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CENTER FOR STOCHASTIC PROCESSES

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ON FUNCTIONAL ESTIMATES FOR ILL-POSED LINEAR PROBLEMS

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On functional estimates for ill-posed linear problems

R. Brigola*) and A. Keller

Abstract: Ill-posed linear problems in Hilbert space are considered as stochastic filtering problems. Functional estimates of the signal x are given for the problem $Ax - y = z$ where A is a linear, not necessarily bounded operator between Hilbert spaces and x, y, z are Hilbert space valued random elements. As an application, functional estimates are given explicitly for Radon transformed signals with additive white noise.

1. Introduction: Let H_1 and H_2 be Hilbert spaces and $A: H_1 \longrightarrow H_2$ a linear operator. By Hadamard's definition, a linear problem $Ax = z$ is well-posed if a solution exists, is unique and depends continuously on the data $z \in H_2$. Otherwise it is called ill-posed.

Deterministic regularization methods for ill-posed problems have been extensively treated in literature, starting from the work of A. N. Tichonov and V. Ya. Arsenin [12]. Further references may be found in [6] or [7], for instance. Since an equation $Ax = z$ often decribes a functional relationship between an unknown state x and an observation z, which may be affected with random additive noise, ill-posed problems have also been considered as stochastic filtering problems $Ax + y = z$, where x, y, z are Hilbert space valued random elements. For bounded linear transformations A, defined everywhere in H_1, stochastic solutions for these equations, depending on various models for the noise y, have been studied in the work of J. N. Franklin [2], V. Friedrich and A. Uhlig [3].

In this work, we will consider linear transformations A which are not necessarily bounded or defined everywhere in H_1, and obtain estimates for a given class of linear functionals of the signal x, given an observation $z = Ax + y$, where y is a noise on H_2 with positive definite covariance. Signal and noise will be assumed to be zero-mean, Gaussian, weak random variables on H_1 resp. H_2.

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2. D_1-best, linear estimates in an operator class Δ

Let H_1 and H_2 be Hilbert spaces and $A: D_A \longrightarrow H_2$ be a linear operator with domain of definition $D_A \subset H_1$, and $A': D_{A'} \longrightarrow H_1$ be its formally adjoint operator, i. e. $\langle Ax,y\rangle = \langle x,A'y\rangle$ for $x \in D_A$ and $y \in D_{A'}$ (cf. [13]).
Here $\langle .,.\rangle$ denotes the inner product in H_1 resp. H_2.
Furthermore, let the signal x be a zero-mean, Gaussian, H_1-valued weak random variable (cf. [1]) with covariance operator B, and the noise y be a zero-mean, Gaussian, H_2-valued weak random variable with positive definite covariance operator C. Signal and noise are assumed to be independent, i. e. to have independent finite dimensional distributions.
Given an observation $z = Ax - y$, we look for a functional $h_2 \in H_2$ of the observation z to estimate a given functional $\langle x,g\rangle$, $g \in H_1$, of the signal. More generally, given a subspace D_1 of H_1, we look for an estimation operator L: $H_1 \longrightarrow H_2$, transforming the given functionals in D_1 into functionals in H_2, such that $\langle z,Lg\rangle$ is a least squares estimate for $\langle x,g\rangle$ simultaneously for all $g \in D_1$. Therefore we define:

Definition (2.1):

Let $\Delta \subset \{ L: H_1 \longrightarrow H_2 \mid L \text{ linear and } D_1 \subset D_{L'} \}$ be a given class of admissible estimation operators.

1) For $g \in D_1$, the risk $r_g(L)$ of $L \in \Delta$ is given by
$$r_g(L) := E(|\langle z,Lg\rangle - \langle x,g\rangle|^2) .$$
2) $L_0 \in \Delta$ is called D_1-best, linear estimator for x in Δ if simultaneously for all $g \in D_1$ and $L \in \Delta$ it holds:
$$r_g(L_0) \le r_g(L) .$$

It will become clear from the following, that certain restrictions have to be imposed on the class of admissible estimation operators to obtain well-defined domains of definition for the estimations, since we deal with unbounded operators, generally not defined everywhere in the Hilbert spaces H_1 or H_2.
Now, let $\Gamma := BA'(ABA' + C)^{-1}$ on its natural domain of definition. If we denote the covariance operator of z by $K := ABA' + C$, then K is one-to-one, since C is positive definite and B is positive semidefinite. Therefore Γ is well-defined and for its domain of definition D_Γ it holds:
$$D_\Gamma = rg(K) = K(D_{ABA'}),$$
since $D_K = D_{ABA'} \subset D_{BA'}$. Also, $\Gamma' = K^{-1}AB = (ABA' + C)^{-1}AB$ is well-de-

fined on its natural domain. Here rg(K) denotes the range of K.

