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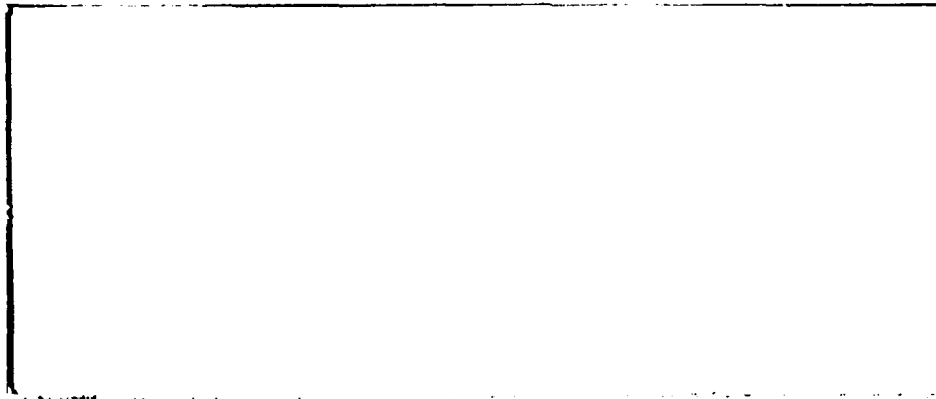
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Efficient Nearly Orthogonal Deletion Designs

by

Subir Ghosh and Joan Mahoney

Technical Report No. 168

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University of California, Irvine and
Hughes Aircraft Company

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Efficient Nearly Orthogonal Deletion Designs

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0. Summary

This article considers single replicate factorial experiments in incomplete blocks. A single replicate $2^{m_1} \times 3^{m_2}$ deletion design in incomplete blocks is obtained from a single replicate 3^m ($m = m_1 + m_2$) preliminary design by deleting all runs (or treatment combinations) with the first m_1 factors at the level two. A systematic method for determining the unbiasedly estimable (u.e.) and not unbiasedly estimable (n.u.e) factorial effects is provided. Although the method is discussed for single replicate $2^{m_1} \times 3^{m_2}$ deletion designs in three incomplete blocks, the method can easily be extended to more than three blocks. It is shown that for $m_2 > 0$ all factorial effects of the type $F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha_{m_1+1}} \dots F_m^{\alpha_m}$, $\alpha_i = 0, 1$ for $i = 1, \dots, m_1$, $\alpha_i = 0, 1, 2$ for $i = m_1+1, \dots, m$, $(\alpha_1, \dots, \alpha_m) \neq (0, \dots, 0)$, $(\alpha_{m_1+1}, \dots, \alpha_m) \neq \alpha(1, \dots, 1)$ where $\alpha = 1$ and 2 , are u.e. and the remaining factorial effects are n.u.e. It is noted that $(2^{m_1} - 1)$

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factorial effects of 2^{m_1} factorial experiments and $(3^{m_2}-3)$ factorial effects of 3^{m_2} factorial experiments, which are embedded in $2^{m_1} \times 3^{m_2}$ factorial experiments, are u.e. The $2 \times 3^{m-1}$ deletion designs were considered in the work of Voss (1986). Defining factorial effects of a $2^{m_1} \times 3^{m_2}$ factorial experiment in a form different than in Voss (1986), a simple representation of u.e. and n.u.e. factorial effects is obtained. In this representation, there are $(2^{m_1+1} + 1)$ n.u.e. factorial effects of the type $F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha_{m_1+1}} \dots F_m^{\alpha_m}$. This number is smaller than the corresponding number of n.u.e. factorial effects in the representation of Voss (1986). The relative efficiencies in the estimation of factorial effects of $2^{m_1} \times 3^{m_2}$ deletion designs are also given.

KEY WORDS: Confounding, Factorial experiment, Single replicate,
Unbiasedly estimable.

1. Introduction

→ There is a vast literature on the construction of orthogonal single replicate factorial designs in incomplete blocks. The reader is referred to Voss (1986) for the list of references. The concept of deletion designs was introduced in Kishen and Srivastava (1959). The deletion technique in deletion designs was then used by many authors, (see Addelman (1962, 1972), Margolin (1969), Sardana and Das (1965), Voss (1986)). This article considers $2^{m_1} \times 3^{m_2}$ deletion designs in three incomplete blocks and then presents a systematic method for finding the u.e. and n.u.e. factorial effects. The smaller values of m_1 and m_2 are the most practically important cases.

