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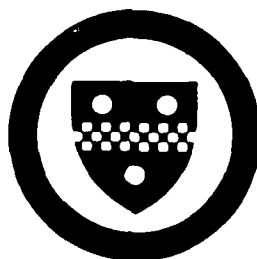
By

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LINEAR TRANSFORMATIONS, PROJECTION OPERATORS AND GENERALIZED INVERSES-A GEOMETRIC APPROACH

ABSTRACT

A generalized inverse of a linear transformation $A: v \rightarrow w$, where v and w are finite dimensional vector spaces, is defined using geometric concepts of linear transformations and projection operators. The inverse is uniquely defined in terms of specified subspaces $m \subset v$, $l \subset w$ and a linear transformation N such that $AN = O$, $NA = O$. Such an inverse which is unique is called the lmN -inverse. A Moore-Penrose type inverse is obtained by putting $N=O$. Applications to optimization problems when v and w are inner product spaces, such as least squares in a general setting, are discussed. The results given in the paper can be extended without any major modification of proofs to bounded linear operators with closed range on Hilbert spaces.

Key Words and Phrases: g -inverse, Linear transformation, Moore-Penrose inverse, Projection operator.

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DEDICATION

Professor K. Nagabhushanam was one of the two inspiring teachers I had when I was pursuing my graduate studies in mathematics at the Andhra University, Waltair. He and Professor V. Ramaswamy not only taught us mathematics but also prepared us to think in terms of mathematics and to use mathematics as an abstract logical method in solving complex problems in any field of inquiry. This training was a great asset to me when I started on my research career. I had kept in touch with Professor Nagabhushanam after I left the Andhra University as he was keenly interested in my activities and often encouraged me in my research work. It is, indeed, a great honor to contribute to the memorial volume of my respected teacher.

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1. INTRODUCTION

Let v and w be two finite dimensional vector spaces and $A: v \rightarrow w$ be a linear transformation from v to w . We denote the range of A by

$$R(A) = \{y: Ax=y, x \in v\} = a \subset w \quad (1.1)$$

and the nullity or kernel of A by

$$K(A) = \{x \in v: Ax = 0\} = k \subset v. \quad (1.2)$$

If c and d are subspaces of v such that

$$c \cap d = 0, \text{ the null vector} \quad (1.3)$$

$$c \oplus d = \{x_1 + x_2: x_1 \in c, x_2 \in d\} = v \quad (1.4)$$

then c and d are said to be direct complements. Given any subspace $c \subset v$, there exists a subspace $d \subset v$, called the direct complement of c , which satisfies the conditions (1.3) and (1.4). The choice of d is not unique. Given any pair c, d satisfying (1.3) and (1.4), any vector $x \in v$ has the unique decomposition

$$x = x_1 + x_2, \quad x_1 \in c, \quad x_2 \in d. \quad (1.5)$$

The transformation $x \rightarrow x_1$, which is linear, is called the projection operator on c along d and is represented by

$$P_{c,d}: v \rightarrow v. \quad (1.6)$$

It is seen that

$$P_{c,d} + P_{d,c} = I, \quad P_{c,d} P_{d,c} = O \quad (1.7)$$

$$R(P_{c,d}) = c, \quad R(P_{d,c}) = d \quad (1.8)$$

where I and O are identity and null transformations on v .

$A: v \rightarrow w$ is called bijective if it is onto and one-to-one. In such a case, there is an inverse of transformation A , denoted by A^{-1} , which is linear and is such that

$$AA^{-1} = I(\text{on } w), \quad A^{-1}A = I(\text{on } v). \quad (1.9)$$

The restriction of $A: v \rightarrow w$ to a subspace $c \subset v$ is also a linear transformation and is denoted by

$$A|_c: c \rightarrow w. \quad (1.10)$$

Let k be the nullity of A as defined in (1.2), m be any direct complement of k , (i.e. $k \oplus m = v$), and a be the range of A . Then

$$A|m: m \rightarrow a \text{ is a bijection.} \quad (1.11)$$

This is easily seen as follows. Choose $y \in a$ and let $x \in v$ be a preimage of y , i.e., x is such that $Ax = y$. Then