Lemma (2.2):

Let $rg(A') \subset D_1 \subset D_{\Gamma'}$. Then the following statements hold:

1) $D_1 \subset D_{AB}$

2) $B(D_1) \subset D_A$

3) $AB(D_1) \subset rg(K)$

4) $D_K = D_{A'}$

5) Γ is formally adjoint to $\Gamma' | D_1$

Proof:

The statements 1) - 3) follow immediately from the assumption $D_1 \subset D_{\Gamma'}$. Statement 4) follows from 1) and the assumption $rg(A') \subset D_1$, and 5) from the definition of Γ and Γ'.

Theorem (2.3):

Let $\Gamma' := (ABA' + C)^{-1} AB$ and $rg(A') \subset D_1 \subset D_{\Gamma'}$.

Then Γ' is the D_1-best, linear estimator for x in

$$\Delta := \{ L: D_1 \longrightarrow D_{A'} \mid L \text{ linear and } rg(ABA' + C) \subset D_{L'} \}.$$

Proof:

First, we state that $\Gamma' \in \Delta$:

Γ' is defined on D_1 and by Lemma (2.2), 3) and 4) we obtain:

$\Gamma'(D_1) = K^{-1} AB(D_1) \subset D_K = D_{A'}$. The formally adjoint Γ of Γ' is defined on $rg(K)$, thus $\Gamma' \in \Delta$.

The risk of an arbitrary $L \in \Delta$ for fixed $g \in D_1$ is given by

$$r_g(L) = E(|\langle x,g\rangle - \langle z,Lg\rangle|^2) = E(|\langle x, g-A'Lg\rangle|^2)$$
$$- E(\langle x, g-A'Lg\rangle \overline{\langle y,Lg\rangle}) - E(\overline{\langle x, g-A'Lg\rangle} \langle y,Lg\rangle) + E(|\langle y,Lg\rangle|^2).$$

By the assumption that x and y are zero-mean and independent, we have

$$r_g(L) = \langle Bg,g\rangle + \langle BA'Lg,A'Lg\rangle - \langle BA'Lg,g\rangle - \langle Bg,A'Lg\rangle + \langle CLg,Lg\rangle .$$

Since by Lemma (2.2), $rg(A') \subset D_1 \subset D_{AB}$ and $AB(D_1) \subset rg(K)$, and since $rg(K) \subset D_{L'}$, we obtain:

$$r_g(L) = \langle Bg,g\rangle + \langle ABA'Lg,Lg\rangle + \langle CLg,Lg\rangle - \langle BA'Lg,g\rangle - \langle L'ABg,g\rangle$$
$$= \langle Bg,g\rangle + \langle L'KLg,g\rangle - \langle BA'Lg,g\rangle - \langle L'ABg,g\rangle .$$

Thus, comparing the risk of L with the risk of Γ, it follows:

$$r_g(L) - r_g(\Gamma') = \langle L'KLg,g\rangle - \langle \Gamma K\Gamma' g,g\rangle - \langle BA'Lg,g\rangle - \langle L'ABg,g\rangle$$
$$- \langle BA'\Gamma' g,g\rangle + \langle \Gamma ABg,g\rangle .$$

Using $\Gamma K = BA'$ on $D_{A'} = D_K$ and $K\Gamma' = AB$ on D_1, we have:

$$r_g(L) - r_g(\Gamma') = \langle L'KLg,g\rangle - \langle L'K\Gamma'g,g\rangle - \langle \Gamma KLg,g\rangle + \langle \Gamma K\Gamma'g,g\rangle$$
$$= \langle (L'-\Gamma)K(L'-\Gamma)'g,g\rangle ;$$

eventually we have $r_g(\Gamma') \le r_g(L)$, since K is positive definite.