For n.u.e. factorial effects, the biased estimators (biased w.r.t block effects) are called the unadjusted estimators. Under the assumption that certain higher order interactions are negligible, the unbiased estimation of block effects contrasts and n.u.e. factorial effects, excluding the general mean, are possible. This makes the deletion design an orthogonal design. The unbiased estimators of n.u.e. factorial effects under the assumption are called the adjusted estimators.

The relative efficiency in the estimation of a factorial effect is the ratio of the variance of the unadjusted estimator divided by the variance of the adjusted estimator. Observe that for u.e. factorial effects there is no need for adjustment and hence the relative efficiency is unity. For n.u.e. factorial effects the relative efficiency is less than unity. The closer the value of the relative efficiency to unity implies the lesser effect of adjustment to the variance of the estimator.

Definition and notation are given in section 2. Section 3 presents the systematic method of determining u.e. and n.u.e. factorial effects. Section 4 discusses the relative efficiency with an illustrative example. Section 5 presents some miscellaneous results.

2. Definition and Notation

Consider a single replicate $2^{m_1} \times 3^{m_2}$ factorial experiment in incomplete blocks. There are m , $m = m_1 + m_2$, factors in the experiment. The runs are denoted by $(x_1, \dots, x_{m_1}, x_{m_1+1}, \dots, x_m)$, where $x_i = 0, 1$, for $i = m_1 + 1, \dots, m$ and $x_i = 0, 1, 2$, for $i = 1, \dots, m_1$. The runs and their effects are denoted by the same notation. The factorial effects are denoted by $F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha_{m_1+1}} \dots F_m^{\alpha_m}$, where $\alpha_i = 0, 1$ for $i = 1, \dots, m_1$ and $\alpha_i = 0, 1, 2$ for $i = m_1 + 1, \dots, m$. The observation on the run $(x_1, \dots, x_m)$ is denoted by $y(x_1, \dots, x_m)$. The fixed effect model assumed is

$$\begin{aligned} E(y(x_1, \dots, x_m)) &= (x_1, \dots, x_m) + \beta_j, \\ V(y(x_1, \dots, x_m)) &= \sigma^2, \\ \text{Cov}(y(x_1, \dots, x_m), y(x'_1, \dots, x'_m)) &= 0, \end{aligned} \quad (1)$$

where β_j is the fixed effect of the j th block containing the run $(x_1, \dots, x_m)$, σ^2 and β_j ($j = 0, 1, 2$) are unknown constants. Recall that the effect of the run $(x_1, \dots, x_m)$ is denoted by the same notation $(x_1, \dots, x_m)$. The notation $\{\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = u_1\}$ represents the sum of all 2^{m_1-1} points $(x_1, \dots, x_{m_1})$ which are solutions of $\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = u_1$ over the Galois Field $GF(2)$, $u_1 = 0, 1$. Again the notation $\{\alpha_{m_1+1} x_{m_1+1} + \dots + \alpha_m x_m = u_2\}$ represents the sum of all 3^{m_2-1} points

$(x_{m_1+1}, \dots, x_m)$ which are solutions of $\alpha_{m_1+1}x_{m_1+1} + \dots + \alpha_m x_m = u_2$ over the Galois Field $GF(3)$, $u_2 = 0, 1$. The Kronecker product of

$\{\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = u_1\}$ and $\{\alpha_{m_1+1} x_{m_1+1} + \dots + \alpha_m x_m = u_2\}$ is denoted by $\{\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = u_1\} \otimes \{\alpha_{m_1+1} x_{m_1+1} + \dots + \alpha_m x_m = u_2\}$ and it represents

the sum of all $2^{m_1-1} 3^{m_2-1}$ run effects $(x_1, \dots, x_{m_1}, x_{m_1+1}, \dots, x_m)$ where $(x_1, \dots, x_{m_1})$ is a solution of $\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = u_1$ over $GF(2)$ and $(x_{m_1+1}, \dots, x_m)$ is a solution of $\alpha_{m_1+1} x_{m_1+1} + \dots + \alpha_m x_m = u_2$ over $GF(3)$.

Example 1. Consider a $2^2 \times 3^2$ factorial experiment. We have $m_1 = 2$, $m_2 = 2$ and $m = m_1 + m_2 = 4$. The notation $\{x_1 + x_2 = 0\}$ represents the sum $(0,0) + (1,1)$. The notation $\{x_3 + 2x_4 = 1\}$ represents the sum $(1,0) + (0,2) + (2,1)$. The Kronecker product $\{x_1 + x_2 = 0\} \otimes \{x_3 + 2x_4 = 1\}$ represents the sum of run effects, $(0,0,1,0) + (0,0,0,2) + (0,0,2,1) + (1,1,1,0) + (1,1,0,2) + (1,1,2,1)$.

