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A Note on Incomplete Integrals of Cylindrical Functions

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Engineering Services Division

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A NOTE ON INCOMPLETE INTEGRALS OF CYLINDRICAL FUNCTIONS

INTRODUCTION

The class of cylindrical functions C includes Bessel functions of the first kind J , modified Bessel functions I , Bessel functions of the second kind or Neumann functions Y (or N), Bessel functions of imaginary argument or MacDonald functions K , and Bessel functions of the third kind that include Hankel functions of the first and second kind, $H^{(1)}$ and $H^{(2)}$.

The general incomplete Lipschitz-Hankel integral of cylindrical functions $C_\nu(z)$ is defined as the function of two complex variables:

$$C_{e_{\mu,\nu}}(a, z) \equiv \int_0^z e^{at} t^\mu C_\nu(t) dt. \quad (1)$$

Here the symbol e denotes the presence of the exponential function and μ, ν may be complex. Analogously, we define integrals that contain the functions $\sin(at)$ and $\cos(at)$ in place of $\exp(at)$:

$$C_{s_{\mu,\nu}}(a, z) \equiv \int_0^z \sin(at) t^\mu C_\nu(t) dt \quad (2)$$

$$C_{c_{\mu,\nu}}(a, z) \equiv \int_0^z \cos(at) t^\mu C_\nu(t) dt. \quad (3)$$

To assure convergence of $C_{e_{\mu,\nu}}(a, z)$ and $C_{c_{\mu,\nu}}(a, z)$, it is necessary that $\operatorname{Re}(\mu + 1) > |\operatorname{Re} \nu|$ when $C = K, Y, H^{(1)}, H^{(2)}$; $\operatorname{Re}(1 + \mu + \nu) > 0$ when $C = I, J$. When $\mu = \nu$, we define, for example, $C_{e_{\mu,\mu}} \equiv C_{e_\mu}$ where for convergence $\operatorname{Re} \mu > -1/2$ for all C .

Integrals of the type given by Eqs. (1) to (3) occur very often in applied mathematics. Agrest and Maksimov [1] have found representations for $C_{e_{\mu,\nu}}(a, z)$, $C_{s_{\mu,\nu}}(a, z)$, and $C_{c_{\mu,\nu}}(a, z)$ using incomplete cylindrical functions. In this report we give representations for $C_{e_{\mu,\nu}}(a, z)$, $C_{s_{\mu,\nu}}(a, z)$, and $C_{c_{\mu,\nu}}(a, z)$ using only the Kampé de Fériet double hypergeometric functions $F_{2:1:0}^{0:2:1}[x, y]$.

PRELIMINARY RESULTS AND DEFINITIONS

To begin, we summarize some results that are found in Ref. 2, p. 85: Let a and b be arbitrary constants,

$$\mathbf{F}_\nu(z) \equiv aI_\nu(z) + be^{i\nu\pi}K_\nu(z)$$

$$\mathbf{G}_\nu(z) \equiv aJ_\nu(z) + bY_\nu(z)$$

$$\alpha \equiv \begin{cases} i: \mathbf{H} = \mathbf{F} \\ 1: \mathbf{H} = \mathbf{G} \end{cases} \quad \beta \equiv \begin{cases} 1: \mathbf{H} = \mathbf{F} \\ 0: \mathbf{H} = \mathbf{G} \end{cases}$$

Then

$$\int_0^z t^\mu \mathbf{H}_\nu(t) dt = e^{-\frac{\pi}{2} i \beta \mu} [(\mu + \nu - 1) z \mathbf{H}_\nu(z) s_{\mu-1, \nu-1}(\alpha z) + (2\beta - 1) \alpha z \mathbf{H}_{\nu-1}(z) s_{\mu, \nu}(\alpha z)], \quad (4)$$

where the Lommel functions $s_{\mu, \nu}$ are given by

$$s_{\mu, \nu}(z) = \frac{z^{\mu+1}}{(\mu - \nu + 1)(\mu + \nu + 1)} {}_1F_2 \left[1; \frac{\mu - \nu + 3}{2}, \frac{\mu + \nu + 3}{2}; \frac{-z^2}{4} \right]. \quad (5)$$