Remark (2.4):

The operator class Δ, which we have used, is maximal in the following sense:

1) For an element $L \in \Delta$, the condition $D_1 \subset D_L$ is necessary to define $\langle z,Lg\rangle$, and $L(D_1) \subset D_{A'}$ is necessary for $\langle x,A'Lg\rangle$ to exist for all $g \in D_1$.
2) For $L \in \Delta$, the existence of a formally adjoint L' with $D_{L'} \supset rg(K)$ is used to ensure comparability of the elements of Δ with respect to the risks r_g, $g \in D_1$.

3. Application: Functional estimates for noisy Radon transformed signals

A well-known ill-posed linear problem is Radon's integral equation (cf. [10]). The Radon transform has technically been used in Computational Axial Tomography for the reconstruction of a density function from its integrals along hyperplanes. This application motivates, in spite of known inversion formulas, a stochastic treatment, because one has only finitely many data, which additionally may be affected with measurement errors, and the unknown density is in time randomly dependent on body functions of the patient, for instance slight motions during measuring.

We will use the following notations.

Definition (3.1):

Let $L^1(\mathbb{R}^n)$ be the space of Lesbegue-integrable functions on $\mathbb{R}^n$, $n \ge 2$, $f \in L^1(\mathbb{R}^n)$, S^{n-1} the unit sphere in $\mathbb{R}^n$ and $H(p,q) := \{ x \in \mathbb{R}^n : \langle x,q\rangle = p \}$, $(p,q) \in \mathbb{R} \times S^{n-1}$, a hyperplane in $\mathbb{R}^n$.

Then the Radon transform Rf of f is defined by

$$Rf(p,q) := \int_{H(p,q)} f(x)\, m(dx),$$

where dm is the $(n-1)$-dimensional Lesbegue measure on $H(p,q)$.

According to [11], the Radon transform is not a continuous operator on the whole of $L^2(\mathbb{R}^n)$, the Hilbert space of square integrable functions.

To use the above concept of functional estimation for the problem $Rx + y = z$, we will make the following assumptions:

i) Let $H_1 := L^2(\mathbb{R}^n)$ and $H_2 := L^2(\mathbb{R} \times S^{n-1})$, endowed with the usual inner products, $n \geq 2$ a fixed integer.

ii) If $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$, we identify $\mathcal{S}(\mathbb{R}^n)$ with a subspace of $L^2(\mathbb{R}^n)$, and consider the Radon transform R as operator from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}\times S^{n-1})$ with $D_R \supset \mathcal{S}(\mathbb{R}^n)$, (cf. [8]).

iii) The observation z is assumed to be $z = Rx + y$, where the signal x is assumed to be a zero-mean, Gaussian, H_1-valued random element, which is stationary, i. e. its covariance operator B is given by:

$$\langle g, Bh\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} b(\|q - s\|)\, g(q)\, \overline{h(s)}\, dq ds \;,\; g,h \in \mathcal{S}(\mathbb{R}\;),$$

for some $b \in \mathcal{S}(\mathbb{R})$, (cf. [5]).

iv) The noise y is assumed to be zero-mean, Gaussian white noise on $L^2(\mathbb{R}\times S^{n-1})$, i. e. a weak random variable with covariance operator $\sigma^2 I$, with I the identity operator on $L^2(\mathbb{R}\times S^{n-1})$.

v) The signal and the noise are assumed to be independent.

vi) Let the class D_1 of functionals, estimates are asked for, be given by $\mathcal{S}(\mathbb{R}^n)$.

We will need the following definitions and relations between Radon, Fourier and Hilbert transforms.