The factorial effects of a $2^{m_1} \times 3^{m_2}$ factorial experiment are defined in terms of run effects by

$$\begin{aligned} & \alpha_1 \dots \alpha_{m_1} \alpha_{m_1+1} \dots \alpha_m \\ & F_1 \dots F_{m_1} F_{m_1+1} \dots F_m \\ & = \left[c_0 \{\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = 0\} + c_1 \{\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = 1\} \right] \\ & \otimes \left[d_0 \{\alpha_{m_1+1} x_{m_1+1} + \dots + \alpha_m x_m = 0\} + d_1 \{\alpha_{m_1+1} x_{m_1+1} + \dots + \alpha_m x_m = 1\} \right. \\ & \quad \left. + d_2 \{\alpha_{m_1+1} x_{m_1+1} + \dots + \alpha_m x_m = 2\} \right], \end{aligned} \quad (2)$$

where the coefficients c_0, c_1, d_0, d_1 and d_2 are given in Table 1.

Table 1

The coefficients c_0 , c_1 , d_0 , d_1 and d_2 in the equation (2)

	c_0	c_1	d_0	d_1	d_2
$(\alpha_1, \dots, \alpha_{m_1})' = \underline{0}, (\alpha_{m_1+1}, \dots, \alpha_m)' = \underline{0}$	1	1	1	1	1
$\neq \underline{0} \quad \quad \quad = \underline{0}$	-1	1	1	1	1
$= \underline{0} \quad \quad \quad \neq \underline{0}$					
(i) the first nonzero element in $(\alpha_{m_1+1}, \dots, \alpha_m)$ is 1.	1	1	-1	0	1
(ii) the first nonzero element in $(\alpha_{m_1+1}, \dots, \alpha_m)$ is 2.	1	1	1	-2	1
$\neq \underline{0} \quad \quad \quad \neq \underline{0}$					
(i) the first nonzero element in $(\alpha_{m_1+1}, \dots, \alpha_m)$ is 1.	-1	1	-1	0	1
(ii) the first nonzero element in $(\alpha_{m_1+1}, \dots, \alpha_m)$ is 2.	-1	1	1	-2	1

Example 2. In Example 1, the factorial effect $F_2 F_3^2$ is defined by

$$\begin{aligned}
 F_2 F_3^2 &= [-\{x_2 = 0\} + \{x_2 = 1\}] \otimes [\{x_3 = 0\} - 2\{x_3 = 1\} + \{x_3 = 2\}] \\
 &= [-(0,0) - (1,0) + (0,1) + (1,1)] \\
 &\otimes [(0,0) + (0,1) + (0,2) - 2(1,0) - 2(1,1) - 2(1,2) \\
 &\quad + (2,0) + (2,1) + (2,2)] \\
 &= - (0,0,0,0) - \dots + 2(0,0,1,0) + \dots - (0,0,2,0) - \dots + (1,1,2,2).
 \end{aligned}$$

A $2^{m_1} \times 3^{m_2}$ deletion design D in three incomplete blocks is described below. The deletion design D is used throughout the discussion. Consider a 3^m factorial experiment in 3 blocks by confounding the two degrees of freedom in $F_1 F_2 \dots F_m$ and $F_1^2 F_2^2 \dots F_m^2$. The block u consists of runs which are solutions of the equation $x_1 + \dots + x_m = u$, $u = 0, 1, 2$. From every block, the runs with the level 2 for the first m_1 factors are deleted. The resulting design is D with $2^{m_1} \times 3^{m_2-1}$ runs in every block. It is assumed that $m_2 \geq 1$. The design D for $m_2 = 0$ is discussed in Section 5.

Example 3. The runs in the three blocks of a $2^2 \times 3^2$ deletion design D are given below.

Block 0	0	0	0	1	1	1	0	0	0	1	1	1
	0	0	0	0	0	0	1	1	1	1	1	1
	0	1	2	2	0	1	2	0	1	1	0	2
	0	2	1	0	2	1	0	2	1	0	1	2
Block 1	0	0	0	1	1	1	0	0	0	1	1	1
	0	0	0	0	0	0	1	1	1	1	1	1
	1	0	2	0	1	2	0	1	2	2	0	1
	0	1	2	0	2	1	0	2	1	0	2	1
Block 2	0	0	0	1	1	1	0	0	0	1	1	1
	0	0	0	0	0	0	1	1	1	1	1	1
	2	0	1	1	0	2	1	0	2	0	1	2
	0	2	1	0	1	2	0	1	2	0	2	1

It is to be noted that in every block there are 12 runs and the columns represent the runs.

