A^T\left[S_k^{-1}+B\left[R+\frac{1}{2}\sum_{i=1}^{s'}\text{ tr }\left(S_kP^i\right)M^i\right]^{-1}B^T\right]^{-1}A\geq 0$$

We already have Q positive definite. Therefore, S_2 is positive definite. Obviously this pattern continues with increasing k. Therefore, starting from $S_0=0$, all subsequent S_k are positive definite.

Next we will show that S_k is a monotonically increasing matrix sequence. We have already shown that starting from $S_0=0$, $S_1=Q$, which is positive definite. Using this as the basis, we will now show that $S_k\geq S_{k-1}$ implies $S_{k+1}\geq S_k$. The following matrix manipulations can be found in [20]. Suppose

$$S_k\geq S_{k-1} \tag{4.6}$$

It follows that

$$[P^i]^{\frac{1}{2}}S_k[P^i]^{\frac{1}{2}}\geq[P^i]^{\frac{1}{2}}S_{k-1}[P^i]^{\frac{1}{2}} \tag{4.7}$$

taking the trace of both sides

$$\text{tr }\left\{[P^i]^{\frac{1}{2}}S_k[P^i]^{\frac{1}{2}}\right\}\geq\text{ tr }\left\{[P^i]^{\frac{1}{2}}S_{k-1}[P^i]^{\frac{1}{2}}\right\} \tag{4.8}$$

using equation (2.37) the terms inside the trace may be rearranged

$$\operatorname{tr}\left\{S_k P^i\right\} \geq \operatorname{tr}\left\{S_{k-1} P^i\right\} \tag{4.9}$$

now, multiply both sides by M^i

$$\operatorname{tr}\left\{S_k P^i\right\} M^i \geq \operatorname{tr}\left\{S_{k-1} P^i\right\} M^i \tag{4.10}$$

next, a sum over all i does not change the inequality

$$\frac{1}{2}\sum_{i=1}^{n'} \operatorname{tr}\left\{S_k P^i\right\} M^i \geq \frac{1}{2}\sum_{i=1}^{n'} \operatorname{tr}\left\{S_{k-1} P^i\right\} M^i \tag{4.11}$$

similarly add R to each side

$$R + \frac{1}{2}\sum_{i=1}^{n'} \operatorname{tr}\left\{S_k P^i\right\} M^i \geq R + \frac{1}{2}\sum_{i=1}^{n'} \operatorname{tr}\left\{S_{k-1} P^i\right\} M^i \tag{4.12}$$

taking the inverse of both sides reverses the inequality

$$\left[R + \frac{1}{2}\sum_{i=1}^{n'} \operatorname{tr}\left\{S_k P^i\right\} M^i\right]^{-1} \leq \left[R + \frac{1}{2}\sum_{i=1}^{n'} \operatorname{tr}\left\{S_{k-1} P^i\right\} M^i\right]^{-1} \tag{4.13}$$

now multiply from the left by B and from the right by B^T

$$B\left[R + \frac{1}{2}\sum_{i=1}^{n'} \operatorname{tr}\left\{S_k P^i\right\} M^i\right]^{-1} B^T \leq B\left[R + \frac{1}{2}\sum_{i=1}^{n'} \operatorname{tr}\left\{S_{k-1} P^i\right\} M^i\right]^{-1} B^T \tag{4.14}$$

inequality (4.6) implies

$$S_k^{-1} \le S_{k-1}^{-1} \tag{4.15}$$

using this fact

$$S_k^{-1} + B\left[R + \frac{1}{2}\sum_{i=1}^{n'} \text{tr}\,\{S_k P^i\} M^i\right]^{-1} B^T \le$$

$$\le S_{k-1}^{-1} + B\left[R + \frac{1}{2}\sum_{i=1}^{n'} \text{tr}\,\{S_{k-1} P^i\} M^i\right]^{-1} B^T \tag{4.16}$$

again taking the inverse reverses the inequality

$$\left[S_k^{-1} + B\left[R + \frac{1}{2}\sum_{i=1}^{n'} \text{tr}\,\{S_k P^i\} M^i\right]^{-1} B^T\right]^{-1} \ge$$

$$\ge \left[S_{k-1}^{-1} + B\left[R + \frac{1}{2}\sum_{i=1}^{n'} \text{tr}\,\{S_{k-1} P^i\} M^i\right]^{-1} B^T\right]^{-1} \tag{4.17}$$

multiply from the left by A^T and from the right by A

$$A^T\left[S_k^{-1} + B\left[R + \frac{1}{2}\sum_{i=1}^{n'} \text{tr}\,\{S_k P^i\} M^i\right]^{-1} B^T\right]^{-1} A \ge$$

$$\ge A^T\left[S_{k-1}^{-1} + B\left[R + \frac{1}{2}\sum_{i=1}^{n'} \text{tr}\,\{S_{k-1} P^i\} M^i\right]^{-1} B^T\right]^{-1} A \tag{4.18}$$

The other summation term can be developed similar to inequalities (4.6) through (4.11), therefore adding Q and the other summation to both sides does not affect the inequality

$$Q + A^T\left[S_k^{-1} + B\left[R + \frac{1}{2}\sum_{i=1}^{n'} \text{tr}\,\{S_k P^i\} M^i\right]^{-1} B^T\right]^{-1} A + \frac{1}{2}\sum_{i=1}^{n'} \text{tr}\,\{S_k P^i\} W^i \geq$$

$$\geq Q + A^T\left[S_{k-1}^{-1} + B\left[R + \frac{1}{2}\sum_{i=1}^{n'} \text{tr}\,\{S_{k-1} P^i\} M^i\right]^{-1} B^T\right]^{-1} A + \frac{1}{2}\sum_{i=1}^{n'} \text{tr}\,\{S_{k-1} P^i\} W^i$$

(4.19)

Comparing inequality (4.19) with equation (4.5) one can see that the left side is just S_{k+1} and the right side is just S_k. Therefore inequality (4.6) implies the following

$$S_{k+1} \geq S_k \qquad (4.20)$$

Therefore, by inductive reasoning, starting from $S_0=0$, S_k is not only positive definite, but it is also monotonically increasing.