Definition (3.2):

1) Let $c := \dfrac{1}{2(2\pi)^{n-1}}$

2) The multiplication operators $M_k\colon \mathcal{S}(\mathbb{R}\times S^{n-1}) \longrightarrow L^2(\mathbb{R}\times S^{n-1})$ resp. $\overline{M}_k\colon \mathcal{S}(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ are defined for $k \geq 1$ by:

$$M_k h(p,q) := |p|^{k-1}\, h(p,q)\,,\; (p,q) \in \mathbb{R}\times S^{n-1}\;,\; h \in \mathcal{S}(\mathbb{R}\times S^{n-1})$$

$$\overline{M}_k g(x) := \|x\|^{k-1}\, g(x),\; x \in \mathbb{R}^n,\; g \in \mathcal{S}(\mathbb{R}^n)$$

3) $$\Phi(x) := \int_{\mathbb{R}^n} b(\|s\|)\, e^{-i\langle s,x\rangle}\, ds\;,\;\; x \in \mathbb{R}^n,$$

i. e. Φ is the power spectral density of the signal x (cf. [5]).

Since $\Phi \in \mathcal{S}(\mathbb{R}^n)$ and depends only on the norm of its argument, $\varphi(r) := \Phi(\|x\|)$, $r = \|x\|$, $x \in \mathbb{R}^n$, is well-defined. Of course, Φ and φ are non-negative.

4) The multiplication operator $M_\Phi : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is defined by:

$$M_\Phi \, g := \Phi \cdot g \,, \quad g \in \mathcal{S}(\mathbb{R}^n).$$

5) By $\hat{H}f(t) := \frac{i}{\pi} \int_{\mathbb{R}} \frac{f(p)}{t-p} \, dp \,, \quad t \in \mathbb{R},\ f \in \mathcal{S}(\mathbb{R}^n)$, the Hilbert transform is denoted (cf. [9]).

6) For $g \in \mathcal{S}(\mathbb{R}\times S^{n-1})$, $(p,q) \in \mathbb{R} \times S^{n-1}$, the operator V is defined by:

$$Vg(p,q) := \begin{cases} c \left(\frac{\partial}{i\partial p}\right)^{n-1} g(p,q) & \text{for odd } n \\ c\, i\, \hat{H}\left(\frac{\partial}{i\partial p}\right)^{n-1} g(p,q) & \text{for even } n \end{cases}$$

(cf. [8]).

Lemma (3.3):

Let R be the Radon transform and $\mathcal{F}$ be the Fourier transform on $L^2(\mathbb{R}^n)$. Then the following statements hold:

1) For $(p,q) \in \mathbb{R} \times S^{n-1}$, $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,

$$Rf(p,q) = (2\pi)^{\frac{n}{2}-1} \int_{\mathbb{R}} \mathcal{F}f(r\cdot q)\, e^{irp}\, dr$$

2) $f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(s)\, e^{i\langle s-x,s'\rangle}\, ds\, ds'$, for $x \in \mathbb{R}^n$, $f \in L^2(\mathbb{R}^n)$.

The proof is straight forward and left to the reader.

Lemma (3.4):

1) $R(\mathcal{S}(\mathbb{R}^n)) \subset \mathcal{S}(\mathbb{R}\times S^{n-1})$

2) The adjoint R' of the Radon transform R is defined on $VR(\mathcal{S}(\mathbb{R}^n))$, where V is the differential operator from definition (3.2), and $R'VR = I$, the identity on $\mathcal{S}(\mathbb{R}^n)$.

For the proof it is referred to [8].

Lemma (3.5):

If $\mathcal{F}_p$ denotes the Fourier transform of a function in the variable $p \in \mathbb{R}$, then:

1) $V = c\, \mathcal{F}_p^{-1} M_n \mathcal{F}_p$ on $\mathcal{S}(\mathbb{R}\times S^{n-1})$

2) $\mathcal{F}_p^{-1} M_n \mathcal{F}_p R = R \mathcal{F}^{-1} \bar{M}_k \mathcal{F}$ on $\mathcal{S}(\mathbb{R}^n)$

3) $VR = c\, R \mathcal{F}^{-1} \bar{M}_n \mathcal{F}$ on $\mathcal{S}(\mathbb{R}^n)$

For the proof of assertion 1), it is referred to [8]. Assertion 2) and 3) follow immediately from Lemma (3.3) and assertion 1).