The least squares estimators of u.e. factorial effects

$F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha_{m_1+1}} \dots F_m^{\alpha_m}$ is $F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha_{m_1+1}} \dots F_m^{\alpha_m}$ which is obtained by replacing the run effect $(x_1, \dots, x_m)$ with the observation $y(x_1, \dots, x_m)$ in (2). For n.u.e. factorial effects, the same method yields biased (non-adjusted) estimators.

Let B_u ($u = 0, 1, 2$) be the sum of all run effects in the u th block. Let $X = -B_1 + B_2$ and $Y = 2B_0 - B_1 - B_2$. Clearly X and Y are confounded with the blocks in D . Let $B_u(\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = i)$, $i = 0, 1$, $u = 0, 1, 2$, denote the sum of all run effects satisfying $\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = i$ in the u th block. Notice that $B_u = B_u(\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = 0) + B_u(\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = 1)$.

Example 4. Consider the block 0 in Example 3. Observe that

$$\begin{aligned} B_0(x_1 + x_2 = 0) &= (0, 0, 0, 0) + (0, 0, 1, 2) + (0, 0, 2, 1) \\ &\quad + (1, 1, 0, 1) + (1, 1, 1, 0) + (1, 1, 2, 2), \\ B_1(x_1 + x_2 = 1) &= (1, 0, 2, 0) + (1, 0, 0, 2) + (1, 0, 1, 1) \\ &\quad + (0, 1, 2, 0) + (0, 1, 0, 2) + (0, 1, 1, 1). \end{aligned}$$

Denote

$$\begin{aligned} F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} X &= - [B_1(\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = 1) - B_1(\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = 0)] \\ &\quad + [B_2(\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = 1) - B_2(\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = 0)], \\ F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} Y &= 2 [B_0(\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = 1) - B_0(\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = 0)] \\ &\quad - [B_1(\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = 1) - B_1(\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = 0)], \\ &\quad - [B_2(\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = 1) - B_2(\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = 0)]. \quad (3) \end{aligned}$$

3. Properties.

In this section the u.e. and n.u.e. factorial effects under D are given. It is assume that $m_2 \geq 1$.

Theorem 1. The factorial effects $F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha_{m_1+1}} \dots F_m^{\alpha_m}$ for $(\alpha_{m_1+1}, \dots, \alpha_m) \neq \alpha(1, \dots, 1)$, $\alpha = 1, 2$ and $(\alpha_1, \dots, \alpha_m) \neq (0, \dots, 0)$, are u.e. under D.

Proof. When $(\alpha_{m_1+1}, \dots, \alpha_m) \neq \alpha(1, \dots, 1)$ and $(\alpha_1, \dots, \alpha_m) \neq (0, \dots, 0)$, it

can be seen that $2^{m_1} 3^{m_2-1}$ runs in a block can be divided into six sets of $2^{m_1-1} 3^{m_2-2}$ runs satisfying $\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = u_1$ and

$\alpha_{m_1+1} x_{m_1+1} + \dots + \alpha_m x_m = u_2$, $u_1 = 0, 1$ and $u_2 = 0, 1, 2$. It now follows from

(1) and (2) that in $E(F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha_{m_1+1}} \dots F_m^{\alpha_m})$, the block effects cancel

and it becomes equal to $F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha_{m_1+1}} \dots F_m^{\alpha_m}$. This completes the proof.

Example 5. In Example 3, the factorial effects $F_1, F_2, F_1 F_2, F_3, F_3^2, F_4, F_4^2, F_3 F_4^2, F_3^2 F_4, F_1 F_3, F_1 F_3^2, F_1 F_4, F_1 F_4^2, F_1 F_3 F_4^2, F_1 F_3^2 F_4, F_2 F_3, F_2 F_3^2, F_2 F_4, F_2 F_4^2, F_2 F_3 F_4^2, F_2 F_3^2 F_4, F_1 F_2 F_3, F_1 F_2 F_3^2, F_1 F_2 F_4, F_1 F_2 F_4^2, F_1 F_2 F_3 F_4^2, F_1 F_2 F_3^2 F_4$ are u.e. under D by Theorem 1.

Theorem 2. The factorial effects $F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha_{m_1+1}} \dots F_m^{\alpha_m}$ and

$F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^2 \dots F_m^2$ are n.u.e. under D (i.e., they are confounded with blocks in D).