All that is left to do is to determine a condition under which the positive definite monotonically increasing matrix sequence S_k remains bounded. First, assume S_k gets large, but that it has not yet exceeded the limit given by inequality (4.21) for a positive scaler α.

$$S_k \leq \alpha I \qquad (4.21)$$

We need to find a condition such that, for large S_k, inequality (4.21) implies

$$S_{k+1} \leq \alpha I \qquad (4.22)$$

so that the limit given by αI is not exceeded by subsequent terms in the matrix sequence. Then, through inductive reasoning we will be able to conclude that this threshold will never be exceeded. Similar to the procedure followed with inequality (4.6), inequality (4.21) can be transformed into inequality (4.23)

$$Q + A^T\left[S_k^{-1} + B\left[R + \frac{1}{2}\sum_{i=1}^{n'} \text{tr}\,(S_k P^i)M^i\right]^{-1} B^T\right]^{-1} A + \frac{1}{2}\sum_{i=1}^{n'} \text{tr}\,(S_k P^i)W^i \leq$$

$$\leq Q + A^T\left[\alpha^{-1}\, I + B\left[R + \frac{1}{2}\alpha\sum_{i=1}^{n'} \text{tr}\,(P^i)M^i\right]^{-1} B^T\right]^{-1} A + \frac{1}{2}\alpha\sum_{i=1}^{n'} \text{tr}\,(P^i)W^i$$

(4.23)

The left side of inequality (4.23) is just S_{k+1}. Therefore if we set the right side of inequality (4.23) less than or equal to αI, then inequality (4.22) is satisfied and S_k will remain bounded and reach a steady-state. Therefore, the system statistics must be such that the following inequality is satisfied

$$Q + A^T\left[\alpha^{-1}\, I + B\left[R + \frac{1}{2}\alpha\sum_{i=1}^{n'} \text{tr}\,(P^i)M^i\right]^{-1} B^T\right]^{-1} A + \frac{1}{2}\alpha\sum_{i=1}^{n'} \text{tr}\,(P^i)W^i \leq \alpha\, I$$

(4.24)

Factoring α out of the inverses

$$A^T\left[\alpha^{-1}\, I + \alpha^{-1}B\left[\alpha^{-1}R + \frac{1}{2}\sum_{i=1}^{n'} \operatorname{tr}\left(P^i\right)M^i\right]^{-1} B^T\right]^{-1} A +$$

$$+Q + \frac{1}{2}\alpha\sum_{i=1}^{n'} \operatorname{tr}\left(P^i\right)W^i \le \alpha\, I \tag{4.25}$$

and

$$Q + \alpha A^T\left[\, I + B\left[\alpha^{-1}R + \frac{1}{2}\sum_{i=1}^{n'} \operatorname{tr}\left(P^i\right)M^i\right]^{-1} B^T\right]^{-1} A + \frac{1}{2}\alpha\sum_{i=1}^{n'} \operatorname{tr}\left(P^i\right)W^i \le \alpha\, I \tag{4.26}$$

Dividing both sides by alpha

$$\alpha^{-1}Q + A^T\left[\, I + B\left[\alpha^{-1}R + \frac{1}{2}\sum_{i=1}^{n'} \operatorname{tr}\left(P^i\right)M^i\right]^{-1} B^T\right]^{-1} A + \frac{1}{2}\sum_{i=1}^{n'} \operatorname{tr}\left(P^i\right)W^i \le I \tag{4.27}$$

As α was an arbitrarily large number, $\alpha^{-1}Q$ and $\alpha^{-1}R$ can be considered negligible, thus completely eliminating α from the equation. If we also assume

$$\frac{1}{2}\sum_{i=1}^{n'} \operatorname{tr}\left(P^i\right)M^i > 0$$

we obtain

$$A^T\left[\, I + B\left[\frac{1}{2}\sum_{i=1}^{n'} \operatorname{tr}\left(P^i\right)M^i\right]^{-1} B^T\right]^{-1} A + \frac{1}{2}\sum_{i=1}^{n'} \operatorname{tr}\left(P^i\right)W^i < I \tag{4.28}$$

This simply means that the magnitude of the eigenvalues of the left hand side of equation (4.28) must be less than one. This is only a sufficient condition for the existence of a steady-state solution to S_2. As was stated before, if the magnitude of the maximum eigenvalue of the left hand side of equation (4.28) exceeds one, a steady-state solution for S_2 may still exist, but, as long as the maximum is less than one, a steady-state solution is guaranteed to exist.

Even though equation (4.28) is only a sufficient condition for the multivariable case, it reduces to the necessary and sufficient condition given in equation (3.16) for the scalar case. If we evaluate equation (4.28) for the scalar case, then $n=1$, which implies $n'=1$. Therefore, we only have a single term for each of the summations, and, if we let $PM=M$ and $PW=W$, then equation (4.28) reduces to

$$\frac{a^2}{1+\frac{b^2}{\frac{M}{2}}}+\frac{W}{2}<1 \tag{4.29}$$

clearing some of the fractions

$$\frac{W}{2}+\frac{a^2M}{2b^2+M}<1 \tag{4.30}$$

adding and subtracting a^2

$$\left(a^2+\frac{W}{2}\right)-a^2+\frac{a^2M}{2b^2+M}<1 \tag{4.31}$$

finally this simplifies to the scalar condition, equation (3.16), less the cross term $N/2$

$$\left(a^2+\frac{W}{2}\right)-\frac{(ab)^2}{b^2+\frac{M}{2}}<1 \tag{4.32}$$

Therefore, as was stated above, even though equation (4.28) is only a sufficient condition, in the scalar case, it does indeed reduce to a necessary and sufficient condition. It should also be noted that, as in the scalar case, we must require $f_k(x_k,u_k,a_k)$ to not contain a purely additive noise term, making P^o and e_k both equal to zero.

4.2 Simulations

In the following simulations, n is equal to two, and S_k is plotted vs k for fifty values of k. With n equal to two, n' is equal to three. Therefore if we let

$$P' = \frac{1}{3}I \qquad M' = \frac{1}{3}m \qquad W' = \frac{1}{3}wI$$

then equations (4.2) and (4.3) reduce to

$$\bar{W}_k = \frac{1}{2} w \ \mathrm{tr}\left(S_k\right) \tag{4.33}$$

$$\bar{M}_k = \frac{1}{2} m \ \mathrm{tr}\left(S_k\right) \tag{4.34}$$

Inserting equations (4.33) and (4.34) into equation (4.1) results in

$$S_{k+1} = Q + A^T\left[S_k - S_k B\left[R + B^T S_k B + \frac{m}{2} \ \mathrm{tr}\left(S_k\right)\right]^{-1} B^T S_k\right] A + \frac{w}{2} \ \mathrm{tr}\left(S_k\right) \tag{4.35}$$

Applying the same parameters to the threshold condition, equation (4.28), results in

$$wI + A^T\left[\ I + \frac{1}{m} B B^T\right]^{-1} A < I \tag{4.36}$$

and similar to the scalar case, the optimal cost to go from time 0 to time k is

$$J_k = x_0^T S_k x_0 \tag{4.37}$$

The first four figures show the evolution of S_k, equation (4.35), for different values of m and w. The maximum eigenvalue of the threshold condition is shown at the top of each figure.

Also, with n equal to two, S_k is a two by two matrix, and, as it is symmetric, $S_k(1,2) = S_k(2,1)$. Therefore only three plots appear in each figure. The next four figures show the respective evolution of the optimal cost given in equation (4.37), with x_0 equal to a unit vector.