Lemma (3.6):

For the covariance operator B of the signal x, it holds:

$$B = \mathcal{F}^{-1} M_\Phi \mathcal{F} \text{ on } \mathcal{S}(\mathbb{R}^n)$$

Proof:

For given $g \in \mathcal{S}(\mathbb{R}^n)$, $h \in L^2(\mathbb{R}^n)$ we have:

$$\langle Bg,h\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} b(\|s-q\|)\, g(q)\, \overline{h(s)}\, dq\, ds$$

$$= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} b(\|s-q\|) \int_{\mathbb{R}^n} \mathcal{F}g(u)\, e^{i\langle u,q\rangle}\, du \int_{\mathbb{R}^n} \overline{\mathcal{F}h(v)}\, e^{-i\langle v,s\rangle}\, dv\, dq\, ds$$

$$= (2\pi)^{-n}\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} \mathcal{F}g(u)\, \Phi(u)\, e^{i\langle u-v,s\rangle}\, du\Big)\, ds\Big)\, \overline{\mathcal{F}h(v)}\, dv\,,$$

where the last equality holds by Fubini's theorem, the well-known substitution rule for integrals, and by definition of Φ.

Thus, by Lemma (3.3), 2) we obtain:

$$\langle Bg,h\rangle = \int_{\mathbb{R}^n} \mathcal{F}g(v)\, \Phi(v)\, \overline{\mathcal{F}h(v)}\, dv = \langle M_\Phi \mathcal{F}g, \mathcal{F}h\rangle\,,$$

which proves the assertion.

Now, we can calculate best functional estimates for functionals in $\mathcal{S}(\mathbb{R}^n)$, according to theorem (2.3), given the problem $Rx + y = z$ under the above assumptions:

Theorem (3.7):

1) The operator $\Gamma' : D_{\Gamma'} \longrightarrow L^2(\mathbb{R}\times S^{n-1})$, defined by $\Gamma' := (RBR' + \sigma^2 I)^{-1} RB$, gives a $\mathcal{S}(\mathbb{R}^n)$-best, linear estimator for x in the sense of definition (2.1) within the class:

   $$\Delta := \{\, L : \mathcal{S}(\mathbb{R}^n) \longrightarrow rg(VR) \;\big|\; L \text{ linear and } rg(RBR' + \sigma^2 I) \subset D_{L'} \,\}.$$

   Here, V is the operator from definition (3.2).

2) For $(p,q) \in \mathbb{R}\times S^{n-1}$ and $g \in \mathcal{S}(\mathbb{R}^n)$, it holds:

$$\Gamma g(p,q) = (2\pi)^{\frac{n}{2}-1} \int_{\mathbb{R}} \frac{|r|^{n-1}\varphi(|r|)}{2(2\pi)^{n-1}\varphi(|r|) + \sigma^2 |r|^{n-1}}\, \mathcal{F}g(rq)\, e^{irp}\, dr\,.$$

Proof:

Clearly, $\sigma^2 I$ ($\sigma > 0$) is positive definite, and by Lemma (3.4):

$rg(R') = R'VR(\mathcal{S}(\mathbb{R}^n)) = \mathcal{S}(\mathbb{R}^n) = D_{\Gamma'}$.

Thus, we have to show $\mathscr{S}(\mathbb{R}^n) = D_1 \subset D_{\Gamma'}$, according to theorem (2.3). Let us state, that:

$$RBR' + \sigma^2 I = RB(VR)^{-1} + \sigma^2 VR(VR)^{-1} = R(B + c\sigma^2 \mathcal{F}^{-1}\bar{M}_n \mathcal{F})(VR)^{-1}$$
$$= R\mathcal{F}^{-1}(M_\Phi + c\sigma^2 \bar{M}_n)\mathcal{F}(VR)^{-1}$$

by Lemma (3.4), (3.5) and (3.6).

Therefore, we have:

i) $(RBR' + \sigma^2 I)VR(\mathscr{S}(\mathbb{R}^n)) = R\mathcal{F}^{-1}(M_\Phi + c\sigma^2\bar{M}_n)\mathcal{F}(\mathscr{S}(\mathbb{R}^n))$

Since $\Phi \in \mathscr{S}(\mathbb{R}^n)$, also the functions h_g on $\mathbb{R}^n$, defined by:

$$h_g(x) := \frac{\Phi(x)}{\Phi(x) + c\sigma^2\|x\|^{n-1}}\, g(x), \quad g \in \mathscr{S}(\mathbb{R}^n),$$

are elements of $\mathscr{S}(\mathbb{R}^n)$.

