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A numerical procedure is given for evaluating P, the integral of the trivariate normal distribution over an offset sphere of radius R. A numerical algorithm is also described for computing R when P is given. The procedures generally give P or R to six decimal digits. Program listings in HP BASIC, for the HP 9845B computer, are included.

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11. INVERSE PROBLEM

FOREWORD

The work described in this report was done in the Space and Surface Systems Division of the Strategic Systems Department. Its purpose was to design software, suitable for a high quality mathematics and/or statistical subroutine library, which yields probabilities and percentage points for 3-dimensional normal distributions over off-set spheres.

The author is grateful to Mel Holloway for his guidance and direct assistance with the technical word processor that was used to produce the draft copy of this report.

Released by:

D. B. Colby

D. B. COLBY, Head
Strategic Systems Department

CONTENTS

I. INTRODUCTION

In the application of the methods of operation research to problems of military strategy, it is sometimes necessary to compute the value of a kill probability expressible in the form

$$P = \iiint_{-\infty}^{\infty} p(m)\ F(x,y,z,u,v,w)\ dx\ dy\ dz, \tag{1}$$

where

$$\begin{cases} F(x,y,z,u,v,w) \equiv \dfrac{1}{2\pi\sqrt{2\pi}\ uvw}\ E(x/u)\ E(y/v)\ E(z/w) \\ \\ E(t) \equiv \exp(-t^2/2). \end{cases} \tag{2}$$

Here P is the kill probability. The shots or bursts are assumed to form an uncorrelated trivariate normal distribution with their mean or expected burst point set at the origin of a rectangular cartesian coordinate system Oxyz. The distribution has standard deviations u, v, w in the x, y, z directions, respectively. This configuration may be assumed without loss of generality for any trivariate normal distribution, since if the distribution is correlated and does not have its mean at the origin, a translation and a rotation of axes will bring about the situation which has been described.

A target T, which may be either a point or area target, is assumed in some arbitrary position in the Oxyz space. The distance m is the radial or slant range miss distance of an arbitrary burst point (x,y,z) from the target. If a target is a point at a position (H,K,L), then m is the ordinary distance $[(x - H)^2 + (y - K)^2 + (z - L)^2]^{1/2}$. If T is an area target, m is the minimum distance from (x,y,z) to points of the target. For area targets m is set to zero for all burst points (x,y,z) lying within the target.

The expression p(m) is a function of m as prescribed on the basis of physical conditions or assumptions of the problem. Possible forms of this function which have practical applicability are: (1) p(m) = 1 if $m \le R$ (a constant), p(m) = 0 otherwise; (2) p(m) is a decreasing linear or exponential function of m; (3) p(m) is a monotonically decreasing step function.

This report describes a procedure for the efficient computation of these kill probabilities by a high speed digital computer for an important practical case, that in which the target is a point at an arbitrary position (H,K,L) where, without loss of generality, H, K, L are nonnegative and the function p(m) has the first form mentioned above, i.e., $p(m) = 0$ for $m > R$, $p(m) = 1$ for $m \leq R$. The constant R may be thought of as the lethal radius of the weapon.

The function p(m) has the effect of reducing the field of integration to the interior of the sphere S of radius R with the target (H,K,L) as center, i.e.,

$$S : (x - H)^2 + (y - K)^2 + (z - L)^2 = R^2. \tag{3}$$

The kill probability P is then given by

$$P = \int_{L-R}^{L+R} \int_{K-Y}^{K+Y} \int_{H-X}^{H+X} F(x,y,z,u,v,w)\, dx\, dy\, dz, \tag{4}$$

where

$$X = \sqrt{R^2 - (y - K)^2 - (z - L)^2}, \qquad Y = \sqrt{R^2 - (z - L)^2}. \tag{5}$$

Rather than carry out the numerical triple integration of Equation 4 directly we make use of an available computer program that yields the probability $P_c(\bar{r},H,K,u,v)$ of a shot falling under a bivariate normal distribution inside a circle in the Oxy-plane of radius $\bar{r}$ and center (H,K). P_c is the two-dimensional analog of P.[4,5,6]

Geometrically, one observes then that P can be obtained by considering circular slices of S parallel to the Oxy-plane. For a fixed z in [L - R, L + R], the xy-integration in Equation 4 over a slice of radius $\bar{r} = \sqrt{R^2 - (z - L)^2}$ yields P_c. Weighting P_c with $1/(\sqrt{2\pi}\, w)\, E(z/w)$ and integrating the result over z in [L - R, L + R] gives P, i.e.,

$$P = \frac{1}{\sqrt{2\pi}\, w} \int_{L-R}^{L+R} E(z/w)\, P_c(\bar{r},H,K,u,v)\, dz, \tag{6}$$

where

$$\bar{r} = \sqrt{R^2 - (z - L)^2}, \tag{7}$$

or

$$P = \frac{1}{\sqrt{\pi}} \int_{(L-R)/\sqrt{2}\,w}^{(L+R)/\sqrt{2}\,w} E(\sqrt{2}t)\ P_c(r,H,K,u,v)\ dt, \tag{8}$$

[From (2), $E(\sqrt{2}t) = \exp(-t^2)$]

$$r = \sqrt{R^2 - (L - \sqrt{2}wt)^2}. \tag{9}$$

Hereafter, we assume P_c is available and the evaluation of P reduces to the single integration indicated in Equation 6 or Equation 8.

The symmetry properties of F (see Equation 2) allow H, K, L to always be taken as nonnegative. Since F > 0 and bounded, the order of integration in Equation 4 is immaterial. Thus, as long as H is associated with u, K with v, and L with w, it does not matter thereafter which is called L and w. For example, if the order of integration is chosen so that the x-integration is performed last, then if initially H = 10, u = 5, L = 20, w = 7, we simply let H = 20, u = 7, L = 10, w = 5. In this way, we can always refer to Equation 6 or Equation 8 as the basic representation for P, where L and w are always associated with the z- or t-integration indicated in Equation 6 or Equation 8 with the understanding that the order of integration may have been changed and the variables H, K, L and u, v, w renamed as in the example above.

II. COMPUTATION OF P

Sometimes we shall write P(R) or P(R,H,K,L,u,v,w) for P. As stated in the previous section, R denotes the radius of the sphere S given by Equation 3. The trivariate normal distribution F is defined in Equation 2 by its mean (0,0,0) and its standard deviations u,v,w in the x, y, and z directions, respectively. By appropriate normalization of the variables R, H, K, L, u, v, w it is easy to show that P is in general a function of six independent variables. The program for computing P is called ELLCOV.

In most cases P will be computed by the numerical integration of Equation 8, where it is assumed P_c is available. There are two special cases however where P can be evaluated in closed form. We shall call these cases A and B. The derivations of the results for these will be given in Appendix A. Case A is found in the literature[10]. Case B is also probably given in the literature, but we have not found it.

CASE A: $\boxed{u = v = w}$ $\quad D \equiv \sqrt{H^2 + K^2 + L^2}$

$$P = \frac{1}{2}\left\{ \operatorname{erf}\left(\frac{D + R}{\sqrt{2}\,w}\right) - \operatorname{erf}\left(\frac{D - R}{\sqrt{2}\,w}\right)\right\} - \sqrt{\frac{2}{\pi}}\,\frac{R}{w}\left[\frac{\exp(d) - 1}{d}\right] E\left(\frac{D - R}{w}\right), \quad \underline{D \neq 0} \tag{10}$$

$$P = \operatorname{erf}\left(R/\sqrt{2}\,w\right) - \sqrt{\frac{2}{\pi}}\,\frac{R}{w}\,E(R/w), \qquad \underline{D = 0}, \tag{11}$$

where

$$\begin{cases} d = -2DR/w^2 \\ \operatorname{erf} x \equiv \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x \exp(-t^2)\,dt. \end{cases} \tag{12}$$

CASE B: $\boxed{u = v \text{ and } H = K = 0.}$

The two situations $u > w$ and $u < w$ are treated separately.