Unlike the scalar case, we do not have large variations in magnitude within each individual figure so there is no reason to take the logarithm. Therefore, the actual values of S_k and J_k are plotted vs k, but one should note that there are still large variations in magnitude between the figures. The following values apply to all simulations. R=0.1, and

$$A = \begin{bmatrix} 0.1 & 0.8 \\ 0.0 & 0.2 \end{bmatrix} \quad Q = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \quad B = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$$

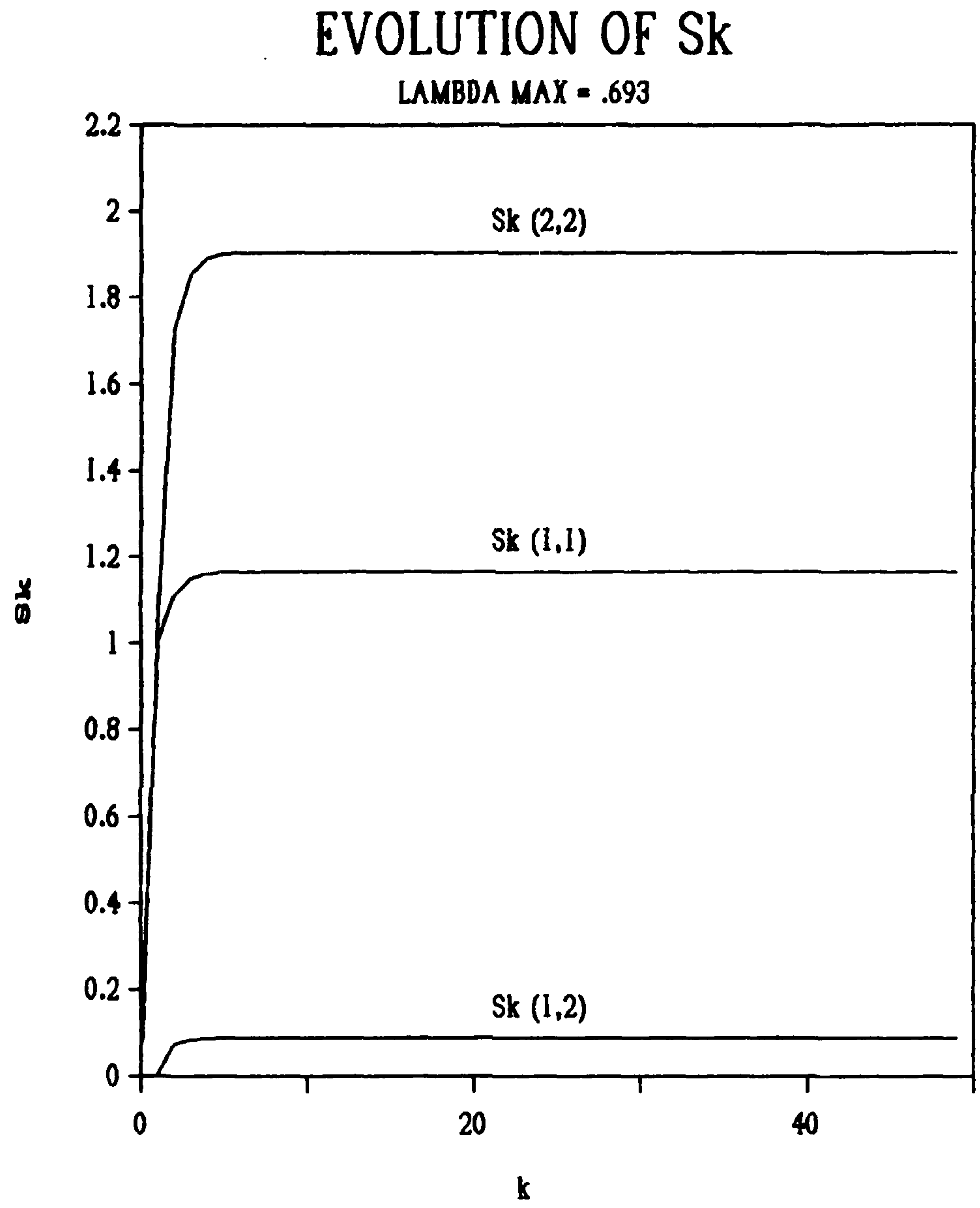

Figure 4.1 Evolution of s_k with s_0=0, $s_k(1,2)$ = $s_k(2,1)$, m=w=0.1, and the threshold condition equal to 0.693

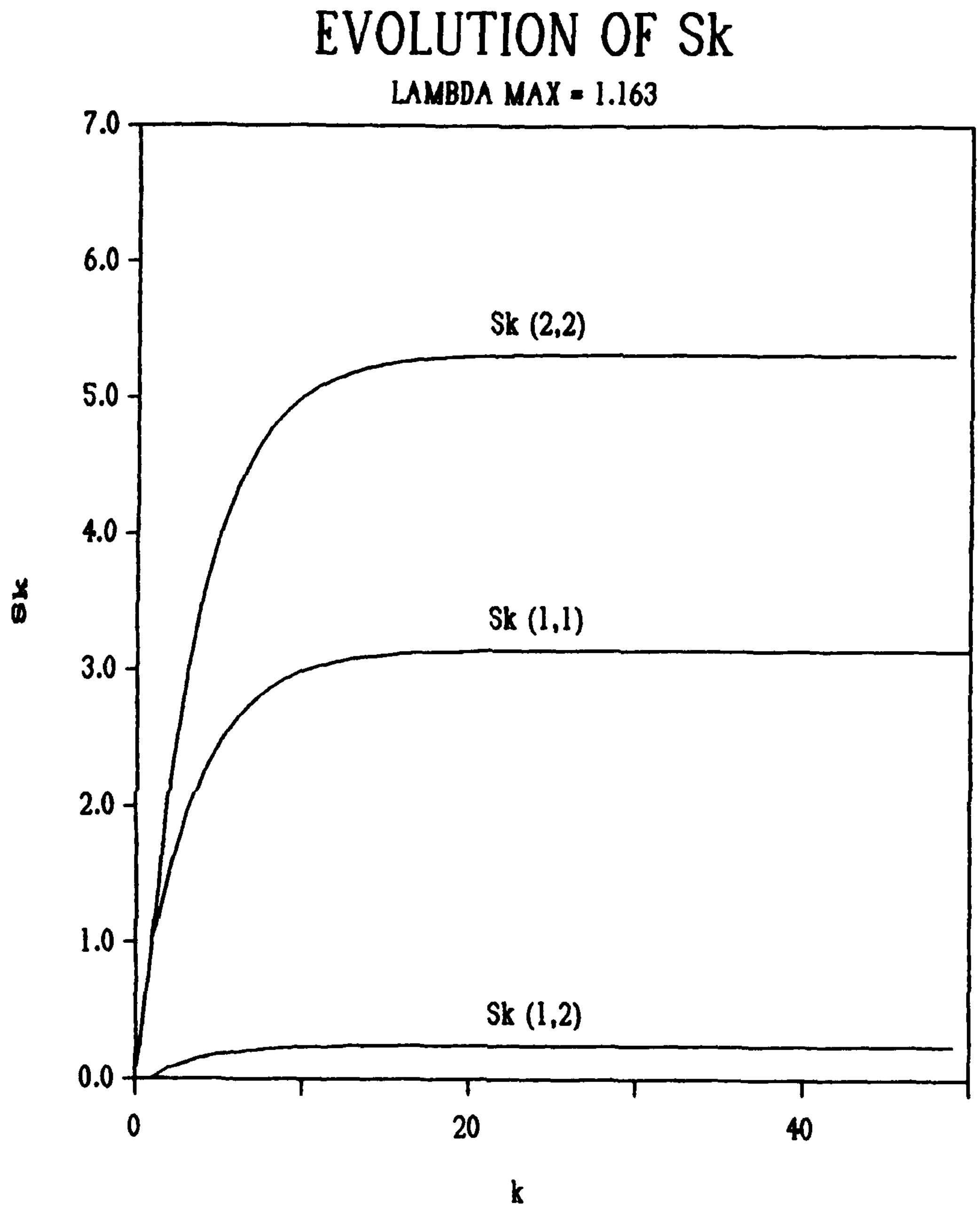

Figure 4.2 Evolution of s_k with $s_0=0$, $s_k(1,2) = s_k(2,1)$, $m=w=0.5$, and the threshold condition equal to 1.163

