Naval Research Laboratory
Washington, DC 20375-5000

NRL Memorandum Report 61

On the Generation of Waveforms Having Comb-Shaped Spectra

BRUCE A. BLACK

ASEE Summer Faculty
Information Technology Division

May 6, 1988

REPORT DOCUMENTATION PAGE

Form Approved OMB No 0704-0188

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY 2b. DECLASSIFICATION / DOWNGRADING SCHEDULE	3 DISTRIBUTION / AVAILABILITY OF REPORT Approved for public release; Distribution unlimited
4. PERFORMING ORGANIZATION REPORT NUMBER(S) NRL Memorandum Report 6190	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION Naval Research Laboratory 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION
6c. ADDRESS (City, State, and ZIP Code) Washington, DC 20375-5000	7b. ADDRESS (City, State, and ZIP Code)
8a. NAME OF FUNDING / SPONSORING ORGANIZATION Air Force Systems Command Electronics Systems Division 8b. OFFICE SYMBOL (If applicable) TCVH	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
8c. ADDRESS (City, State, and ZIP Code) Hanscom AFB, MA 01731	10. SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO: 27423F PROJECT NO: (75)-0173-00 TASK NO: FY7620-84-00181 WORK UNIT ACCESSION NO: DN630-141

11. TITLE (Include Security Classification)
On the Generation of Waveforms Having Comb-Shaped Spectra

12. PERSONAL AUTHOR(S)
Black, Bruce A.

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15 PAGE COUNT
Final	FROM 6/85 TO 8/85	1988 May 6	41

16 SUPPLEMENTARY NOTATION

17 COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)	
FIELD	GROUP	SUB-GROUP	Jamming	Frequency Hopping
			Spread spectrum	Multitone signal
			Anti-jamming	Comb spectra

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

Two techniques, direct-sequence spread-spectrum and frequency hopping are widely used to provide an anti-jam capability for communication systems. To test the effectiveness of such systems, suitable jammers must be devised. A frequency hopping system can be jammed by wideband noise, by a following frequency hopped carrier (if the hopper is slow enough), or by a multitone jamming signal. Multitone jamming signals are considered advantageous in that jamming power is not wasted on frequencies never visited by the frequency hopper, as is the case when wideband noise is used. A multitone signal can be generated by summing the outputs of a set of oscillators, but when a large number of oscillators would by required, it is more realistic to generate a single jamming waveform having a comb-shaped spectrum. This report considers several classes of waveforms having power spectra that are approximately comb-shaped. The report is tutorial in nature and an effort is made to relate the properties of the comb spectrum to the time-domain properties of the waveforms. Two classes of waveforms are of particular interest. These are the repeated pseudorandom

(continues)

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21 ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL Nelson M. Head	22b TELEPHONE (Include Area Code): (202) 767-2148 22c OFFICE SYMBOL: Code 5572

19. Abstract (continued)

sequence, used to modulate a carrier, and the frequency-swept sinusoid. A good quality comb spectrum should have uniformly strong "teeth" over its passband, and the spectrum should fall away rapidly at the band edges. To minimize the distortion caused by saturating power amplifiers, it is helpful if the time-domain function has a constant envelope. It is shown that a pseudorandom sequence that is low-pass filtered prior to modulation can produce a very uniform comb with sharp band edges, provided that the required width of the comb is not excessive. A swept-frequency sinusoid can produce a comb of very wide bandwidth, also with sharp band edges. The swept-frequency sinusoid has the additional advantage of a constant envelope, buy it is found that the comb itself may not be very uniform. For some cases of practical interest the variation in power from tooth to tooth of the comb can be as great as 35 dB.

CONTENTS

ON THE GENERATION OF WAVEFORMS HAVING COMB-SHAPED SPECTRA

1. INTRODUCTION

1.1. Jamming strategies

Two techniques, direct-sequence spread-spectrum and frequency hopping, are widely used to provide an anti-jam capability for communication systems [1, 2, 3]. To test the effectiveness of such anti-jam systems, suit able jammers must be devised. Traditionally, a jammer transmits a tone or a noise waveform at the highest feasible power level on the carrier frequency of the target system. The object is to render voice communications unintelligible, or in the case of a digital target system, to increase the bit error rate to an unacceptable level. Many variations on the waveform transmitted by the jammer are possible. For instance the jamming carrier may be modulated, or the jamming carrier frequency may be offset from the carrier frequency of the target system.

Direct-sequence systems and frequency hopping systems use different mechanisms to provide resistance to jamming. The "de-spreading" process used in a direct-sequence receiver will spread the spectrum of a jamming carrier, turning the jamming signal into a wideband noise-like waveform. Subsequently most of the power in the jamming signal will be rejected by the receiver's IF filter. A well designed direct-sequence system will have the ability to spread a modulated jamming signal as well as as a pure carrier, even if the modulation attempts to mimic the spreading signal itself.

In a frequency hopping system the transmitter changes its carrier frequency at frequent intervals in a seemingly random manner, hopping among a large number of possible frequencies. The receiver knows the hopping pattern, and tunes in synchronism with the transmitter allowing continuous reception of the transmitted data. A single-tone or narrow-band jammer on the other hand, will be able to jam only one of the possible carrier frequencies, and will be effective only a small fraction of the time. Thus, as in the case of the direct-sequence system, only a small fraction of the available jamming power will actually affect the receiver.

In considering strategies to be used in jamming a frequency hopping system, it is important to be aware that there can be a limit to the amount of jammer power that it is worth transmitting on any given frequency. To see why, consider the case of a data link using binary frequency shift keying (FSK). A noncoherent FSK receiver will measure the power at the mark and space frequencies during each bit interval, and will decide between a one and a zero depending on whether more power is measured at the mark frequency or at the space frequency. If a jammer were to introduce enough power at, say, the mark frequency, it could force the receiver to decode every bit as a binary one. The resulting error rate of fifty percent would be adequate to overwhelm even the most robust error-correcting code. Clearly, once a jammer has sufficient power to disrupt a receiver in this way, an increase in jammer power will not cause any additional disruption in communications. When a single tone jammer is used against a frequency hopper, the worst effect of the jammer would be to completely disrupt one of the set of carrier frequencies. Jammer power in excess of that necessary to cause this disruption would be wasted, and could be better used in some other way.

Two more effective strategies are available for attacking a frequency hopping system. If the hopping rate is slow enough, it may be possible for a jammer to detect each transmission frequency while it is in use, and to tune the jammer in response. A rapidly following jammer of this sort would render the frequency hopping ineffective, and this strategy could be used regardless of whether or not

Manuscript approved December 8, 1987.

the jammer had sufficient power to overwhelm the communications receiver. When the hopping rate is too fast to allow a following jammer, the alternative strategy is to try to simultaneously jam as many as possible of the carrier frequencies used by the frequency hopper. This necessitates use of a wideband jamming signal, and bandlimited white noise is one example of a possible jamming signal.

A jammer wishing to generate an effective wideband jamming signal must overcome two problems. First is the generation of a possibly complicated wideband jamming waveform. Second is the effective use of the limited power available to the jammer. A bandlimited white noise jamming signal, for example, is easy to generate but wasteful of power. Power is wasted, probably unavoidably, at each of the carrier frequencies used by the frequency hopper because the hopper uses each frequency for only a small percentage of the time, while the jammer transmits on each frequency continuously. Power is also wasted because the bandlimited white noise includes frequency components that lie between frequencies used by the frequency hopper. A more effective use of jammer power can be made by using a multitone jamming signal. Here the jamming signal is a sum of sinusoidal components at carrier frequencies used by the hopper. No jammer power is wasted on frequencies that are unused by the frequency hopping system. Ideally, only enough power will be carried on each tone to disrupt communications on the corresponding hopper frequency. The optimum number of tones in the multitone signal will then be a function of the total jammer power.

A multitone jamming signal has a comb-shaped spectrum. Such a signal can conceptually be generated by summing the outputs of a set of oscillators tuned to the appropriate carrier frequencies. A frequency hopping system may hop among hundreds or even thousands of carrier frequencies, however, and it is likely to prove impractical to generate the multitone signal in this way. This report presents and discusses the attributes of several waveforms which possess more or less comb-shaped spectra and may be practical to generate. The presentation relies heavily on Fourier analysis to describe the candidate waveforms and to relate the waveform parameters to the parameters of their spectra. Of particular interest are waveforms generated by modulating a carrier by a pseudorandom binary sequence and frequency sweeping a carrier over the band of interest using triangular or ramp-shaped sweeping signals. It is envisioned that waveforms such as those described in this report will be useful in testing the effectiveness of frequency hopping anti-jam systems.

1.2. The multitone jammer

An example of a comb-shaped spectrum is given in Figure 1 along with

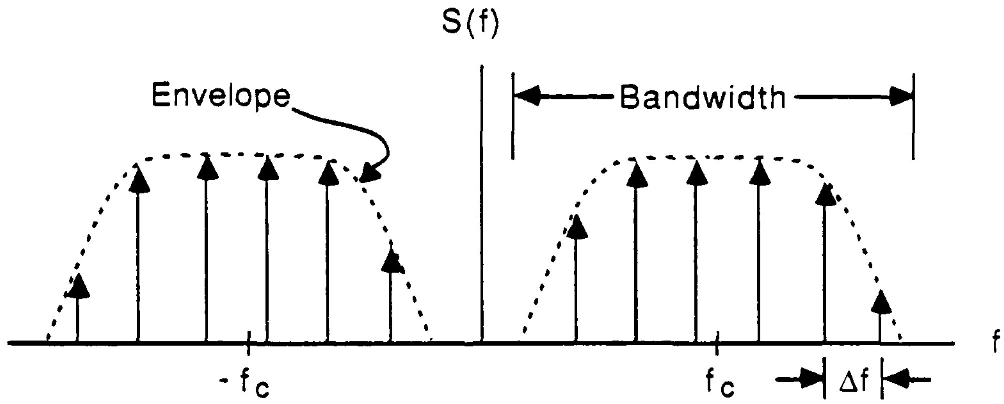

Figure 1: A Comb-Shaped Power Spectrum

some terminology that will be used to describe such spectra. Only the power spectrum is shown, as the phase spectrum is not relevant to the effectiveness of a jammer. In the sequel, power spectra will usually be designated $S(f)$, with a subscript used to distinguish specific spectra when necessary. The "tooth spacing," or frequency interval between spectral lines, will be denoted Δf (delta f). We will consider only waveforms for which the tooth spacing is uniform, even though it is certainly possible for a frequency hopper to hop among non-uniformly spaced carrier frequencies. The frequency shown as

f_c denotes the center of the spectrum or, where appropriate, the carrier frequency. There may or may not be a spectral line at f_c and f_c may or may not fall at an integral multiple of Δf.

We can now specify the desirable properties of a multitone jamming signal. Clearly, the envelope of the spectrum should be as rectangular as possible. In the same way that a bandpass filter is specified, we can identify a "passband," lower and upper "stopbands," and transition regions. Within the passband, the comb should have teeth of approximately equal spectral height. In the stopbands the spectral level should be low enough so that the multitone jamming signal does not interfere with radio services in these bands. To avoid interference, it may be necessary that the transition bands occupy spectral space used by the frequency hopper. It is desirable, then, that the transitions be as sharp as possible.

In field applications, the multitone waveform will be applied to an RF power amplifier. Unless the amplifier is linear, intermodulation will occur between components of the multitone waveform. As the waveforms under consideration can contain a great many components, a significant fraction of the power amplifier output power can be expended on intermodulation terms that do not contribute to the jamming capability of the transmitted signal. A multitone signal that has a constant envelope will be resistant to intermodulation caused by amplifier nonlinearity, and so constant envelope signals will be of special interest in the sequel.

1.3. Organization of the report

The remainder of this report is organized as follows. Section 2, entitled Preliminaries, contains the Fourier transform background showing how waveforms having comb-shaped power spectra can be generated and how the parameters of the waveforms are related to the parameters of their spectra. Section 2.1 considers the use of periodic waveforms to generate comb-shaped spectra. Section 2.2 explores the relationship between Fourier transform and power spectrum for periodic waveforms. Section 2.3 introduces the use of modulation to shift the center frequency of a spectrum. Section 3, entitled Candidate Waveforms, contains detailed analyses of some waveforms of particular interest. Section 3.1 considers the rectangular pulse train as an impractical but instructive example. Section 3.2 contains the important case of the repeated pseudonoise sequence. Examples of pseudonoise combs are given in Section 3.2.1, and an examination of the use of minimum shift keying to increase bandwidth efficiency while maintaining constant envelope is given in Section 3.2.2. Section 3.3 presents the second important case of the frequency-swept sinusoid. Triangle modulation is presented in Section 3.3.1, sawtooth modulation is presented in Section 3.3.2. The use of the fast Fourier transform to evaluate the power spectra is discussed in Section 3.3.3 and examples of spectra of various bandwidths and line spacings are given in Section 3.3.4. Section 4 contains the final summary and conclusions.

2. PRELIMINARIES

2.1. The comb-shaped spectrum

A comb spectrum can be generated by a periodic waveform. The tooth spacing Δf is the reciprocal of the period. The simplest example is a periodic train of impulses:

2.1.1. Impulse train

Let

$$g(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT). \tag{1}$$

This waveform is shown in Figure 2.

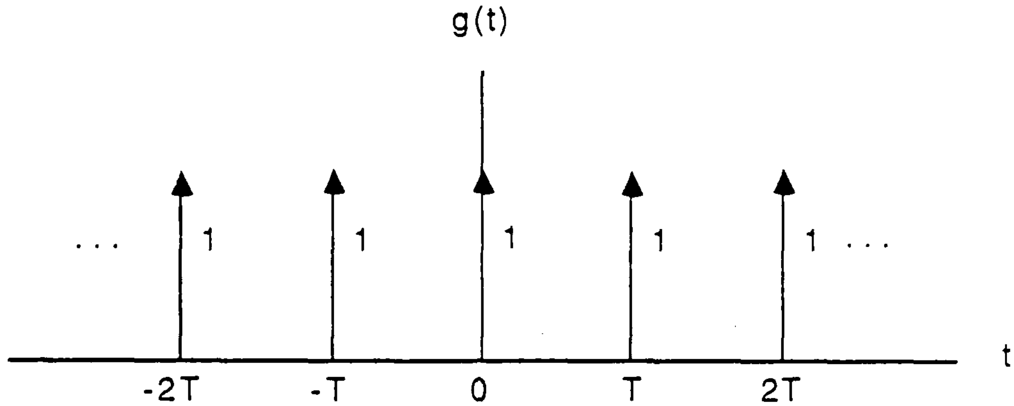

Figure 2: An Impulse Train

The spectrum of $g(t)$ can be determined by obtaining the Fourier transform $G(f)$. It turns out that $g(t)$ is not Fourier integrable, and so an indirect approach through the Fourier series is called for. Expanding $g(t)$ in its Fourier series gives

$$g(t) = \sum_{n=-\infty}^{\infty} G_n e^{j\frac{2\pi}{T}nt}, \tag{2}$$

where for each n,

$$G_n = \frac{1}{T} \int_{-T/2}^{T/2} g(t) e^{-j\frac{2\pi}{T}nt} \, dt. \tag{3}$$

Substituting for $g(t)$ gives

$$G_n = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-j\frac{2\pi}{T}nt} \, dt = \frac{1}{T}, \tag{4}$$

so that

$$g(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{j\frac{2\pi}{T}nt}. \tag{5}$$

Now

$$F[1] = \delta(f) \tag{6}$$

and using the frequency shift theorem,

$$F\left[1\cdot e^{j\frac{2\pi}{T}nt}\right] = \delta(f - \frac{n}{T}). \tag{7}$$

We can then transform both sides of equation (5) to yield

$$G(f) = \sum_{n=-\infty}^{\infty} \frac{1}{T}\delta(f - \frac{n}{T}). \tag{8}$$

This spectrum is shown in Figure 3.