$\underline{u > w}$

$$P = \frac{1}{2}\left\{\operatorname{erf} L_4 - \operatorname{erf} L_5 - \frac{1}{L_2}\,E(R/u)\,\exp\left\{.5[L/(L_2 u)]^2\right\} \cdot (\operatorname{erf} F_1 - \operatorname{erf} S_0)\right\}, \quad \underline{S_0 < 0} \tag{13}$$

$$P = \frac{1}{2}\left\{\operatorname{erf} L_4 - \operatorname{erf} L_5 - \frac{1}{L_2}\exp(-L_5{}^2)\left[\exp(S_0{}^2)\operatorname{erfc} S_0 - \exp(-2RL/w^2)\,\exp(F_1{}^2)\operatorname{erfc} F_1\right]\right\}, \quad \underline{S_0 > 0} \tag{14}$$

$$P = \operatorname{erf} L_4 - \frac{1}{L_2}\,E(R/u)\,\operatorname{erf}(RL_2/\sqrt{2}\,w), \ \underline{L = 0} \tag{15}$$

$\underline{u < w}$

$$P = \frac{1}{2}\left\{\operatorname{erf} L_4 - \operatorname{erf} L_5 - \frac{2}{\sqrt{\pi}\,L_2}\exp(-L_5{}^2)\left[\operatorname{daws}(F_1) - \exp\left(-\frac{2RL}{w^2}\right)\operatorname{daws}(S_0)\right]\right\} \tag{16}$$

$$P = \operatorname{erf} L_4 - 2/(\sqrt{\pi}\,L_2)\,E(R/w)\,\operatorname{daws}(RL_2/\sqrt{2}\,w), \qquad \underline{L = 0}, \tag{17}$$

where

$$\begin{cases} \text{daws } x \equiv E(\sqrt{2}\,t)\displaystyle\int_0^x \exp(t^2)\,dt & \text{(Dawson's integral [1; p. 298], [3])} \\ \text{erfc } x \equiv 2/\sqrt{\pi}\displaystyle\int_x^\infty \exp(-t^2)\,dt = 1 - \text{erf } x, & \end{cases} \tag{18}$$

$$L_2 = \sqrt{|1 - (w/u)^2|} \tag{19.1}$$

$$L_4 = (L + R)/(\sqrt{2}\,w) \tag{19.2}$$

$$L_5 = (L - R)/(\sqrt{2}\,w) \tag{19.3}$$

$$S_0 = L/(\sqrt{2}\,wL_2) - RL_2/(\sqrt{2}\,w) \tag{19.4}$$

$$F_1 = L/(\sqrt{2}\,wL_2) + RL_2/(\sqrt{2}\,w). \tag{19.5}$$

In order to achieve improved accuracy, Equations 10-19, are used in modified form in the computing program, ELLCOV. It is also not difficult to show that Equations 13, 14, 16 approach Equation 10 as $u \to w$.

If cases A and B do not apply an attempt is made to improve efficiency by sensing if P_c is numerically constant over the effective range of the t-integration in Equation 8. Call this case C. We discuss this case below after specifying the effective range of the t-integration, (see Equation 29).

In general, excluding cases A, B and C, P is computed by numerical integration. The 24-point Gaussian quadrature rule [1, p. 916] is used for this purpose with, in some cases, an additional 24-point Gaussian.

The most time consuming part of computing the integrand in Equation 8 is the evaluation of P_c, the elliptic coverage function[4,5]. In most cases P_c is obtained from the subprogram function ELLCV. In the special situations that $u = v$ or $H = K = 0$, (referring to Equation 8) P_c is called the circular coverage function, CIRCV. and the generalized circular error function, GCEF, respectively[4,5,6]. Both are computed by the subprogram CIRCV. In these special situations, a parameter V is set to 1 and 0, respectively, otherwise V = 2. When V = 0 or 1, P_c can be computed about 10 times faster than the case when V = 2. Hence, since the order of integration in Equation 8 can be changed, the appropriate interchanges are always made to make V = 1, if possible. If this cannot be done ($u \neq v$, $v \neq w$, $w \neq u$) then an attempt is made to obtain V = 0. However some instability can appear in this case, when computing GCEF, if

$$1(-5) > u/v > 1(5). \qquad (1(N) \equiv 10^N) \tag{20}$$

Hence if V = 0 and Inequality 20 holds then the computation of P_c is carried out as described below for the case V = 2.

If neither V = 1 nor V = 0 can be brought about then the order of integration is determined by finding the largest ratio $a/b < 1$, where a, b take the values of u, v, w. The pair from u, v, w giving the largest ratio (<1) is then used in the first two integrations (the evaluation of P_c) with the corresponding H, K, L values. For example if u = 2, H = 0, v = 5, K = 2, w = 3, L = 6, then the a/b ratios are 2/5, 3/5, 2/3. Hence v and w, K and L, are interchanged, the integrand in Equation 8 becomes

$$\exp(\sqrt{2}\,t)\ P_c(r,0,6,2,3), \tag{21}$$

where

$$r = \sqrt{R^2 - (2 - \sqrt{2}\,5t)^2}\ ; \tag{22}$$

the limits of integration become $(2 - R)/5\sqrt{2}$, $(2 + R)/5\sqrt{2}$. If the two largest ratios are equal, then the one with the largest denominator is used to fix the first two integrations.

Choosing the integration order in this way permits the most accurate computation of P_c. This remark is based on an empirical study in Reference 4, where it was noted that the numerical quadrature of P_c is best when the standard deviations are nearly equal and not small. We mention at this point that the program ELLCV has been up-graded over its description in Reference 4. Namely, the accuracy for P_c has been increased from 6 to approximately 8-decimal digits. This was required in order to deal effectively with the inverse problem discussed in Section III.

It is possible, in some cases, by the tests given below to determine if $P \leq 2.5(-8)$ or $P \geq 1 - 2.5(-8)$. In such situations P is set to zero and one, respectively. The tests below are discussed in Appendix B.

Let Z4 = 2.5(-8), $M \equiv \max(u, v, w)$.

Test #1:

$$\boxed{P \text{ set to 0 if } R^3 \le 1.5\sqrt{2\pi}\, uvwZ4}$$

Test #2:

$$\boxed{\begin{array}{c} P \text{ set to 0 if } R^2 < D^2 \text{ and } R^3 \exp(-\alpha^2) \le 1.5\sqrt{2\pi}\, uvwZ4, \\ \alpha = (D - R)/\left(\sqrt{2}\,M\right), \quad D = \sqrt{H^2 + K^2 + L^2} \end{array}}$$

Test #3:

$$\boxed{\begin{array}{c} P \text{ set to 0 if } h - R > \sqrt{2}\,TF2, \text{ where } F2 = 4.262,\ h = H, K, L, \\ \text{and } T \text{ equals the corresponding } u, v, \text{ or } w. \end{array}}$$

Test #4:

$$\boxed{\begin{array}{c} P \text{ set to 1 if } \bar{P} = P(R,H,K,L,M,M,M) \ge 1 - Z4, \text{ where} \\ \bar{P} \text{ is computed from Equation 10 or Equation 11.} \end{array}}$$

For efficiency it is important to try to reduce the number of times P_c is called by ELLCOV. We note that $\bar{r}$ in Equation 7 has the same value for two values of z, i.e., $z = L \pm (R - \gamma)$, where $0 \le \gamma < R$. This observation in many cases reduces by half the number of times P_c needs to be computed. Indeed, we express Equation 6 as the sum of two integrals, the first has the limits of integration L - R to L and the second L to L + R. In the second integral, the substitution Z = 2L - z is made with the following result

$$P = \frac{1}{\sqrt{2\pi}\, w} \int_{L-R}^{L} [E(z/w) + E[(2L - z)/w]\, P_c(\bar{r},H,K,u,v)\, dz, \qquad (23)$$

or, in terms of Equation 8,

$$P = \frac{1}{\sqrt{\pi}} \int_{A5}^{B5} \left\{\exp(-t^2) + \exp[-(\sqrt{2}\,L/w - t)^2]\right\} P_c(r,H,K,u,v)\, dt, \qquad (24)$$

where

$$\begin{cases} r = \sqrt{R^2 - (L - \sqrt{2}wt)^2} \\ A5 = (L-R)/(\sqrt{2}\,w) \\ B5 = L/(\sqrt{2}\,w). \end{cases} \tag{25}$$

The number of Gaussian abscissae used to integrate Equation 24 is fixed at 24, and in some cases at 48. Greater accuracy may be obtained if the total integration interval, B5 - A5 (= $R/(\sqrt{2}\ w)$), can be reduced. Indeed, consider the infinite volume Ve bounded by the planes $|z| = aw$, where a is uniquely defined by

$$1/(2\pi\sqrt{2\pi}uvw) \int_{-aw}^{aw} E(z/w)\ dz \int_{-\infty}^{\infty} E(y/v)\ dy \int_{-\infty}^{\infty} E(x/u)\ dx = 1 - \varepsilon, \quad \varepsilon > 0, \tag{26}$$

or

$$\operatorname{erfc}\ (a/\sqrt{2}) = \varepsilon. \tag{27}$$

The entire region above the plane z = aw contains $\varepsilon/2$ of the distribution F given in Equation 2. Some typical values for a as a function of ε are

$$\begin{cases} \varepsilon = 5(-8) & a \cong 3.8545\sqrt{2} \cong 5.45109 \\ \varepsilon = 5/3(-9) & a \cong 4.262\sqrt{2} \cong 6.02738. \end{cases} \tag{28}$$

In ELLCOV we use the latter. The integration over the sphere S, as indicated in Equation 6, therefore can be reduced to the intersection of S with Ve ($S \cap Ve$). In many cases this permits the z (or t) interval of integration in Equation 6 (or Equation 24) to be shortened. The final results for Equation 24 are summarized by

$$\begin{cases} B5 = \begin{cases} L/\sqrt{2}\,w & \text{if } L \le aw \\ a/\sqrt{2} & \text{if } L > aw \end{cases} \\ A5 = \begin{cases} -a/\sqrt{2} & \text{if } L-R < -aw \\ (L - R)/\sqrt{2}\,w & \text{if } |L - R| \le aw \end{cases} \end{cases} \tag{29}$$

$$W9 = \begin{cases} 0 & \text{if } L < aw \\ 1 & \text{otherwise.} \end{cases} \tag{30}$$

(Note: P set to 0 if $L - R \geq aw$.--Reduces to Test #3, $a = \sqrt{2}$ F2)

The parameter W9 is used in ELLCOV to indicate if the second exponential in Equation 24 can be ignored. The case W9 = 1 implies the exponential can be dropped and W9 = 0 implies it must be retained. The case for W9 = 1 occurs when the integration from L to L + R in Equation 6 is negligible since the center of S lies outside $S \cap Ve$, i.e., $L \geq aw$.