$$P_{m,k} x \in m \quad \text{and} \quad Ax = A(P_{m,k} + P_{k,m} x) = AP_{m,k} x = y$$

so that the mapping (1.11) is onto. Further if $x_1 \in m$ and $x_2 \in m$, then $x_1 - x_2 \in m$. If now $Ax_1 = Ax_2$, then $A(x_1 - x_2) = 0 \Rightarrow x_1 - x_2 \in k$ which is a contradiction unless $x_1 = x_2$. Thus $x_1 \neq x_2 \Rightarrow Ax_1 \neq Ax_2$, which shows that the mapping (1.11) is one to one.

The result (1.11) implies the existence of an inverse

$$(A|m)^{-1} = T_m: a \rightarrow m \quad (1.12)$$

such that

$$T_m A|m = I(\text{on } m), \quad AT_m A = A, \quad T_m A = P_{m,k} \quad (1.13)$$

$$AT_m | a = I(\text{on } a), \quad T_m AT_m = T_m. \quad (1.14)$$

Note that T_m is defined only on $a \subset w$ and not the whole of w .

In this paper, we develop the concept of a generalized inverse (g-inverse) of A when it is not necessarily bijective and provide some characterizations and applications. This is accomplished by extending T_m as a linear transformation on any complement of a in w .

2. G-INVERSE OF A LINEAR TRANSFORMATION

Let $A: v \rightarrow w$ be a linear transformation with range $a \subset w$ and nullity $k \subset v$. Consider a consistent linear equation $Ax = y$ ($y \in a$) and denote the set of solutions by

$$S_y = \{x \in v : Ax = y\}. \quad (2.1)$$

Suppose that there exists a linear transformation $G: w \rightarrow v$ such that

$$Gy \in S_y \text{ for any } y \in a \quad (2.2)$$

i.e., Gy is a solution of $Ax = y$ for any $y \in a$. Then substituting Gy for x in $Ax = y$

$$AGy = y \quad \forall y \in a \quad \text{or} \quad AG|_a = I(\text{on } a) \quad \Rightarrow \quad AGA = A. \quad (2.3)$$

Conversely, if there exists a G such that (2.3) holds and $Ax = y$ is a consistent equation, i.e., there exists an x such that $Ax = y$, then

$$AGA = A \Rightarrow AGAx = Ax \quad \text{or} \quad AGy = y \quad (2.4)$$

so that Gy is a solution of $Ax = y$. Any G satisfying the condition (2.3) is called a g -inverse of A ; this enables us to solve a consistent equation $Ax = y$. Such a G which may not be unique is denoted by A^- and the set of all G satisfying (2.3) is denoted by $\{A^-\}$. The following lemmas characterize g -inverses.

Lemma 2.1 Let k be the nullity of A , a the range of A , and G be a linear transformation such that $AGA = A$, i.e., $G \in \{A^-\}$. Then the following hold:

- (i) $m = R(GA)$ is a direct complement of k , and $T_m : a \rightarrow m = G|_a$ or $GA|_m = I(\text{on } m)$, where $T_m = (A|_m)^{-1}$.
- (ii) If $N = G - GAG$, then $AN = O$ and $NA = O$.
- (iii) $l = R(I - AG)$ is a direct complement of a , and $N|_l = G|_l$.

Proof of (i). Let $x \in m \cap k$. Then $x = GAu$ for some u and $Ax = AGAu = Au = 0 \Rightarrow x = 0$, i.e., $m \cap k = 0$. Also, if $x \in v$, then

$$x = GAx + (I-GA)x = x_1 + x_2$$

where $x_1 \in m$ and $x_2 \in k$ since $Ax_2 = 0$. Thus the conditions (1.3) and (1.4) are satisfied so that m is a direct complement of k . Then there exists T_m as defined in (1.12) - (1.14), and in particular $T_m A|m = I$. Now

$$T_m Ax = T_m AGAx = T_m A (GAx) = GAx$$

since $GAx \in m$, which implies $T_m = Gl_a$ or $GA|m = I$.

Proof of (ii). Obvious.