Proof. Consider the u th ($u = 0, 1, 2$) block in D. Out of $2^{m_1} 3^{m_2-1}$ runs in the u th block, $2^{m_1-1} 3^{m_2-1}$ runs satisfy $x_1 + \dots + x_{m_1} = 0$ over GF(2) and

$x_{m_1+1} + \dots + x_m = u$ over $GF(3)$. The remaining $2^{m_1-1} 3^{m_2-1}$ runs satisfy $x_1 + \dots + x_{m_1} = 1$ over $GF(2)$ and $x_{m_1+1} + \dots + x_m = u - 1$ over $GF(3)$. Out of $2^{m_1-1} 3^{m_2-1}$ runs satisfying $x_1 + \dots + x_{m_1} = i$, $i = 0, 1$, $2^{m_1-2} 3^{m_2-1}$ runs satisfy $\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = j$, $j = 0, 1$, and $(\alpha_1, \dots, \alpha_{m_1}) \neq (1, \dots, 1)$, $(0, \dots, 0)$. It is now clear from the definition (2) of $F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha_{m_1+1}} \dots F_m^{\alpha_m}$ with $\alpha_{m_1+1} = \dots = \alpha_m = \alpha$, $\alpha = 1, 2$, that in $E(F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha_{m_1+1}} \dots F_m^{\alpha_m})$ the block effects do not cancel. This completes the proof.

Example 6. In Example 3, the factorial effects $F_3 F_4$, $F_3^2 F_4^2$, $F_1 F_3 F_4$, $F_2 F_3 F_4$, $F_1 F_2 F_3 F_4$, $F_1 F_3^2 F_4^2$, $F_2 F_3^2 F_4^2$ and $F_1 F_2 F_3^2 F_4^2$ are not u.e. in addition to the general mean μ .

Theorem 3. Under D , $F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} X$ and $F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} Y$ with $(\alpha_1, \dots, \alpha_{m_1}) \neq (0, \dots, 0)$, defined in (3) are u.e.

Proof. In the u th ($u = 0, 1, 2$) block of D , $2^{m_1-1} 3^{m_2-1}$ runs can be divided into 2 sets of $2^{m_1-2} 3^{m_2-1}$ runs each satisfying $\alpha_1 x_1 + \dots + \alpha_{m_1} x_{m_1} = i$, $i = 0, 1$. It now follows from (1) that in $E(F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} X)$ and $E(F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} Y)$ the block effects cancel. The rest is clear. This completes the proof.

Observe that μ , X , Y are confounded with blocks in D . The $(2^{m_1-1} (3^{m_2-1} - 1))$ factorial effects $F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha_{m_1+1}} \dots F_m^{\alpha_m}$ with $(\alpha_{m_1+1}, \dots, \alpha_m) \neq \alpha(1, \dots, 1)$, $\alpha = 1, 2$ and $(\alpha_1, \dots, \alpha_{m_1}) \neq (0, \dots, 0)$, are

u.e. under D. The $(2^{m_1-1}-1)2$ linear functions of factorial effects $F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} X$ and $F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} Y$ with $(\alpha_1, \dots, \alpha_{m_1}) \neq (0, \dots, 0)$, are u.e. under

D. The above $[3 + (2^{m_1}(3^{m_2-2}-1) + (2^{m_1-1}-1)2)] = 2^{m_1} 3^{m_2}$ linear functions of factorial effects are othogonal to each other.

4. Relative Efficiency

In this section the relative efficiencies of n.u.e. factorial effects are calculated. First note that

$$E(\widehat{F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha} \dots F_m^{\alpha}}) = F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha} \dots F_m^{\alpha} + (d_0 \beta_0 + d_1 \beta_1 + d_2 \beta_2), \quad (4)$$

where d_0 , d_1 and d_2 depends on the values of α_i , $i = 1, \dots, m_1$ and

α , $\alpha = 1, 2$. The estimator $\widehat{F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha} \dots F_m^{\alpha}}$ is called the unadjusted

estimator of $F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha} \dots F_m^{\alpha}$ and it is denoted by

$(\widehat{F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha} \dots F_m^{\alpha}})_{\text{unadj}}$. It can be checked that

$$\text{Var}(\widehat{F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha} \dots F_m^{\alpha}})_{\text{unadj}} = \begin{cases} \sigma^2 2^{m_1+1} 3^{m_2-1} & \text{for } \alpha = 1, \\ \sigma^2 2^{m_1+1} 3^{m_2} & \text{for } \alpha = 2. \end{cases} \quad (5)$$