A further effort is made to reduce B5-A5 by attempting to increase A5. Let the integrand in Equation 24 be denoted by G(t) and the increasing Gaussian abscissae by t_j, $j = 1,2,...,24$. If $G(A5) \leq 1(-10)$ then $G(t_j)$ is evaluated for increasing j until $G(t_n) > 1(-10)$. Then A5 is set to t_{n-1} and new t_j are generated based on the new A5.

Since G(t) has an infinite slope at $t = (L - R)/(\sqrt{2}w)$, it may happen that accuracy is lost when G(A5) and $G(t_1)$ differ significantly. Once A5 is determined, as described above, if $G(t_1) > 1(-7)$ a 24-point Gaussian integration is carried out from A5 to t_1, followed by another 24-point Gaussian integration from t_1 to B5. In most cases $G(t_1) < 1(-7)$, in which case only the 24-point Gaussian integration from A5 to B5 is used.

Another sensing to expedite the computation determines when P_c can be set to one from some $\bar{t}$ to B5. Using the procedure given in Reference 4 for testing whether $P_c \geq 1 - \varepsilon$, we have $P_c \geq 1 - 1.3(-8)$ if

$$r = [R^2 - (L - \sqrt{2}wt)^2]^{1/2} \geq \sqrt{H^2 + K^2} + bs \equiv G, \qquad |\sqrt{2}wt - L| < R, \tag{31}$$

where

$$s = \max(u,v), \qquad b = 6.02738.$$

Therefore,

$$|L - \sqrt{2}wt| < (R^2 - G^2)^{1/2} \equiv U,$$

or

$$(L - U)/(\sqrt{2}w) < t < (L + U)/(\sqrt{2}w), \qquad \bar{t} = (L - U)/\sqrt{2}w. \tag{32}$$

Again, the interval of integration over which the Gaussian quadrature is needed can be reduced when Relation 31 holds and Equation 24 becomes

$$P = 1/\sqrt{\pi} \int_{A5}^{\bar{t}} \left\{ E(\sqrt{2}\,t) + E[(2L - \sqrt{2}\,wt)/w] \right\} P_c(r,H,K,u,v)\,dt$$

$$+ 1/\sqrt{\pi} \int_{(L-U)/\sqrt{2}\,w}^{(L+U)/\sqrt{2}\,w} E(\sqrt{2}\,t)\,dt. \qquad (33)$$

We also note that

$$1/\sqrt{\pi} \int_{(L-U)/\sqrt{2}\,w}^{(L+U)/\sqrt{2}\,w} \exp(-t^2)\,dt =$$

$$\begin{cases} 1/2\ \mathrm{erfc}[(L - U)/\sqrt{2}\,w] & \text{if } (L + U)/\sqrt{2}\,w > F2 \\ 1/2\left\{\mathrm{erf}[(L + U)/\sqrt{2}\,w] - \mathrm{erf}[(L - U)/\sqrt{2}\,w]\right\}, & \text{otherwise.} \end{cases}$$

Now that the effective range of integration for Equation 8 has been defined by Equation 29 we can return to case C. If $A5 < -F2$ then it is possible for P_c to remain essentially constant over the effective range of integration.

Let $r(t) \equiv \sqrt{R^2 - (L - \sqrt{2}wt)^2}$. Then since r(t) is an increasing function of t, we take P_c as a constant if

$$|r(A5) - r(B5)| < 1(-8)r(A5), \quad A5 < -F2. \ (-4.262)$$

In this case P is given by

$$P = \mathrm{erf}(F2)\ P_c\left\{.5[r(A5) + r(B5)],H,K,u,v\right\}.$$

After much experimentation which included an extensive look at Simpson's rule, the 24-point Gaussian integration scheme was chosen as the basic quadrature procedure. For small P, a lower Gaussian procedure would have sufficed, nevertheless it was decided for simplicity to use the 24-point Gaussian throughout. The integrating routine is called GQUAD. As a matter of interest, GQUAD allows for the interval [A5, B5] to be subdivided into a set of equal subintervals with the same Gaussian order formula 6,8,12,16,20, or 24 applied to each subinterval. This feature of GQUAD however is not used in ELLCOV.

Extensive testing of ELLCOV showed that P can be obtained with approximately 8-decimal digits, even though we conservatively claim 6-decimal digits. There does not appear to be any significant limitations, for realistic values of the normalized off-set distance $\sqrt{(H/u)^2 + (K/v)^2 + (L/w)^2}$, and the quantities u/v, v/w, R/w can have values as large as 10^5.

Below, Table 1 contains some numerical results from ELLCOV. It gives values of P to compare with approximations given previously be F.E. Grubbs.[8,9] A listing of ELLCOV in HP-BASIC is given in Appendix C.

TABLE 1

		R = 1	K = L = 0		
		$\sigma^2 = 1/2$	$\sigma^2 = 1$	$\sigma^2 = 2$	$\sigma^2 = 3$
			u = v = w		
H = 0	WH P MC	.890 .888390 .885	.607 .608375 .615	.313 .317730 .320	.204 .198748 .210
H = .5	WH P MC	.807 .736060 .755	.530 .501231 .505	.307 .277180 .315	.205 .179569 .195
H = 1	WH P MC	.337 .337133 .353	.261 .269975 .320	.174 .183329 .205	.127 .132299 .135
H = 2	WH P MC	.000 3.0986(-3) .000	.001 1.59356(-2) .020	.004 3.3474(-3) .035	.006 3.85359(-2) .040
H = 3	WH P MC	.000 1.48(-7) .000	.000 7.569(-5) .000	.000 1.7473(-3) .000	.000 4.7661(-3) .010
			u = 2v = 2w		
H = 0	WH P MC	.868 .876323 .860	.630 .647542 .670	.388 .375392 .400	.279 .246952 .250
H = .5	WH P MC	.739 .747518 .770	.554 .569818 .555	.364 .346921 .400	.269 .233510 .285
H = 1	WH P MC	.432 .437088 .435	.372 .385964 .405	.287 .273711 .290	.233 .197402 .235
H = 2	WH P MC	.019 3.08781(-2) .040	.055 7.57294(-2) .090	.092 .105474 .125	.104 .100705 .105
H = 3	WH P MC	.000 1.679(-4) .000	.001 4.2614(-3) .000	.011 2.11601(-2) .035	.023 3.26805(-2) .050

$$\sigma^2 = u^2 + v^2 + w^2$$

WH--Grubbs' approximation, MC--Monte Carlo (200 shots),
WH and MC results taken from [8], [9]. P computed by ELLCOV.

TABLE 1 (Cont)

R = 1, K = L = 0		$\sigma^2 = 1/2$	$\sigma^2 = 1$	$\sigma^2 = 2$	$\sigma^2 = 3$
			$\boxed{u = v = 2w}$		
H = 0	WH	.878	.618	.350	.236
	P	.878302	.625938	.351727	.229160
	MC	.890	.635	.365	.275
H = .5	WH	.738	.524	.314	.219
	P	.735871	.531421	.315323	.211615
	MC	.735	.520	.395	.270
H = 1	WH	.378	.308	.221	.171
	P	.378261	.319040	.226834	.166578
	MC	.380	.355	.265	.210
H = 2	WH	.005	.021	.044	.055
	P	9.5532(-3)	3.40282(-2)	5.94781(-2)	6.36591(-2)
	MC	.015	.040	.050	.070
H = 3	WH	.000	.000	.002	.006
	P	4.86(-6)	5.545(-4)	6.0278(-3)	1.26241(-2)
	MC	.000	.000	.000	.010
			$\boxed{u = 2v = 3w}$		
H = 0	WH	.740	.640	.415	.318
	P	.869274	.659413	.407256	.279152
	MC	.774	.662	.412	.275
H = .5	WH	.739	.568	.390	.299
	P	.747340	.586033	.378893	.265242
	MC	.750	.584	.366	.266
H = 1	WH	.447	.392	.312	.259
	P	.452305	.409374	.305022	.227522
	MC	.452	.408	.306	.223
H = 2	WH	.025	.068	.109	.123
	P	3.96655(-2)	9.20037(-2)	.127553	.123078
	MC	.044	.087	.122	.120
H = 3	WH	.000	.002	.015	.030
	P	3.428(-4)	6.6584(-3)	2.94238(-2)	4.40764(-2)
	MC	.001	.009	.029	.046

$\sigma^2 = u^2 + v^2 + w^2$

WH--Grubbs' approximation, MC--Monte Carlo (200 shots),
Last MC Group (u = 2v = 3w) 1000 shots,
WH and MC results taken from [8], [9]. P computed by ELLCOV.

