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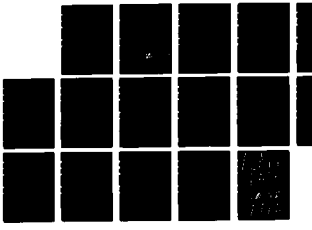

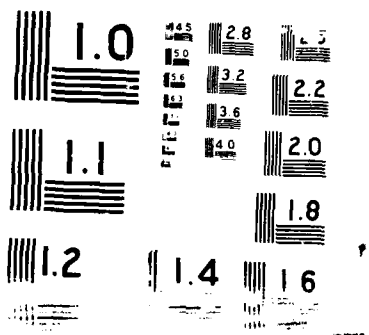
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NONPARAMETRIC ESTIMATION OF A LATENT TRAIT DENSITY

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A method for estimating the density of the latent trait (or ability) in an item response model is given based on the logspline density estimator of Stone and Koo. In the implementation presented here, families of densities whose logarithms are quadratic splines are used to estimate the unknown latent trait density. The number of knots in the spline is variable permitting arbitrary densities to be well approximated from the logspline family. Because the family is exponential and contains all normal distributions, the likelihood ratio test can be used to test for normality. An ad hoc method is proposed for choosing the number of knots, and the method is illustrated with two simulated data sets.

NONPARAMETRIC ESTIMATION OF A LATENT TRAIT DENSITY

by

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1. INTRODUCTION

The purpose of this report is to describe a technique for estimating the latent trait density in an item response model. In this study, the item distributions are assumed known, and the latent traits are regarded as a random sample from an unknown distribution with a smooth density.

This work is directly related to the Bayesian models of Bock and Aitken (1981) and Rigdon and Tsutakawa (1983). Bock and Aitken assumed a somewhat crude discrete prior at specified ability levels. In effect they used a histogram for the estimator of the latent trait density. In the parametric empirical Bayes approach of Rigdon and Tsutakawa, the prior was assumed to be normal and parameter estimates were obtained. The approach here is to obtain estimates from a large (nonstandard) class of densities in which the number of parameters is allowed to grow with the sample size. This is analogous to the traditional way to estimate curves of unspecified form in regression, for example. The purpose is to achieve at least consistent estimates regardless of the form of the density. There are inherent theoretical difficulties in the problem, but we show by example that good estimates are possible.

The notation here follows Rigdon and Tsutakawa. Given n examinees and m test items, let $\mathbf{y}_i$ denote the vector of responses of the ith examinee. Let $\boldsymbol{\beta}$ denote the item parameters, and assume $\mathbf{y}_i$ has conditional pdf

$$p(\mathbf{y}_i|\theta_i,\boldsymbol{\beta}),$$

where θ_i is a real–valued ability (or latent trait) parameter of the ith examinee. The ability parameters $\boldsymbol{\theta} = (\theta_1,\cdots,\theta_n)$ are assumed to be drawn at random from an unknown distribution with density $g_0(\theta)$. Thus $\mathbf{y}$ and $\boldsymbol{\theta}$ have the joint pdf

$$\begin{aligned} f(\mathbf{y},\boldsymbol{\theta}|\boldsymbol{\beta}) &= \prod_{i=1}^{n} p(\mathbf{y}_i|\theta_i,\boldsymbol{\beta})g_0(\theta_i) \\ &= p(\mathbf{y},\boldsymbol{\theta}|\boldsymbol{\beta})g_0(\boldsymbol{\theta}). \end{aligned}$$

Since $\boldsymbol{\theta}$ is unobservable, inference on g_0 must be based on the marginal likelihood

$$L(\mathbf{y}|g) = \int f(\mathbf{y},\boldsymbol{\theta}|\boldsymbol{\beta})g(\boldsymbol{\theta})d\boldsymbol{\theta}. \tag{1.1}$$

The latent trait density g_0, a prior density in the Bayesian framework, is also known as a mixing density in this situation.

For simplicity, details are carried out here for the Rasch model assuming local independence. In principle the techniques apply as well to other parametric item distributions. We assume throughout that the item parameters are known. Letting $y_i = (y_{i1},\cdots,y_{im})$ denote the vector of observations for the ith subject, the Rasch model specifies

$$P(y_{ij} = 1|\theta_i,\beta_j) = 1 - P(y_{ij} = 0|\theta_i,\beta_j) = e^{\theta_i - \beta_j}/(1 + e^{\theta_i - \beta_j}).$$

Assuming local independence and using the sufficient statistics $r_i = \Sigma_{j=1}^{m} y_{ij}$ (the raw score for the ith subject) and $q_j = \Sigma_{i=1}^{n} y_{ij}$ (the total number of correct responses on the jth item), we then have

$$p(\mathbf{y}|\boldsymbol{\theta},\boldsymbol{\beta}) = \prod_{i=1}^{n}\prod_{j=1}^{m} P(y_{ij} = 1|\theta_i,\beta_j)$$

$$= \frac{e^{\Sigma r_i \theta_i - \Sigma q_j \beta_j}}{\prod_i \prod_j (1 + e^{\theta_i - \beta_j})}.$$

If we let

$$p_r(\theta) = \frac{e^{r\theta}}{\prod_j (1 + e^{\theta - \beta_j})}, \quad r = 0,\cdots,m,$$

then the logarithm of (1.1) is given by

$$\log(L(g)) = -\Sigma q_j \beta_j + \sum_{r=0}^{m} n_r \log\left\{\int p_r(\theta)g(\theta)d\theta\right\}, \tag{1.2}$$

where $n_r = \#\{i : r_i = r, 1 \le i \le n\}$. (The n_r are the bin frequencies for the m + 1 possible raw scores.)