It can be seen that out of 2^{m_1-1} points $(x_1, \dots, x_{m_1})$ satisfying $x_1 + \dots + x_{m_1} = 0$ over $\text{GF}(2)$, n_{0u} points satisfy $x_1 + \dots + x_{m_1} = u$, $u = 0, 1, 2$,

over $\text{GF}(3)$. Again, out of 2^{m_1-1} points $(x_1, \dots, x_{m_1})$ satisfying $x_1 + \dots + x_{m_1} = 1$ over $\text{GF}(2)$, n_{1u} points satisfy $x_1 + \dots + x_{m_1} = u$, $u = 0, 1, 2$,

over $\text{GF}(3)$. Clearly, $n_{00} + n_{01} + n_{02} = n_{10} + n_{11} + n_{12} = 2^{m_1-1}$. It can be check that

$$\begin{aligned}
 n_{00} &= \sum_{\substack{w \geq 0 \\ w \text{ even integer}}} \binom{m_1}{3w}, & n_{01} &= \sum_{\substack{w \geq 0 \\ w \text{ odd integer}}} \binom{m_1}{3w+1}, \\
 n_{02} &= \sum_{\substack{w \geq 0 \\ w \text{ even integer}}} \binom{m_1}{3w+2}, & n_{10} &= \sum_{\substack{w \geq 0 \\ w \text{ odd integer}}} \binom{m_1}{3w}, \\
 n_{11} &= \sum_{\substack{w \geq 0 \\ w \text{ even integer}}} \binom{m_1}{3w+1}, & n_{12} &= \sum_{\substack{w \geq 0 \\ w \text{ odd integer}}} \binom{m_1}{3w+2}.
 \end{aligned} \tag{6}$$

Under the assumption that the factorial effects $F_1 \dots F_{m_1} F_{m_1+1}^\alpha \dots F_m^\alpha$,

$\alpha = 1, 2$, are negligible, it follows that

$$\begin{aligned}
 E(\widehat{F_1 \dots F_{m_1} F_{m_1+1}^\alpha \dots F_m^\alpha})_{\text{unadj}} &= 3^{m_2-1} [(n_{10} - n_{12} - n_{00} + n_{02})\beta_2 \\
 &+ (n_{12} - n_{11} - n_{02} + n_{01})\beta_1 + (n_{11} - n_{10} - n_{01} + n_{00})\beta_0], \\
 E(\widehat{F_1 \dots F_{m_1}^2 F_{m_1+1}^\alpha \dots F_m^\alpha})_{\text{unadj}} &= 3^{m_2-1} [(n_{10} - 2n_{11} + n_{12} - n_{00} + 2n_{01} - n_{02})\beta_2 \\
 &+ (n_{12} - 2n_{10} + n_{11} - n_{02} + 2n_{00} - n_{01})\beta_1 + (n_{11} - 2n_{12} + n_{10} - n_{01} + 2n_{02} - n_{00})\beta_0].
 \end{aligned} \tag{7}$$

For $(\alpha_1, \dots, \alpha_{m_1}) \neq (1, \dots, 1)$, the adjusted estimators of factorial

effects $F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^\alpha \dots F_m^\alpha$ are

$$\begin{aligned}
 (\widehat{F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^\alpha \dots F_m^\alpha})_{\text{adj}} &= (\widehat{F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^\alpha \dots F_m^\alpha})_{\text{unadj}} \\
 &+ w_1 (\widehat{F_1 \dots F_{m_1} F_{m_1+1}^\alpha \dots F_m^\alpha})_{\text{unadj}} + w_2 (\widehat{F_1 \dots F_{m_1}^2 F_{m_1+1}^\alpha \dots F_m^\alpha})_{\text{unadj}},
 \end{aligned} \tag{8}$$

where w_1 and w_2 are constants depending on α and $(\alpha_1, \dots, \alpha_{m_1})$. Notice

that under the assumption that $F_1 \dots F_{m_1} F_{m_1+1}^\alpha \dots F_m^\alpha$ are negligible, the

factorial effects $F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^\alpha \dots F_m^\alpha$, $(\alpha_1, \dots, \alpha_{m_1}) \neq (1, \dots, 1)$,