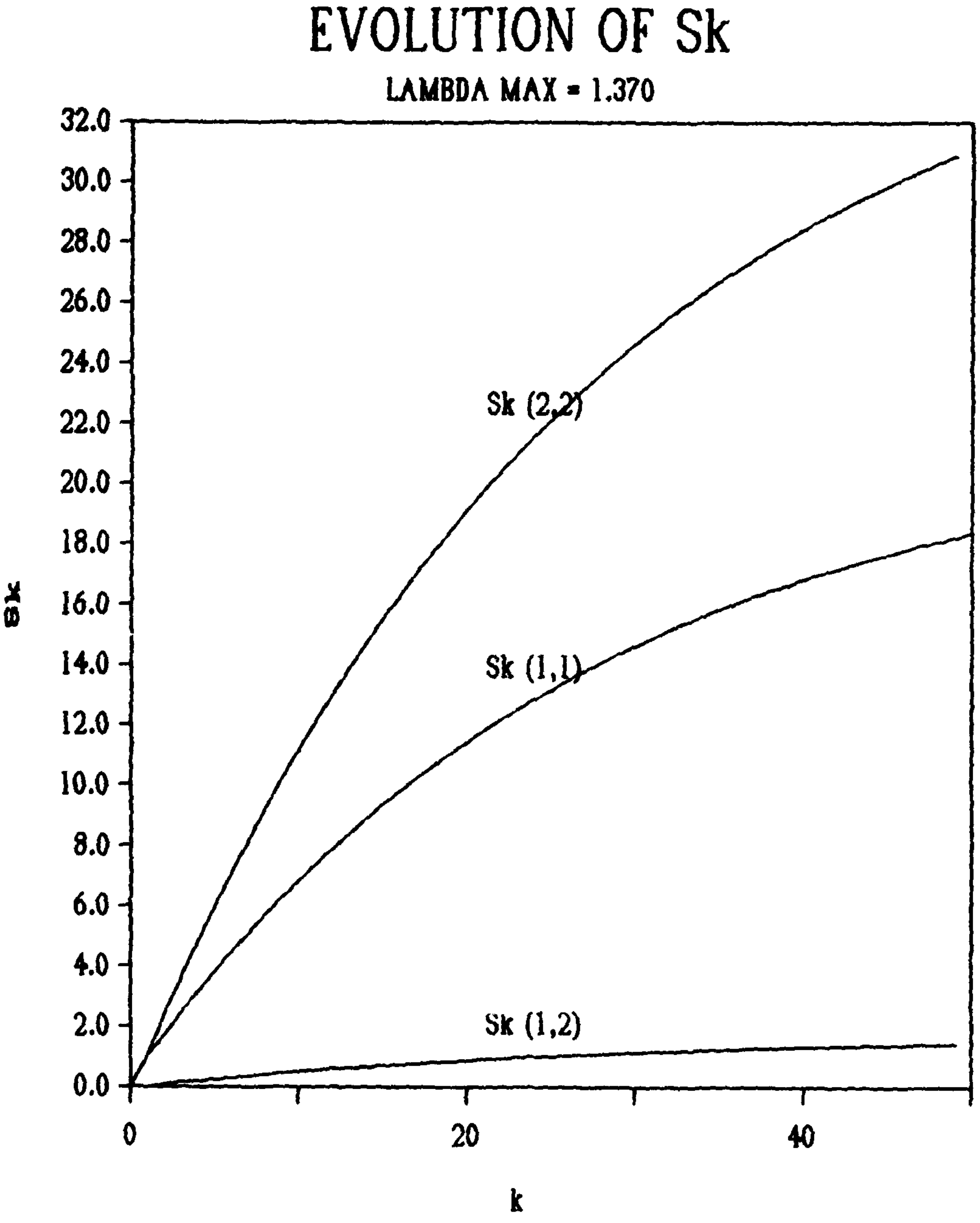

Figure 4.3 Evolution of s_k with s_0=0, $s_k(1,2)$ = $s_k(2,1)$, m=w=0.7, and the threshold condition equal to 1.370