Now defining

$$\xi \equiv \begin{cases} 1: & C = I, K \\ -1: & C = H, J, Y \end{cases} \quad \eta \equiv \begin{cases} 1: & C = K \\ -1: & C = H, I, J, Y \end{cases}$$

we may deduce from Eqs. (4) and (5) the result

$$\begin{aligned} \int_0^z t^\mu C_\nu(t) dt &= \frac{z^{\mu+1}}{\mu - \nu + 1} \left\{ C_\nu(z) {}_1F_2 \left[1; \frac{\mu - \nu + 3}{2}, \frac{\mu + \nu + 1}{2}; \frac{\xi z^2}{4} \right] \right. \\ &\quad \left. + \frac{\eta z C_{\nu-1}(z)}{\mu + \nu + 1} {}_1F_2 \left[1; \frac{\mu - \nu + 3}{2}, \frac{\mu + \nu + 3}{2}; \frac{\xi z^2}{4} \right] \right\}. \end{aligned} \quad (6)$$

We define the Kampé de Fériet double hypergeometric functions L and Q and give associated generating relations [3, 4]:

$$\begin{aligned} L[\alpha, \beta; \gamma, \delta; x, y] &\equiv F_{2,0,0}^{0,1,1} \left[\begin{array}{c} \text{---}; \quad \alpha; \quad \beta; \\ \gamma, \delta: \quad \text{---}; \quad \text{---}; \end{array} x, y \right], \quad |x| < \infty, \quad |y| < \infty \\ Q[\alpha, \beta, \gamma; \mu, \nu, \lambda; x, y] &\equiv F_{2,1,0}^{0,2,1} \left[\begin{array}{c} \text{---}; \quad \alpha, \beta; \quad \gamma; \\ \mu, \nu: \quad \lambda; \quad \text{---}; \end{array} x, y \right], \quad |x| < \infty, \quad |y| < \infty \\ L[\alpha, \beta; \gamma, \delta; x, y] &= \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(\gamma)_m (\delta)_m} \frac{x^m}{m!} {}_1F_2[\beta; m + \gamma, m + \delta; y] \\ Q[\alpha, \beta, \gamma; \mu, \nu, \lambda; x, y] &= \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\mu)_m (\nu)_m (\lambda)_m} \frac{x^m}{m!} {}_1F_2[\gamma; m + \mu, m + \nu; y]. \end{aligned} \quad (7)$$

It is easy to see that the function L is a special case of Q :

$$Q[\alpha, \lambda, \beta; \gamma, \delta, \lambda; x, y] = L[\alpha, \beta; \gamma, \delta; x, y].$$

For brevity we define the parameter lists

$$A_1(\mu, \nu) \equiv \frac{\mu + \nu + 1}{2}, \frac{\mu - \nu + 1}{2}, 1; \frac{\mu + \nu + 1}{2}, \frac{\mu - \nu + 3}{2}, \frac{1}{2}$$

$$A_2(\mu, \nu) \equiv \frac{\mu + \nu + 1}{2}, \frac{\mu - \nu + 1}{2}, 1; \frac{\mu + \nu + 3}{2}, \frac{\mu - \nu + 3}{2}, \frac{1}{2}$$

$$B_1(\mu, \nu) \equiv \frac{\mu + \nu + 2}{2}, \frac{\mu - \nu + 2}{2}, 1; \frac{\mu + \nu + 2}{2}, \frac{\mu - \nu + 4}{2}, \frac{3}{2}$$

$$B_2(\mu, \nu) \equiv \frac{\mu + \nu + 2}{2}, \frac{\mu - \nu + 2}{2}, 1; \frac{\mu + \nu + 4}{2}, \frac{\mu - \nu + 4}{2}, \frac{3}{2}$$

$$D_1(\mu) \equiv \frac{1}{2} + \mu, 1; \frac{1}{2} + \mu, \frac{3}{2}$$

$$D_2(\mu) \equiv \frac{1}{2} + \mu, 1; \frac{3}{2} + \mu, \frac{3}{2}$$

$$E_1(\mu, \nu) \equiv \frac{1 + \mu + \nu}{2}, \frac{2 + \mu + \nu}{2}, 1; \frac{2 + \mu + \nu}{2}, \frac{3 + \mu + \nu}{2}, 1 + \nu$$

$$E_2(\mu, \nu) \equiv \frac{1 + \mu + \nu}{2}, \frac{2 + \mu + \nu}{2}, 1; \frac{3 + \mu + \nu}{2}, \frac{4 + \mu + \nu}{2}, 1 + \nu$$

$$F_1(\mu) \equiv \frac{1}{2} + \mu, 1; 1 + \mu, \frac{3}{2} + \mu$$

$$F_2(\mu) \equiv \frac{1}{2} + \mu, 1; 2 + \mu, \frac{3}{2} + \mu$$

REPRESENTATIONS FOR $C_{e_{\mu}}(a, z)$, $C_{s_{\mu}}(a, z)$, $C_{c_{\mu}}(a, z)$