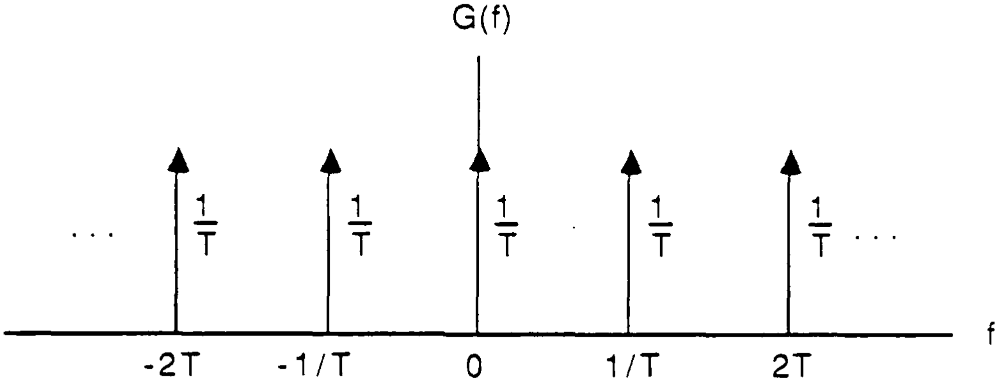

Figure 3: Spectrum of an Impulse Train

The impulse train has a perfectly uniform comb spectrum, and would be an ideal multitone jamming signal, except for its infinitely wide bandwidth and for the difficulty in generating impulses. The case of a train of "impulse-like" signals is a special case of the general periodic waveform.

2.1.2. General periodic waveform

Now consider a waveform $g(t)$ with period T as shown in Figure 4.

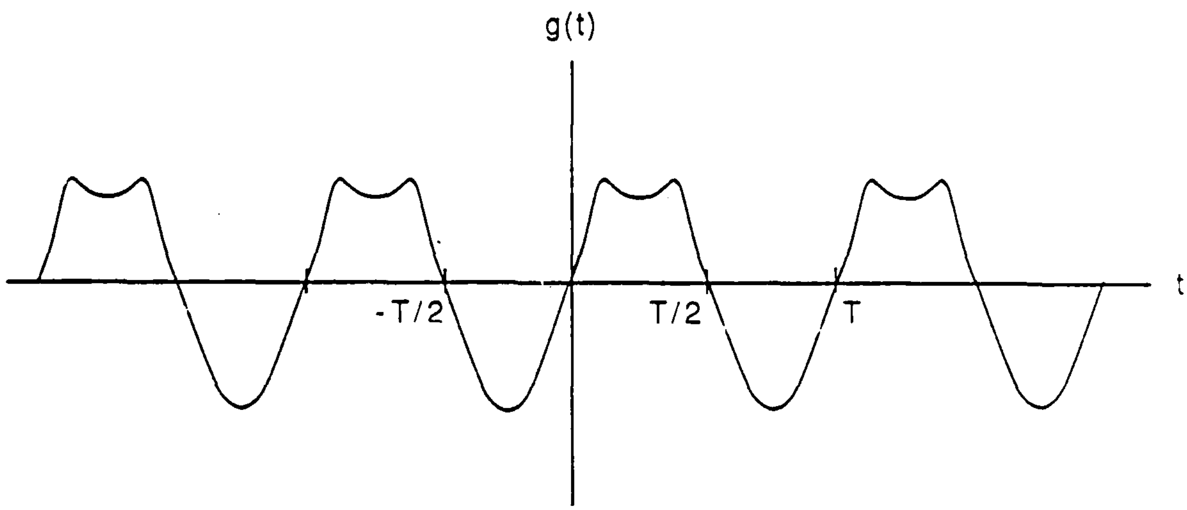

Figure 4: A Periodic Waveform

Suppose one period of the waveform $g(t)$ is identified as the pulse $p(t)$ as shown in Figure 5. We can write $g(t)$ in terms of the pulse $p(t)$ as

$$g(t) = \sum_{k=-\infty}^{\infty} p(t - kT). \tag{9}$$

That is,

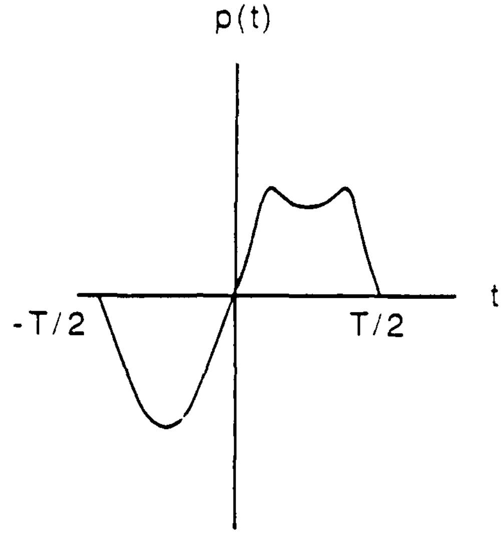

Figure 5: A Pulse

$$g(t) = p(t) * \sum_{k=-\infty}^{\infty} \delta(t - kT), \tag{10}$$

where the symbol * denotes convolution. Fourier transforming gives

$$\begin{aligned} G(f) &= P(f) \cdot F\left[\sum_{k=-\infty}^{\infty} \delta(t - kT)\right] \\ &= P(f) \sum_{n=-\infty}^{\infty} \frac{1}{T} \delta(f - \frac{n}{T}), \end{aligned} \tag{11}$$

or

$$G(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} P(\frac{n}{T}) \delta(f - \frac{n}{T}). \tag{12}$$

Equations (9) through (11) make no use of the property that $p(t)$ extends only over one period from $-\frac{T}{2}$ to $\frac{T}{2}$. In fact, $p(t)$ may be a pulse of any duration and $g(t)$ may be formed in equation (9) as a sum of shifted overlapping pulses.

The spectrum $G(f)$ given by equation (12) is represented in Figure 6. Since $p(t)$ is arbitrary, $G(f)$ will be complex valued. A magnitude and a phase spectrum are shown. Since Figure 6 represents the general case, certain features will be typical of any periodic waveform. First, the line or "tooth" spacing is given by $\Delta f = \frac{1}{T}$ and depends only on the period of the periodic $g(t)$. Next the lines in the magnitude spectrum are weighted by the magnitude of the spectrum $P(f)$ of the pulse $p(t)$, so that $|P(f)|$ forms the envelope of the line spectrum. In our quest for a multitone jamming signal having a broad passband of uniform spectral height and rapid transition regions, we can expect the passband and transition region shape to come from proper selection of $p(t)$. The line spacing Δf can be independently fixed by the choice of period T.

2.2. Power spectrum

In this section a connection is established between Fourier transform and power spectrum. The Fourier transform is essential as a calculation aid, but results are more meaningfully presented as power spectra. This presentation allows us to suppress the phase information and to display predicted results in a form that can be compared with spectrum analyzer measurements of actual waveforms.

Consider first a simple sinusoid

$$g(t) = A\cos(2\pi f_0 t + \theta). \tag{13}$$

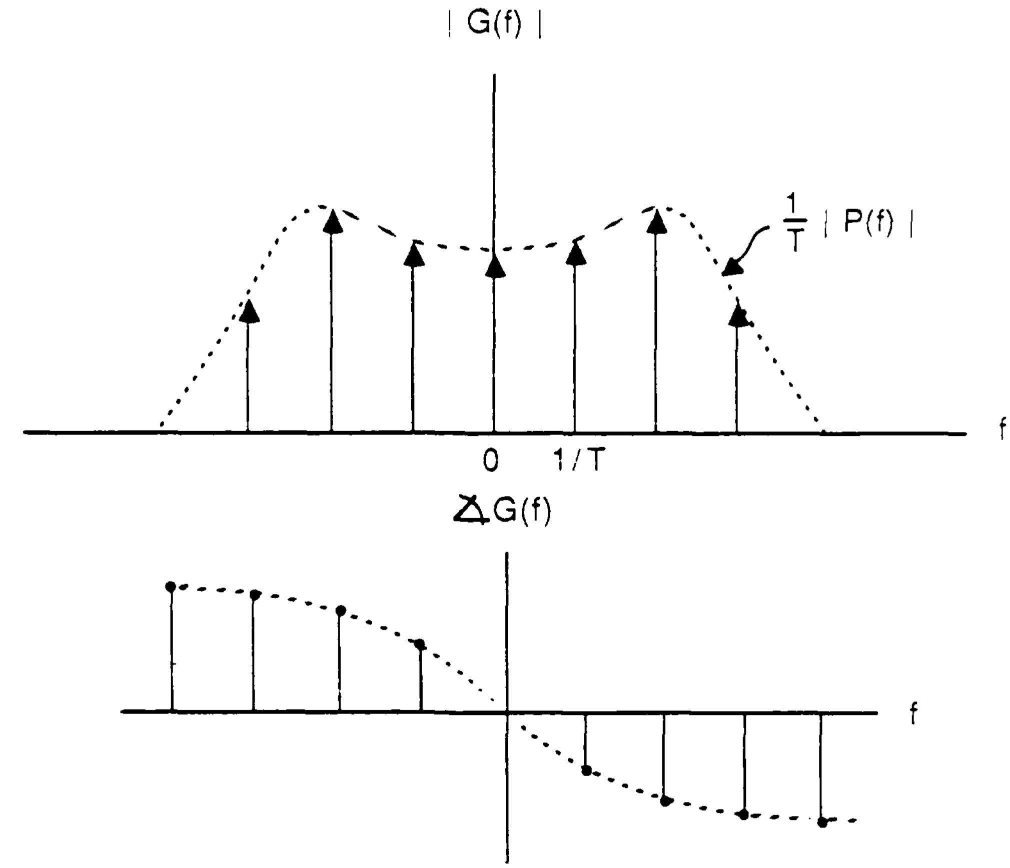

Figure 6: Spectrum G(f) of an Arbitrary Periodic g(t)

That is,

$$g(t) = \frac{A}{2}e^{j\theta}e^{j2\pi f_0 t} + \frac{A}{2}e^{-j\theta}e^{-j2\pi f_0 t}. \tag{14}$$

Using (7) we can Fourier transform $g(t)$ to obtain

$$G(f) = \frac{A}{2}e^{j\theta}\delta(f - f_0) + \frac{A}{2}e^{-j\theta}\delta(f + f_0). \tag{15}$$

The power spectrum of the sinusoid given by (13) is a density function that represents the way in which power is distributed with frequency. First we note that a sinusoid of peak value A has average power equal to $\frac{A^2}{2}$. This power is entirely associated with the frequency f_0, but it is customary to represent half the power as belonging to the negative frequency $-f_0$, thus producing a two-sided spectrum. The power spectrum $S_g(f)$ of $g(t)$ consists of two impulses of area $\frac{A^2}{4}$, one at frequency f_0 and one at frequency $-f_0$, as shown in Figure 7.

Let us now return to the arbitrary periodic signal $g(t)$ given by equation (9). Using (2) and (3) we can write $g(t)$ as a Fourier series, where for each n the coefficient G_n is given by

$$G_n = \frac{1}{T}P(\frac{n}{T}), \tag{16}$$

where $P(\frac{n}{T})$ is the Fourier transform $P(f)$ of the pulse $p(t)$ evaluated at the frequency $f = \frac{n}{T}$. The Fourier series represents $g(t)$ as a sum of sinusoids. This is made explicit in equation (17):

$$g(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T}P(\frac{n}{T})e^{j\frac{2\pi}{T}nt}$$

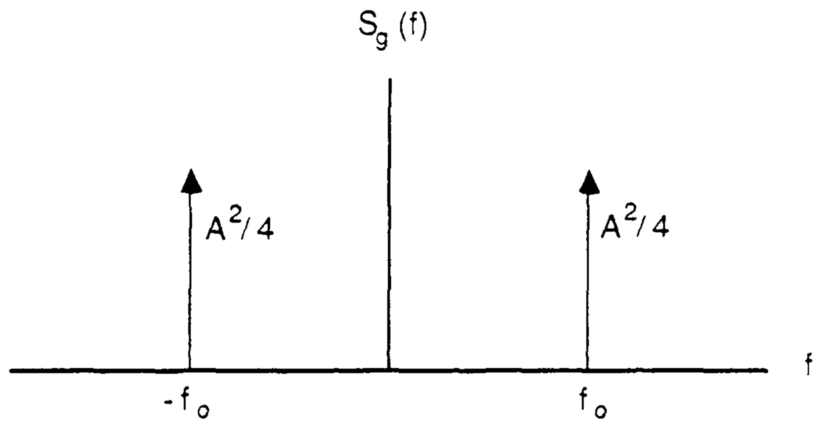

Figure 7: Power Spectrum of a Sinusoidal Signal

$$= \frac{1}{T}P(0) + \sum_{n=1}^{\infty}\frac{2}{T} \mid P(\frac{n}{T}) \mid \cos[2\pi\frac{n}{T}t + angle\ P(\frac{n}{T})]. \tag{17}$$

Parseval's theorem asserts that the average power in $g(t)$ is the sum of the powers in each of the sinusoidal components given in (17). We can then identify the DC power as $\frac{P^2(0)}{T^2}$ and the average power in the component at $f = \frac{n}{T}$ as $\frac{2 \mid P(\frac{n}{T}) \mid^2}{T^2}$. The power spectrum of $g(t)$ will then appear as a set of impulses located at $f = \frac{n}{T}$ for $n = \ldots,-2,-1,0,1,2,\ldots$. The area under the impulse located at $f = \frac{n}{T}$ will be $\frac{\mid P(\frac{n}{T}) \mid^2}{T^2}$. Figure 8 shows the magnitude of the Fourier transform of the pulse $p(t)$ and the power spectrum $S_g(f)$ of the periodic signal formed by repeating $p(t)$ every T seconds.

2.3. Modulation

The center frequency f_c of the comb waveform shown in Figure 1 can be established in two ways. First, the pulse $p(t)$ can be designed to provide a spectral envelope for the periodic $g(t)$ centered on f_c. Alternatively, the entire power spectrum $S_g(f)$ can be shifted in frequency if $g(t)$ is taken as a baseband signal and used to modulate a carrier $\cos 2\pi f_c t$. The second method is advantageous when f_c is to be much greater than the comb bandwidth. Also, modulation allows f_c to be chosen independently of the tooth spacing Δf. That is, in $S_g(f)$ all of the spectral lines lie at frequencies that are multiples of Δf. If $g(t)$ modulates a carrier, the resulting spectrum will still have lines spaced Δf apart, but the frequencies at which the lines fall can be arbitrarily chosen.

If $g(t)$ is an arbitrary baseband signal, let $s(t)$ be given by

$$s(t) = g(t)\cos 2\pi f_c t. \tag{18}$$

Then the power spectrum $S_s(f)$ of $s(t)$ is given by

$$S_s(f) = \frac{1}{4}S_g(f - f_c) + \frac{1}{4}S_g(f + f_c). \tag{19}$$

The spectrum $S_s(f)$ is depicted in Figure 9.