$\alpha = 1, 2$, are u.e. and the adjusted estimators of $F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha} \dots F_m^{\alpha}$, $(\alpha_1, \dots, \alpha_{m_1}) \neq (1, \dots, 1)$, $\alpha = 1, 2$, are in fact unbiased estimators. The unbiased estimators of factorial effects (except the general mean) are orthogonal to each other and hence the deletion design is orthogonal under the assumption that $F_1 \dots F_{m_1} F_{m_1+1}^{\alpha} \dots F_m^{\alpha}$, $\alpha = 1, 2$, are negligible. The effect of adjustment is now evaluated in terms of the variance of the estimators. It can be seen from (8) that for $(\alpha_1, \dots, \alpha_{m_1}) \neq (1, \dots, 1)$,

$$V(\widehat{F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha} \dots F_m^{\alpha}})_{adj} = \begin{cases} \sigma^2 2^{m_1+1} 3^{m_2-1} (1 + w_1^2 + 3w_2^2) & \text{for } \alpha = 1, \\ \sigma^2 2^{m_1+1} 3^{m_2-1} (3 + w_1^2 + 3w_2^2) & \text{for } \alpha = 2. \end{cases} \quad (9)$$

The relative efficiency in the estimation of $F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha} \dots F_m^{\alpha}$, $(\alpha_1, \dots, \alpha_{m_1}) \neq (1, \dots, 1)$ is

$$RE = \frac{V(\widehat{F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha} \dots F_m^{\alpha}})_{unadj}}{V(\widehat{F_1^{\alpha_1} \dots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha} \dots F_m^{\alpha}})_{adj}} = \begin{cases} \frac{1}{1 + w_1^2 + 3w_2^2} & \text{for } \alpha = 1, \\ \frac{3}{3 + w_1^2 + 3w_2^2} & \text{for } \alpha = 2. \end{cases} \quad (10)$$

Notice that $0 < RE \leq 1$. For u.e. factorial effects $RE = 1$ and for n.u.e. factorial effects $RE < 1$. Further the value of E away from 1 the more is the effect of the adjustment to the variance of the estimator.

Example 7. In Example 3, m_1 equals to 2 and moreover, $n_{00} = \binom{2}{0} = 1$, $n_{01} = 0$, $n_{02} = \binom{2}{2} = 1$, $n_{10} = 0$, $n_{11} = \binom{2}{1} = 2$ and $n_{12} = 0$. Under the assumption that $F_1 F_2 F_3 F_4$ and $F_1 F_2 F_3^2 F_4^2$ are negligible, it follows from (7) that

$$E(\widehat{F_1 F_2 F_3 F_4})_{\text{unadj}} = 9(-\beta_1 + \beta_0),$$

$$E(\widehat{F_1 F_2 F_3^2 F_4^2})_{\text{unadj}} = 9(-2\beta_2 + \beta_1 + \beta_0).$$

It can be seen that

$$E(\widehat{F_1 F_3 F_4})_{\text{unadj}} = F_1 F_3 F_4 + 3(-2\beta_2 + \beta_1 + \beta_0).$$

Thus

$$(\widehat{F_1 F_2 F_4})_{\text{adj}} = (\widehat{F_1 F_3 F_4})_{\text{unadj}} - \frac{1}{3} (\widehat{F_1 F_2 F_3^2 F_4^2})_{\text{unadj}}.$$

Therefore, from (8), $\alpha = 1$, $w_2 = -\frac{1}{3}$ and $w_1 = 0$. Hence from (10),

$$RE = \frac{1}{1+3\left(\frac{1}{3}\right)^2} = \frac{3}{4} = .75.$$

Table 2 presents the values of w_1 , w_2 and the relative efficiencies for factorial effects. It is to be noted that the relative efficiencies for all 6 factorial effects are more than .75 and therefore the adjustments do not have large effects on the variances of the estimators. The deletion design with such high relative efficiencies can be considered as a near orthogonal design.

Table 2
Efficiencies for $2^2 \times 3^2$ deletion designs

Factorial Effects	α	w_1	w_2	RE
$F_3 F_4$	1	$-\frac{1}{3}$	0	.90
$F_3^2 F_4^2$	2	0	$-\frac{1}{3}$	.90
$F_1 F_3 F_4$	1	0	$-\frac{1}{3}$	.75
$F_2 F_3 F_4$	1	0	$-\frac{1}{3}$	.75
$F_1 F_3^2 F_4^2$	2	1	0	.75
$F_2 F_3^2 F_4^2$	2	1	0	.75

5. Miscellaneous Results

In this section the case $m_2 = 0$ i.e., $m_1 = m$ is considered for the sake of completeness. The u.e. and n.u.e. factorial effects for a 2^m deletion design are displayed. It is a feeling that the deletion design for the case $m_2 = 0$ is of lesser practical importance than the deletion designs for the case $m_2 > 0$.