There is a fairly large literature on methods for estimating a mixing distribution

itself without assuming the existence of a density going back at least to Deely and Kruse (1968). It is known that the unrestricted marginal maximum likelihood estimate of the mixing distribution is discrete. Laird (1978) used the EM algorithm to find the unrestricted marginal maximum likelihood estimate, although convergence in these problems is generally quite slow. Padgett and Tsokos (1979) considered mixing distribution estimates in the context of empirical Bayes estimation similar to the situation considered here. In contrast, although extensive work has been done on density estimation (see for example Silverman (1986)), there has been relatively little work on estimating a mixing density. O'Bryan and Susarla (1975) proposed a Fourier–transform method in the case of deconvolution for empirical Bayes problems, but their method appears to be computationally intractable in the general setting of item response models. Mendelsohn and Rice (1982) obtained good results by deconvolving a smoothed estimate of the marginal density, and more recently Levine and Williams (1987) used a linear estimate with constraints for latent trait models. Both of these techniques require quadratic programming.

A different approach will be used here. In Section 2 we follow Stone and Koo (1986) and introduce an exponential family of densities whose logarithms are spline functions parameterized by an N–vector $\varphi = (\varphi_1 \cdots, \varphi_N)$. The parameterization will be variable to allow good approximation of arbitrary smooth densities with enough smoothing to stay away from the discrete marginal mle of the distribution, and estimates will be obtained by maximizing the marginal likelihood in φ. In addition, because the families of densities contain all normal distributions, the likelihood ratio test will be available to test for normality.

2. LOG SPLINE DENSITIES

In order to estimate densities $g(\theta)$ of unspecified form, we consider densities of the

form $e^{s(\theta)}$, where $s(\theta)$ is a "spline function" as defined below. For further information on splines, the reader is referred to de Boor (1978). Splines are known to have excellent approximation properties for large classes of functions, hence $\log(g_0(\theta))$ should be well approximated by a spline $s(\theta)$. In addition, the log spline densities are automatically nonnegative, simplifying the estimation algorithm.

For computational purposes, we truncate the support of $g(\theta)$ to a bounded interval $\theta \in [a,b]$, where $-\infty < a < b < \infty$. A spline function $s(\theta)$ on $[a,b]$ of order k is a piecewise polynomial of order k (or degree $k-1$) with possible discontinuities in the function or its derivatives at a finite number of points $\mathbf{t} = \{t_1,\cdots,t_u\} \subset [a,b]$, where $a = t_1 \le t_2 \le \cdots \le t_u = b$. These points are called the "knots" of $s(\theta)$. A repeated knot, say $t_i = t_{i+1} = \cdots = t_{i+M-1}$, is said to have multiplicity M. Each knot (including multiplicity) removes one continuity constraint on $s(\theta)$ and its derivatives, so that $s(\theta)$ is assumed continuous with $(k-M-1)$ continuous derivatives at a knot t_i when t_i has multiplicity M. With this notation, a function $s(\theta)$ is called a spline of order k if it satisfies the following properties:

(i) $s(\theta)$ is a polynomial of degree $(k-1)$ on (t_i,t_{i+1}) for $i = 1,\cdots, u-1$.

(ii) If t_i has multiplicity M_i, $s^{(j)}(\theta)$ is continuous at t_i for $j = 0,\cdots, k-M_i-1$.

The space of all such functions will be denoted by $S_{k,\mathbf{t}}$.

Because $S_{k,\mathbf{t}}$ is linear and finite dimensional, it has a basis. Following de Boor (1978), we use the B–spline basis, a basis consisting of splines $\{B_i(\theta;\mathbf{t}): i = 1,\cdots,u\} \in S_{k,\mathbf{t}}$ which have a minimal support property,

$$B_i(\theta;\mathbf{t}) = 0, \ \theta \notin [t_i,t_{i+k}]. \tag{2.1}$$

This property characterizes the basis and makes it especially convenient to work with. With this notation, the Curry Schoenberg theorem states that

$$S_{k,\mathbf{t}} = \left\{s(\theta): s(\theta) = \sum_{i=1}^{u} \varphi_i B_i(\theta;\mathbf{t}), \ \varphi_i \text{ real}\right\}. \tag{2.2}$$

Now let $G_{k,\mathbf{t}}$ denote the class of log spline densities whose exponents are in $S_{k,\mathbf{t}}$. Then we have explicitly

$$G_{k,t} = \left\{ e^{\Sigma \varphi_i B_i(\theta;t)} / \int e^{\Sigma \varphi_i B_i(x;t)} dx: \varphi_i \text{ real} \right\}.$$

From this representation we see that for a fixed knot set t, $G_{k,t}$ is actually an exponential family with natural parameter φ. Note that $G_{k,t}$ is overparameterized since $S_{k,t}$ contains all constants, hence there are actually $N - 1$ free parameters here.

The underlying assumption of the analysis in many item response applications is that $g(\theta)$ is standard normal. Thus it is reasonable to take a symmetric interval such as $[a,b] = [-3.5,3.5]$ for the support set. As in de Boor (1978, Chapter XII), we assume knots of multiplicity k at the boundaries in order to free $s(\theta)$ from boundary conditions but simple knots in the interior. Throughout, we will let

(2.3a) $$t_1 = t_2 = \cdots = t_k = a,\ t_N = t_{N+1} = \cdots = t_{N+k} = b,$$

and

(2.3b) $$t_{k+u} = \Phi^{-1}(u/(N-k+1)),\ u = 1,\cdots,N-k+1.$$

Here Φ is the cdf for the standard normal distribution. By (2.1), this defines N B–splines of order k, and the multiple knots at a and b imply that $s(\theta)$ has no constraints at either boundary. There is no theoretical reason for the interior knot placement given here, but the strategy seems reasonable for placing ample knots in the region where $g_0(\theta)$ might be expected to have interesting features. (In general, the issue of optimal knot placement is a delicate problem which seldom has satisfactory closed form solution.) For this knot set t, the absence of boundary conditions implies that $S_{k,t}$ also has the alternate representation

(2.3) $$s(\theta) = \sum_{j=1}^{k} \alpha_j \theta^{j-1} + \sum_{u=1}^{N-k} \gamma_u (\theta - t_{k+u})_+^{k-1}$$

for constants α_j and γ_u, where $(\theta - t)_+^{k-1} = (\theta - t)^{k-1}$ for $\theta \geq t$, and 0 otherwise. Thus with $N = k$, it is clear that $s(\theta)$ is a polynomial of degree $k - 1$. In particular, when $k = 3$, $S_{k,t}$ contains all quadratic polynomials and $G_{k,t}$ contains all normal distributions.