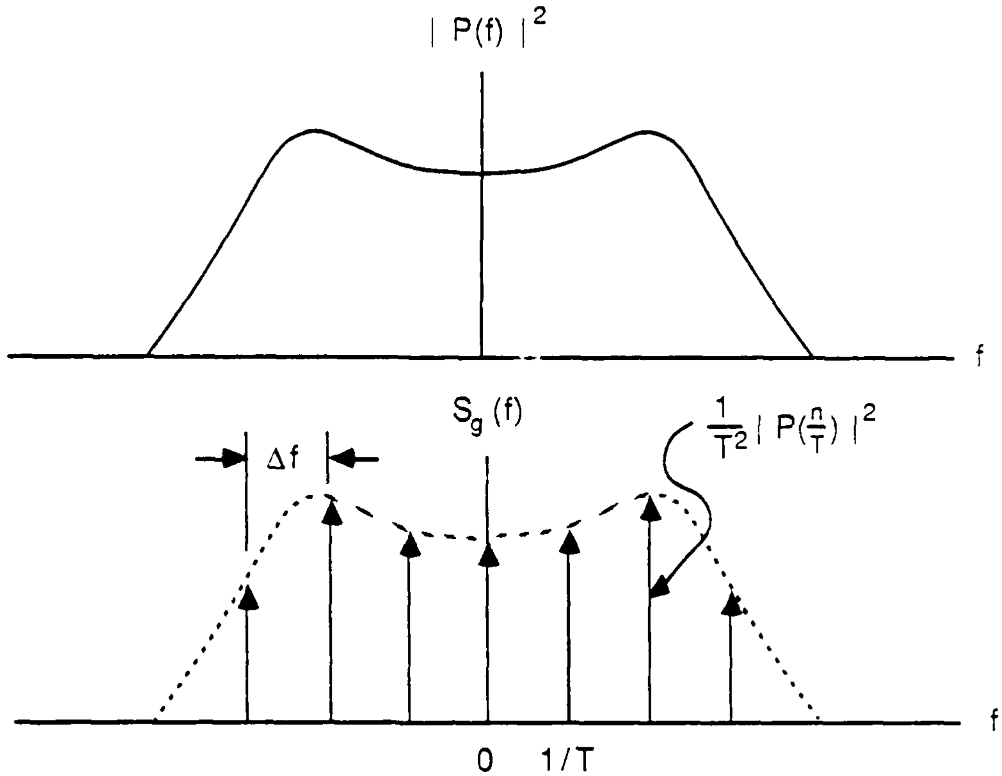

Figure 8: Transform of p(t) and Power Spectrum of g(t)

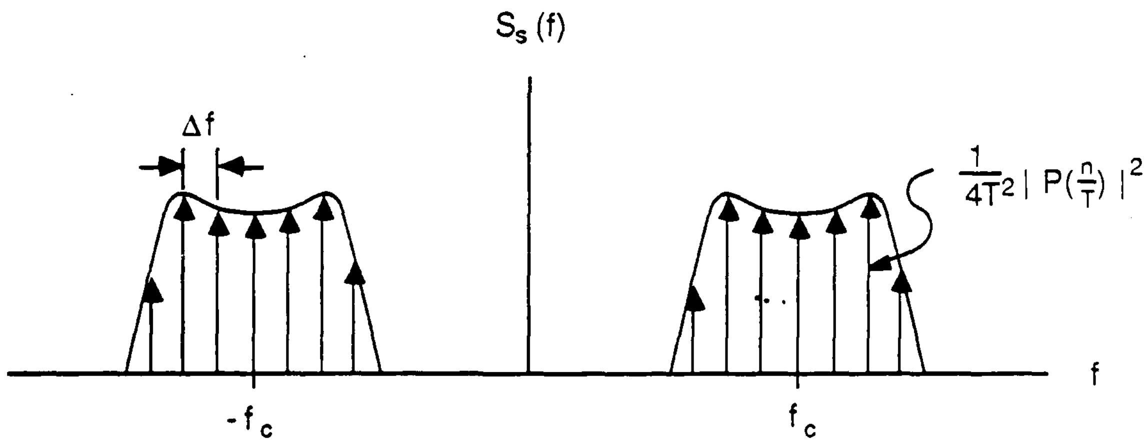

Figure 9: Power Spectrum of a Modulated Carrier

3. CANDIDATE WAVEFORMS

The preliminary relationships developed in the previous section show that the design of a waveform whose power spectrum has the shape of a comb with a specified number of uniformly strong teeth consists principally in design of a pulse $p(t)$ having a sufficiently rectangular magnitude spectrum of the appropriate bandwidth. Subsequently, a repetition period T can be chosen to provide the desired tooth spacing of $\Delta f = \frac{1}{T}$ and modulation can provide the desired center frequency f_c. In this section several candidates for the pulse waveform $p(t)$ are examined in detail. First, by way of introduction, we consider a single rectangular pulse. This waveform has the advantage of being easy to generate, but the periodic repetition of it suffers from a low duty cycle. Thus a high peak pulse value is required if the comb is to contain significant power. As a second candidate we consider a "pulse" in the form of a pseudonoise (PN) shift register sequence. This waveform has the same spectrum as the rectangular pulse, but has a much more favorable peak-to-average power ratio. Several alternative forms of modulation are considered for use with the PN pulse. Binary phase shift keying (BPSK) provides a constant envelope, but produces a comb with a broad transition region and significant sidelobes. Minimum shift keying (MSK) also has a constant envelope and provides lower sidelobes, but is more difficult to generate. If the waveform is filtered prior to BPSK, the sidelobes can be eliminated and the transition regions reduced, but the resulting RF signal will no longer have a constant envelope.

The next candidate waveform is the swept-frequency sinusoid. This waveform can be generated by a voltage-tuned oscillator (VTO) driven by a triangle or sawtooth waveform. The swept-frequency sinusoid turns out to have an approximately rectangular spectrum. The approximation improves as the frequency range of the sweep increases and the sweep rate decreases. Thus the swept-frequency waveform is most useful for generating very wide bandwidth combs. This is precisely the circumstance in which a pulse-type waveform modulating a carrier is difficult to generate. It turns out, however, that the spectrum of a swept-frequency sinusoid contains a certain amount of unavoidable ripple. In consequence, the periodic repetition will have a spectrum whose teeth are uneven. In the worst case, some of the teeth may be missing. We shall see that sawtooth modulation produces a more uniform passband than does triangle modulation, but this occurs at the expense of the transition regions, which are much sharper for a triangle modulating waveform. We will also find that the transition regions become sharper as the number of teeth in the comb increases.

3.1. Rectangular pulse

A single pulse $p(t)$ is shown in Figure 10 along with its Fourier transform P(f). Figure 11 shows the periodic repetition $g(t)$ and its power spectrum $S_g(f)$. Figure 12 shows a carrier modulated by $g(t)$ and its power spectrum $S_s(f)$.

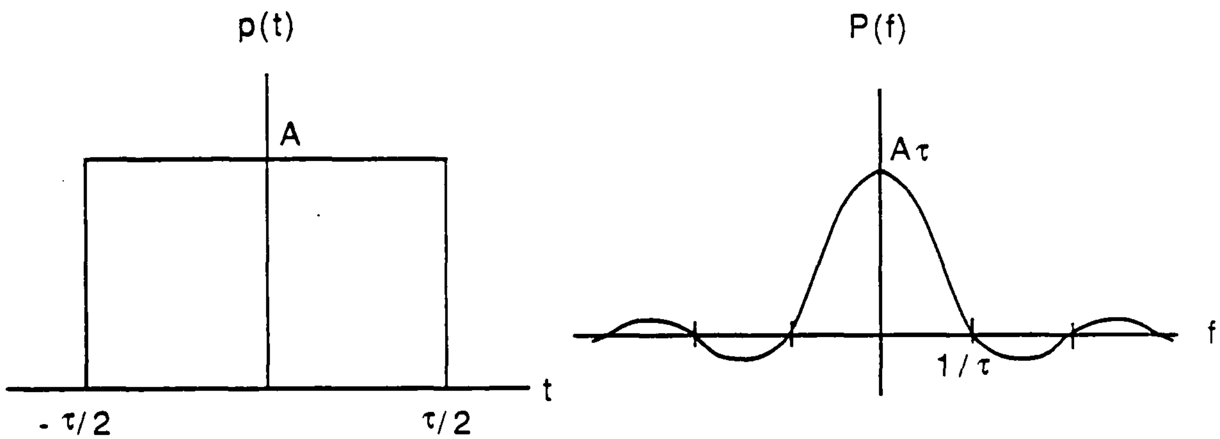

Figure 10: A Pulse and its Fourier Transform

If we write

$$p(t) = A\Pi(\frac{t}{\tau}), \tag{20}$$

then

$$P(f) = A\tau\frac{\sin\pi\tau f}{\pi\tau f} = A\tau sinc\tau f. \tag{21}$$

The spectrum has a $sinc^2$ envelope whose bandwidth is determined by the pulse width τ. A more uniform comb requires a narrower pulse. The area under each impulse in the power spectrum is

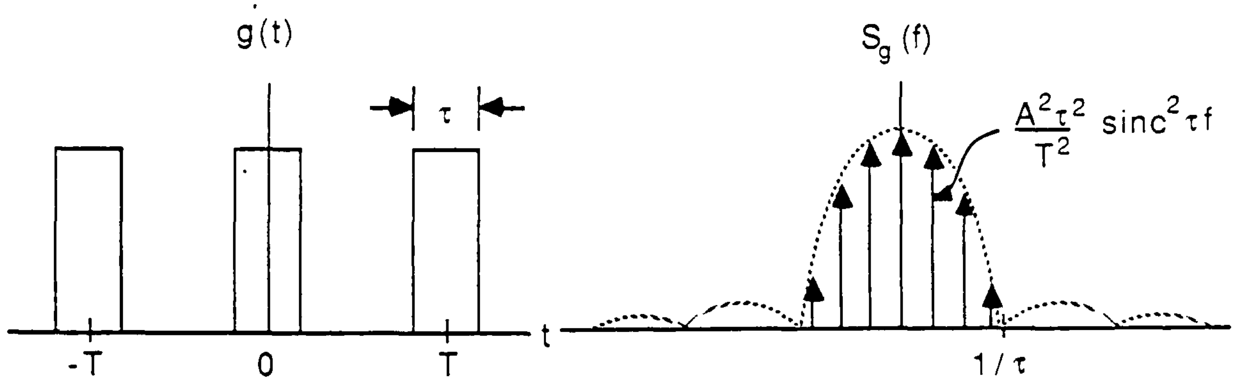

Figure 11: A Pulse Train and its Power Spectrum

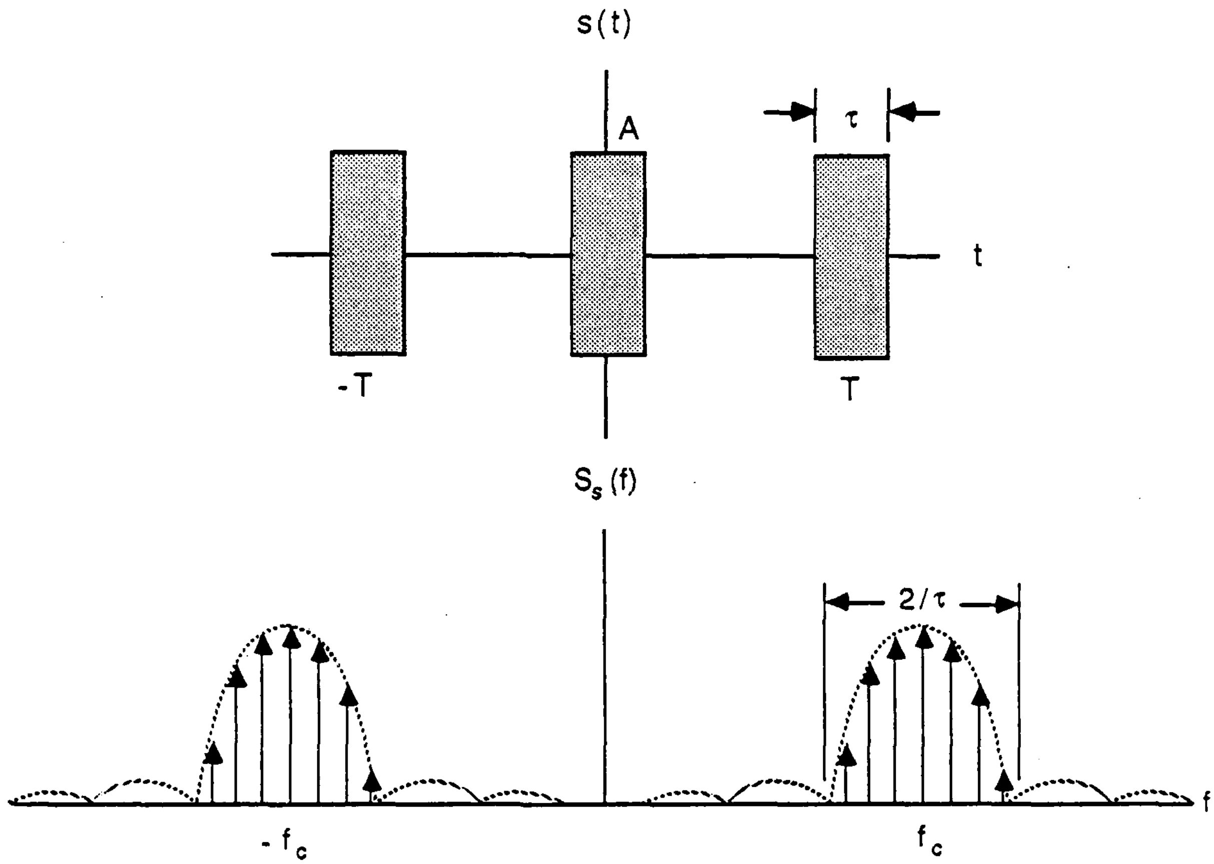

Figure 12: A Pulse-Modulated Carrier and its Power Spectrum

weighted by $\frac{\tau^2}{T^2}$, the square of the pulse duty cycle. Thus the total power in the pulse train depends on the duty cycle, and for a given amplitude A the power drops rapidly as τ is reduced.

Two variations on the pulse train are discussed next. First we consider the special case of 50% duty cycle, or a square wave. Then we return to the narrow pulses, but consider the possibility of filtering them to produce a more uniform comb.

3.1.1. Square wave

Figure 13 shows a square wave and its power spectrum. In this case, $\tau = \frac{T}{2}$. The interesting feature in this case is that every second spectral line is missing, and the nonzero comb teeth are thus spaced $2\Delta f$ apart.

3.1.2. Filtered pulses

The pulse $p(t)$ can be filtered to improve its special shape. If the pulse train $g(t)$ shown in Figure 11 is filtered by passing it through a low-pass filter of bandwidth W, the sidelobes of the spectrum can be truncated. If the filter passband is shaped to enhance the frequencies near cutoff, a comb can be produced that has a nearly rectangular spectral envelope. In the ideal case, in which the envelope of the comb is actually rectangular, the pulse $p(t)$ will be a *sinc* function. Figure 14 is a sketch depicting $g(t)$ and the power spectrum $S_g(f)$ in the ideal case.