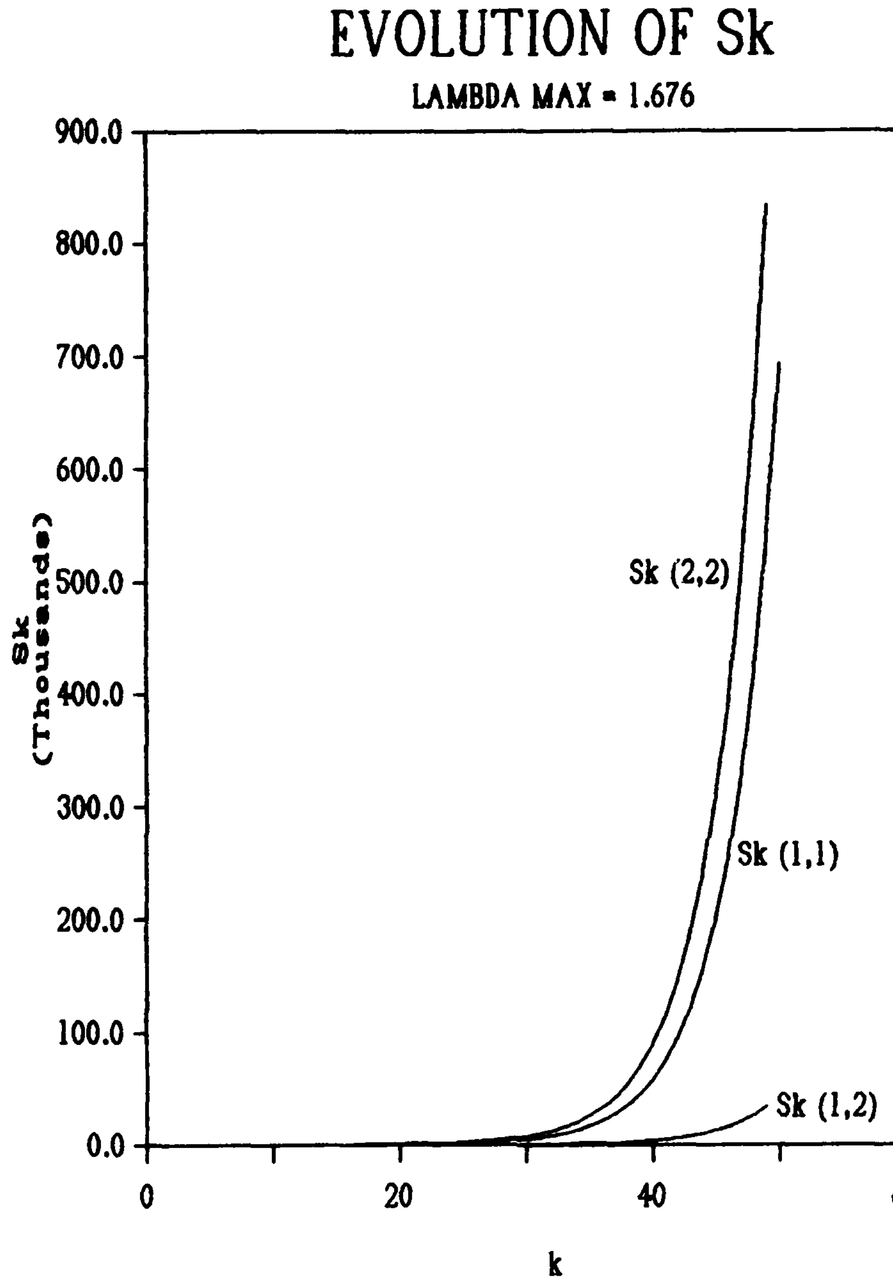

Figure 4.4 Evolution of s_k with $s_0=0$, $s_k(1,2) = s_k(2,1)$, $m=w=1.0$, and the threshold condition equal to 1.676

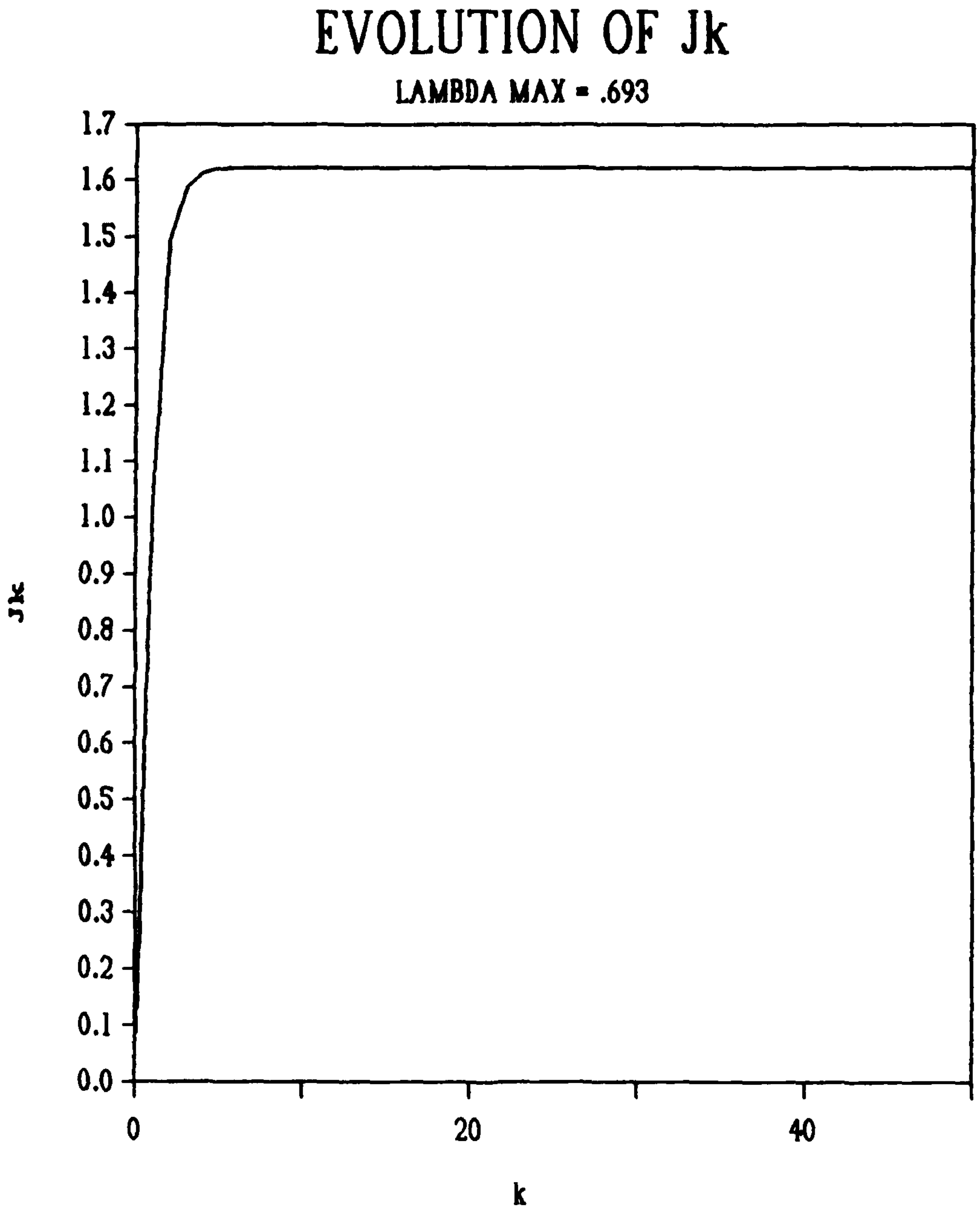

Figure 4.5 Evolution of J_k with $x_0=[1 \;\; 1]^T$, $m=w=0.1$, and the threshold condition equal to 0.693