III. COMPUTATION OF R (The inverse problem)

In this section we describe a procedure for finding R given P, H, K, L, u, v, w. For fixed H, K, L, u, v, w the radius R is a monotone increasing function of P and thus unique for a given P. The procedure for finding R requires the computation of P, as described in the previous section. Since that basically involves a time-consuming numerical triple integration a large effort is made to obtain good early approximations for R. Once such approximations are found uniform stepping, halving, Regula-falsi, and King's root finding procedure[11] are used to refine the estimates for R. The objective is to obtain R correctly to approximately 6-decimal digits.

In the discourse some details are omitted. They may be obtained from a detailed study of INVELLCOV, the BASIC program for computing R. A listing is given in Appendix C and a flowchart is shown in Figure 2 on page 18.

The unknown value of R will always be contained in a known interval. Initially crude lower and upper bounds, R_{min} and R_{max}, are found for R.

Let $I \equiv \max\left[3\sqrt{\pi/2}\ Puvw,\ (\sqrt{\pi/2}\ PM)^3\right]$, $M = \max(u, v, w)$. Then

$$R \geq R_{min} = \begin{cases} I^{1/3} & \text{if } P \leq 1/2 \\ I^{1/3} & \text{if } P > 1/2 \text{ and } I > D^3 = (H^2 + K^2 + L^2)^{3/2} \\ D & \text{if } P > 1/2 \text{ and } I \leq D^3. \end{cases} \tag{34}$$

The derivation of Equation 34 is similar to the methods used for deriving Tests #1 and #2 of Section II. See Appendix B.

For R_{max}, two arrays A6(j), B6(j), j = 1,2,...10 are used.

A6(1) = 5(-6)	B6(1) = .0266
A6(2) = 1(-4)	B6(2) = .0723
A6(3) = .01	B6(3) = .339
A6(4) = .1	B6(4) = .765
A6(5) = .3	B6(5) = 1.1933
A6(6) = .6	B6(6) = 1.717
A6(7) = .9	B6(7) = 2.5005
A6(8) = .999	B6(8) = 4.0335
A6(9) = .999999	B6(9) = 5.538
A6(10) = 1	B6(10) = 10

The element B6(j)M gives the radius of a sphere centered at (0,0,0) for which A6(j) = P(B6(j)M,0,0,0,M,M,M). Then

$$R \le R_{max} = D + B6(J)M, \qquad M = \max(u,v,w), \tag{35}$$

where J is the minimum integer j for which P ≤ A6(j). The plausibility of Equation 35 is easily seen from its 2-dimensional analog. In Figure 1, the inner ellipse contains P of the distribution. The outer ellipse with semi-axes qu, qv contains A6(J) of the distribution. Then, from the figure, one easily concludes $R < \sqrt{H^2 + K^2} + qs$, where s = max(u,v) and q corresponds to B6(J).

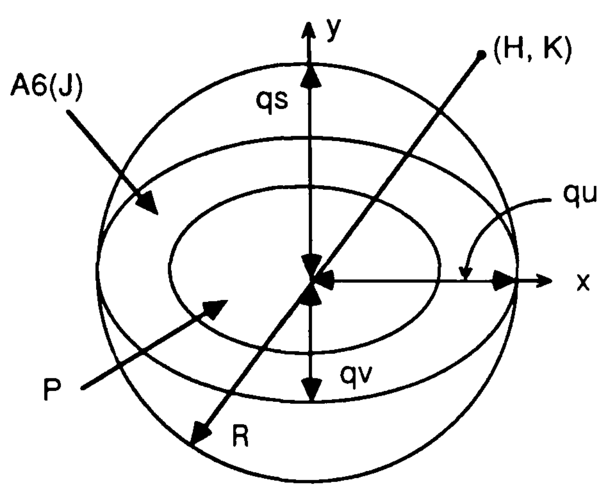


FIGURE 1: USED WITH (35)

R_{min} and R_{max} establish crude bounds for R. An improvement is generally obtained by using an estimate for R, call it Rg, given by F.E. Grubbs.[8,9] His estimate depends on a percentage point of the chi-squared distribution which is available through the subprogram GAMINV. This subprogram is described in Reference 7. The quantity actually given by Grubbs is Rg^2 which estimates R^2 and is given by

$$Rg^2 = W_4\left\{\left[x(4/V_5, P) - 4/V_5\right]W_2 + N\right\}, \tag{36}$$

where

$$W_4 = u^2 + v^2 + w^2 \tag{37.1}$$

$$N = 1 + D^2/W_4 \tag{37.2}$$

$$V_4 = 2\left\{\frac{u^4}{W_4{}^2}\left[1 + 2\left(\frac{H}{u}\right)^2\right] + \frac{v^4}{W_4{}^2}\left[1 + 2\left(\frac{K}{v}\right)^2\right] + \frac{w^4}{W_4{}^2}\left[1 + 2\left(\frac{L}{W}\right)^2\right]\right\} \tag{37.3}$$

$$T_5 = 8\left\{\frac{u^6}{W_4{}^3}\left[1 + 3\left(\frac{H}{u}\right)^2\right] + \frac{v^6}{W_4{}^3}\left[1 + 3\left(\frac{K}{v}\right)^2\right] + \frac{w^6}{W_4{}^3}\left[1 + 3\left(\frac{L}{W}\right)^2\right]\right\} \tag{37.4}$$

$$W_2 = T_5/(2V_4) \tag{37.5}$$

$$V_5 = T_5{}^2/V_4{}^3, \tag{37.6}$$

and x(A,P) satisfies

$$P = \frac{1}{\Gamma(A)}\int_0^{x(A,P)} \exp(-t)\, t^{A-1}\, dt, \quad A = 4/V_5. \quad [1;\ p.\ 260] \tag{38}$$

If

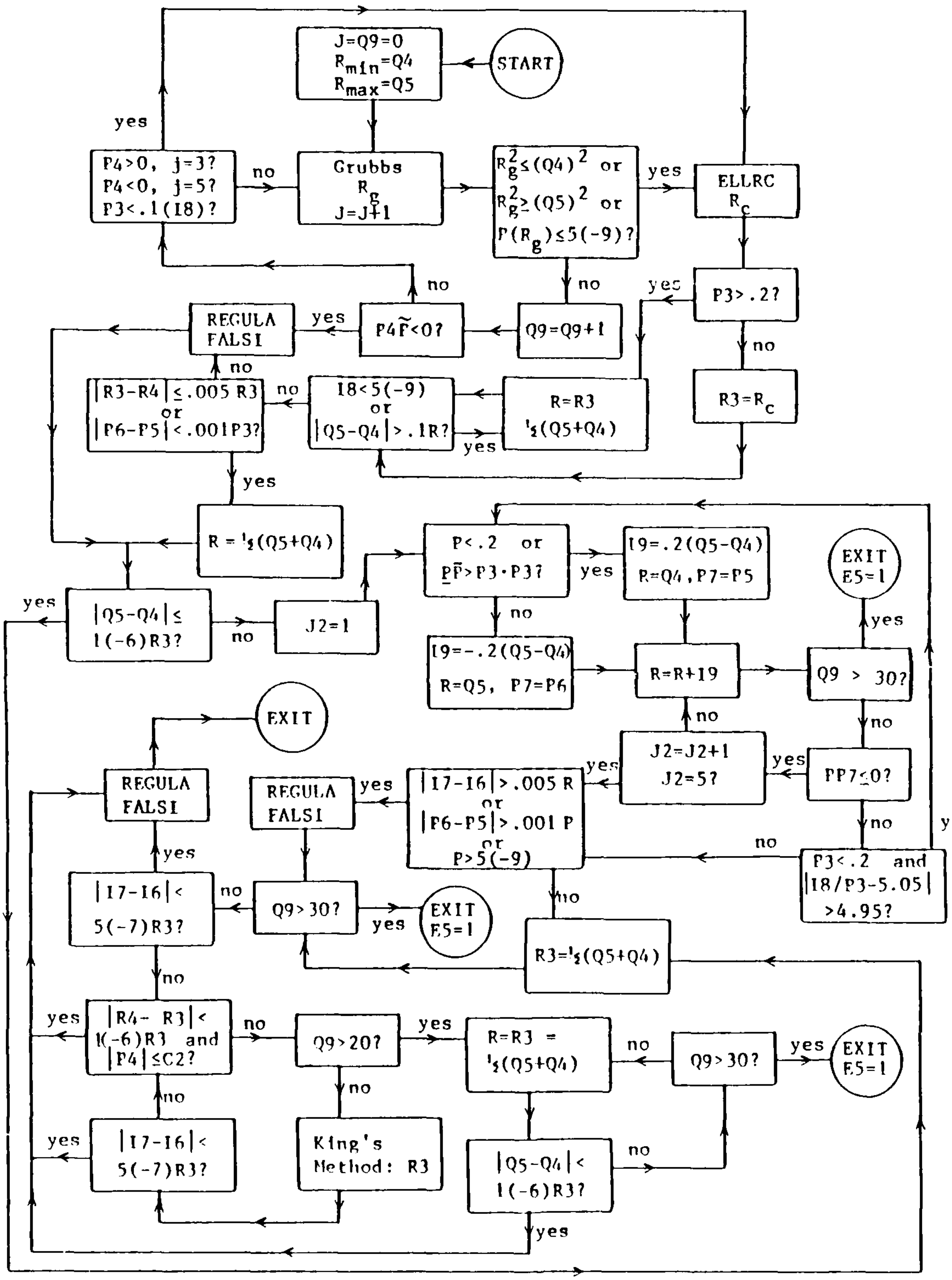
$$R_{min}{}^2 < Rg^2 < R_{max}{}^2, \tag{39}$$

then Rg is accepted as an improved estimate for R. If Equation 39 holds and P(Rg) < P, then Rg is stored in Q4 and R_{max} is stored in Q5; if P(Rg) > P, then Rg is stored in Q5 and R_{min} is stored in Q4. Generally Q4 and Q5 bound R with Q4 $\leq$ R $\leq$ Q5 (When King's procedure is used Q4 and Q5 still bound R, but their roles as lower and upper bounds may be reversed.). Refinements to Q4 and Q5 are obtained by using the procedures mentioned above. Some of the details are shown in the flow chart of Figure 2. A flow chart for King's root finding method is given in Figure 3 on page 19.