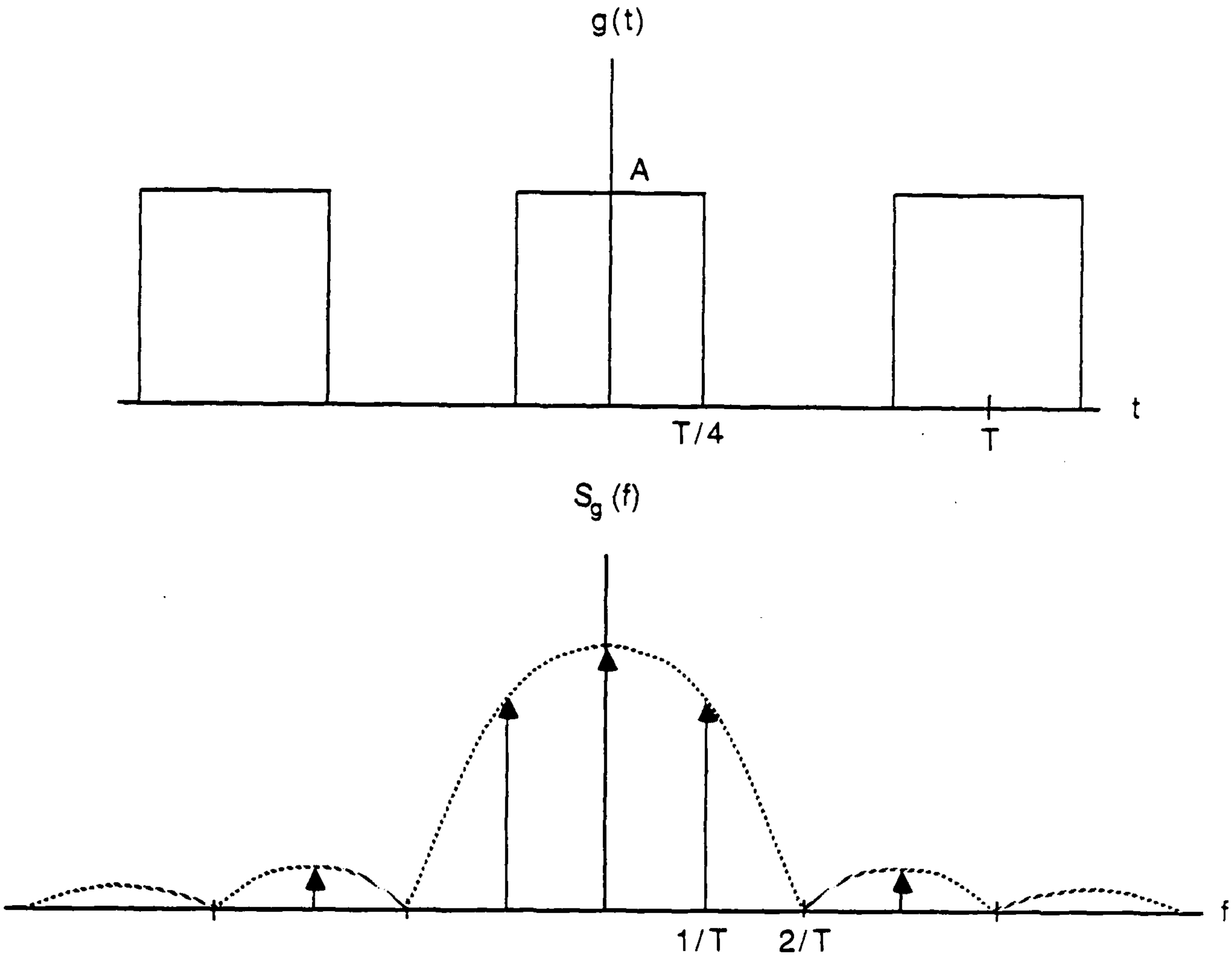

Figure 13: A Square Wave and its Power Spectrum

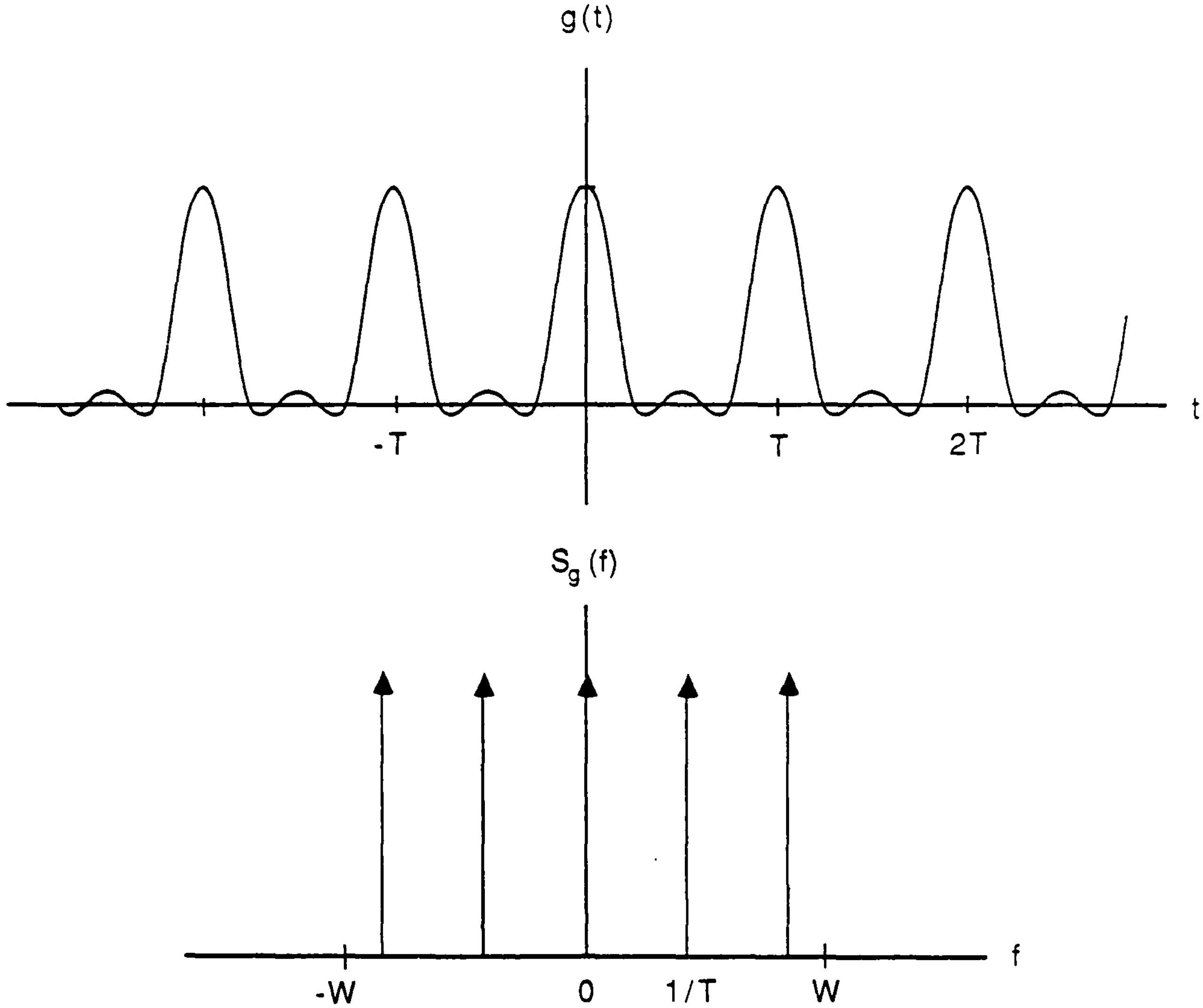

Figure 14: Filtered Pulse Train and its Uniform Comb Spectrum

3.2. Pseudonoise sequence

Suppose the pulse $p(t)$ is one period of a pseudonoise (PN) binary sequence. Let the sequence appear as a train of rectangular pulses of width τ and amplitude ± 1. The sequence length N can take any value $N = 2^n - 1$ for $n = 1,2,3,\ldots$. Figure 15 shows the "pulse" $p(t)$ for $N = 15$.

PN sequences are normally generated using shift registers. Feedback connections for generating sequences of various lengths are available from a variety of sources [1,3,4].

Let $g(t)$ be the periodic signal formed by repeating $p(t)$. The period T of $g(t)$ is given by $T = N\tau$. The power spectrum $S_g(f)$ is worked out in Ziemer and Peterson [3, p. 37], and is given by

$$S_g(f) = \sum_{n=-\infty}^{\infty} P_n \delta(f - \frac{n}{t}), \tag{22}$$

where

$$P_n = \frac{N+1}{N^2} sinc^2 \frac{n}{N} \quad \text{for } n \neq 0,$$

and

$$P_0 = \frac{1}{N^2}.$$

This spectrum is sketched in Figure 16, and should be compared with the spectrum of a repeated

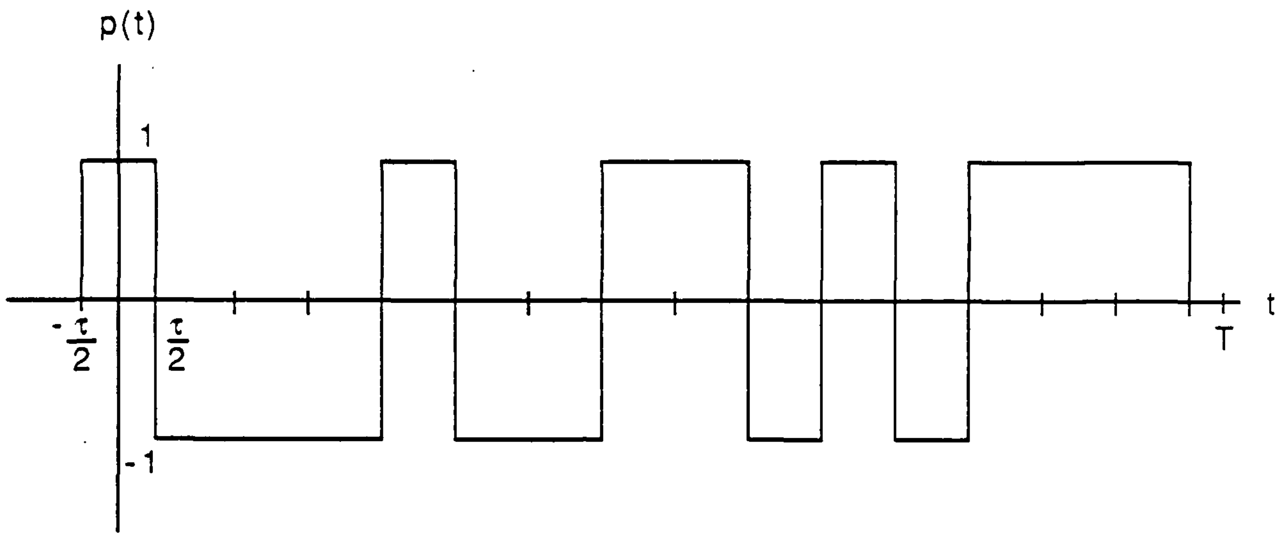

Figure 15: Pseudorandom Sequence for N = 15

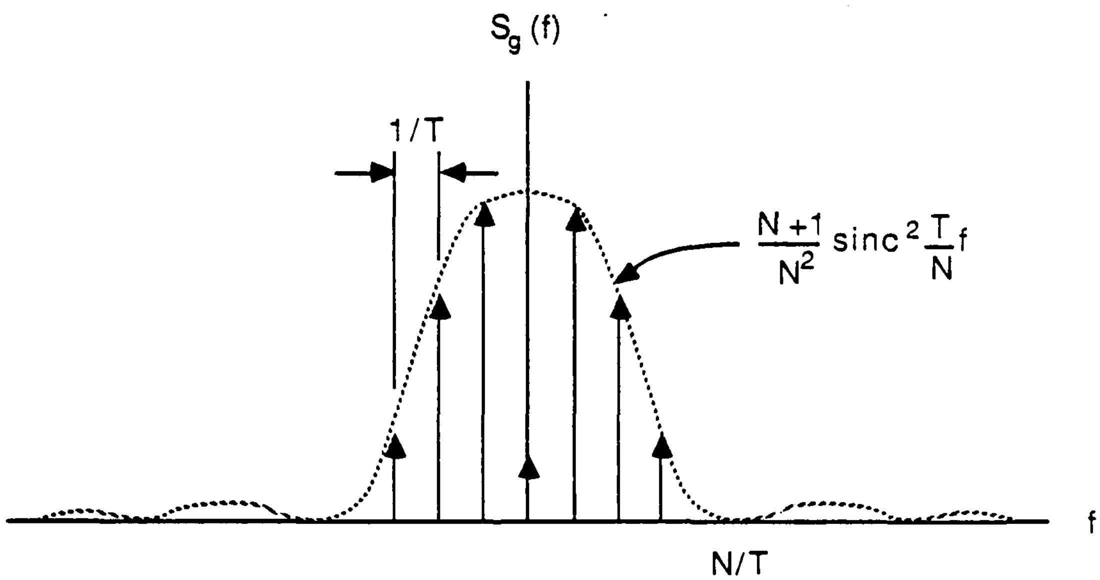

Figure 16: Power Spectrum of a Pseudorandom Sequence

single pulse shown in Figure 11.

The pseudorandom sequence can be used to modulate a carrier. For binary phase shift keying, the modulated signal $s(t)$ is given by

$$s(t) = g(t)\cos 2\pi f_c t. \quad (23)$$

The spectrum of $s(t)$ is as shown in Figure 12, with the exception of the spectral line at the carrier frequency, which in the present case is reduced in strength. To review the relationship between the parameters of the time domain signal and the parameters of the spectrum, the location of the first spectral null is given by $\frac{N}{T} = \frac{1}{\tau}$. The spacing between the spectral lines is $\frac{1}{T}$.

It is likely in an application that the signal (23) would be filtered to reduce the height of the spectral sidelobes. Filtering can be carried out either on the baseband signal $g(t)$ or on the RF signal $s(t)$. As in the case of the rectangular pulse train, it is also possible to shape the filter passband to make the comb more rectangular.

3.2.1. Examples

Suppose a comb is needed with teeth spaced 3 MHz apart over an approximate 100 MHz bandwidth. If we generate a PN sequence at a bit rate of 93 Mbits/s, the bit interval τ will be $\frac{1}{93MHz} = (.01075 \times 10^{-6})$sec. The null-to-null bandwidth of the RF spectrum will be twice 93 MHz, which is about twice the width required. This will allow filtering to remove the sidelobes and leave a comb 93 MHz wide that deviates from uniformity by a factor of $sinc^2(\frac{\tau}{2\tau})$, or about 4 decibels. To achieve the 3 MHz line spacing, select the PN sequence length N to be 31. Then the line spacing, Δf, is given by

$$\Delta f = \frac{1}{T} = \frac{1}{N\tau} = \frac{1}{31}(93\ MHz) = 3\ MHz.$$

Finally, the carrier frequency f_c can be chosen to center the spectrum in whatever frequency band is desired.

Now suppose a comb is needed with teeth spaced 50 kHz apart over an approximate 50 MHz bandwidth. This time generate the PN sequence at a bit rate of 50 Mbits/s. This will give a bit interval $\tau = (\frac{1}{50})\mu$sec, and a null-to-null bandwidth of 100 MHz. Once again, the signal can be filtered, and the filtered RF bandwidth can be reduced to 50 MHz. To achieve the 50 kHz tooth spacing, let the PN sequence have length $N = 1024$. The waveform period T is then $N\tau = \frac{1024}{50MHz}$, giving a spacing Δf of $\frac{1}{T} = 50$ kHz as required.

3.2.2. Constant Envelope - Minimum Shift Keying

The use of BPSK modulation in (23) produces a modulated signal $s(t)$ having a constant envelope. The constant envelope may be useful when the signal is to be amplified using a power amplifier of questionable linearity. Once the signal $s(t)$ is filtered to reduce the sidelobes, the envelope will no longer be constant. In recent years considerable research has centered on development of digital modulation schemes that produce waveforms having a constant envelope and a high "spectral efficiency". The term "spectral efficiency" implies a spectrum having low sidelobes. In the best case, RF filtering to provide bandlimiting may not be needed, allowing the constant envelope to be carried through to the transmitted signal.

Detailed discussions of the minimum shift keying (MSK) modulation method can be found in Pasupathy [5] and in Ziemer & Peterson [3]. There are at least three ways of perceiving MSK modulation, each of which leads to its own particular insights and implementation methods. First, MSK can be viewed as a special case of frequency shift keying (FSK) in which the mark and space frequencies differ by a suitably defined "minimum" amount, and in which the data bits are differentially encoded. Alternatively, MSK can be seen as a special case of quadrature phase shift keying (QPSK) in which data bits are alternately transmitted using BPSK over in-phase and quadrature channels. Transmissions over the two channels are staggered by a bit time, and the amplitudes of the transmitted bits are shaped by a cosine shaping. The QPSK perspective leads to effective implementations, and because QPSK is a linear modulation scheme, analysis of performance is simplified. It is also possible to produce MSK by a so-called "serial" method [3,6] in which a BPSK signal is passed through a bandpass conversion filter. As Ziemer observes [3], this method is "somewhat more subtle to grasp" than the QPSK approach, but it has implementation advantages for transmission at high data rates.