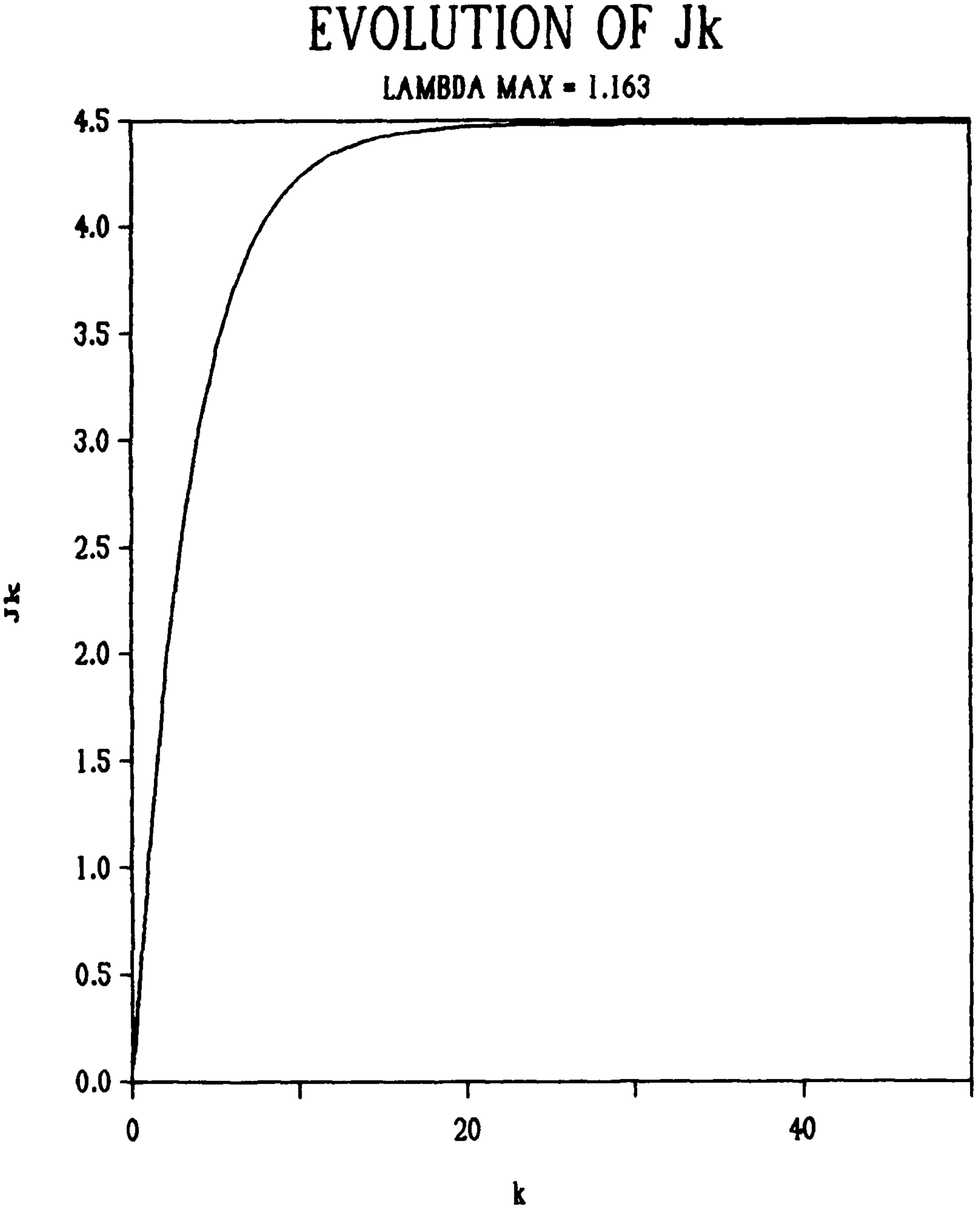

Figure 4.6 Evolution of J_k with $x_0=[1 \quad 1]^T$, $m=\omega=0.5$, and the threshold condition equal to 1.163

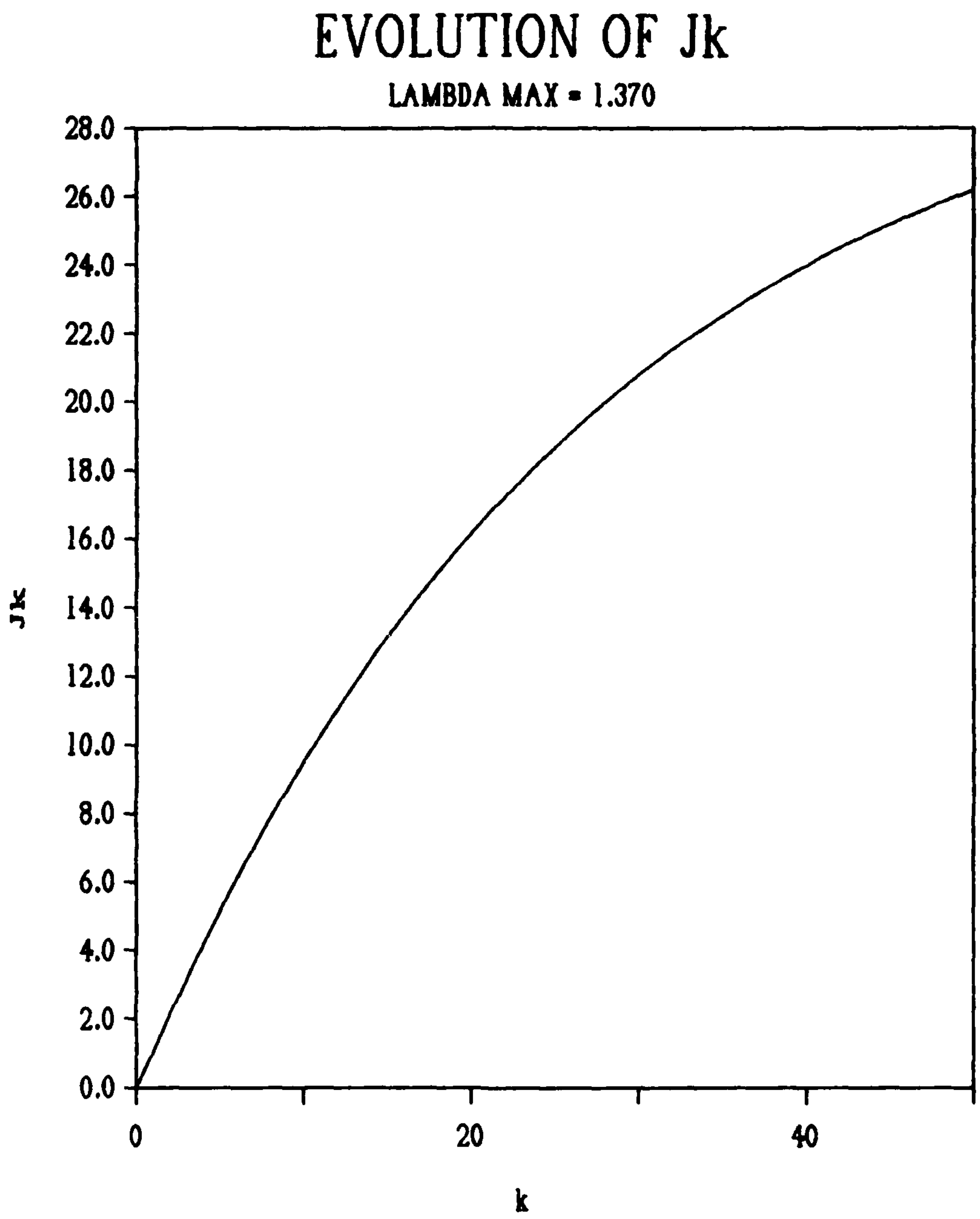

Figure 4.7 Evolution of J_k with $x_0=[1 \;\; 1]^T$, $m=w=0.7$, and the threshold condition equal to 1.370