Proof of (iii). Let $y \in l \cap a$. Then, for some x and u

$$y = Ax = (I-AG)u \Rightarrow AGAx = AG(I-AG)u = 0 \Rightarrow Ax = 0$$

so that $l \cap a = 0$. Further

$$y = AGy + (I-AG)y = y = y_1 + y_2$$

where $y_1 \in a$ and $y_2 \in l$. Thus the conditions (1.3) and (1.4) are satisfied so that l is a direct complement of a . Consider

$$N(I-AG)u = Nu = (G-GAG)u = G(I-AG)u$$

for any u , so that $Nl = Gl$. Lemma 1 is proved.

Definition. Let m be any direct complement of k (the nullity of A) and l be any direct complement of a (the range of A) and $N: w \rightarrow v$ be any linear transformation such that $AN = 0$ and $NA = 0$. Then G is said to be an lmN -inverse of A if

$$(i) \ G \text{ is linear, } (ii) \ GA|m = I \quad \text{and} \quad (iii) \ Gl = Nl. \quad (2,5)$$

Such a G , if it exists, is denoted by G_{lmN} . Lemma 2.2 characterizes such inverses.

Lemma 2.2.

$$G_{lmN} = T_m P_{a,l} + N \quad (2.6)$$

where T_m is as defined in (1.12) with respect to a chosen m , so that G_{lmN} is unique for any given l, m and N .

Proof. Let $y = y_1 + y_2$, $y_1 \in a$ and $y_2 \in l$. Then

$$\begin{aligned} G_{lmN} y &= G_{lmN} y_1 + G_{lmN} y_2 \\ &= T_m y_1 + N y_2, \text{ using (2.5)} \\ &= T_m P_{a,l} y + N y \\ &= (T_m P_{a,l} + N) y \quad \forall y \in w. \end{aligned}$$

which establishes Lemma 2.

Lemma 2.3. $G_{lmN} \in \{A^-\}$ and the mapping

$$(l, m, N) \rightarrow \{A^-\} \text{ is bijective.} \quad (2.7)$$

Proof. Using the representation (2.6)

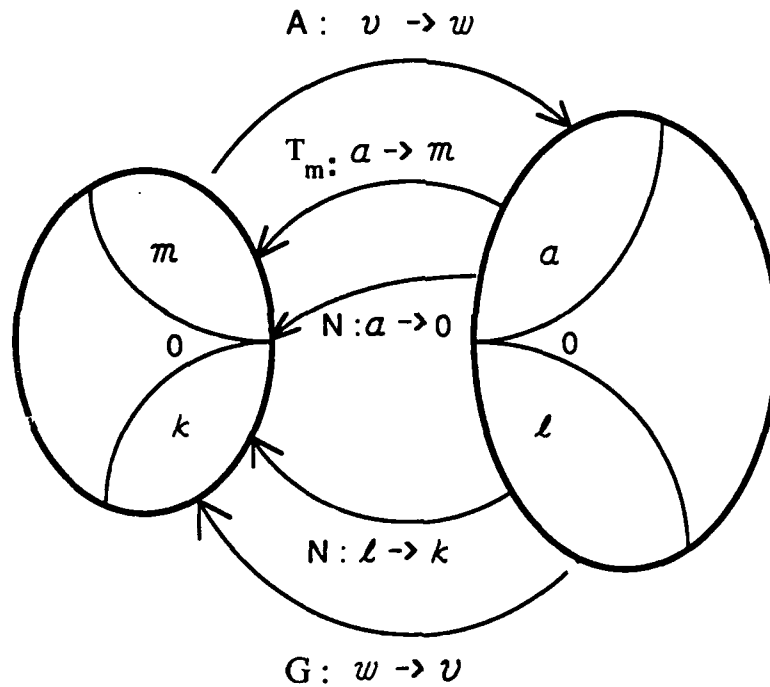
$$\begin{aligned} A G_{lmN} A &= A(T_m P_{a,l} + N)A \\ &= A T_m P_{a,l} A = A T_m A = A, \text{ using (1.13)} \end{aligned}$$

so that $G_{lmN} \in \{A^-\}$.