A block diagram showing a serial MSK transmitter is shown in Figure 17. The input is the PN shift register sequence discussed in Section 3.2. The multiplier produces BPSK which is then spectrally shaped in the bandpass conversion filter. The filter has an impulse response given by

$$g(t) = \frac{1}{\tau}\Pi\left(\frac{t-\tau/2}{\tau/2}\right)\sin 2\pi\left(f_c + \frac{1}{4\tau}\right)t, \tag{24}$$

where f_c is the MSK carrier frequency and τ is the bit interval. This filter has a frequency response shaped like a sinc function centered $\frac{1}{2\tau}$ Hz above the BPSK carrier. Ziemer and Ryan [6] report that such filters can be constructed using SAW devices for data rates up to about 100 Mbits/s, and using stripline transmission line filters for higher data rates.

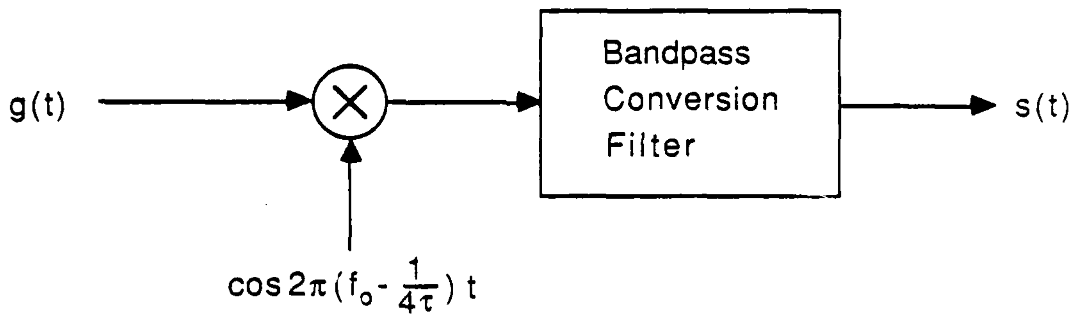

Figure 17: A Serial MSK Modulator

The power spectrum of a PN sequence is shown in Figure 16. The power spectrum after BPSK modulation is similar to that shown in Figure 12. If $S_a(f)$ represents the BPSK power spectrum, then the power spectrum $S_s(f)$ of the MSK signal is given by

$$S_s(f) = |G(f)|^2 S_a(f)$$

$$= \begin{cases} \sum_{n=-\infty}^{\infty} \frac{N+1}{8N^2}\left[\frac{16}{\pi^2}\,\frac{\cos 2\pi\tau(f-f_c)}{1-16\tau^2(f-f_c)^2}\right]^2 \delta\left(f - f_c + \frac{1}{4\tau} - \frac{n}{N\tau}\right), & f>0 \\ \sum_{n=-\infty}^{\infty} \frac{N+1}{8N^2}\left[\frac{16}{\pi^2}\,\frac{\cos 2\pi\tau(f+f_c)}{1-16\tau^2(f+f_c)^2}\right]^2 \delta\left(f + f_c - \frac{1}{4\tau} - \frac{n}{N\tau}\right), & f<0, \end{cases} \tag{25}$$

where $G(f)$ is the frequency response of the conversion filter. The spectrum of the resulting MSK signal is shown in Figure 18.

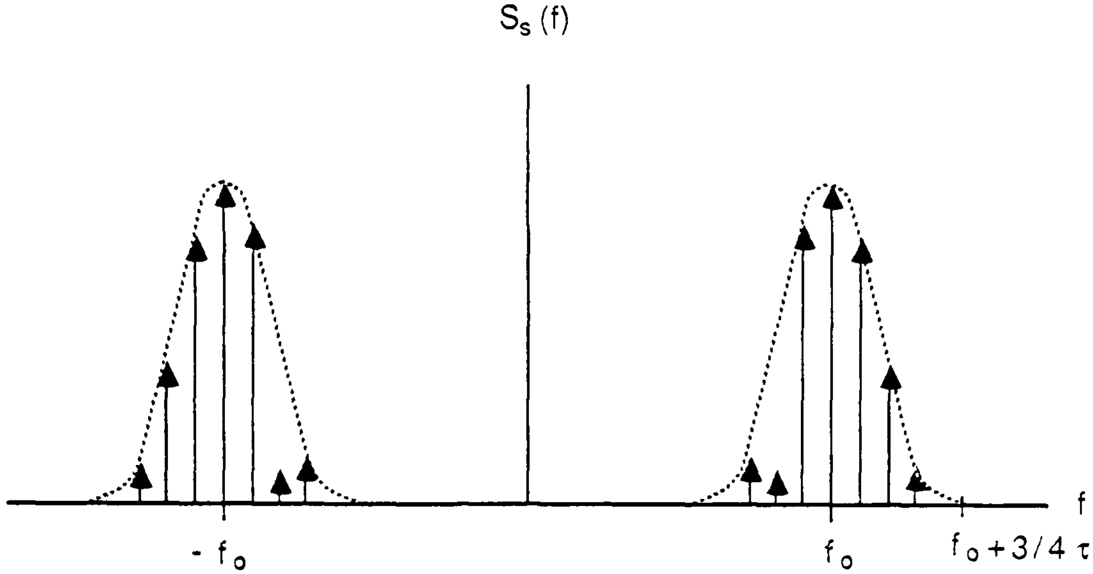

Figure 18: Power Spectrum of an MSK PN Sequence

Referring to Figure 18, several observations can be made about the spectrum of the MSK modulated comb. First, the tooth spacing Δf is given by $\Delta f = \frac{1}{T}$ as has been the case in all of the combs considered so far. As previously, T is the period of the PN sequence. Second, there is an attenuated spectral line at the carrier frequency of the BPSK signal. which is slightly lower in frequency than the MSK carrier. The first spectral nulls occur $\pm\frac{0.75}{\tau}$ from the carrier, and so for a given bit interval τ the spectral width is slightly smaller than the $\frac{1}{\tau}$ that applies for BPSK modulation. Finally, and most important, the sidelobes decrease with frequency as $\frac{1}{f^4}$ compared with $\frac{1}{f^2}$ for BPSK.

We can summarize the advantages and disadvantages of MSK modulation as follows. The principal advantage MSK has over BPSK is its greater spectral efficiency. The spectral efficiency manifests itself in two ways. When compared with BPSK, MSK has a narrower main lobe and sidelobes that decrease more rapidly with distance from the carrier. The narrower main lobe, although a distinct advantage in data communication applications where it allows a higher data rate to be used with a given channel bandwidth, is a disadvantage in the multitone jammer application. This is because a higher pulse rate will be needed to produce a comb with a given bandwidth. In Section 3.2.1 a 93 Mbit/s pulse rate was needed to produce a 93 MHz comb. If the comb had not been filtered to reduce its sidelobes, a 45 Mbit/s pulse rate would have been adequate. Using MSK, a 60 Mbit/s rate would be needed to produce a spectrum having the same null-to-null bandwidth.

The rapid spectral decline with frequency is the main advantage of MSK. If it is found necessary to filter the MSK waveform to further narrow its spectrum, the amplitude will no longer remain constant. Austin and Chang [7] found little difference between the envelopes of signals produced by MSK and other quadrature modulation methods when the modulated signals were filtered. Finally, it should be noted that MSK is more difficult to generate that BPSK, as more sophisticated hardware is required.

3.3. Swept-frequency sinusoid

In this section we consider a sinusoidal carrier that is frequency modulated by a triangle or sawtooth waveform. As was noted at the beginning of Section 3, such a signal has a comb spectrum that is approximately rectangular, with the spectral shape depending primarily on an index of modulation that is proportional to the number of teeth in the comb passband. In the following we present a derivation of the power spectrum of the frequency modulated carrier. It was found that formulas for power spectrum are not particularly enlightening in this case, and some numerical evaluation was necessary. The fast Fourier transform (FFT) algorithm was used to evaluate the power spectrum for several values of index of modulation.

The power spectrum of a carrier modulated by a sawtooth becomes rectangular in the limit as the peak frequency deviation becomes very large and the sweep rate becomes very small. This can be explained by the following argument. For large enough deviation and slow enough sweep rate the modulated waveform can be interpreted as a sequence of steady-state sinusoids at frequencies determined by the modulating voltage. The value of the power spectrum of the modulated waveform at any given frequency is proportional to the length of time that the waveform spends at that frequency. For a sawtooth modulating signal, the modulated waveform spends an equal fraction of time at each frequency, and so the power spectrum is constant over the frequency interval swept. For an arbitrary modulating signal, the result is that the power spectrum of the modulated signal has the same shape as the probability density function of the modulation.

Although the above argument seems to promise a rectangular spectrum for both sawtooth and triangle modulating functions, it turns out that for frequency deviations and sweep rates in ranges of interest the actual spectra deviate from rectangles in significant ways. In particular, the comb teeth will be seen to vary significantly in strength over the passband, particularly in the case of triangle modulation, and the transition regions will not be arbitrarily sharp, particularly in the case of sawtooth modulation.

In the following subsections the spectrum for triangle modulation is derived first, followed by the spectrum for sawtooth modulation Next, graphical results are shown for several examples and finally conclusions are summarized. A listing of the computer program used to evaluate the Fourier transforms is included in the Appendix.

3.3.1. Triangle modulation

Let the modulated signal be given by

$$s(t) = cos[2\pi f_c t + 2\pi F \int^{t} m(\alpha) d\alpha], \tag{26}$$

where $m(t)$ is the periodic triangle wave show in Figure 19. We may parenthetically note that this triangle forms a straight line approximation to the sinusoidal modulating signal $\cos\frac{(2\pi t)}{T}$. The derivation we shall present is very close to the classical derivation, found in most communication systems text books [e.g. 9], of the spectrum of a carrier that is frequency modulated by a sinusoidal modulating signal.

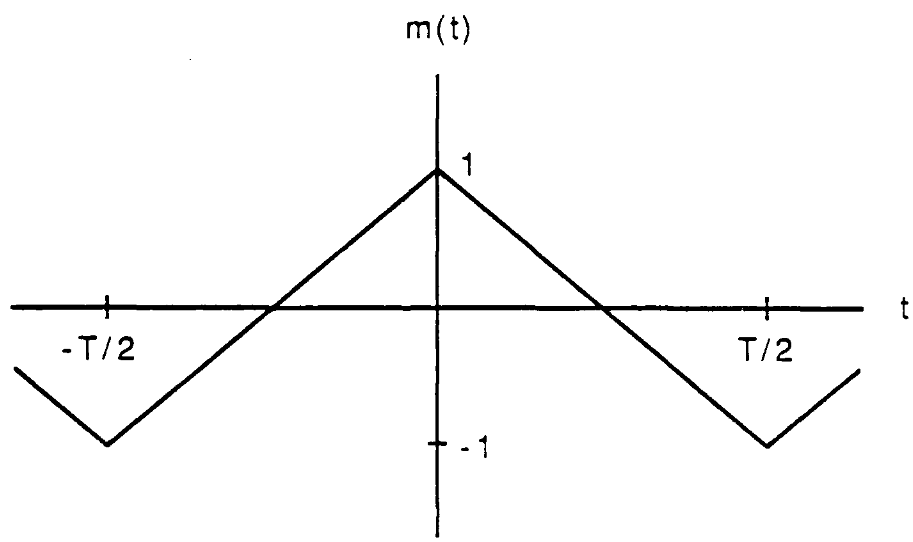

Figure 19: A Triangle Modulating Signal

The excess phase function for the signal (26) is given by

$$\int^{t} m(\alpha)d\alpha = \begin{cases} T\left[2(\frac{t}{T} + \frac{1}{4})^2 - \frac{1}{8}\right], & -\frac{T}{2} \le t < 0 \\ T\left[\frac{1}{8} - 2(\frac{t}{T} - \frac{1}{4})^2\right] & 0 \le t < \frac{T}{2}, \end{cases} \tag{27}$$

where the constant of integration has been chosen to make $\int^{t} m(\alpha)d\alpha$ a quadratic approximation to $\frac{T}{2\pi}\sin\frac{2\pi}{T}t$. Figure 20 shows the excess phase function $\int^{t} m(\alpha)d\alpha$.

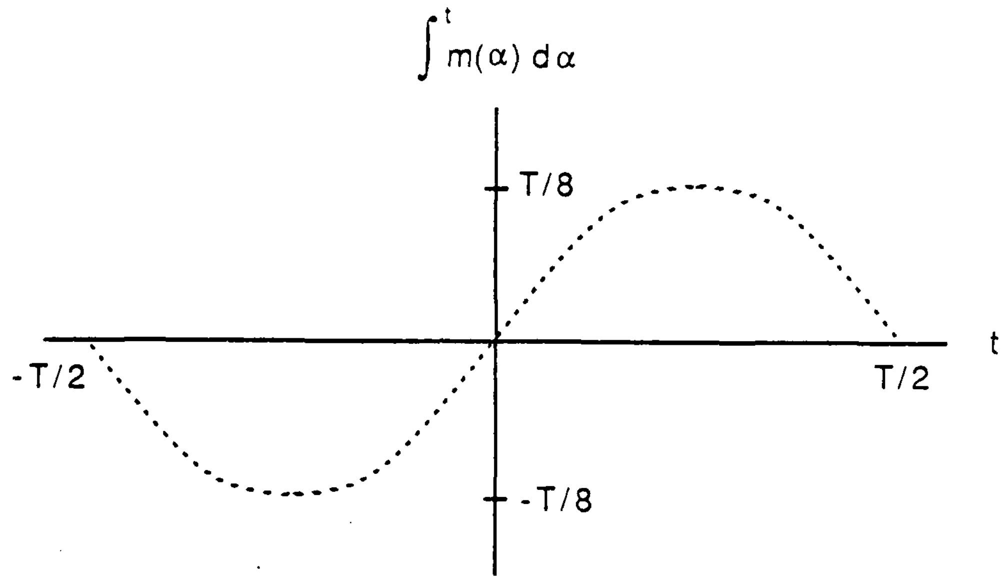

Figure 20: The Excess Phase $\int^{t} m(\alpha)\, d\alpha$

The instantaneous frequency $f_i(t)$ of $s(t)$ is given by

$$2\pi f_i(t) = \frac{d}{dt}[2\pi f_c t + 2\pi F \int^{t} m(\alpha)d\alpha]$$

$$f_i(t) = f_c + Fm(t). \tag{28}$$

The frequency deviation is

$$f_i(t) - f_c = Fm(t). \tag{29}$$

and the peak frequency deviation is

$$\mid f_i(t) - f_c \mid_{PEAK} = F. \tag{30}$$

since the peak value of $m(t)$ is unity. When sinusoidal modulation is used, the modulation index β is defined as the peak frequency deviation divided by the modulation frequency. Using the same definition here,

$$\beta = \frac{F}{1/T} = FT. \tag{31}$$

In terms of β we can write $s(t)$ as

$$s(t) = \begin{cases} \cos\left[2\pi f_c t + 2\pi\beta[2(\frac{t}{T} + \frac{1}{4})^2 - \frac{1}{8}]\right], & -\frac{T}{2} \le t < 0 \\ \cos\left[2\pi f_c t + 2\pi\beta[\frac{1}{8} - 2(\frac{t}{T} - \frac{1}{4})^2]\right], & 0 \le t < \frac{T}{2} \end{cases} \tag{32}$$

The objective now is to find the power spectrum $S(f)$ of $s(t)$. Writing the cosine in exponential form, we obtain

$$s(t) = \frac{1}{2} e^{j[2\pi f_c t + 2\pi F \int^t m(\alpha) d\alpha]} + \frac{1}{2} e^{-j[2\pi f_c t + 2\pi F \int^t m(\alpha) d\alpha]}. \tag{33}$$

In most cases of interest the bandwidth of $s(t)$ is a fraction of the carrier frequency f_c. If this true, then the first term in equation (33) represents positive frequency components, while the second term represents negative frequency components. Let the $s_+(t)$ be twice the positive-frequency part of $s(t)$. That is,

$$s_+(t) = e^{j[2\pi f_c t + 2\pi F \int^t m(\alpha) d\alpha]}. \tag{34}$$

The complex envelope $v(t)$ of $s(t)$ is defined as

$$v(t) = e^{-j2\pi f_c t} s_+(t). \tag{35}$$

In the frequency domain, $v(t)$ represents a shifting of the spectrum of $s_+(t)$ to the left by f_c Hz. We have

$$v(t) = e^{j2\pi F \int^t m(\alpha) d\alpha}. \tag{36}$$

The complex envelope $v(t)$ is a complex-valued periodic signal with period T. It plays the same role in the present derivation that the periodic $g(t)$ played in the derivations of spectra of the pulse train and the PN sequence. By analogy with the pulse $p(t)$ that was repeated periodically to form $g(t)$, let $v_T(t)$ represent one period of $v(t)$. We found in the discussion following equation (17) that the power spectrum $S_g(f)$ of $g(t)$ can be expressed in terms of the Fourier transform $P(f)$ of $p(t)$. In the same way we can write

$$S_v(f) = \sum_{n=-\infty}^{\infty} \frac{1}{T^2} \mid V_T(\frac{n}{T}) \mid^2 \delta(f - \frac{n}{T}), \tag{37}$$

where $S_v(f)$ is the power spectrum of $v(t)$ and $V_T(f)$ is the Fourier transform of $v_T(t)$. In the classical derivation for the case of sinusoidal modulation, the spectrum $V_T(f)$ evaluated at $f = \frac{n}{T}$ can be expressed in terms of the Bessel function $J_n(\beta)$.