The basis for inference using the family $G_{k,t}$ is the following.

THEOREM. If $g_0(\theta) \to 0$ as $\theta \to \pm\infty$ and $\log(g)$ is continuously differentiable, then for any $\epsilon > 0$, there exists a knot set t and a density $g(\theta|\varphi) \in G_{k,t}$ such that

$$\sup_{-\infty < \theta < \infty} |g_0(\theta) - g(\theta|\varphi)| < \epsilon.$$

Proof. One can choose the interval [a,b] such that $|g_0(\theta)| < \epsilon/2$ for $\theta \notin [a,b]$. Under these conditions, t can be chosen to make $|s(\theta) - \log(g_0(\theta))|$ uniformly small on [a,b], hence the result.

3. MARGINAL MAXIMUM LIKELIHOOD

From (1.2), the marginal mle over $G_{k,t}$ is obtained by maximizing

$$\log(L(\varphi)) = -\Sigma q_j \beta_j + \sum_{r=0}^{m} n_r \log\left\{\int p_r(\theta) g(\theta|\varphi) d\theta\right\}.$$

If we renormalize by dividing by n, let $w_r = n_r/n$, substitute (2.2) for $g(\theta|\varphi)$, and ignore terms not depending on φ, the marginal mle problem is to maximize

$$\sum_{r=0}^{m} w_r \log\left\{\int p_r(\theta) e^{\Sigma \varphi_i B_i(\theta;t)} d\theta\right\}$$

subject to

$$\int e^{\Sigma \varphi_i B_i(\theta;t)} d\theta = 1.$$

This is simplified by the following variant of a result of Silverman (1982).

LEMMA. If $w_r \geq 0$ and $\Sigma w_r = 1$, the maximum value of

$$K(s) = \sum_{r=0}^{m} w_r \log\left\{\int p_r(\theta) e^{s(\theta)} d\theta\right\} \tag{3.1}$$

over $s(\theta) \in S_{k,t}$ subject to

$$\int e^{s(\theta)} d\theta = 1 \tag{3.2}$$

is the same as the value of K(s) at the unconditional maximizer over $S_{k,t}$ of

$$\sum_{r=0}^{m} w_r \log\left\{\int p_r(\theta) e^{s(\theta)} d\theta\right\} - \int e^{s(\theta)} d\theta. \tag{3.3}$$

Proof. For $s \in S_{k,t}$, let $s^*(\theta) = s(\theta) - \log\left\{\int e^{s(x)} dx\right\}$, so that $\int e^{s^*(\theta)} d\theta = 1$. Then

$$\sum_{r=0}^{m} w_r \log\left\{\int p_r(\theta) e^{s^*(\theta)} d\theta\right\} - \int e^{s^*(\theta)} d\theta$$

$$= \sum_{r=0}^{m} w_r \log\left\{\int p_r(\theta)e^{s(\theta)}d\theta\right\} - \sum_{r=0}^{m} w_r \log\left\{\int e^{s(\theta)}d\theta\right\} - 1$$
$$= \sum_{r=0}^{m} w_r \log\left\{\int p_r(\theta)e^{s(\theta)}d\theta\right\} - \log\left\{\int e^{s(\theta)}d\theta\right\} - 1.$$

But $1 + \log(t) \leq t$ with strict inequality for $t \neq 1$, so the last term is bounded below by

$$\sum_{r=0}^{m} w_r \log\left\{\int p_r(\theta)e^{s(\theta)}d\theta\right\} - \int e^{s(\theta)}d\theta,$$

with strict inequality unless $\int e^{s(\theta)}d\theta = 1$. Thus the maximizer of (3.3), say $s_1(\theta)$, must satisfy (3.2), so within this class $\max K(s) \geq K(s_1)$. But for any s satisfying (3.2), $K(s_1) - 1 \geq K(s) - 1$ by the optimality of s_1 since (3.2) holds for s_1. This proves the lemma.

REMARK. The lemma shows that "one" is the appropriate Lagrange multiplier for the problem.

We have implemented the marginal maximum likelihood using Newton–Raphson iteration. To describe the algorithm, let

$$\ell(\varphi) = \sum_{r=0}^{m} w_r \log\left\{\int p_r(\theta)e^{\Sigma\varphi_i B_i(\theta;t)}d\theta\right\} - \int e^{\Sigma\varphi_i B_i(\theta;t)}d\theta$$

denote the function to be maximized. Recalling that $g(\theta|\varphi) = e^{\Sigma\varphi_i B_i(\theta;t)}$, we define

$$\int p_r(\theta)g(\theta|\varphi)d\theta = E_g p_r$$

and for notational purposes write

$$\frac{\partial}{\partial\varphi_u}\int p_r(\theta)g(\theta|\varphi)d\theta = \int B_u(\theta;t)p_r(\theta)g(\theta|\varphi)d\theta = E_g B_u p_r.$$

With similar notation, define

$$\frac{\partial^2}{\partial\varphi_u\partial\varphi_v}\int p_r(\theta)g(\theta|\varphi)d\theta = E_g B_u B_v p_r,$$
$$\frac{\partial}{\partial\varphi_u}\int g(\theta|\varphi)d\theta = E_g B_u,$$

and

$$\frac{\partial^2}{\partial\varphi_u\partial\varphi_v}\int g(\theta|\varphi)d\theta = E_g B_u B_v p_r.$$