Again the representation (2.6) shows that G_{lmN} is unique for given l, m, N . On the other hand, it is shown in Lemma 2.1 that given a $G \in \{A^-\}$, there exists an l, m, N for which it is the lmN -inverse, and the result (2.7) is established.

Incidentally, the existence of a g-inverse G satisfying the condition $AGA = A$ is established through the representation (2.6).

The anatomy of a g-inverse is exhibited in the following diagram.



Lemma 2.4 . Let k, m, a, l and N be as given in the definition above the equation (2.5). Then the following statements on a linear transformation G are equivalent.

- (i) $GA|m = I, \quad G|l = N|l$ (as defined in (2.5)).
- (ii) $GA = P_{m.k}, \quad GP_{l.a} = N.$
- (iii) $GA = P_{m.k}, \quad AG = P_{a.l}, \quad P_{k.m} G = N.$
- (iv) $GA = P_{m.k}, \quad AG = P_{a.l}, \quad G - GAG = N.$

$$(v) \quad AGA = A, \quad R(G|a) = m, \quad GP_{l,a} = N.$$

$$(vi) \quad AGA = A, \quad R(G|A) = m, \quad R(G|l) \subset k, \quad G - GAG = N.$$

Proof. (i) $\Rightarrow$ (ii).

$$GAX = x \text{ if } x \in m, \quad \text{using the condition } GA|m = I,$$

$$GAX = 0 \text{ if } x \in k, \quad \text{using the condition } Ax = 0$$

which shows that $GA = P_{m,k}$. Further

$$G|l = N|l \Rightarrow GP_{l,a} = NP_{l,a} = NP_{l,a} + NP_{a,l} = N.$$

(ii) $\Rightarrow$ (iii).

$$GP_{l,a} = N \Rightarrow AGP_{l,a} = 0, \quad \text{and} \quad A = AP_{m,k} = AGA$$

which together imply $AG = P_{a,l}$. Further

$$N = GP_{l,a} = G(I - P_{a,l}) = G(I - AG)$$

$$= (I - GA)G = (I - P_{m,k})G = P_{k,m}G.$$

(iii) $\Rightarrow$ (iv), since

$$P_{k,m} = I - GA \quad \text{and} \quad P_{k,m}G = G - GAG.$$

(iv) $\Rightarrow$ (v), since

$$GA = P_{m,k} \Rightarrow AGA = AP_{m,k} = A,$$

$$GA = P_{m,k} \Rightarrow R(GA) = R(P_{m,k}) = m$$

and

$$GP_{l,a} = G - GP_{a,l} = G - GAG = N.$$

(v) $\Rightarrow$ (vi), since

$$GP_{l,a} = N \Rightarrow AGP_{l,a} = AN = O \Rightarrow R(GP_{l,a}) \subset k.$$

Further

$$(G-GAG)P_{l,a} = N, (G-GAG)P_{a,l} = O \Rightarrow G - GAG = N.$$

Finally we show that (vi) $\Rightarrow$ (i). Let $x_1 \in m$. Then

$$R(G|a) = m \Rightarrow GAx_1 \in m.$$

Now let $x_1 + s = GAx_1$. Then $s \in m$, and

$$Ax_1 + As = A(x_1 + s) = AGAx_1 = Ax_1 \Rightarrow As = 0 \Rightarrow s \in k$$

using the condition $AGA = A$. Since m and k have only the null vector as intersection, $s = 0$ which shows that $GAx_1 = x_1$ or $GA|m = I$. Further noting that

$$R(G|l) \subset k \Rightarrow AGP_{l,a} = 0$$

we have

$$N = G - GAG \Rightarrow$$

$$NP_{l,a} = GP_{l,a} - GAGP_{l,a} = GP_{l,a} \Rightarrow N|l = G|l.$$

Lemma 2.4 is established.

When $N = O$, the statement (iv) of Lemma 2.4 reduces to the definition of a g-inverse given by Nashed and Votruba (1976), so that their g-inverse is G_{lmO} which is discussed in the next section.