The procedure for finding the power spectrum of the triangle modulated FM signal $s(t)$ can now be stated as follows:

1) Find the power spectrum $S_v(f)$ of $v(t)$ using (37).
2) Shift all of the frequency components of $S_v(f)$ to the right by f_c Hz to produce the power spectrum $S_+(f)$ of $s_+(t)$.
3) Recall that $s_+(t)$ is twice the positive frequency part of $s(t)$. The power spectrum $S_+(f)$ must therefore be divided by four. Finally, include negative frequency components that form a mirror image of the positive frequency components.

We postpone the evaluation of the Fourier transform $V_T(f)$ until after consideration of sawtooth modulation.

3.3.2. Sawtooth modulation

Suppose $s(t)$ is as given in equation (26), but now let the modulating signal $m(t)$ be the sawtooth waveform given by

$$m(t) = \begin{cases} \frac{2}{T(1-\delta)}t + \frac{\delta}{(1-\delta)}, & -\frac{T}{2} \leq t < \frac{T}{2} - \delta T \\ -\frac{2}{\delta T}t + (\frac{1}{\delta} - 1), & \frac{T}{2} - \delta T \leq t < \frac{T}{2}. \end{cases} \quad (38)$$

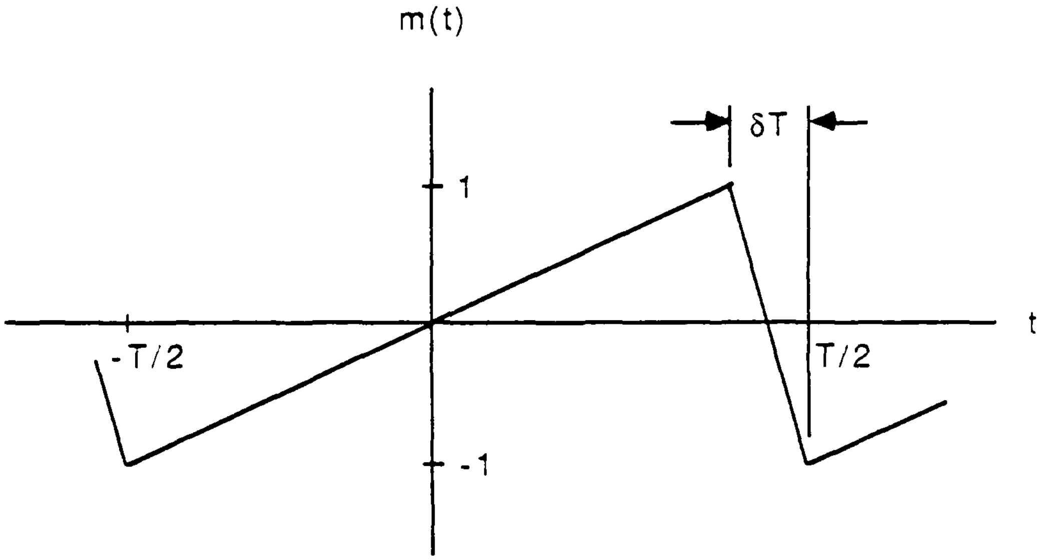

Figure 21: A Sawtooth Modulating Signal

The sawtooth is depicted in Figure 21. For generality, the flyback time has been written as δT, where δ is the fraction of the modulating period allocated to flyback. This waveform $s(t)$ is similar to the one used in a chirp radar and an analysis of the spectrum is presented in Cook [8] for the case of zero flyback time. The analysis presented here differs from that of Cook in providing results that are more amenable to numerical evaluation. The excess phase $\int^t m(\alpha)d\alpha$ for the sawtooth modulating signal is given by

$$\int^t m(\alpha)d\alpha = \begin{cases} T\left[\frac{1}{1-\delta}(\frac{t}{T})^2 + \frac{\delta}{1-\delta}(\frac{t}{T}) - \frac{1/2-\delta}{2(1-\delta)}\right], & -\frac{T}{2} \leq t < \frac{T}{2} - \delta T \\ T\left[-\frac{1}{\delta}(\frac{t}{T})^2 + (\frac{1}{\delta} - 1)(\frac{t}{T}) + \frac{1}{2}(1 - \frac{1}{2\delta})\right], & \frac{T}{2} - \delta T \leq t < \frac{T}{2}. \end{cases} \quad (39)$$

The excess phase is sketched in Figure 22. The complex envelope is now given by

$$v(t) = \cos 2\pi F \int^{t} m(\alpha) d\alpha + j\sin 2\pi F \int^{t} m(\alpha) d\alpha. \tag{40}$$

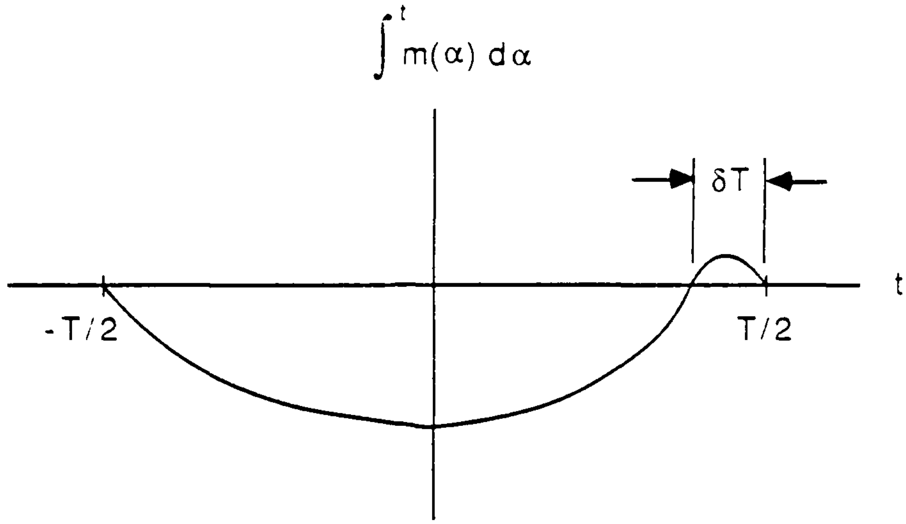

Figure 22: Excess Phase for Sawtooth Modulation

3.3.3. Transforming the complex envelope

To complete evaluation of the power spectrum of the frequency modulated signal $s(t)$ we must evaluate the power spectrum $S_v(f)$ of the complex envelope $v(t)$ as given in equation (37). Calculation of $S_v(f)$ requires calculation of the Fourier transform of the "pulse" $v_T(t)$. A satisfactory approximation to the transform $V_T(f)$ can be obtained using the fast Fourier transform algorithm (FFT). The FFT accepts values of $v_T(t)$ at discrete intervals Δt of time and provides spectral values at discrete intervals Δf of frequency. The time interval Δt and the frequency interval Δf are related by

$$\Delta t \Delta f = \frac{1}{N}, \tag{41}$$

where N is the number of points in the transform. Since the signal $v_T(t)$ has a T second duration. $T = N\Delta t$. It follows then that $\Delta f = \frac{1}{T}$ will be the spacing between values of the spectrum produced by the transform, and this is exactly the spacing needed to provide the values of $V_T(\frac{n}{T})$ required by equation (37).

Aside from errors associated with the numerical calculations, the only source of error in using the FFT to evaluate the spectrum $V_T(f)$ is aliasing. To avoid aliasing it is required that the sampling interval Δt be small enough so that the total frequency span provided by the FFT, $N\Delta f = \frac{1}{\Delta t}$, will be greater than four times the bandwidth of $v(t)$ (two times for positive frequencies and two times for negative frequencies). Taking Δf fixed at $\frac{1}{T}$, we can make Δt small only by making N large. Now to a first approximation the bandwidth of $v(t)$ is F, the peak frequency deviation. For good accuracy, then, we must insure that the total frequency span provided by the transform satisfies $N\Delta f > 4(F)$, giving $N > 4(F)T$. We have previously defined the index of modulation β to be $\beta = FT$. Thus to avoid aliasing we require

$$N > 4\beta. \tag{42}$$

A program has been written in APPLESOFT BASIC to calculate $V_T(f)$. The program requires specification of N and β and generates input to the FFT in the form of samples of $v(t)$ from $t = 0$ to $t = T$. That is,

$$\text{Re } v(k\Delta t) = \begin{cases} \cos\left\{2\pi\beta[\frac{1}{8} - 2(\frac{k}{N} - \frac{1}{4})^2]\right\} & 0 \le k < \frac{N}{2} \\ \cos\left\{2\pi\beta[2(\frac{k}{N} - \frac{3}{4})^2 - \frac{1}{8}]\right\} & \frac{N}{2} \le k < N \end{cases}$$

$$\text{Im } v(k\Delta t) = \begin{cases} \sin\left\{2\pi\beta[\frac{1}{8} - 2(\frac{k}{N} - \frac{1}{4})^2]\right\} & 0 \le k < \frac{N}{2} \\ \sin\left\{2\pi\beta[2(\frac{k}{N} - \frac{3}{4})^2 - \frac{1}{8}]\right\} & \frac{N}{2} \le k < N \end{cases} \quad (43)$$

for the case of triangle modulation. The program produces $V_T(\frac{n}{T})$. The output array holds values for $n = 0,1,...,(\frac{N}{2})-1$ followed by values for $n = -(\frac{N}{2}),-((\frac{N}{2})-1),...,-1$.

To interpret the FFT output, observe that except for a scale factor the values of $|V_T(\frac{n}{T})|^2$ for $n = 0,...,(\frac{N}{2})-1$ correspond to values of the power spectrum $S(f)$ at frequencies $f = f_c, f_c+\frac{1}{T},..., f_c+\frac{N/2-1}{T}$. Values of $|V_T(\frac{n}{T})|^2$ for negative n correspond to values of $S(f)$ at frequencies below f_c. Because $v_T(t)$ is a complex valued function of time, the spectrum $|V_T(f)|^2$ may not be symmetrical about $f = 0$. However, for the special cases of triangular modulation and sawtooth modulation with instant flyback ($\delta = 0$), the symmetry of the modulating waveforms produces a spectrum $|V_T(f)|^2$ that is symmetric about $f = 0$ and thus a spectrum $S(f)$ that is symmetric about f_c.

3.3.4. Examples

Figures 23 through 25 show $|V_T(\frac{n}{T})|$ plotted in decibels versus n for $n = 0,1,...,(\frac{N}{2})-1$ for the case of triangle modulation. The three graphs correspond to values of modulation index β of 17,80, and 500 respectively. The value $\beta = 17$ results from a comb of bandwidth 100 MHz having a line spacing of 3 MHz. To achieve the 100 MHz bandwidth, the peak deviation F was set at 50 MHz and to achieve the required line spacing, the triangle period T was at $\frac{1}{3\ MHz}$. The FFT was computed using $N = 128$, which is greater than 4 x 17 = 68. For the second case, using $\beta = 80$, the FFT was computed using $N = 256$. This value is less than 4 x 80 = 320, and some aliasing is evident in the plot. The third case, using $\beta = 500$ corresponds to a comb of bandwidth 50 MHz having a line spacing of 50 kHz. To achieve the 50 MHz bandwidth, the peak deviation F was set at 25 MHz and to achieve the required line spacing, the triangle period T was set at $\frac{1}{50\ kHz}$. The FFT was computed using $N = 2048$, which is greater than 4 x 500 = 2000. Only every sixteenth point is plotted in the figure. In all three cases the plots are normalized so that zero decibels corresponds to the value of $|V_T(0)|$.

Examinations of Figures 23, 24, and 25 yield some general observations. In each case the one-sided bandwidth is approximately $n = \beta$, or $f = \frac{n}{T} = \frac{\beta}{T} = F$. In each case the passband of the comb is not very uniform, with the strength of the comb teeth varying by as much as 35 dB. Examination of the passband-stopband transition regions shows that the band edges are reasonably sharp,