Then the gradient $\nabla\ell(\varphi)$ is the N–vector with components

$$\frac{\partial\ell(\varphi)}{\partial\varphi_u} = \sum_r w_r(E_g B_u p_r)/E_g p_r - E_g B_u,$$

and the Hessian matrix $H(\varphi)$ is the N × N matrix whose (u,v)th element is

$$\frac{\partial^2 \ell(\varphi)}{\partial\varphi_u \partial\varphi_v} = \sum_r w_r \left\{ (E_g B_u B_v p_r) E_g p_r - (E_g B_u p_r)(E_g B_v p_r) \right\} / (E_g p_r)^2 - E_g B_u B_v.$$

Starting with initial guess $\hat{\varphi}^{(0)}$, the Newton–Raphson method for convergence to $\hat{\varphi}$ is to iteratively compute

$$\hat{\varphi}^{(i+1)} = \hat{\varphi}^{(i)} - H(\hat{\varphi}^{(i)})^{-1} \nabla \ell(\hat{\varphi}^{(i)})$$

until convergence is obtained. The algorithm was implemented with a step halving modification: if at some stage $\ell(\hat{\varphi}^{(i+1)}) \le \ell(\hat{\varphi}^{(i)})$, $\hat{\varphi}^{(i+1)}$ was replaced by $\hat{\varphi}^{(i+1)} = \hat{\varphi}^{(i)} - cH(\hat{\varphi}^{(i)})^{-1} \nabla \ell(\hat{\varphi}^{(i)})$ for $c = 1/2, 1/4, \cdots$ until improvement resulted or until $c = 1/32$. The starting value $\hat{\varphi}^{(0)}$ was taken to satisfy $\log(g(\theta|\hat{\varphi}^{(0)})) = s(\theta|\varphi) = -\theta^2/2 - \log(2\pi)/2$ thus forcing $g(\theta|\hat{\varphi}^{(0)})$ to be standard normal.

Convergence of the Newton–Raphson algorithm is assured if $H(\varphi)$ is a negative definite matrix for all φ. Unfortunately, $H(\hat{\varphi})$ need not be negative definite as the following argument shows. Let $B(\theta) = \Sigma \tau_i B_i(\theta)$ for arbitrary constants τ_i, and let

$$h_r(\theta) = p_r(\theta) g(\theta|\varphi) / \int p_r(u) g(u|\varphi) du$$

and $\mu_r = \int B(\theta) h_r(\theta) d\theta$. Then $\tau^t H(\varphi) \tau = \sum_r w_r \int [B(\theta) - \mu_r]^2 h_r(\theta) d\theta - \int B(\theta)^2 g(\theta|\varphi) d\theta$. There is no theoretical reason for $\tau^t H(\varphi) \tau$ to be either positive or negative. However, in the test cases, $H(\hat{\varphi}^{(i)})$ appeared to be negative definite, and convergence was very fast (at most 10 steps) when φ had less than 9 components. For severely overparameterized cases, the step–halving modification to the Newton–Raphson algorithm greatly improved convergence. Using the results of the next section, this argument can be modified to show that $H(\varphi)$ becomes be negative definite in a neighborhood of the maximizing φ^* as $n \to \infty$ if g_0 is well approximated from $G_{k,t}$.

In the test cases reported below, the support of g_0 was assumed to be $[a,b] = [-3.5,3.5]$ following the assumption that g_0 should be roughly standard normal. Simpson's rule with 51 points was used to approximate the integrals, and all computations were performed in double precision arithmetic on an IBM 4381 computer. The spline

evaluations were performed using public domain FORTRAN routines from de Boor (1978). SPLINT for initially computing $\hat{\varphi}^{(0)}$ and BVALUE for all B–spline evaluations.

As an experiment, a program was written to implement the EM algorithm to obtain $\hat{\varphi}$. Although essentially the same estimates were obtained, convergence was much slower. Our conclusions were that the EM algorithm could be used if one wanted to estimate item parameters along with g_0 as in Bock and Aitken (1981) or Rigdon and Tsutakawa (1983), but if the primary purpose is to estimate g_0 with fixed item parameters, the Newton–Raphson method is preferable.

4. IDENTIFIABILITY AND CONSISTENCY

By the Strong Law of Large Numbers,

$$w_r = n_r/n \to \pi_r,$$

where

$$\pi_r = P[\sum_{j=1}^{m} y_{ij} = r] = \int P[\sum_{j=1}^{m} y_{ij} = r|\theta] g_0(\theta) d\theta$$
$$= \left\{ \sum_{y_{i\cdot}=r} e^{-y_{ij}\beta_j} \right\} \int p_r(\theta) g_0(\theta) d\theta, \; r = 0, \cdots, m$$

with $y_{i\cdot} = \sum_j y_{ij}$. Thus in the limit, marginal mle is equivalent to maximizing

$$M(g) = \sum_{r=0}^{m} \pi_r \log \left\{ \int p_r(\theta) g(\theta|\varphi) d\theta \right\}$$

over $G_{k,t}$. Let $\tilde{\pi}_r = \int P[\sum_{j=1}^{m} y_{ij} = r|\theta] g(\theta|\varphi) d\theta$ so that $M(g(\theta|\varphi)) = \Sigma \pi_r \log \tilde{\pi}_r$ + constant. The information inequality (cf. Rao (1973), chapter 5) states that $\Sigma \pi_r \log \tilde{\pi}_r \leq \Sigma \pi_r \log \pi_r$ with equality if and only if $\pi_r = \tilde{\pi}_r$ for $r = 0, \cdots, m$. Thus if $g_0(\theta) = g(\theta|\varphi^*)$, one might expect $\hat{\varphi}$ to provide a consistent estimate of π_r. An analysis similar to that of Section 3 can be used to show that the Hessian matrix for $M(g(\theta|\varphi))$ is nonpositive definite at φ^*.