Hence, we obtain:

ii) $M_\Phi \mathcal{F}(\mathscr{S}(\mathbb{R}^n)) \subset (M_\Phi + c\sigma^2\bar{M}_n)\mathcal{F}(\mathscr{S}(\mathbb{R}^n))$.

Therefore, application of $R\mathcal{F}^{-1}$ on both sides, Lemma (3.6) and i) yield:

$$RB(\mathscr{S}(\mathbb{R}^n)) \subset R\mathcal{F}^{-1}(M_\Phi + c\sigma^2\bar{M}_n)\mathcal{F}(\mathscr{S}(\mathbb{R}^n)) = (RBR' + \sigma^2 I)VR(\mathscr{S}(\mathbb{R}^n)).$$

Thus, we have shown, that $RB(\mathscr{S}(\mathbb{R}^n)) \subset rg(RBR' + \sigma^2 I)$, which implicates: $\mathscr{S}(\mathbb{R}^n) \subset D_{\Gamma'}$.

Since $D_{R'} = VR(\mathscr{S}(\mathbb{R}^n)) = rg(VR)$, the class Δ equals that one used in theorem (2.3), from which we now obtain assertion 1).

To prove assertion 2), remark that by the above we have for $g \in \mathscr{S}(\mathbb{R}^n)$:

$$\Gamma' g = (RBR' + \sigma^2 I)^{-1} RBg = [R\mathcal{F}^{-1}(M_\Phi + c\sigma^2\bar{M}_n)\mathcal{F}(VR)^{-1}]^{-1} RBg$$
$$= VR\mathcal{F}^{-1}(M_\Phi + c\sigma^2\bar{M}_n)^{-1}\mathcal{F}R^{-1}RBg.$$

Using Lemma (3.5) and (3.6), we get:

$$\Gamma' g = cR\mathcal{F}^{-1}\bar{M}_n(M_\Phi + c\sigma^2\bar{M}_n)^{-1}M_\Phi \mathcal{F}g.$$

Eventually, the representation of the Radon transform in Lemma (3.3) implicates for $(p,q) \in \mathbb{R} \times S^{n-1}$, $g \in \mathscr{S}(\mathbb{R}^n)$:

$$\Gamma' g(p,q) = (2\pi)^{\frac{n}{2}-1} \int_{\mathbb{R}} \frac{c\|rq\|^{n-1}\Phi(rq)}{\Phi(rq) + c\sigma^2\|rq\|^{n-1}}\, \mathcal{F}g(rq)\, e^{irp}\, dr.$$

$$\Gamma' g(p,q) = (2\pi)^{\frac{n}{2}-1} \int_{\mathbb{R}} \frac{|r|^{n-1}\varphi(|r|)}{2(2\pi)^{n-1}\varphi(|r|) + \sigma^2|r|^{n-1}}\, \mathcal{F}g(rq)\, e^{irp}\, dr.$$

<u>Remark (3.8):</u>

1) Thus, the theorem says, that the evaluation $\int \Gamma' g(p,q)\overline{z(p,q)}\,dp\,dq$ of the observation $z = Rx - y$ gives the best estimate for the weighted, total density $\int g(s)x(s)\,ds$; the local density $x(s_0)$, $s_0 \in \mathbb{R}^n$ can be approximated

using the approximate identity as weight functional $g \in \mathcal{S}(\mathbb{R}^n)$. With regard to the representation of Γ^* in the proof of theorem (3.7), the estimation error is given by:

$$E(|\langle z,\Gamma^* g\rangle - \langle x,g\rangle|^2) = \int_{\mathbb{R}^n} \Phi(x) \left(1 - \frac{2(2\pi)^{n-1}\,\Phi(x)}{2(2\pi)^{n-1}\,\Phi(x) + \sigma^2 \|x\|^{n-1}}\right) \mathcal{F}g(x)\,\overline{\mathcal{F}g(x)}\,dx$$

2) If the Radon transform is considered as a mapping between weighted L^2-spaces, deterministic regularizations have been worked out by A. K. Louis [7]. Using the above stochastic filtering method, for this case solutions can also be obtained and be related to the result of Tichonov regularization. This will be done elsewhere.

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