NOTATION FOR FLOWCHART OF FIGURE 2 AND BASIC LISTING

Rt denotes true value of R for a given value P = P3. P3 = P(Rt)

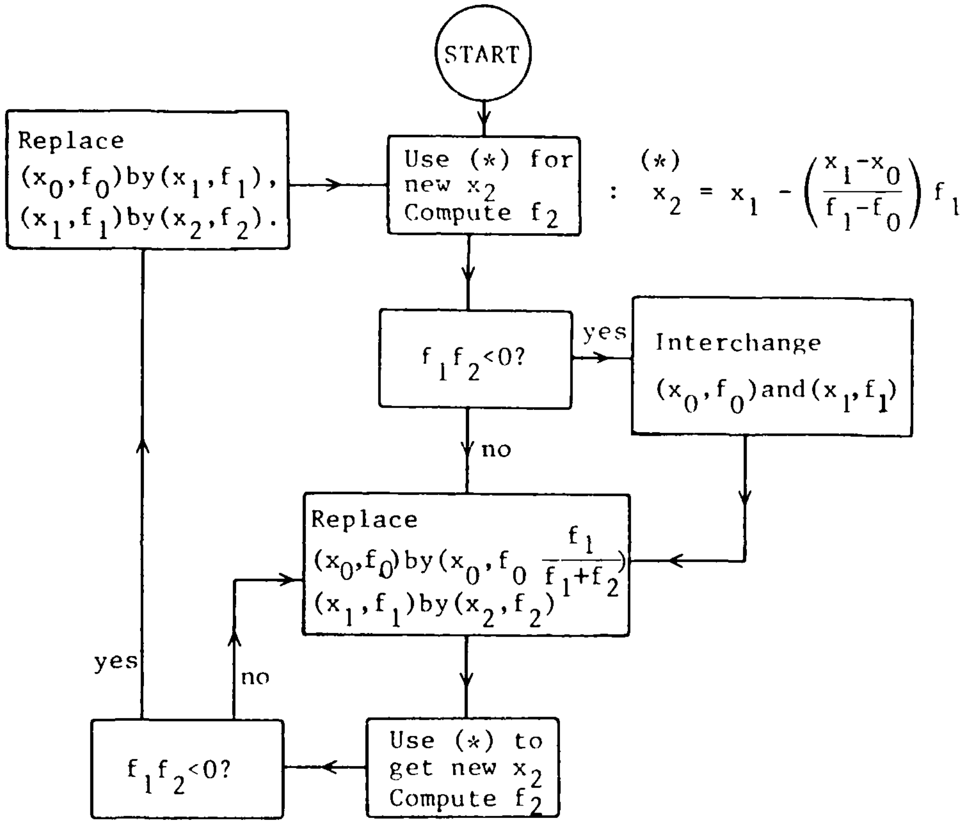
C2 = 10 X8 (1 - P3), P3 < .5; C2 = 10 X8 P3, P3 ≥ .5
E4 = 0, 1(-6) ≤ P3 ≤ .999999; E4 = 1, 1(-6) > P3 > .999999
E5 = 0, Less than 30 iterations to find R3; E5 = 1, 30 iterations are used to find best estimate for Rt. Final estimate R3 may not be accurate to 6-decimal digits.
Hx - H of text; Hy - K of text; Hz - L of text.
I6 - R3 with P4 < 0
I7 - R3 with P4 > 0
I8 - P(R3), latest value of P
I9 - Increment used in stepping procedure. I9 = ±.2 (Q_5 - Q_4).
KO = $H^2 + K^2 + L^2$; K2 = $\sqrt{KO}$
$\tilde{P}$ - Previous value of I8 - P3
P3 - Input value of P
P4 - P(R3) or P(R3) - P3
P5 - P(Q4) - P3
P6 - P(Q5) - P3
P7 - P5 or P6
$\bar{P}$ - Latest value of P > P3
$\underline{P}$ - Latest value of P ≤ P3
Q4 - See Section III. Latest estimate for Rt such that P(Q_4) - P3 < 0.
Q5 - See Section III. Latest estimate for Rt such that P(Q_5) - P3 > 0.
Q9 - Contains number of iterations used to estimate Rt.
R - Latest estimate for Rt
Rg - See Section III. Estimate from Grubb's approximation for Rt.
Rc - See Section III. Lower estimate for Rt from ELLRC.
R3 - The latest estimate for Rt.
R4 - The next to latest estimate for Rt.
R_{min} - Starting lower bound for Rt. R_{max} - Starting upper bound for Rt^2.
Sx - u of text
Sy - v of text
Sz - w of text
M - max(u, v, w)
T6 - Number of iterations used to obtain Rc using ELLRC.
X8 - See Section III. Used to exit INVELLCOV if |P4(R3) - P3| ≤ X8.



Not Shown: P(R) computed by ELLCOV. If $|P(R)-P3| \leq X8$ then R is an acceptable estimate for Rt. INVELLCOV exited.

FIGURE 2. FLOW CHART FOR INVELLCOV

Input: x_0, f_0, x_1, f_1, $(f_0 f_1 < 0)$



$x_2 \equiv$ latest estimate for $\bar{x}$, $f_n \equiv f(x_n)$, $f(\bar{x}) = 0$

FIGURE 3. KING'S ROOT FINDING METHOD FOR f(x)

In most cases Rg gives a very good estimate for R. In fact, Rg in many cases can be accepted as the final estimate for R, with no iterations required. If however Inequality 39 does not hold, (the right hand side of Equation 36 may even be negative) then we proceed as follows.

The constant R_p is determined so that

$$P = \int_{L-R_p}^{L+R_p} \int_{K-R_p}^{K+R_p} \int_{H-R_p}^{H+R_p} F(x,y,z,u,v,w)\ dx\ dy\ dz. \tag{40}$$

The subroutine ELLRC, using the Newton-Raphson method, is used to compute R_p. The right hand side of Equation 40 yields the probability P over the smallest cube, with center at (H,K,L) and sides parallel to the x, y, and z axes, which contains the sphere of radius R_p. Hence $R_p < R < \sqrt{3}\ R_p$; if $R_p > R_{min}$, then $Q4 = R_p$, and if $\sqrt{3}\ R_p < R_{max}$ then $Q5 = \sqrt{3}\ R_p$.

In ELLRC the maximum number of iterations for R_p is set at 40. A maximum of 37 iterations has been observed.

Again, once R_p is computed, refinements are made by using halving, stepping, Regula-falsi and King's method to obtain a final estimate for R.

The iterations for R are generally terminated when Q4 and Q5 agree to at least 6 significant digits or if the latest estimate for R, say R_3, yields $P(R_3)$ such that $|P - P(R_3)| \leq X_8$, where X_8 is a small number which depends on P. For example, if $P \leq 1(-5)$ then $X_8 = 5(-10)$, whereas if $.001 < P < .999995$ then $X_8 = 5(-7)$. When exit conditions, some of which have just been described, are satisfied, then one more application of Regula-falsi is carried out before exiting INVELLCOV. See Figure 2 for more details, page 18.

The maximum number of iterations for R in INVELLCOV is set at 30. Extensive checking has never shown more than 20 iterations for any case. The average number of iterations is between 4 and 5. When P is very small (<.001) more than the average number of iterations is generally needed.

INVELLCOV can be used with any reasonable input values of H, K, L, u, v, w. The input variable P should satisfy $|P - 1/2| \leq .499999$. If P is in (0,1) and does not satisfy the above inequality, INVELLCOV may still give a result for R, but there is no assurance near 6 digit accuracy will be obtained.

Table 2 below contains some numerical results from INVELLCOV. It indicates the wide range of applicability of INVELLCOV.