Unfortunately, in the nonparametric setting where g_0 is completely unspecified, g_0

may not be identifiable given M(g). To see this, note that if $h(\theta)$ is a function such that $\int p_r(\theta)h(\theta)d\theta = 0$ for all r, $\int h(\theta)d\theta = 0$, and $g + h \geq 0$, then $(g + h)$ is a density and $M(g + h) = M(g)$. It seems plausible that these conditions actually force h to be small, perhaps in some norm, if m is moderately large, but we are unable to prove this. Further analytical work needs to be done in this area.

Identifiability did not appear to be a problem in the test cases we studied, and we conjecture that it is not a problem in general for tests of moderate length m. Intuitively, if g_0 is smooth, the Theorem of Section 2 guarantees that there is a member $g(\theta|\varphi^*)$ of some equivalence class $\{\bar{g}: H(\bar{g}) = H(g)\}$ very close to the true g_0 which can be consistently estimated. We believe that it should be possible to prove identifiability either by letting $m \to \infty$ or by assuming a prior distribution on the item parameters.

It is possible to carry out at least a formal asymptotic theory to get approximate pointwise confidence intervals (ignoring bias) for $g_0(\theta)$. Stone and Koo (1986) did this for the ordinary density estimation problem. The details in the present context will be provided elsewhere.

5. A LIKELIHOOD RATIO TEST FOR NORMALITY

Here we focus on quadratic splines with k = 3, hence from (2.4)

$$\sum_{i=1}^{u} \varphi_i B_i(\theta;t) = \alpha_1 + \alpha_2\theta + \alpha_3\theta^2 + \sum_{u=1}^{N-3} \gamma_u(\theta - t_{3+u})_+^2. \tag{5.1}$$

Note that α_1 is determined by (3.2), so that there are actually $N - 1$ free parameters. Moreover, if N = 3, $\log(g(\theta|\varphi))$ is quadratic hence $g(\theta|\varphi)$ is normal. If t is fixed with N > 3 large enough so that g_0 is well approximated from $G_{k,t}$, it is reasonable to test that g_0 is normal by testing H_0: $\gamma_1 = \cdots = \gamma_{N-3} = 0$. The likelihood ratio test statistic $\lambda_N = 2[\log L(\hat{\varphi}_N) - \log L(\hat{\varphi}_3)]$ thus has an approximate chi–square distribution with $N - 3$ degrees of freedom under H_0.

A fundamental question here is the proper choice of N. For N too large, the marginal mle will tend to estimate the discrete solution and appear too rough, but if N is too small, $g(\theta|\hat{\varphi})$ will not be able to differentiate features in the true g_0. The situation is anlaogous to ordinary density estimation where the proper "bandwidth" parameter must be chosen (cf. Silverman, 1986). From limited experience with simulated data, it appears that N between 6 and 8 appears to work well.

We have experimented with an ad hoc data–based approach to the choice of N based on likelihood ratio test statistics. Define

$$\Delta_N = 2[\log L(\hat{\varphi}_N) - \log L(\hat{\varphi}_{N-1})],$$

the change in the likelihood ratio statistic due to the addition of the Nth parameter. As a criterion for choosing N, we propose adding parameters until the increase in Δ_N is negligible. Since the families $G_{k,\mathbf{t}}$ are not nested as N is increased when $\mathbf{t}$ is defined by (2.3), the likelihood ratio test does not apply to Δ_N. However, it does appear that a strategy of adding parameters until Δ_N fails to change appreciably is effective in choosing a suitable N. The theoretical implications of this proposal are topics for further investigation.

While successive models are not nested, there are models which are. Suppose two models have $\tilde{N} > N$ with respective knot sets $\tilde{\mathbf{t}}$ and $\mathbf{t}$. If $\mathbf{t} \subset \tilde{\mathbf{t}}$, then from (5.1) the models are nested, so the likelihood ratio test does apply. This is always the case when using the knots given by (2.3) if $(N-1)$ divides $(\tilde{N}-1)$.

6. SIMULATION RESULTS

In order to assess the feasibility of the proposed method, several experiments were conducted with simulated data. The results of two trials are reported here. In both cases the β_j's were chosen equally spaced on $[-1.5,1.5]$ with $\beta_j = 3(j-1)/(n-1) - 1.5$.

CASE 1. In the first experiment, there were $n = 2000$ subjects, $m = 20$ items, and g_0 was

taken to be standard normal. The results presented in Table 1 clearly show that there is no evidence that g_0 is not normal. Figure 1 shows graphs of $g(\theta|\hat{\varphi}_N)$ for selected N. As N increases, the marginal mle appears to be converging to a discrete distribution with perhaps 4 point masses at roughly −1.8, −0.7, 0.2, and 1.2. This phenomenon is to be expected and demonstrates the potential problem for overfitting. On the other hand, the λ_N and Δ_N statistics presented in the table show that there is no reason based on the data to choose N larger than 3.

CASE 2. For the second experiment, we again chose n = 2000 and m = 20, but took g_0 to be a mixture of normal densities,

$$g_0(\theta) = .6n(s(\theta - a)) + .4n(s(\theta + 1.5a)) \tag{6.1}$$

where $n(\theta) = e^{-\theta^2/2}/(2\pi)^{1/2}$, a = 0.65, and $s = (1 - 1.5a^2)^{1/2}$. The constants a and s were chosen so that $\int \theta g_0(\theta)d\theta = 0$ and $\int \theta^2 g_0(\theta)d\theta = 1$. The results given in Table 2 show large increases in λ_N when N changes from 3 to 4 and 4 to 5, but negligible changes thereafter. Thus the statistics indicate that the choice N = 5 is best. Figure 2 giving the plots of selected fits in comparison with the true density shows that N = 5 is indeed a good choice. Note that the log likelihood actually decreased in going from N = 5 to N = 6. This is possible since the two models are not nested.