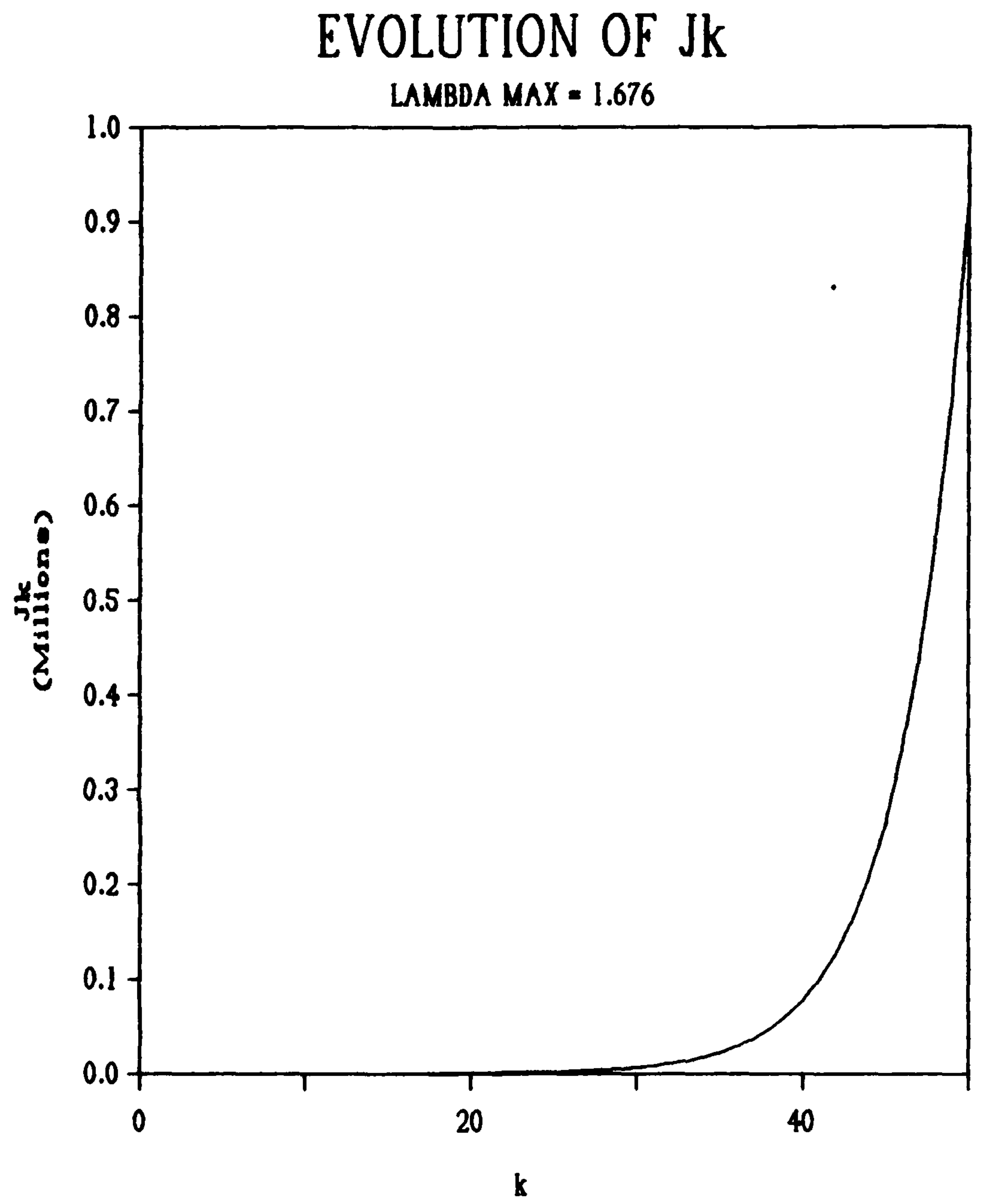

Figure 4.8 Evolution of J_k with $x_0=[1 \quad 1]^T$, $m=w=1.0$, and the threshold condition equal to 1.676

CHAPTER 5

CONCLUSIONS

We began by introducing a very general system description. We showed that this system description can represent anything from a simple deterministic system to a very complex nonlinear stochastic system, and gave several examples. After giving the optimal finite-horizon solution for several known simpler cases, we then stated the finite-horizon optimal controller for the proposed multivariable nonlinear stochastic system, subsequently proving the general result.

The practicality of the infinite-horizon control approach was pointed out, prompting an investigation into the steady-state characteristics of the Riccati-like equations involved in the finite-horizon control solutions. A necessary and sufficient condition for the existence of a steady-state solution was established for the scalar case. Due to the complexities involved in the matrix case, only a sufficient condition was developed that guaranteed the existence of a steady-state solution. This sufficient condition for the multivariable case was shown to reduce to the necessary and sufficient condition for the scalar case. Extensive simulations were provided for both cases to verify these conditions.

Possible future work in this area might investigate the continuous-time case. The continuous-time counterpart of the general controller proposed in this work would most likely be needlessly complex. Therefore, it would seem advantageous to develop a sampling procedure for continuous-time systems, such that, the resulting discretized system would fit into the framework of the discrete-time solution already given in this work.

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APPENDIX A.1

```
C     * * * * * * * * * * * * * * * * * * * * * * * * *
C     *  MARK CIANCETTA                               *
C     *  20 JUL 87                                    *
C     *  THIS PROGRAM COMPUTES THE THRESHOLD VALUE,    *
C     *  AND SK FOR THE SCALAR CASE                   *
C     * * * * * * * * * * * * * * * * * * * * * * * * *
C
      REAL M,N
C
      A = 1.1
      B = 1.0
      Q = 1.0
      T = .2
      SK = 0.
      R = 1.0
C
C     * * * * INPUT VALUES FOR M, N, AND W * * * *
C
      WRITE (5,5)
5     FORMAT (1X,'M = ?')
      READ (5,10) M
10    FORMAT (F5.2)
      WRITE (5,15)
15    FORMAT (1X,'W = ?')
      READ (5,10) W
      WRITE (5,17)
17    FORMAT (1X,'N = ?')
      READ (5,10) N
C
C     * * * * CALCULATE THE THRESHOLD CONDITION * * * *
C
      g = (A**2 + W/2) - ((A*B + n/2)**2/(B**2 + M/2))
C
C     * * * * OUTPUT VALUE AND PROMPT TO CONTINUE * * * *
C
      WRITE (5,20) g
20    FORMAT (1X,'THE THRESHOLD VALUE g = ',F6.3)
      WRITE (5,25)
25    FORMAT (1X,'ENTER 1 TO CONTINUE')
      READ (5,30) I
30    FORMAT (I2)
      IF (I.NE.1) GO TO 50
C
C     * * * * OPEN A FILE AND INITIALIZE IT * * * *
C
      OPEN (UNIT=10,FILE='SK',STATUS='NEW')
      WRITE (5,35) g
      WRITE (10,35) g
      WRITE (5,35) SK
      WRITE (10,35) SK
C
```

```
C      * * * * CALCULATE THE LOG OF SK * * * *
C
       DO 40 I=1,50
             SK = Q + SK*(A**2 + W/2)
      *      -(((T + SK*(A*B + N/2))**2)/
      *      (R + SK*(B**2 + M/2)))
C
C      * * * * SET VALUES <1 TO 0 TO AVOID NEG OUTPUT * * *
C
             IF (SK.GT.1.)THEN
                   S = ALOG10(SK)
             ELSE
                   S = 0.
             ENDIF
C
C      * * * * OUTPUT RESULTS * * * *
C
             WRITE (5,35) S
             WRITE (10,35) S
35           FORMAT (3X,F15.5)
C
40     CONTINUE
50     STOP
       END
```