TABLE 2

Item	u/v	v/w	R/w	H/u	K/v	L/w	P
1	1/2	2/3	3.183274/6	5/2	5/2	10/3	5(-6)
2	1/2	2/3	10.28906/6	5/2	5/2	10/3	5(-3)
3	1/2	2/3	16.60908/6	5/2	5/2	10/3	.1
4	1/2	2/3	23.39931/6	5/2	5/2	10/3	.5
5	1/2	2/3	30.48622/6	5/2	5/2	10/3	.9
6	1/2	2/3	40.77124/6	5/2	5/2	10/3	.999
7	1/2	2/3	48.44240/6	5/2	5/2	10/3	.999995
8	1(-3)	.05	1.171111	4000	20	5	5(-6)
9	1(-3)	.05	2.629994	4000	20	5	5(-3)
10	1(-3)	.05	3.855994	4000	20	5	.1
11	1(-3)	.05	5.103200	4000	20	5	.5
12	1(-3)	.05	6.364060	4000	20	5	.9
13	1(-3)	.05	8.155921	4000	20	5	.999
14	1(-3)	.05	9.472573	4000	20	5	.999995
15	2/5	5	6.184626	0	1	10	5(-6)
16	2/5	5	8.185500	0	1	10	5(-3)
17	2/5	5	9.733491	0	1	10	.1
18	2/5	5	11.72659	0	1	10	.5
19	2/5	5	15.45659	0	1	10	.9
20	2/5	5	22.95502	0	1	10	.999
21	2/5	5	29.02551	0	1	10	.999995
22	1(-5)	2(9)	2.1376073(4)	0	1(-4)	0	5(-6)
23	1(-5)	2(9)	1.2533239(7)	0	1(-4)	0	5(-3)
24	1(-5)	2(9)	2.5132270(8)	0	1(-4)	0	.1
25	1(-5)	2(9)	1.3489795(9)	0	1(-4)	0	.5
26	1(-5)	2(9)	3.2897073(9)	0	1(-4)	0	.9
27	1(-5)	2(9)	6.5810535(9)	0	1(-4)	0	.999
28	1(-5)	2(9)	9.1295755(9)	0	1(-4)	0	.999995

Values of R as computed by INVELLCOV.

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APPENDIX A

DERIVATION OF (10 - 11) and (13 - 17)

Derivation of Equations 10, 11 and 13-17

Equations 10 and 11 refer to Case A in Section II. We have

$$\boxed{u = v = w} \qquad D \equiv \sqrt{H^2 + K^2 + L^2}, \tag{A-1}$$

and

$$P = \int_S F(x,y,z,u,v,w)\ dx\ dy\ dz, \tag{A-2}$$

where the integration is carried out over the interior of the sphere

$$S : (x - H)^2 + (y - K)^2 + (z - L)^2 = R^2. \tag{A-3}$$

Let

$$x = H + \tau \cos\theta \sin\phi$$
$$y = K + \tau \sin\theta \sin\phi$$
$$z = L + \tau \cos\phi,$$
$$0 \le \tau \le R,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.$$

Then Equation A-2 becomes

$$P = \frac{1}{2\pi\sqrt{2\pi}\ uvw}\int_0^R \int_{\phi=0}^{\pi} \int_{\Theta=0}^{2\pi} E(D/u)\ \exp\left\{-\frac{\tau}{u^2}[H\cos\theta\cos\phi + K\sin\theta\sin\phi\right.$$

$$\left. + L\cos\phi]\right\}\ E(\tau/u)\ \tau^2 \sin\phi\ d\theta\ d\phi\ d\tau.$$

Taking advantage of spherical symmetry, we take H = K = 0, L = D. Hence

$$P = \frac{1}{\sqrt{2\pi}\ u^3}\int_0^R \int_0^{\pi} E(D/u)\ \exp(-D\tau\cos\phi/u^2)\ E(\tau/u)\ \sin\phi\ d\phi\ \tau^2\ d\tau. \tag{A-4}$$

With $\cos\phi = s$,

$$P = \frac{1}{\sqrt{2\pi}\ u^3}\int_0^R E(D/u)\ E(\tau/u)\left[\int_{-1}^{1} \exp(-D\tau s/u^2)\ ds\right]\tau^2\ d\tau,$$

where

$$\int_{-1}^{1} \exp(-D\tau s/u^2)\ ds = \frac{u^2}{D\tau}\ [\exp(D\tau/u^2) - \exp(-D\tau/u^2)].$$

Thus

$$P = \frac{1}{\sqrt{2\pi}\ Du} \int_0^R \left\{E[(D - \tau)/u] - E[(D + \tau)/u]\right\} \tau \, d\tau,$$

and

$$P = \frac{1}{2} \left\{ \operatorname{erf}\left(\frac{D + R}{\sqrt{2}\,u}\right) - \operatorname{erf}\left(\frac{D - R}{\sqrt{2}\,u}\right) - \sqrt{\frac{2}{\pi}}\ \frac{u}{D} \left[E\left(\frac{D - R}{u}\right) - E\left(\frac{D + R}{u}\right)\right]\right\}, \quad \underline{D \neq 0.} \qquad \text{(A-5)}$$

If D = 0, then

$$P = \operatorname{erf}\left(\frac{R}{\sqrt{2}\,u}\right) - \sqrt{\frac{2}{\pi}}\ \frac{R}{u}\ E(R/u), \quad \underline{D = 0}. \qquad \text{(A-6)}$$

Now let

$$Y = \frac{E\left(\frac{D - R}{u}\right) - E\left(\frac{D + R}{u}\right)}{2DR/u^2} = E\left(\frac{D - R}{u}\right) \left\{\frac{1 - \exp\left(\frac{-2DR}{u^2}\right)}{2DR/u^2}\right\}$$

Using the result in Equation A-5 gives Equation 10; then Equation A-6 or Equation 11 follows since Y = E(R/u) when D = 0.

Derivation of Equations 13-17

We have

$$\boxed{u = v \text{ and } H = K = 0} \qquad u \neq w. \qquad \text{(A-7)}$$

It is easily shown that

$$P_c(\bar{r},0,0,u,u) = 1 - E(\bar{r}/u). \qquad [5;\ \text{Eq. } 2] \qquad \text{(A-8)}$$

Therefore with Relations A-7 and A-8 we have from Equation 6

$$P = \frac{1}{\sqrt{2\pi}\ w} \int_{L-R}^{L+R} E(z/w)[1 - E(\bar{r}/u)]\, dz,$$

where

$$\bar{r}^2 = R^2 - (L - z)^2 \geq 0.$$

Therefore

$$P = \frac{1}{2}\,[\mathrm{erf}\ L_4 - \mathrm{erf}\ L_5] - T,\ L_4 = \frac{L + R}{\sqrt{2}\ w},\ L_5 = \frac{L - R}{\sqrt{2}\ w} \tag{A-9}$$

$$T \equiv \frac{E\left(\sqrt{R^2 - L^2}/u\right)}{\sqrt{2\pi}\ w} \int_{L-R}^{L+R} E(z/w)\ \exp\left[-\frac{1}{2}(2Lz - z^2)/u^2\right] dz. \tag{A-10}$$

Two cases are now treated separately, namely $u > w$ and $u < w$.

Let <u>$u > w$</u>. Then Equation A-10 becomes, after straight-forward algebra,

$$T = \frac{1}{\sqrt{2\pi}\ w} \exp\left\{-\frac{1}{2}\left[\left(\frac{R}{u}\right)^2 - \frac{L^2}{u^2 - w^2}\right]\right\} \frac{\sqrt{2}}{\alpha} \int_{S_0 = [\alpha(L-r)+\beta]/\sqrt{2}}^{F_1 = [\alpha(L+R)+\beta]/\sqrt{2}} \exp(-t^2)\ dt, \tag{A-11}$$

where

$$\alpha = L_2/w, \qquad \beta = \frac{Lw}{u^2\ L_2}, \qquad L_2 = \sqrt{|1 - (w/u)^2|}\ .$$

We also have

$$F_1 \equiv [\alpha(L + R) + \beta]/\sqrt{2} = \frac{(L + R)L_2}{\sqrt{2}\ w} + \frac{Lw}{\sqrt{2}\ L_2 u^2} = \frac{1}{\sqrt{2}\ w}\,(L/L_2 + RL_2) \tag{A-12.1}$$

$$S_0 \equiv [\alpha(L - R) + \beta]/\sqrt{2} = \frac{1}{\sqrt{2}\ w}\,(L/L_2 - RL_2), \tag{A-12.2}$$

with L_2 given by Equation 19.1. Then from Equation A-9, using Equation A-11 and Equation A-12, we get

$$P = \frac{1}{2}\left\{\mathrm{erf}\ L_4 - \mathrm{erf}\ L_5 - \frac{1}{L_2}\,E(R/u)\ E[L/(uL_2)]\ (\mathrm{erf}\ F_1 - \mathrm{erf}\ S_0)\right\} \tag{A-13}$$

which is Equation 13.