In Figure 3, we attempted to see how well the mixture density of (6.1) could be approximated from $G_{k,t}$. There are potentially many ways to choose the parameter φ to obtain a good fit. We decided to approximate the Kullback–Leibler distance

$$\int \log(g_0(\theta)/g(\theta|\varphi))g_0(\theta)d\theta$$

by minimizing

$$\int \left\{\log(g_0(\theta)) - s(\theta|\varphi)\right\}^2 g_0(\theta)d\theta$$

subject to $\int e^{s(\theta)}d\theta = 1$. Selected fits are shown in Figure 3. It is obvious that the bias in this example from approximating g_0 from $G_{k,t}$ is negligible for $N \geq 5$.

7. CONCLUSIONS

In this report the log spline method of Stone and Koo is used to estimate the latent trait density in item response models in which the item parameters are assumed known. Despite the identifiability problem inherent in the model, we show that consistent estimates of good approximations to the unknown density are possible. In addition, a likelihood ratio test for normality is introduced. Several examples are given which show that the method can work well in practice, both for testing that a density is normal and for estimating the density in a test case where it is not normal. In addition, a heuristic method is proposed for choosing the appropriate number of parameters to put into the model.

The results here extend density estimation methodology to a case where the random variables, namely latent traits, are unobservable. The method as proposed is primarily a data–analytic tool rather than an inferential one, and the intended purpose is to provide researchers with information similar to that obtained from histograms for usual observable data. The problem of choosing suitable N is analogous to that of choosing the correct number of bins for a histogram. A nonnormal ability density could indicate a mixture of populations as in the second example, or it could be a sign of multidimensionality. In either case, a visual realization of the density could be a useful exploratory tool.

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Table 1

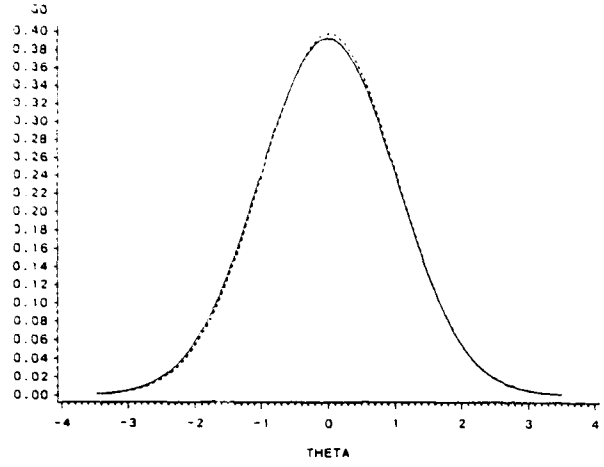
Case 1: Standard normal density

N	λ_N	df	Δ_N
4	0.091	1	0.091
5	0.300	2	0.209
6	1.686	3	1.386
7	1.967	4	0.281
8	4.286	5	2.282
9	4.239	6	0.375
10	6.620	7	2.000

Table 2

Case 2: Mixture density (6.1)

N	λ_N	df	Δ_N
4	10.213	1	10.213
5	27.100	2	16.887
6	26.930	3	–0.169
7	27.370	4	0.439
8	27.433	5	0.064
9	28.140	6	0.706
10	28.217	7	0.077



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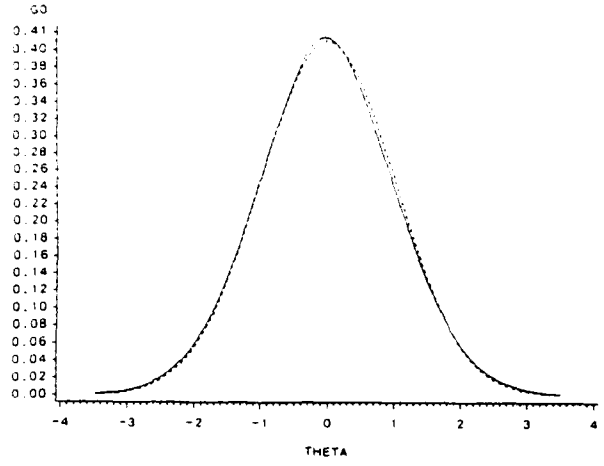

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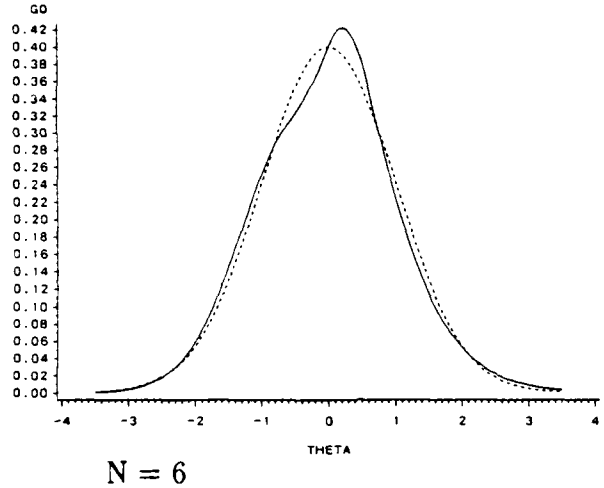

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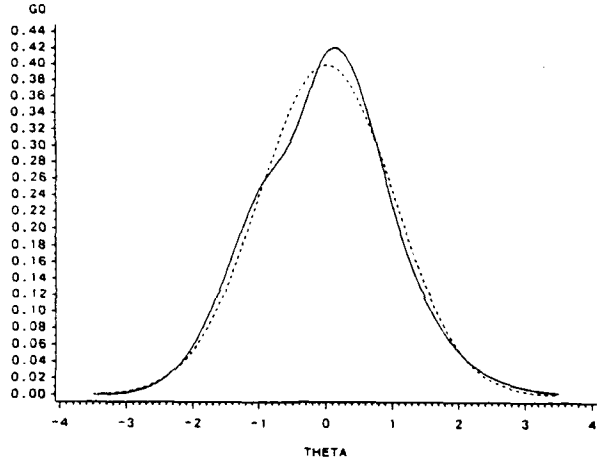