APPENDIX A.2

```
C     * * * * * * * * * * * * * * * * * * * * * * * * * *
C     * MARK CIANCETTA                                    *
C     * 19 JUN 87                                         *
C     * THIS PROGRAM COMPUTES THE THRESHOLD VALUE, SK,   *
C     * AND JK FOR THE MULTIVARIABLE CASE                 *
C     * * * * * * * * * * * * * * * * * * * * * * * * * *
C
      DIMENSION A(2,2),AT(2,2),B(2),BBT(2,2),Z(2,2)
      DIMENSION Q(2,2),SKT(2,2),SK(50,4),ZI(2,2)
      REAL M, JK(50)
C
      DATA A/.1,0.,.8,.2/,B/.1,.2/,SK/200*0./,
     *     Q/1.,0.,0.,1./,JK/50*0./
      R = .1
C
C     * * * * INPUT m AND w * * * *
C
      WRITE (5,5)
5     FORMAT (1X,'m = ?')
      READ (5,10) M
10    FORMAT (F5.2)
      WRITE (5,15)
15    FORMAT (1X,'w = ?')
      READ (5,10) W
C
C     * * * * CALCULATE AT(ranspose) * * * *
C
      AT(1,1) = A(1,1)
      AT(1,2) = A(2,1)
      AT(2,1) = A(1,2)
      AT(2,2) = A(2,2)
C
C     * * * * CALCULATE BBT(ranspose) * * * *
C
      BBT(1,1) = B(1)**2
      BBT(1,2) = B(1)*B(2)
      BBT(2,1) = B(1)*B(2)
      BBT(2,2) = B(2)**2
C
C     * * * * CALCULATE Z = I + 1/m BBT * * * *
C
      Z(1,1) = 1 + BBT(1,1)/M
      Z(1,2) = BBT(1,2)/M
      Z(2,1) = BBT(2,1)/M
      Z(2,2) = 1 + BBT(2,2)/M
C
C     * * * * DET. OF Z * * * *
C
      D = Z(1,1)*Z(2,2) - Z(1,2)*Z(2,1)
C
C     * * * * CALCULATE INVERSE * * * *
```

```
C
      ZI(1,1) = Z(2,2)/D
      ZI(1,2) = -Z(1,2)/D
      ZI(2,1) = -Z(2,1)/D
      ZI(2,2) = Z(1,1)/D
C
C     * * * * CALCULATE AT(ranspose)ZI(nverse)A -> Z * * *
C
C
      CALL MULT(AT,ZI,Z)
      CALL MULT(Z,A,Z)
C
C     * * * * ADD wI * * * *
C
      Z(1,1) = Z(1,1) + W
      Z(2,2) = Z(2,2) + W
C
C     * * * * CALCULATE MAG OF THE EIGENVALUES * * * *
C
      C = Z(1,1) + Z(2,2)
      D = 4*(Z(1,1)*Z(2,2) - Z(1,2)*Z(2,1))
      D = C**2 - D
C
      IF (D.GE.0.)THEN
         E1 = ABS((C + SQRT(D))/2)
         E2 = ABS((C - SQRT(D))/2)
      ELSE
         D = -D
         E1 = SQRT(C**2 + D)/2
         E2 = SQRT(C**2 - D)/2
      ENDIF
C
C     * * * * FIND LARGEST EIGENVALUE * * * *
C
      IF (E1.GT.E2)THEN
         E = E1
      ELSE
         E = E2
      ENDIF
C
C     * * * * OUTPUT THRESHOLD VALUE AND CONTINUE? * * * *
C
      WRITE (5,30) E
30    FORMAT (1X,'THE MAXIMUM EIGENVALUE IS ',F6.3)
      WRITE (5,32)
32    FORMAT (1X,'ENTER 1 TO CONTINUE')
      READ (5,33) N
33    FORMAT (I2)
      IF (N.NE.1) GO TO 50
C
C     * * * * OPEN FILE AND OUTPUT S(0) AND J(0) * * * *
```

```
C
      OPEN (UNIT=10,FILE='SK',STATUS='NEW')
      WRITE (5,35) SK(1,1),SK(1,2),SK(1,3),SK(1,4),JK(1)
      WRITE (10,35) SK(1,1),SK(1,2),SK(1,3),SK(1,4),JK(1)
C
C     * * * * STORE PREVIOUS SK IN SKT(emp) * * * *
C
      DO 40 I=1,50
            SKT(1,1) = SK(I,1)
            SKT(1,2) = SK(I,2)
            SKT(2,1) = SK(I,3)
            SKT(2,2) = SK(I,4)
C
C     * * * * CALCULATE tr [SKT] /2 * * * *
C
            TR = (SKT(1,1) + SKT(2,2))/2
C
C     * * * * CALCULATE B(transpose)SKB * * * *
C
            BTSKB = B(1)**2*SKT(1,1) + B(1)*B(2)*SKT(2,1)
     *            + B(1)*B(2)*SKT(1,2) + B(2)**2*SKT(2,2)
C
C     * * * * THE INVERSE ASSOCIATED WITH S(k+1) * * * *
C
            C = 1/(R + BTSKB + M*TR)
            CALL MULT(SKT,BBT,Z)
            CALL MULT(Z,SKT,Z)
C
C     * * * * SUBTRACT THIS FROM SK * * * *
C
            SKT(1,1) = SKT(1,1) - C*Z(1,1)
            SKT(1,2) = SKT(1,2) - C*Z(1,2)
            SKT(2,1) = SKT(2,1) - C*Z(2,1)
            SKT(2,2) = SKT(2,2) - C*Z(2,2)
C
C     * * * * MULT FROM THE LEFT BY A(transpose) * * * *
C
            CALL MULT(AT,SKT,Z)
C
C     * * * * MULT FROM THE RIGHT BY A * * * *
C
            CALL MULT(Z,A,Z)
C
C     * * * * ADD Q AND (w/2) * tr (SK) * * * *
C
            SK(I+1,1) = Z(1,1) + 1. + W*TR
            SK(I+1,4) = Z(2,2) + 1. + W*TR
            SK(I+1,2) = Z(1,2)
            SK(I+1,3) = Z(2,1)
C
C     * * * * CALCULATE JK * * * *
```

```
C
            JK(I+1) = (SK(I+1,1)+SK(I+1,2)+
     *                 SK(I+1,3)+SK(I+1,4))/2
C
C     * * * * OUTPUT THE RESULTS * * * *
C
         J = I+1
         WRITE(5,35)SK(J,1),SK(J,2),SK(J,3),SK(J,4),JK(J)
         WRITE(10,35)SK(J,1),SK(J,2),SK(J,3),SK(J,4),JK(J)
35       FORMAT (1X,5(2X,F15.5))
C
40    CONTINUE
50    STOP
      END
```

APPENDIX A.3

```
C     * * * * * * * * * * * * * * * * * * * * * * * *
C     *  MARK CIANCETTA                              *
C     *  15 JUNE 1987                                *
C     *  THIS PROGRAM MULTIPLIES TWO 2X2 MATRICES,   *
C     *  A AND B, AND RETURNS THE PRODUCT IN C       *
C     * * * * * * * * * * * * * * * * * * * * * * * *
C
      SUBROUTINE MULT(A,B,C)
C
      DIMENSION A(2,2),B(2,2),C(2,2)
C
      DO 4 I = 1,2
            DO 3 J = 1,2
                  C(I,J)  = A(I,1)*B(1,J)  + A(I,2)*B(2,J)
3           CONTINUE
4     CONTINUE
C
      RETURN
      END
```