Substituting the Maclaurin series for $\exp(at)$ in Eq. (1) and splitting into even and odd terms we obtain on integrating term by term

$$C_{e_{\mu}}(a, z) = \sum_{n=0}^{\infty} \frac{a^{2n}}{(2n)!} \int_0^z t^{\mu+2n} C_{\nu}(t) dt + \sum_{n=0}^{\infty} \frac{a^{1+2n}}{(1+2n)!} \int_0^z t^{1+\mu+2n} C_{\nu}(t) dt.$$

Then using Eq. (6) and the generating relation Eq. (7) we obtain after a tedious but straightforward computation the principal result of this note

$$\begin{aligned}
 C_{e_{\mu, \nu}}(a, z) = & z^{1+\mu} C_{\nu}(z) \left\{ \frac{1}{\mu - \nu + 1} Q \left[A_1; \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] + \frac{az}{\mu - \nu + 2} Q \left[B_1; \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right\} \\
 & + \eta z^{2+\mu} C_{\nu-1}(z) \left\{ \frac{1}{(\mu + \nu + 1)(\mu - \nu + 1)} Q \left[A_2; \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right. \\
 & \left. + \frac{az}{(\mu + \nu + 2)(\mu - \nu + 2)} Q \left[B_2; \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right\}. \quad (8)
 \end{aligned}$$

Since

$$C_{s_{\mu, \nu}}(a, z) = \frac{1}{2i} \left\{ C_{e_{\mu, \nu}}(ia, z) - C_{e_{\mu, \nu}}(-ia, z) \right\}$$

$$C_{c_{\mu, \nu}}(a, z) = \frac{1}{2} \left\{ C_{e_{\mu, \nu}}(ia, z) + C_{e_{\mu, \nu}}(-ia, z) \right\}$$

we may write

$$\begin{aligned}
 C_{s_{\mu, \nu}}(a, z) = & \frac{az^{2+\mu}}{\mu - \nu + 2} \left\{ C_{\nu}(z) Q \left[B_1; \frac{-a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right. \\
 & \left. + \frac{\eta z}{\mu + \nu + 2} C_{\nu-1}(z) Q \left[B_2; \frac{-a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right\} \quad (9)
 \end{aligned}$$

$$\begin{aligned}
 C_{c_{\mu, \nu}}(a, z) = & \frac{z^{1+\mu}}{\mu - \nu + 1} \left\{ C_{\nu}(z) Q \left[A_1; \frac{-a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right. \\
 & \left. + \frac{\eta z}{\mu + \nu + 1} C_{\nu-1}(z) Q \left[A_2; \frac{-a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right\}. \quad (10)
 \end{aligned}$$

For $\mu = \nu$, Eqs. (8) to (10) reduce to

$$\begin{aligned}
 C_e(a, z) = & z^{1+\mu} C_{\mu}(z) \left\{ L \left[D_1; \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] + \frac{az}{2} Q \left[B_1(\mu, \mu); \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right\} \\
 & + \eta z^{2+\mu} C_{\mu-1}(z) \left\{ \frac{1}{1+2\mu} L \left[D_2; \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] + \frac{az}{4(1+\mu)} Q \left[B_2(\mu, \mu); \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right\} \quad (11)
 \end{aligned}$$

$$\begin{aligned}
 C_s(a, z) = & \frac{1}{2} az^{2+\mu} \left\{ C_{\mu}(z) Q \left[B_1(\mu, \mu); \frac{-a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right. \\
 & \left. + \frac{\eta z}{2(1+\mu)} C_{\mu-1}(z) Q \left[B_2(\mu, \mu); \frac{-a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right\} \quad (12)
 \end{aligned}$$

$$C_{\nu}(a, z) = z^{1+\mu} \left\{ C_{\mu}(z) L \left[D_1; \frac{-a^2 z^2}{4}, \frac{\xi z^2}{4} \right] + \frac{\eta z}{1+2\mu} C_{\mu-1}(z) L \left[D_2; \frac{-a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right\}. \quad (13)$$