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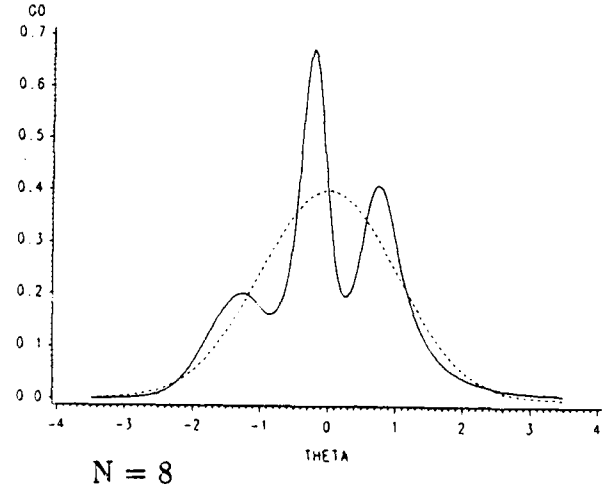

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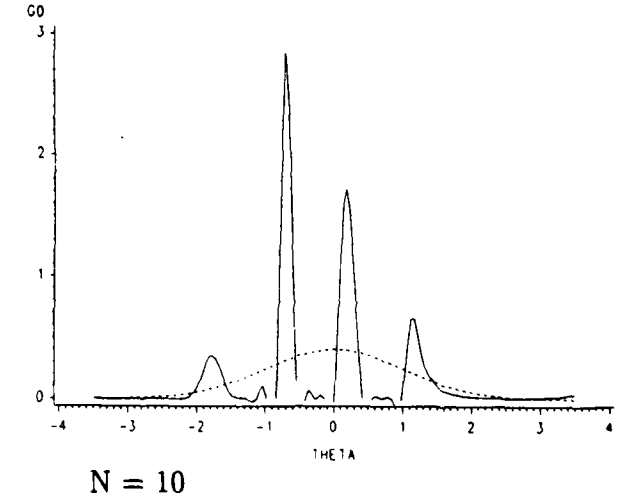

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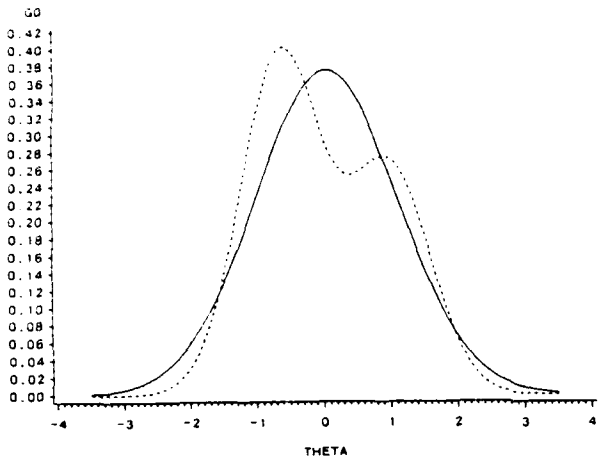
Figure 1. Density estimates for case 1 corresponding to $\hat{\varphi}_N$ for N = 3, 5, 6, 7, 8, and 10 respectively.



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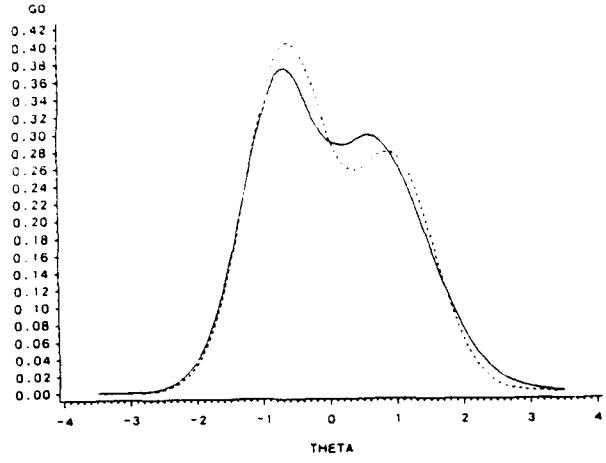

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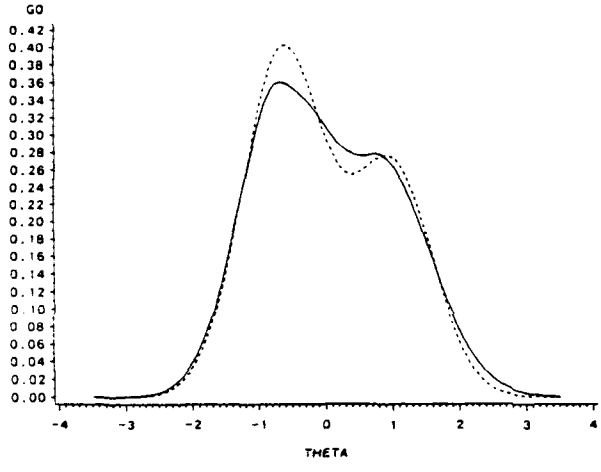

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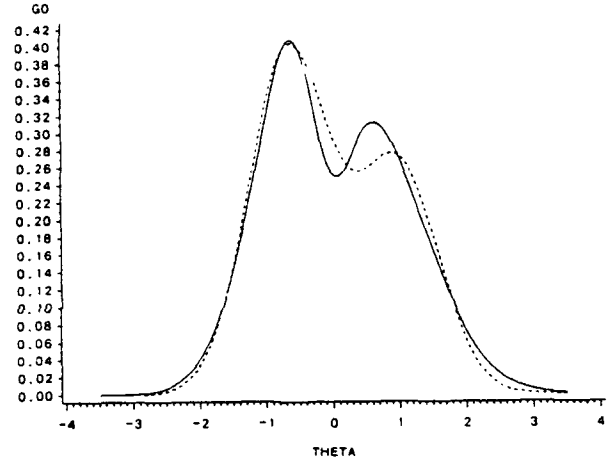