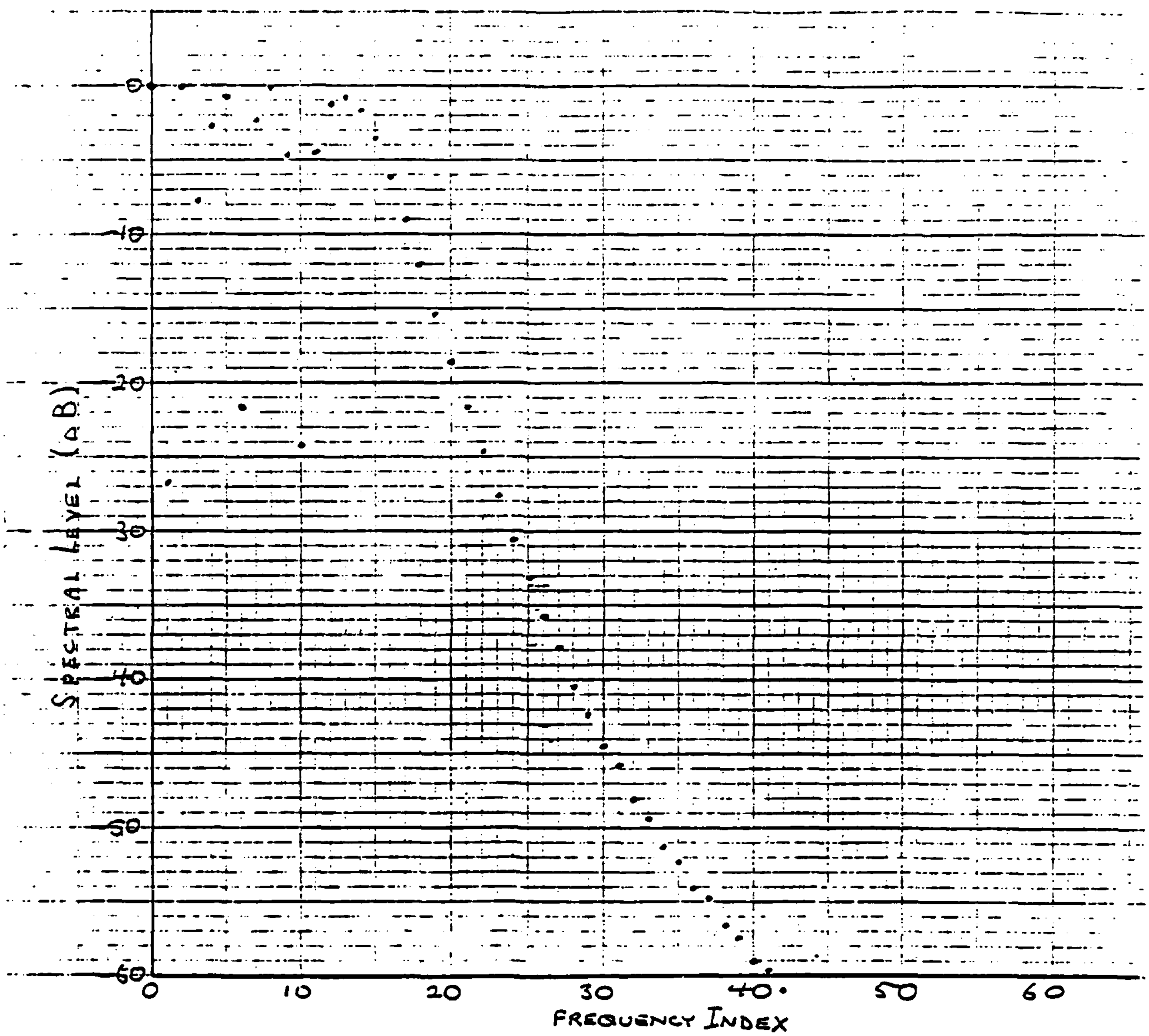

Figure 23: Power Spectrum for Triangle Modulation, β = 17

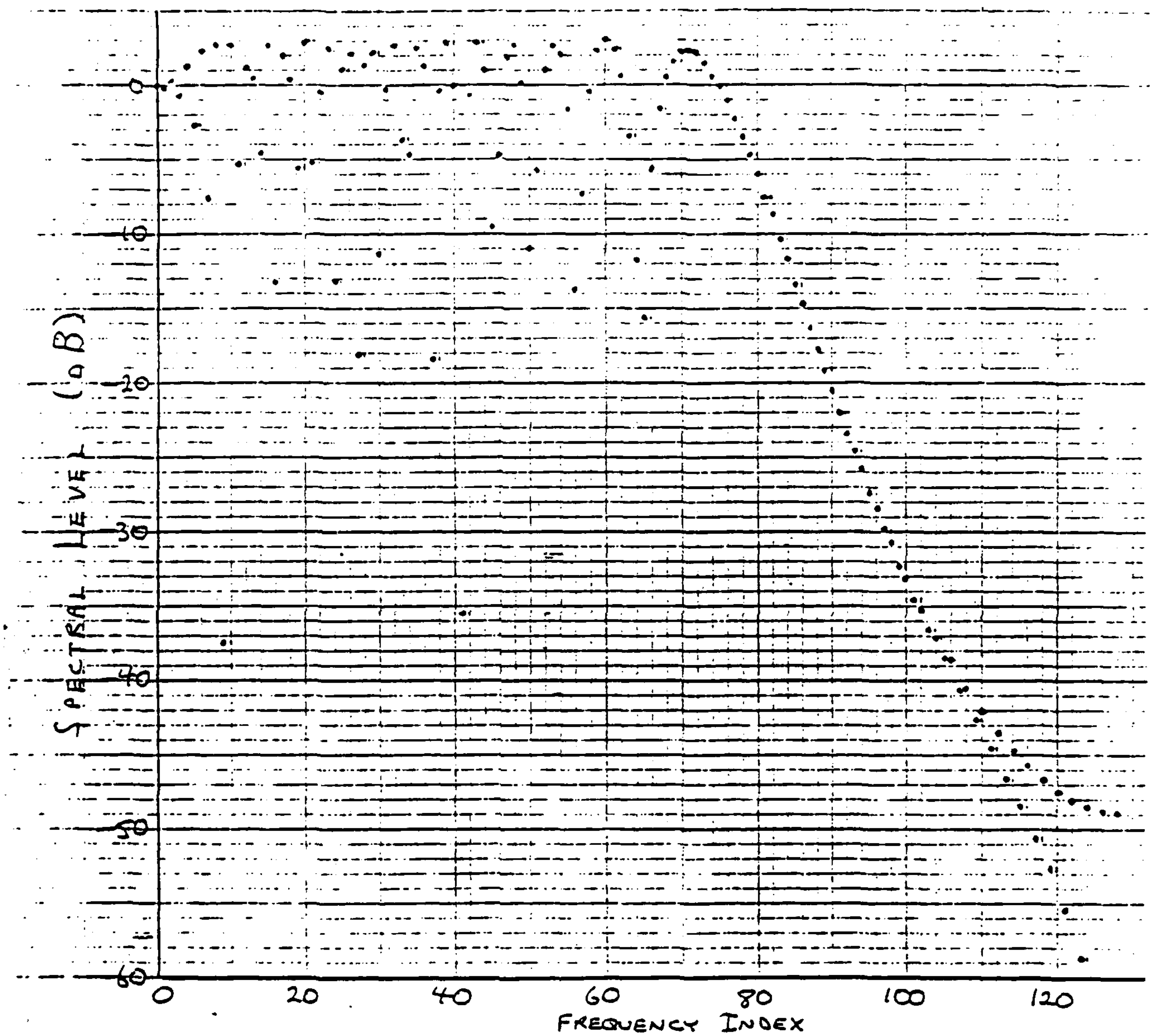

Figure 24: Power Spectrum for Triangle Modulation, β = 80

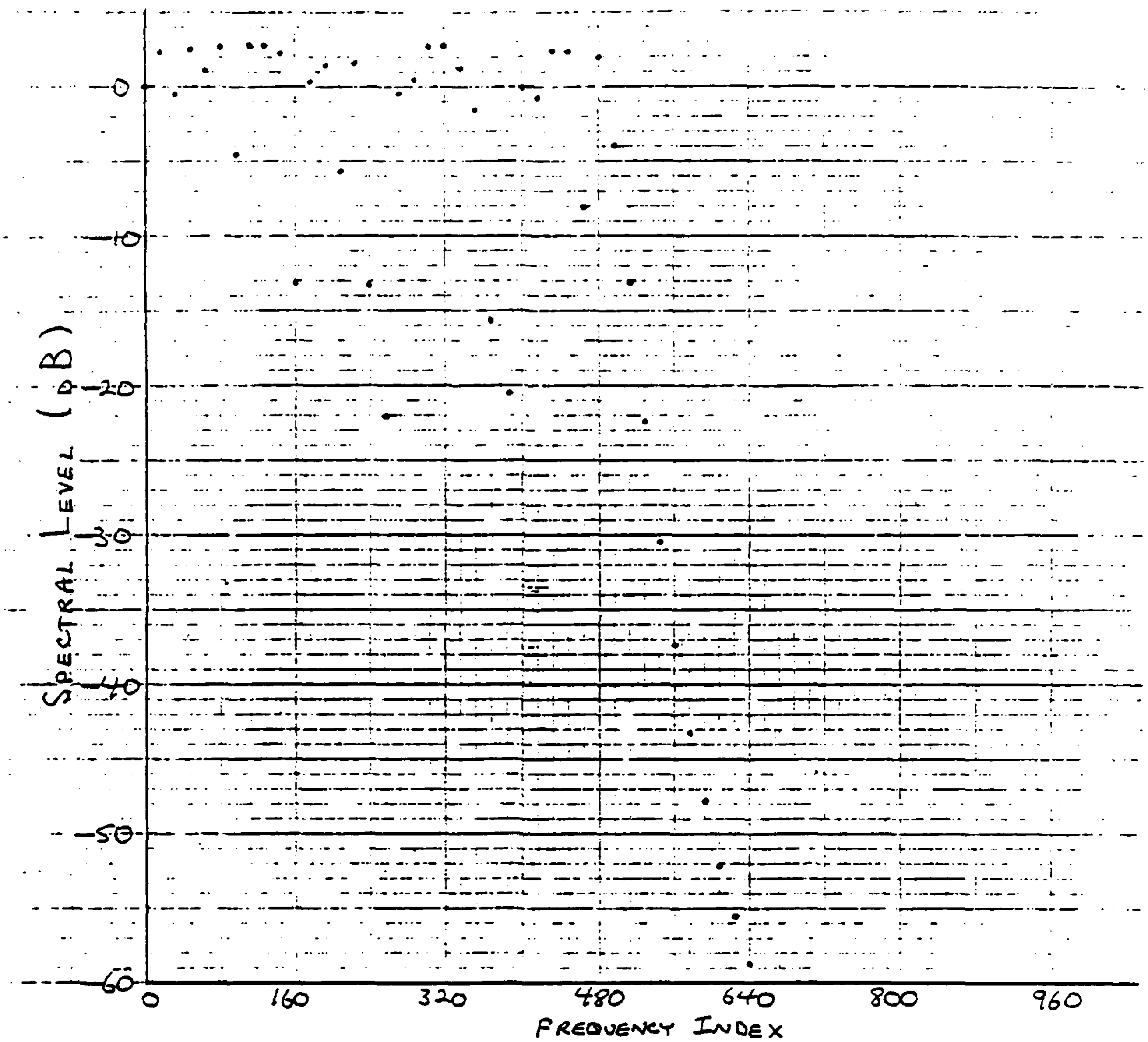

Figure 25: Power Spectrum for Triangle Modulation, β = 500

but are sharper for larger values of β. The sharpness of the band edges can be characterized by comparing them with the cutoff rate of a Butterworth filter of comparable bandwidth. For $\beta = 17$ the cutoff rate corresponds to a seven section filter. For $\beta = 80$ the band edge corresponds to a thirteen section filter, and for $\beta = 500$ the band edge corresponds to a twenty-five section filter.

Figures 26 and 27 show $| V_T(\frac{n}{T}) |$ plotted in decibels versus n for $n = 0,1...,(\frac{N}{2})-1$ triangle and sawtooth modulation respectively. The index of modulation β is forty for both figures, corresponding to a comb of 240 MHz bandwidth with a line spacing of 3 MHz. The sawtooth modulation corresponds to $\delta = 0$, or instantaneous flyback. In both cases the FFT was computed using $N = 512$. Examination of the plots reveals differences in both the passbands and the transition regions. The triangle modulation case shown in Figure 26 reveals a variation in spectral height of more that 30 dB from tooth to tooth within the passband. The sawtooth case shown in Figure 27 shows much less variation; the comb teeth vary over only a few decibels. The penalty for the uniform passband is apparent in the transition region. The spectrum for the sawtooth case shows a much more gradual cutoff than does the spectrum for the triangle case.

4. Conclusions

The principal contributions of this document are the identification of the properties of waveforms which lead to comb-shaped spectra of "good quality," and the examination of certain classes of waveforms that might be useful in producing such combs. A good quality comb spectrum should have uniformly strong teeth over its passband and should have transition regions that fall away in the shortest possible frequency interval. It is also desirable, to minimize effects of distortion caused by saturating power amplifiers, that the time-domain waveform have a constant envelope. The constant envelope feature is only useful for comb spectra that have narrow transition regions, as filtering of a waveform destroys the constant envelope.

A comb spectrum having teeth equally spaced in frequency can be generated by a periodic waveform. The reciprocal of the period is the tooth spacing. Looking at the periodic waveform as a train of pulses, it is the shape of the pulses that determines the envelope of the comb spectrum. Thus the uniform passband and sharp transition regions can be achieved by shaping the basic pulse. Once a pulse shape has been found that produces a satisfactory baseband comb, the periodic pulse train can be used to modulate a carrier. The carrier frequency can be chosen to locate the comb in a convenient part of the spectrum. Further, the type of modulation will have a bearing on the sharpness of the transition regions.

Two classes of waveforms were identified in the report as being suitable for the generation of combs. These were the repeated pseudonoise sequence and the frequency-swept sinusoid. The PN sequence produces a comb having a baseband spectrum with a sinc function shape. This comb can be positioned anywhere in the RF spectrum by using BPSK modulation. The resulting spectrum forms a very uniform comb at frequencies near the carrier, and the time-domain waveform has a constant envelope. The comb has significant sidelobes, however, leading to very broad transition regions. There is also a limit imposed by the hardware on how broad the comb bandwidth can be made. The bandwidth depends on the chip rate of the PN sequence. It will probably be necessary to use ECL logic to achieve chip rates much above 70 MHz.

The transition regions of the PN sequence comb can be improved to a great extent if the baseband PN sequence is filtered prior to modulation. If a high quality low-pass filter is employed, a very uniform comb with very sharp transition regions can be produced. The filtered PN sequence will not produce an RF signal having a constant envelope.

If constant envelope is essential, minimum shift keying (MSK) can be used instead of BPSK to locate the center frequency of the comb. The MSK comb will have only three quarters of the bandwidth of the BPSK comb for a given PN sequence bit rate, but the sidelobes of the spectrum will be greatly reduced, leading to sharper transition regions. There are two potential disadvantages to the use of MSK modulation. First, MSK is more difficult to generate than BPSK, particularly at high bit rates. Second, although the transition regions of the MSK comb are sharper than those of the unfiltered BPSK comb, they are not as sharp as those of the filtered comb and may not be sharp enough for some applications. If filtering is used to sharpen the transitions, the constant envelope of the time function will be lost, and with it any advantage over filtered BPSK.