3. REFELXIVE G-INVERSE

Definition. G is said to be a reflexive (or an outer inverse) of A if

$$AGA = A \text{ and } G = GAG. \quad (3.1)$$

The class of G satisfying (3.1) is represented by $\{A_T^-\}$ and any member of it by A_T^- .

The following lemma characterizes reflexive g-inverses

Lemma 3.1

$$\{G_{lmO}\} = \{A_T^-\} \subset \{A^-\} \quad (3.2)$$

i.e., all reflexive g-inverses are generated by choosing $N = O$, and $\{l, m\}$ as the set of all direct complements of a and k respectively.

Proof. From the representation (2.6)

$$G_{lmO} = T_m P_{a.l}.$$

Then

$$A T_m P_{a.l} A = A T_m A = A, \quad \text{using (1.13)}$$

$$T_m P_{a.l} A T_m P_{a.l} = T_m A T_m P_{a.l} = T_m P_{a.l}, \quad \text{using (1.14)}$$

which shows that $G_{lmO} \in \{A_T^-\}$.

Conversely let $AGA = A$ and $G = GAG$. Choose $m = R(GA)$ and $l = R(I-AG)$. Then as in (i) of Lemma 2.1

$$Gla = T_m.$$

Further, if $y \in l$, then $y = (I-AG)u$ for some u so that

$$\begin{aligned} Gy &= G(I-AG)u = 0, \quad \text{since } G = GAG \\ &\Rightarrow Gl = 0. \end{aligned}$$

Thus G satisfies the definition of G_{lmO} and Lemma 3.1 is proved.

Lemma 3.2

$$G_{lmN} = G_{lmO} + N \quad (3.3)$$

Proof. The result (3.3) follows from the representation (2.6).

Lemma 3.3 . The following statements are equivalent.

- (i) G is the lmO -inverse
- (ii) $GA = P_{m.k}, Gl = 0$
- (iii) $GA = P_{m.k}, AG = P_{a.l}, G = GAG.$
- (iv) $GA = P_{m.k}, g = R(G) = m, AG = P_{a.l}.$
- (v) $AGA = A, g = R(G) = m, R(Gl) = 0$

The results are proved on the same lines as in Lemma 2.4.

If v and w are inner product spaces and if we choose m as the orthogonal complement of k , and l as that of a , then the condition (iv) of Lemma 3.3 can be written as

$$GA = I - P_k = P_g, AG = P_a \quad (3.4)$$

where $g = R(G)$, P_k is the orthogonal projector on k and P_a is the orthogonal projector on a , which is equivalent to the definition given by Moore (1920) and Penrose (1955).

4. LM -INVERSE

If in the definition of the lm N-inverse, we do not specify N but only require that $Gl \subset k$, then we can write the conditions for a g -inverse in the form

$$GA \mid m = I, \quad AG \mid l = O. \quad (4.1)$$

We represent the solution of (4.1) by G_{lm} , which may not be unique and call it an lm -inverse.

Lemma 4.1. The following statements are equivalent for given l and m , any direct complements of a and k respectively.

- (i) G is an lm -inverse.
- (ii) $GA = P_{m.k}, \quad AG = P_{a.l}.$
- (iii) $AGA = A, \quad R(Gla) = m, \quad R(Gl) \subset k.$

The proof of Lemma 4.1 follows on the same lines as that of Lemma 2.4.

The definition given in (ii) of Lemma 4.1 was proposed by Langehop (1967).

5. OTHER INVERSES

5.1 M-inverse

An m -inverse is defined by specifying m only as in the following lemma.

Lemma 5.1. The following conditions are equivalent for an m -inverse.

- (i) $GA_m = I.$
- (ii) $GA = P_{m.k}$
- (iii) $AGA = A, R(GA) = m.$

We represent an m -inverse of A , which may not be unique, by A_m^- to be consistent with the notation developed earlier by the author (Rao, 1967).

If v is a vector space endowed with an inner product, then we may choose m to be an orthogonal complement of k . In such a case, if $Ax = y$ is a consistent equation, then

$$\min_{Ax=y} \|x\| = \|A_m^- y\| \quad (5.1)$$

so that $A_m^- y$ is the minimum norm solution of $Ax = y$. Note that any particular solution of $Ax = y$ is given by $A^- y$ where A^- is any g -inverse satisfying the condition $AA^-A = A$.