Another form for Equation A-13, namely Equation 14, is obtained by replacing $\mathrm{erf}\ F_1 - \mathrm{erf}\ S_0$ by $\mathrm{erfc}\ S_0 - \mathrm{erfc}\ F_1$, and then multiplying $\mathrm{erfc}\ S_0$ by $\exp(S_0{}^2)\ \exp(-S_0{}^2)$, and $\mathrm{erfc}\ F_1$ by $\exp(F_1{}^2)\ \exp(-F_1{}^2)$. Combining the results with the remaining exponentials in Equation A-13 gives Equation 14, where use is made of the relation

$$\frac{1}{2}\left\{(R/u)^2 - L^2/(u^2 - w^2) + [\alpha(L \pm R) + \beta]^2\right\} = \frac{1}{2}[(L \pm R)/w]^2.$$

When L = 0, Equation A-13 reduces to

$$P = \mathrm{erf}(R/\sqrt{2}\,w) - (1/L_2)\,E(R/u)\,\mathrm{erf}(RL_2/\sqrt{2}\,w). \qquad \text{(A-14)}$$

Now consider <u>w > u</u>. Starting with Equation A-10, we have

$$T = E(R/u)\,E(L/\sqrt{w^2 - u^2})\,\frac{1}{\sqrt{2\pi}\,w}\int_{L-R}^{L+R} \exp\left[\frac{1}{2}\,(\alpha z - \beta)^2\right] dz. \qquad \text{(A-15)}$$

Note here that the argument of the exponential in the integrand is positive. Proceeding as above, T is obtained in terms of Dawson's integral denoted here by daws(x).

$$T = \frac{1}{\sqrt{\pi}\,L_2}\,E(R/u)\,E\left[L/\sqrt{(w^2 - u^2)}\right]\left[\exp(F_1{}^2)\mathrm{daws}(F_1) - \exp(S_0{}^2)\mathrm{daws}(S_0)\right],$$

$$\mathrm{daws}(x) \equiv \exp(-x^2)\int_0^x \exp(t^2)\,dt, \quad \mathrm{daws}(-x) = -\mathrm{daws}(x), \quad [1;\ p.\ 298],\ [3].$$

We also have using Equation A-12

$$S_0 = \frac{-1}{\sqrt{2}\,uw\sqrt{|u^2 - w^2|}}\,[R(w^2 - u^2) - Lu^2]$$

$$F_1 = \frac{1}{\sqrt{2}\,uw\sqrt{|u^2 - w^2|}}\,[R(w^2 - u^2) + Lu^2]\ .$$

Thus using the relation

$$\frac{1}{2}\left\{\left(\frac{R}{u}\right)^2 + \frac{L^2}{(w^2 - u^2)} - \frac{1}{u^2w^2(w^2 - u^2)}\,[R^2(w^2 - u^2)^2 \pm 2RLu^2(w^2 - u^2) + u^4L^2]\right\}$$

$$= \frac{1}{2}\left\{\left(\frac{R}{u}\right)^2\left(1 - \frac{(w^2 - u^2)}{w^2}\right) \pm \frac{2RL}{w^2} + \frac{L^2}{(w^2 - u^2)}\left(1 - \frac{u^2}{w^2}\right)\right\} = \left[\frac{L \pm R}{\sqrt{2}\,w}\right]^2$$

Equation A-15 reduces to

$$T = \frac{1}{2L_2}\,\frac{2}{\sqrt{\pi}}\,E\left(\frac{L - R}{w}\right)\left[\mathrm{daws}(F_1) - \exp(-\,2RL/w^2)\,\mathrm{daws}(S_0)\right], \qquad \text{(A-16)}$$

which agrees with the corresponding term in Equation 16.

For $L = 0$, with $w > u$, it is easy to conclude from Equation A-9 and Equation A-16 that

$$P = \operatorname{erf} L_4 - \frac{2}{\sqrt{\pi}} \frac{1}{L_2} E(R/w) \operatorname{daws}(RL_2/\sqrt{2}w), \qquad \text{(A-17)}$$

which is Equation 17.

APPENDIX B

DISCUSSION OF TESTS #1, #2, #3, #4

Discussion of Tests #1, #2, #3, #4

Test #1: From Equation 4 with $E(x/u)\ E(y/v)\ E(z/w) < 1$, we have

$$P < \frac{4}{3}\pi\ R^3/(2\pi\sqrt{2\pi}\ u\ v\ w) \leq Z4, \qquad Z4 = 2.5(-8),$$

or

$$R^3 \leq 1.5\sqrt{2\pi}\ uvw\ Z4 \quad (\Rightarrow P < Z4). \tag{B-1}$$

The relations in Inequality B-1 lead to Test #1.

Also, from Equation 6, setting $E(z/w)\ P_c = 1$ with $w = \max(u,v,w)$

$$P < \frac{1}{\sqrt{2\pi}\ M}[(L + R) - (L - R)] = \sqrt{\frac{2}{\pi}}\ \frac{R}{M} \leq Z4, \qquad M = \max(u,v,w),$$

$$R < \sqrt{\frac{\pi}{2}}\ Z_4 M. \tag{B-2}$$

The last inequality and Inequality B-1 are used in Equation 34.

Test #2: Note if $D > R$ then $P < 1/2$ and the sphere S does not contain the origin. Therefore, we have for $D > R$

$$E(x/u)\ E(y/v)\ E(z/w) \leq E(x/M)\ E(y/M)\ E(z/M) \leq E[(D - R)/M]$$

and it follows

$$P < \frac{1}{2\pi\sqrt{2\pi}\ uvw}\int_S E\left(\frac{D - R}{M}\right) dx\ dy\ dz = \frac{4\pi R^3}{6\pi\sqrt{2\pi}\ uvw}\ E\left(\frac{D - R}{M}\right) \leq Z4. \tag{B-3}$$

Then Test #2 follows directly from Inequality B-3.

Test #3: This follows by recognizing that the sphere S is located outside the region which contains $1 - Z4$ of the distribution. It is shown by Equation 26 and Equation 27 that if a is determined by $\mathrm{erfc}(a/\sqrt{2}) = \varepsilon$ then the entire region $\bar{U}$ above the plane $z = a\ w$ contains $\varepsilon/2$ of the distribution. Since $z = L - R > 0$ locates the point of the sphere closest to the xy-plane, and if $L - R > \sqrt{2}\ w\ F_2$ ($\sqrt{2}\ F2 = a$), then the sphere S must be entirely contained in the region $\bar{U}$. Here L has been used for h, and w has been used for T.

Test #4: This test is based on the fact that if $P \approx 1$ and u, v or w increases, then for fixed R, P must decrease. Thus if we replace u, v, and w by M to obtain $\bar{P}$, then $\bar{P} \le P$. If $\bar{P} \ge 1 - Z4$, then $P \ge 1 - Z4$.

A heuristic argument follows that indicates if

$$R^2 > H^2 + K^2 + L^2, \tag{B-4}$$

then

$$P = P(R,H,K,L,u,v,w) \ge P(R,H,K,L,M,M,M,), \quad M = \max(u,v,w) \tag{B-5}$$

Without loss of generality, assume $w < M$ and let

$$P = P_0 \equiv P(R_0,H,K,0,u,v,w). \tag{B-6}$$

Then it is easy to show from Equation 8 (with $L = 0$) that

$$\frac{\partial P_0}{\partial w} = \int_{-R/\sqrt{2}\,w}^{R/\sqrt{2}\,w} \frac{\partial P_c}{\partial \xi}\frac{\partial \xi}{\partial w}\, dt < 0, \tag{B-7}$$

where

$$\xi = \sqrt{R^2 - 2w^2t^2} \quad \text{and} \quad \frac{\partial P_c(\xi,H,K,u,v)}{\partial \xi} > 0 \tag{B-8}$$

Thus from Equation B-7,

$$P = P_0(R_0,H,K,0,u,v,w) > P(R_0,H,K,0,u,v,M). \tag{B-9}$$

Then by continuity, we have for sufficiently small $\bar{\delta}$

$$P = P(\bar{R},H,K,\bar{\delta},u,v,w) > P(\bar{R},H,K,\bar{\delta},u,v,M). \tag{B-10}$$

Now consider a sequence of spheres $\{S_i\}$ with centers (H,K,δ_i) and radii R_i, such that δ_i is an arbitrary increasing set of positive numbers with $0 \le \delta_i \le L$, $R_0 \le R_i \le R$ and

$$P = P(R_i,H,K,\delta_i,u,v,w). \tag{B-11}$$

Note that by Inequality B-4 all S_i contain the origin.