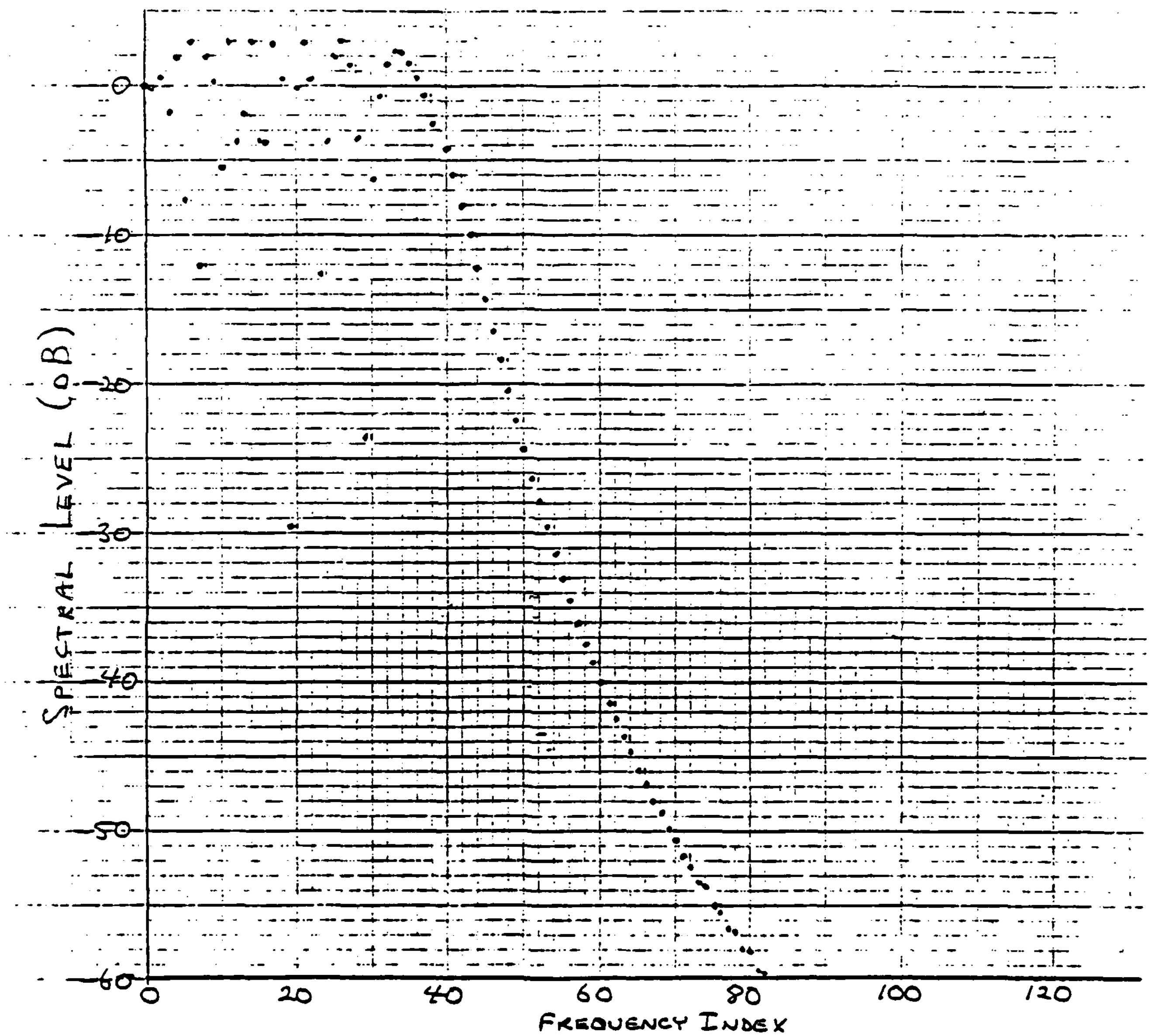

Figure 26: Power Spectrum for Triangle Modulation, β = 40

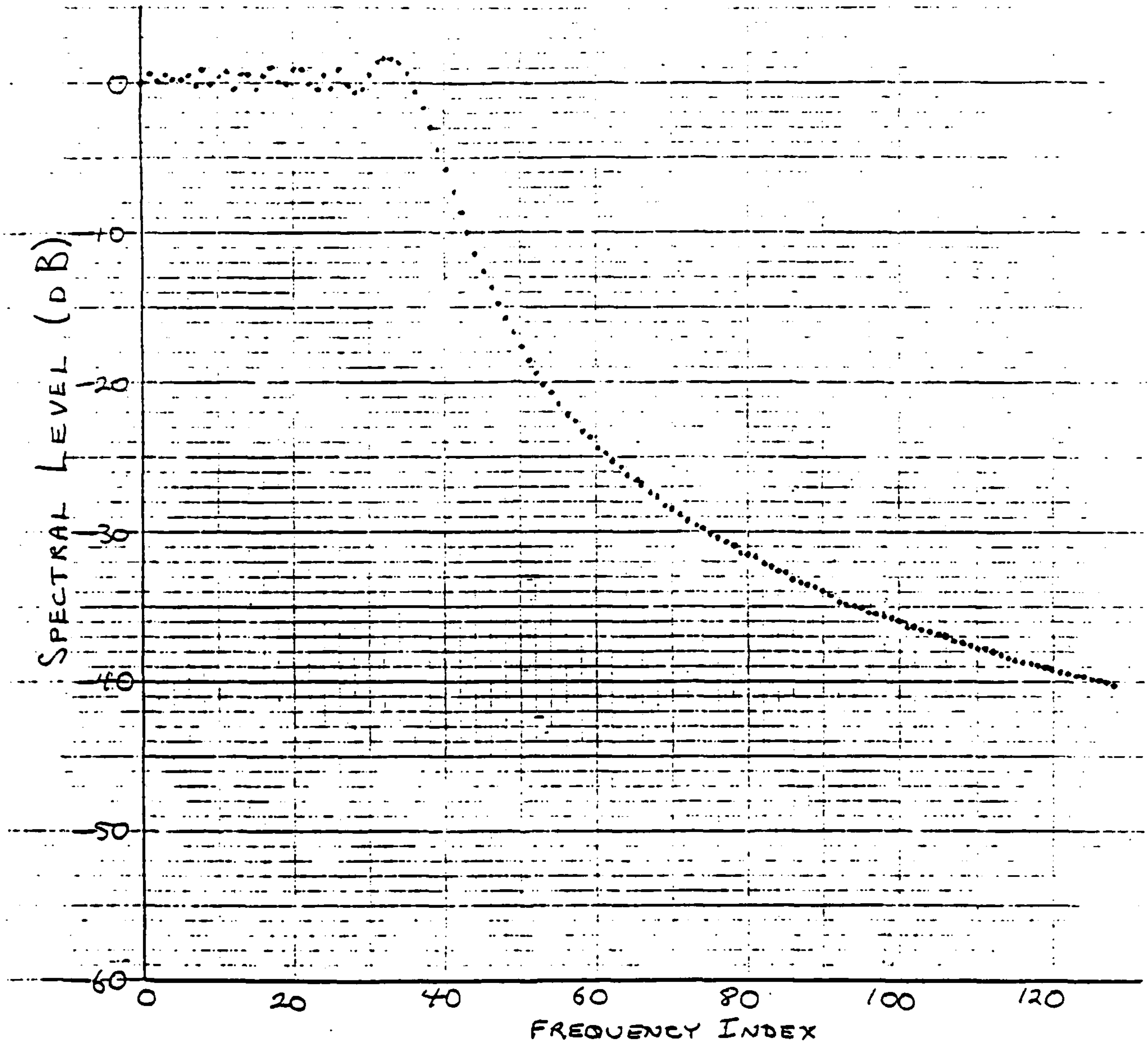

Figure 27: Power Spectrum for Sawtooth Modulation, $\beta = 40$

The frequency-swept sinusoid is capable of producing combs having very wide bandwidths. The time domain functions will in all cases have constant envelopes. The shape of the comb passband is determined by the shape of the modulating waveform. Two waveforms were considered: the triangle wave and the sawtooth. Both modulating waveforms produce combs that are nonuniform in their passbands. The triangle-modulated waveform is particularly flawed in this way. Examples showed variations in the comb from tooth to tooth of over 30 dB. Variations for the sawtooth are considerably less, but the indications are that production of a uniform comb requires a sawtooth having a nearly instantaneous flyback.

The shape of the transition regions of the frequency-swept combs is a function of an index of modulation which is the product of the comb bandwidth and the sweep period. This index is also approximately equal to the number of teeth in the comb passband. Combs having higher indexes of modulation turn out to have sharper transition regions. The triangle-modulated combs have sharper transition regions than do the sawtooth-modulated combs. In some cases of practical interest the transitions may be as sharp as can be produced by filters, thus rendering filtering unnecessary and allowing preservation of the time-domain constant envelope.

Appendices:

Program Listings

Appendix A

This appendix contains a program in APPLESOFT BASIC to compute the spectrum of a sinusoid frequency modulated by a triangle wave. The program generates the complex envelope of the modulated sinusoid, computes the spectrum using the FFT algorithm, and outputs the magnitude of the spectrum in decibels.

The program accepts as inputs the parameters M, ß, and Z. The parameter ß is the index of modulation. The output of the program is written to a file called RESULTSZ, where the "Z" is the third parameter. RESULTSZ can be listed using a program called RETRIEVE.

Lines 50 through 140 compute the complex envelope of the modulated sinusoid. Lines 1000 through 1350 are the FFT. Lines 155 through 205 produce the output.

```
5  DIM RX(1024),IX(1024)
10 D$ = CHR$ (4)
15 INPUT "FILENUMBER: ";Z
20 PI = 3.14159265358979
30 INPUT "M: ";M
35 M1 = 2 ↑ M:M2 = M1 / 2
40 INPUT "BETA: ";BETA
50 FOR I = 1 TO M2
60 RX(I) = COS (2 * PI * BETA *
     (.125 - 2 * ((I - 1) / M1 -
     .25) ↑ 2))
70 IX(I) = SIN (2 * PI * BETA *
     (.125 - 2 * ((I - 1) / M1 -
     .25) ↑ 2))
80 NEXT I
100 FOR I = M2 + 1 TO M1
110 RX(I) = COS (2 * PI * BETA *
     (2 * ((I - 1) / M1 - .75) ↑
     2 - .125))
120 IX(I) = SIN (2 * PI * BETA *
     (2 * ((I - 1) / M1 - .75) ↑
     2 - .125))
140 NEXT I
150 GOSUB 1000
153 M0 = RX(1) ↑ 2 + IX(1) ↑ 2
155 PRINT D$;"OPEN RESULTS";Z
157 PRINT D$;"DELETE RESULTS";Z
158 PRINT D$;"OPEN RESULTS";Z
159 PRINT D$;"WRITE RESULTS";Z
160 FOR I = 1 TO M1
170 MAG = RX(I) ↑ 2 + IX(I) ↑ 2
190 PRINT 10 * LOG (MAG / M0) /
     LOG (10)
200 NEXT I
205 PRINT D$;"CLOSE RESULTS";Z
210 END
990 REM THIS SUBROUTINE COMPUTE
     S THE FFT
1000 N = 2 ↑ M
1030 FOR QL = 1 TO M
1040 LE = 2 ↑ (M + 1 - QL)
1050 L1 = LE / 2
1060 RU = 1:IU = 0
1070 RW = COS (PI / L1):IW = -
     SIN (PI / L1)
1080 FOR QJ = 1 TO L1
1090 FOR QI = QJ TO N STEP LE
1100 IP = QI + L1
1110 RT = RX(QI) + RX(IP):IT = IX
     (QI) + IX(IP)
1120 TR = (RX(QI) - RX(IP)) * RU -
     (IX(QI) - IX(IP)) * IU
1130 TI = (RX(QI) - RX(IP)) * IU +
     (IX(QI) - IX(IP)) * RU
1140 RX(IP) = TR:IX(IP) = TI
1150 RX(QI) = RT:IX(QI) = IT
1160 NEXT QI
1170 TR = RU * RW - IU * IW:TI =
     RU * IW + IU * RW
1180 RU = TR:IU = TI
1190 NEXT QJ: NEXT QL
1200 N2 = N / 2
1210 N1 = N - 1
1220 QJ = 1
1230 FOR QI = 1 TO N1
1240 IF QI >  = QJ THEN  GOTO 12
     60
1250 RT = RX(QJ):IT = IX(QJ)
1260 RX(QJ) = RX(QI):IX(QJ) = IX(
     QI)
1270 RX(QI) = RT:IX(QI) = IT
1280 QK = N2
1290 IF QK >  = QJ THEN  GOTO 13
     30
1300 QJ = QJ - QK
1310 QK = QK / 2
1320 GOTO 1290
1330 QJ = QJ + QK
1340 NEXT QI
1350 RETURN
```

Appendix B

This appendix contains a program in APPLESOFT BASIC to compute the spectrum of a sinusoid frequency modulated by a sawtooth wave. The program generates the complex envelope of the modulated sinusoid, computes the spectrum using the FFT algorithm, and outputs the magnitude of the spectrum in decibels.

The program accepts as inputs the parameters M and ß. The parameter M determines the number of points N in the FFT, as $N=2^M$. The parameter ß is the index of modulation. The output of the program is written to a file called RESULTS1. RESULTS1 can be listed using a program called RETRIEVE.

Lines 60 through 146 compute the complex envelope of the modulated sinusoid. Lines 1000 through 1350 are the FFT. Lines 155 through 205 produce the output. The parameter DE in line 40 is the ∂ that determines the flyback interval. This parameter can be changed if value other than zero is desired.

```
[illegible] DIM [illegible]
[illegible] D$ = CHR$ (4)
[illegible] PI = 3.14159[illegible]
[illegible] INPUT "M = ";M
[illegible] M1 = 2 ↑ M:M2 = M1 / 2
[illegible] N = M1
[illegible] DE = 0:D1 = 1 - DE:D2 = [illegible] - D
    E
[illegible] INPUT "BETA = ";BETA
[illegible] FOR I = 1 TO M2
[illegible] ARG = ((I - 1) / N) ↑ 2 / D1 +
    ((I - 1) / N) * DE / D1 - D2
    / (2 * D1)
[illegible] RX(I) = COS (2 * PI * BETA *
    ARG)
[illegible] IX(I) = SIN (2 * PI * BETA *
    ARG)
[illegible] NEXT I
[illegible] FOR I = M2 + 1 TO M1
[illegible] ARG = ((I - 1) / N - 1) ↑ 2 /
    D1 + ((I - 1) / N - 1) * DE /
    D1 - D2 / (2 * D1)
140 RX(I) = COS (2 * PI * BETA *
    ARG)
[illegible] IX(I) = SIN (2 * PI * BETA *
    ARG)
144 NEXT I
150 GOSUB 1000
155 M0 = RX(1) ↑ 2 + IX(1) ↑ 2
156 PRINT D$;"OPEN RESULTS1"
157 PRINT D$;"DELETE RESULTS1"
158 PRINT D$;"OPEN RESULTS1"
159 PRINT D$;"WRITE RESULTS1"
160 FOR I = 1 TO M1
170 MAG = RX(I) ↑ 2 + IX(I) ↑ 2
190 PRINT 10 * LOG (MAG / M0) /
    LOG (10)
[illegible] NEXT I
[illegible] PRINT D$;"CLOSE RESULTS1"
[illegible] END
```

```
1000 N = 2 ↑ M
1010 FOR QL = 1 TO M
[illegible] PRINT CHR$ (7)
1040 LE = 2 ↑ (M + 1 - QL)
1050 L1 = LE / 2
1060 RU = 1:IU = 0
1070 RW = COS (PI / L1):IW = -
    SIN (PI / L1)
1080 FOR QJ = 1 TO L1
1090 FOR QI = QJ TO N STEP LE
1100 IP = QI + L1
1110 RT = RX(QI) + RX(IP):IT = IX
    (QI) + IX(IP)
1120 TR = (RX(QI) - RX(IP)) * RU -
    (IX(QI) - IX(IP)) * IU
1130 TI = (RX(QI) - RX(IP)) * IU +
    (IX(QI) - IX(IP)) * RU
1140 RX(IP) = TR:IX(IP) = TI
1150 RX(QI) = RT:IX(QI) = IT
1160 NEXT QI
1170 TR = RU * RW - IU * IW:TI =
    RU * IW + IU * RW
1180 RU = TR:IU = TI
1190 NEXT QJ: NEXT QL
1200 N2 = N / 2
1210 N1 = N - 1
1220 QJ = 1
1230 FOR QI = 1 TO N1
1240 IF QI > = QJ THEN GOTO 12
    80
1250 RT = RX(QJ):IT = IX(QJ)
1260 RX(QJ) = RX(QI):IX(QJ) = IX(
    QI)
1270 RX(QI) = RT:IX(QI) = IT
1280 QK = N2
1290 IF QK > = QJ THEN GOTO 13
    30
1300 QJ = QJ - QK
1310 QK = QK / 2
1320 GOTO 1290
1330 QJ = QJ + QK
1340 NEXT QI
1350 RETURN
```

References

1. R.C. Dixon, Spread Spectrum Systems. New York: Wiley-Interscience, 1976.

2. R.C. Dixon, ed, Spread Spectrum Techniques. New York: IEEE Press, 1976.

3. R.E. Ziemer & R.L. Peterson, Digital Communications & Spread Spectrum Systems. New York: MacMillan Publishing Company, 1985.

4. W.W. Peterson & E.J. Weldon, Error Correcting Codes. Cambridge, Mass: MIT Press, 1972.

5. S. Pasupathy, "Minimum Shift Keying: A Spectrally Efficient Modulation", IEEE Communications Magazine, vol 17, pp 14-22, July 1979.

6. R.E. Ziemer & C.E. Ryan, "Minimum-Shift Keyed Modem Implementations for High Data Rates", IEEE Communications Magazine, vol 21, pp 28-37, October, 1983.

7. M.C. Austin & M.U. Chang, "Quadrature Overlapped Raised-Cosine Modulation", IEEE Transactions on Communications, vol COM-29, no 3, March 1981.

8. C.E. Cook, "Pulse Compression - Key to More Efficient Radar Transmission", Proc. IRE, vol 48, pp 310-316, March 1960.

9. A. B. Carlson, Communication Systems. New York: McGraw-Hill, 1975.