Defining $J^+ \equiv J$, $J^- \equiv I$, it is interesting to note that we may also write [6]

$$J_{\nu}^{\pm}(a, z) = \frac{z^{1+\mu+\nu} e^{az}}{2^{\nu}(1+\mu+\nu)\Gamma(1+\nu)} \left\{ Q \left[E_1; \frac{\mp z^2}{4}, \frac{a^2 z^2}{4} \right] - \frac{az}{2+\mu+\nu} Q \left[E_2; \frac{\mp z^2}{4}, \frac{a^2 z^2}{4} \right] \right\}$$

$$J_{\nu}^{\pm}(a, z) = \frac{z(z^2/2)^{\mu} e^{az}}{(1+2\mu)\Gamma(1+\mu)} \left\{ L \left[F_1; \frac{\mp z^2}{4}, \frac{a^2 z^2}{4} \right] - \frac{az}{2(1+\mu)} L \left[F_2; \frac{\mp z^2}{4}, \frac{a^2 z^2}{4} \right] \right\}. \quad (14)$$

Here the Bessel functions $J_{\nu}^{\pm}$ do not appear.

REDUCTION FORMULAS FOR L AND Q

Many special cases of Eqs. (11) to (14) may be obtained in one form or another, provided we know a reduction formula for either L or Q . We summarize some known relevant reduction formulas [3-6]:

$$L[\alpha, \beta; \gamma, \delta; z, z] = {}_1F_2[\alpha + \beta; \gamma, \delta; z]$$

$$L \left[D_2; \frac{z^2}{4}, \frac{z^2}{4} \right] = \frac{\sinh z}{z}$$

$$L \left[D_1; \frac{z^2}{4}, \frac{z^2}{4} \right] = \frac{2\mu}{1+2\mu} \frac{\sinh z}{z} + \frac{\cosh z}{1+2\mu}$$

$$Q \left[B_2(\mu, \mu); \frac{z^2}{4}, \frac{z^2}{4} \right] = \frac{1+\mu}{1+2\mu} \frac{4}{z^2} \left\{ \cosh z - \left(\frac{2}{z} \right)^{\mu} \Gamma(1+\mu) I_{\mu}(z) \right\}$$

$$Q \left[B_1(\mu, \mu); \frac{z^2}{4}, \frac{z^2}{4} \right] = \frac{2}{1+2\mu} \frac{1}{z} \left\{ 2\mu \frac{\cosh z}{z} + \sinh z - \left(\frac{2}{z} \right)^{\mu} \Gamma(1+\mu) I_{\mu-1}(z) \right\}.$$

Other properties and reduction formulas for L and Q are found in Refs. 3-6.

APPLICATIONS

Of interest in applications are the functions $J_{e_0}(a, z)$, $I_{e_0}(a, z)$, $Y_{e_0}(a, z)$, and $K_{e_0}(a, z)$. $J_{e_0}(a, z)$ and $Y_{e_0}(a, z)$ occur in problems in the theory of diffraction in optical apparatus [1, p. 227]. The function $I_{e_0}(a, z)$ plays an important role in the study of oscillating wings in supersonic flow and arises in the study of resonant absorption in media with finite dimensions [1, p. 195]. $K_{e_0}(a, z)$ occurs when the statistical distribution of the maxima of a random function is applied to the amplitude of a sine wave in order to calculate the distribution of its ordinate. This latter distribution is of

interest in the study of the scattered coherent reflected field from the sea surface [7, 8]. Since the functions $C_{e_0}(a, z)$ are of some importance, by using Eq. (11) and defining

$$L_1(x, y) \equiv L \left[\frac{1}{2}, 1; \frac{1}{2}, \frac{3}{2}; x, y \right]$$

$$L_0(x, y) \equiv L \left[\frac{1}{2}, 1; \frac{3}{2}, \frac{3}{2}; x, y \right]$$