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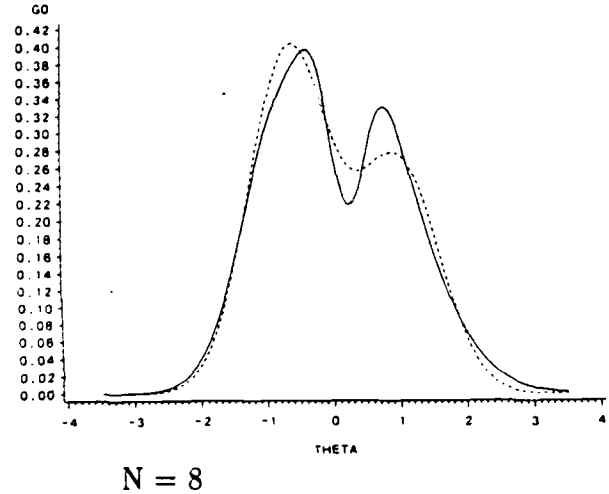

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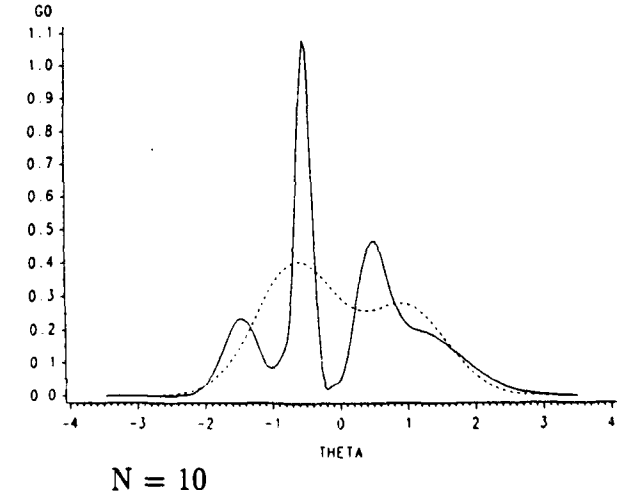
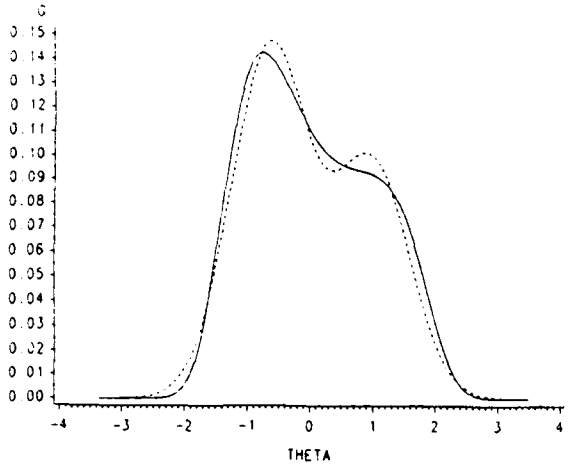


N = 10

Figure 2. Density estimates for case 2 corresponding to $\hat{\varphi}_N$ for N = 3, 5, 6, 7, 8, and 10 respectively.



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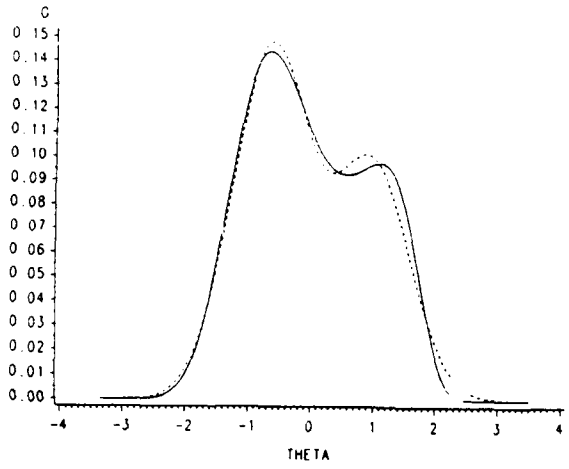

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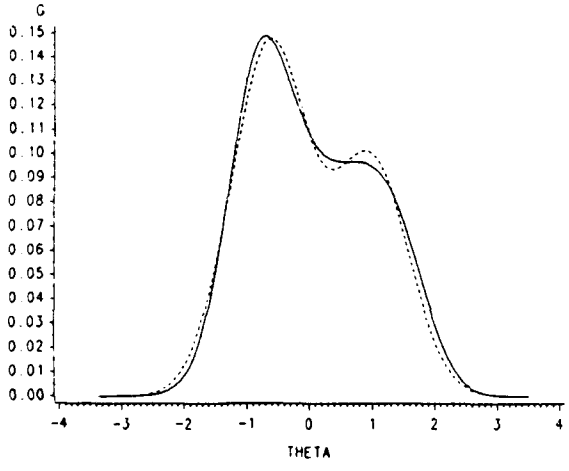

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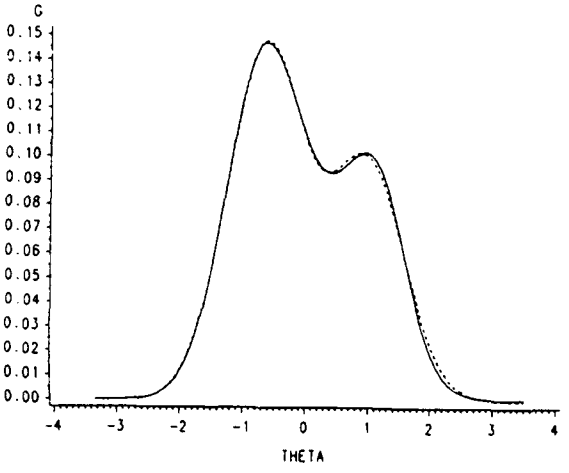

N = 7

Figure 3. The mixture density (6.1) and approximating $g(\theta|\varphi_N)$ for N = 4,5,6,and 7.

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