Theorem 4. Under a 2^m deletion design D, the factorial effects

$F_1^{\alpha_1} \dots F_m^{\alpha_m}$ for all $\alpha_1, \dots, \alpha_m$ are not u.e.

Proof. First observe that three blocks in D can not be of equal sizes and therefore the block sizes can not all be even. The rest is clear from the definition of $F_1^{\alpha_1} \dots F_m^{\alpha_m}$. This completes the proof.

Denote the number of nonzero elements in a vector $(\alpha_1, \dots, \alpha_m)$ by $W(\alpha_1, \dots, \alpha_m)$. For $w = 0, 1, \dots, m$, denote

$$A_w = \{F_1^{\alpha_1} \dots F_m^{\alpha_m}; W(\alpha_1, \dots, \alpha_m) = w\}. \quad (11)$$

Notice that A_0 consists of the general mean, A_1 consists of all main effects, A_2 consists of all two factor interactions and so on.

Theorem 5. For a $w (\neq 0, m)$ all contrasts of the elements in A_w are u.e.

Proof. Consider two vectors $(\alpha_1, \dots, \alpha_m)$ and $(\alpha_1^*, \dots, \alpha_m^*)$ so that $W(\alpha_1, \dots, \alpha_m) = W(\alpha_1^*, \dots, \alpha_m^*) = w (\neq 0)$. It can now be seen that in every block, the number of runs satisfying $\alpha_1 x_1 + \dots + \alpha_m x_m = u$ is exactly identical to the number of runs satisfying $\alpha_1^* x_1 + \dots + \alpha_m^* x_m = u$ for $u = 0, 1$. The rest is clear from the definition of factorial effects and the model (1).

Example 8. The three blocks in a 2^4 deletion design are given below.

Block 0	0	1	1	1	0
	0	1	1	0	1
	0	1	0	1	1
	0	0	1	1	1

Block 1	1	0	0	0	1
	0	1	0	0	1
	0	0	1	0	1
	0	0	0	1	1

Block 2	1	1	1	0	0	0
	1	0	0	1	1	0
	0	1	0	1	0	1
	0	0	1	0	1	1

Notice that the Blocks 0 and 1 are of the same size 5 and the Block 2 is of the size 6. For the set $A_1 = \{F_1, F_2, F_3, F_3^2, F_4, F_4^2\}$, it follows from Theorems 4 and 5 that all the elements in A_1 are n.u.e. but every contrast of elements in A_1 is u.e.

Theorem 6.

(a) For a w , $\sum_{A_w} F_1^{\alpha_1} \dots F_m^{\alpha_m}$ is n.u.e. under D.

(b) The linear function of factorial effects $c_0 B_0 + c_1 B_1 + c_2 B_2$ with $c_0 + c_1 + c_2 = 0$ is n.u.e. under D.

(c) For a $w(\neq 0, m)$,

$\sum_{A_w} F_1^{\alpha_1} \dots F_m^{\alpha_m} + (c_0 B_0 + c_1 B_1 + c_2 B_2)$
with $c_0 + c_1 + c_2 = 0$, is u.e. under D.

Proof. The part (a) can be seen from Theorems 4 and 5. The part (b) is obvious. The part (c) follows from the block structures in D. This completes the proof.

References

- Addleman, S. (1962). Orthogonal main-effect plans for asymmetrical factorial experiments. *Technometrics*, 4, 21-46.
- Addleman, S. (1972). Recent developments in the design of factorial experiments. *J. Am. Stat. Assoc.*, 67, 103-111.
- Bose, R. C. (1947). Mathematical theory of the symmetrical factorial design. *Sankhyā*, 8, 107-166.
- Kishen, K. and Srivastava, J. N. (1959), Mathematical theory of confounding in asymmetrical and symmetrical factorial designs. *Journal of the Indian Society of Agricultural Statistical*, 11, 73-110.
- Margolin, B. H. (1969). Orthogonal main-effect plans permitting estimation of all two-factor interactions for the $2^n 3^m$ factorial series of designs. *Technometrics*, 11, 747-762.
- Sardana, M. G. and Das, M. N. (1965). On the construction and analysis of some confounded asymmetrical factorial designs. *Biometrics*, 21, 948-956.
- Voss, D. T. (1986). First-order deletion designs and the construction of efficient nearly orthogonal factorial designs in small blocks J. *Am. Stat. Assoc.*, 81, 813-818.
- Voss, D. T. and Dean, A. M. (1987). A comparison of classes of single replicate factorial design. *Ann. Statist.* 15, 376-384.

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