$$Q_1(x, y) \equiv Q \left[1, 1, 1; 1, 2, \frac{3}{2}; x, y \right]$$

$$Q_0(x, y) \equiv Q \left[1, 1, 1; 2, 2, \frac{3}{2}; x, y \right]$$

we obtain

$$\begin{aligned} K_{e_0}(a, z) &= zK_0(z) \left\{ L_1 \left[\frac{a^2 z^2}{4}, \frac{z^2}{4} \right] + \frac{az}{2} Q_1 \left[\frac{a^2 z^2}{4}, \frac{z^2}{4} \right] \right\} \\ &\quad + z^2 K_1(z) \left\{ L_0 \left[\frac{a^2 z^2}{4}, \frac{z^2}{4} \right] + \frac{az}{4} Q_0 \left[\frac{a^2 z^2}{4}, \frac{z^2}{4} \right] \right\} \\ Y_{e_0}(a, z) &= zY_0(z) \left\{ L_1 \left[\frac{a^2 z^2}{4}, \frac{-z^2}{4} \right] + \frac{az}{2} Q_1 \left[\frac{a^2 z^2}{4}, \frac{-z^2}{4} \right] \right\} \\ &\quad + z^2 Y_1(z) \left\{ L_0 \left[\frac{a^2 z^2}{4}, \frac{-z^2}{4} \right] + \frac{az}{4} Q_0 \left[\frac{a^2 z^2}{4}, \frac{-z^2}{4} \right] \right\} \\ J_{e_0}(a, z) &= zJ_0(z) \left\{ L_1 \left[\frac{a^2 z^2}{4}, \frac{-z^2}{4} \right] + \frac{az}{2} Q_1 \left[\frac{a^2 z^2}{4}, \frac{-z^2}{4} \right] \right\} \\ &\quad + z^2 J_1(z) \left\{ L_0 \left[\frac{a^2 z^2}{4}, \frac{-z^2}{4} \right] + \frac{az}{4} Q_0 \left[\frac{a^2 z^2}{4}, \frac{-z^2}{4} \right] \right\} \\ I_{e_0}(a, z) &= zI_0(z) \left\{ L_1 \left[\frac{a^2 z^2}{4}, \frac{z^2}{4} \right] + \frac{az}{2} Q_1 \left[\frac{a^2 z^2}{4}, \frac{z^2}{4} \right] \right\} \\ &\quad - z^2 I_1(z) \left\{ L_0 \left[\frac{a^2 z^2}{4}, \frac{z^2}{4} \right] + \frac{az}{4} Q_0 \left[\frac{a^2 z^2}{4}, \frac{z^2}{4} \right] \right\}. \end{aligned}$$

The equations for $H_{e_0}^{(1)}$ and $H_{e_0}^{(2)}$ are the same as those for Y_{e_0} or J_{e_0} with Y or J replaced by $H^{(1)}$ or $H^{(2)}$. Further, from Eq. (14) we have

$$J_{e_0}(a, z) = ze^{az} \left\{ L \left[1, \frac{1}{2}; \frac{3}{2}, 1; \frac{a^2 z^2}{4}, \frac{-z^2}{4} \right] - \frac{az}{2} L \left[1, \frac{1}{2}; \frac{3}{2}, 2; \frac{a^2 z^2}{4}, \frac{-z^2}{4} \right] \right\}$$

$$I_{e_0}(a, z) = ze^{az} \left\{ L \left[1, \frac{1}{2}; \frac{3}{2}, 1; \frac{a^2 z^2}{4}, \frac{z^2}{4} \right] - \frac{az}{2} L \left[1, \frac{1}{2}; \frac{3}{2}, 2; \frac{a^2 z^2}{4}, \frac{z^2}{4} \right] \right\}.$$

Here we have used the properties of L that

$$L[\alpha, \beta; \gamma, \delta; x, y] = L[\alpha, \beta; \delta, \gamma; x, y] = L[\beta, \alpha; \gamma, \delta; y, x].$$

The latter results for $C_{e_0}(a, z)$ should prove useful in numerical computation of these functions.

SUMMARY

Representations for incomplete Lipschitz-Hankel integrals of cylindrical functions using only the Kampé de Fériet functions in two variables $F_{2;1;0}^{0;2;1}[x, y]$ are given. In addition, known relevant reduction formulas for these functions are provided.

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