To prove 5.1, we observe that if x is any solution of $Ax = y$, then

$$A(x - A_m^- y) = 0 \Rightarrow x - A_m^- y \in k$$

and by definition $A_m^- y \in m$, and since m and k are orthogonal complements

$$\|x\|^2 = \|x - A_m^- y + A_m^- y\|^2 = \|x - A_m^- y\|^2 + \|A_m^- y\|^2, \quad \text{by Pythagorean theorem}$$

$$\geq \|A_m^- y\|^2.$$

5.2 l -inverse

An l -inverse of A , denoted by A_l^- is a linear transformation G as characterized in Lemma 5.2.

Lemma 5.2. The following conditions are equivalent for an l -inverse.

- (i) $AG = P_{a,l}$.
- (ii) $AGA = A$, $R(G|l) \subset k$.

If w is an inner product space, we may choose l to be the orthogonal complement of a . In such a case

$$\min_x \|y - Ax\| = \|y - A A_l^- y\| \quad (5.2)$$

so that $A_l^- y$ is a general least squares solution of a possibly inconsistent equation. Observe that

$$y - A A_l^- y = (I - P_{a,l})y = P_{l,a} y \in l$$

$$A A_l^- y = P_{a,l} y \in a \Rightarrow A A_l^- y - Ax \in a$$

so that by virtue of orthogonality of l and a

$$\begin{aligned} \|y - Ax\|^2 &= \|y - A A_l^- y + A A_l^- y - Ax\|^2 \\ &= \|y - A A_l^- y\|^2 + \|A A_l^- y - Ax\|^2 \geq \|y - A A_l^- y\|^2. \end{aligned}$$

5.3 lmO -inverse

The lmO -Inverse, specified by l , m and $N = O$ and denoted by A_{lmO} , is already characterized in Lemma 5.3. When v and w are inner product spaces, we may choose m to be the orthogonal complement of k , and l to be the orthogonal complement of a . In such a case the lmO -inverse can be characterized by

$$GA = (1 - P_k) = P_g, \quad AG = P_a \quad (5.3)$$

where P_x denotes the orthogonal projection operator on x . The inverse satisfying the conditions (5.3) is the Moore-Penrose inverse and is represented by A^+ . We show that

$$\min_{x_l} \{ \|x_l\| : \|y - Ax_l\| = \min_x \|y - Ax\| \} = \|A^+y\| \quad (5.4)$$

where the norms associated with $x \in v$ and $y \in w$ may be different, so that A^+y is the minimum norm least squares solution of $Ax = y$ (possibly inconsistent equation).

Observe that

$$\|y - Ax\|^2 = \|P_a y - Ax\|^2 + \|P_l y\|^2 \quad (5.5)$$

where the second member does not involve x , and the minimum of $\|y - Ax\|$ is attained if $Ax = P_a y$ so that x_l is a solution of $Ax = P_a y$ which is a consistent equation. The problem (5.4) demands the solution of $Ax = P_a y$ with the minimum norm, which is supplied by the equation (5.1)

$$\hat{x} = A_m^- P_a y = Gy.$$

Now

$$GA = A_m^- P_a A = A_m^- A = P_{m.k} = (I - P_k)$$

$$AG = A A_m^- P_a = P_a, \quad \text{since } R(P_a) = a$$

so that G satisfies the conditions (5.3) and hence it is the lmO -inverse.

Thus the l and lmO -inverses provide solutions to the least squares problem in the most general setting. When v and w are Euclidian spaces of m and n dimensions respectively, the linear transformation A can be

represented as an $n \times m$ matrix in which case explicit representations can be obtained for various generalized inverses as shown in Rao (1955, 1962, 1967, 1973) and Rao and Mitra (1971). The geometric approach in the case of a general linear transformation was developed in Rao and Yano (1985). All the results of this paper can be extended without any major modification of the proofs to bounded linear operations with closed range on Hilbert spaces.

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