If $P = P(R_i,H,K,\delta_i,u,v,w) > P(R_i,H,K,\delta_i,u,v,M)$ holds for all elements S_i, then a simple argument establishes Inequality B-5. Suppose however for some $R_i = \hat{R}$, $\delta_i = \hat{\delta}$ the last inequality does not hold. Since Inequality B-9 always holds, there must exist by continuity a sphere $\overset{*}{S}$ with $\overset{*}{R}$, $\overset{*}{\delta}$, $(\bar{R} < \overset{*}{R} < \hat{R} \leq R,\ \bar{\delta} < \overset{*}{\delta} < \hat{\delta} \leq L)$ such that

$$P = P(\overset{*}{R},H,K,\overset{*}{\delta},u,v,w) = P(\overset{*}{R},H,K,\overset{*}{\delta},u,v,M). \tag{B-12}$$

But Equation B-12 is impossible, since it implies that the two different cumulative normal distributions over the same sphere can give the same value of P. Indeed, denote the two different density functions of Equation (B-12) by I_w and I_M. These functions, with 3 independent variables and hence not easily visualized, intersect along a surface T bounded by a simple closed curve. From Equation B-12 and Inequaltiy B-4 with the integrations of I_w and I_M carried out over the interioι of $\overset{*}{S}$, we have

$$P = J_w + A_w = J_M + A_M, \tag{B-13}$$

where $J_w(J_M)$ denotes the integral of $I_w(I_M)$ over that part of $\overset{*}{S}$ which contains the origin and is bounded by T; $A_w(A_M)$ denotes the remaining contribution from the integration of $I_w(I_M)$ over that part of $\overset{*}{S}$ bounded by T and not containing the origin. For $w < M$, then $I_w > I_M$ at the origin and

$$J_w > J_M, \text{ or } J_w = J_M + a \qquad a \geq 0; \tag{B-14}$$

$$A_w < A_M, \text{ or } A_w = A_M - b \qquad b \geq 0. \tag{B-15}$$

But $a > b$, since the integration of I_w and I_M over all xyz-space must equal one. This leads to a contradiction of Equation B-13.

This argument is easily seen in the one dimensional case. We use the same notation as above for the corresponding quantities in one dimension. See the figure below.

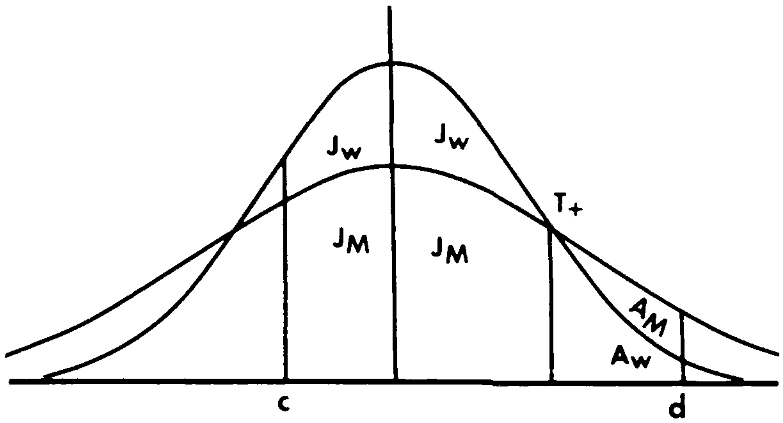

FIGURE 4: NORMAL CURVES WITH UNEQUAL STANDARD DEVIATIONS

Let

$$D(t/y) \equiv \frac{1}{\sqrt{2\pi}\, y} E(t/y) \tag{B-16}$$

$$J_w(c,T_+) = \int_c^{T_+} D(t/w)\, dt, \quad J_M(c,T_+) = \int_c^{T_+} D(t/M)\, dt$$

$$A_w = \int_{T_+}^{d} D(t/w)\, dt, \quad A_M = \int_{T_+}^{d} D(t/M)\, dt.$$

Clearly with $c < 0$, $d > 0$ (corresponding to $R^2 > H^2 + K^2 + L^2$) we have

$J_w > J_M$, $A_w < A_M$, and $J_w(c,d) > J_M(c,d)$.

Thus by the contradiction above we conclude that

$$P(R,H,K,L,u,v,w) > P(R,H,K,L,u,v,M)$$

Then Inequality B-5 easily follows by carrying out the same argument using the min(u,v), say v, in place of w to show the last inequality below,

$$P(R,H,K,L,u,v,w) > P(R,H,K,L,M,v,M) > P(R,H,K,L,M,M,M).$$

APPENDIX C

H-P BASIC LISTINGS OF ELLCOV AND INVELLCOV

```
1925  ! ARRAYS X(*), Y(*), A6(*), B(*) MUST BE CALLED WITH BASIC SUBROUTINE
1926  ! IN MAIN PROGRAM IN ORDER TO USE ELLCOV OR INVELLCOV.
1930  DIM S2(20),Hx(20),H(3),S(3),A5(3),B5(3),T5(3),P(7),W8(3)
1935  COM X(43),Y(43),A6(10),B6(10),Q9,V,A7,P4,Ve,T6,P1,T9
1940  ! GAUSSIAN CONSTANTS STORED IN "GAUSS". USED IN "SUBGquad".
1945  X(1)=.238619186083
1950  X(2)=.661209386466
1955  X(3)=.932469514203
1960  X(4)=.183434642496
1965  X(5)=.525532409916
1970  X(6)=.796666477414
1975  X(7)=.960289856498
1980  X(8)=.125233408511
1985  X(9)=.367831498998
1990  X(10)=.587317954287
1995  X(11)=.769902674194
2000  X(12)=.90411725637
2005  X(13)=.981560634247
2010  X(14)=9.50125098376E-2
2015  X(15)=.281603550779
2020  X(16)=.458016777657
2025  X(17)=.617876244403
2030  X(18)=.755404408355
2035  X(19)=.865631202388
2040  X(20)=.944575023073
2045  X(21)=.989400934992
2050  X(22)=7.65265211335E-2
2055  X(23)=.227785851142
2060  X(24)=.373706088715
2065  X(25)=.510867001951
2070  X(26)=.636053680727
2075  X(27)=.74633190646
2080  X(28)=.839116971822
2085  X(29)=.912234428251
2090  X(30)=.963971927278
2095  X(31)=.993128599185
2100  X(32)=6.40568928626E-2
2105  X(33)=.191118867474
2110  X(34)=.315042679696
2115  X(35)=.433793507626
2120  X(36)=.545421471389
2125  X(37)=.648093651937
2130  X(38)=.740124191579
2135  X(39)=.820001985974
2140  X(40)=.886415527004
2145  X(41)=.938274552003
2150  X(42)=.974728555971
2155  X(43)=.995187219997
2160  Y(1)=.467913934573
2165  Y(2)=.360761573048
2170  Y(3)=.171324492379
2175  Y(4)=.362683783378
2180  Y(5)=.313706645878
2185  Y(6)=.222381034453
2190  Y(7)=.10122853629
2195  Y(8)=.249147045813
2200  Y(9)=.233492536538
```

```
2205  Y(10)=.203167426723
2210  Y(11)=.160078328543
2215  Y(12)=.106939325995
2220  Y(13)=4.71753363865E-2
2225  Y(14)=.189450610455
2230  Y(15)=.182603415045
2235  Y(16)=.169156519395
2240  Y(17)=.149595988817
2245  Y(18)=.124628971256
2250  Y(19)=9.51585116825E-2
2255  Y(20)=6.22535239386E-2
2260  Y(21)=2.71524594118E-2
2265  Y(22)=.152753387131
2270  Y(23)=.149172986473
2275  Y(24)=.142096109318
2280  Y(25)=.131688638449
2285  Y(26)=.118194531962
2290  Y(27)=.101930119817
2295  Y(28)=8.32767415767E-2
2300  Y(29)=6.26720483341E-2
2305  Y(30)=4.06014298004E-2
2310  Y(31)=1.76140071392E-2
2315  Y(32)=.127938195347
2320  Y(33)=.125837456347
2325  Y(34)=.121670472928
2330  Y(35)=.115505668054
2335  Y(36)=.107444270116
2340  Y(37)=9.76186521041E-2
2345  Y(38)=.086190161532
2350  Y(39)=7.33464814111E-2
2355  Y(40)=5.92985349154E-2
2360  Y(41)=4.42774388174E-2
2365  Y(42)=2.85313886289E-2
2370  Y(43)=.0123412298
2375  A6(1)=5E-6
2380  A6(2)=.0001
2385  A6(3)=.01
2390  A6(4)=.1
2395  A6(5)=.3
2400  A6(6)=.6
2405  A6(7)=.9
2410  A6(8)=.999
2415  A6(9)=.999999
2420  A6(10)=1
2425  B6(1)=.0266      !.0372
2430  B6(2)=.0723      !.15
2435  B6(3)=.339       !.474
2440  B6(4)=.765       !.9
2445  B6(5)=1.1933     !1.686
2450  B6(6)=1.717
2455  B6(7)=2.5005     !3.662
2460  B6(8)=4.0335     !6.215
2465  B6(9)=5.538      !8.84
2470  B6(10)=10
2475  RETURN
```

```
2480  ! *********START OF INVERSE PROCEDURE*****
2485  ! THIS PROGRAM IS CALLED INVELLCOV. (X,Y,Z) IS A POINT IN A CARTESIAN CO-
2486  ! ORDINATE SYSTEM. INVELLCOV RETURNS THE RADIUS OF THE SPHERE WITH CENTER
2490  ! (Hx,Hy,Hz) WHICH HAS P3 OF THE NORMAL ELLIPSOIDAL DISTRIBUTION WITH
2491  ! MEAN (0,0,0) AND STANDARD DEVIATIONS Sx,Sy,Sz IN THE X,Y,Z DIRECTIONS,
2495  ! RESPECTIVELY. THE RADIUS R IS GENERALLY GIVEN CORRECTLY TO AT LEAST 6-
2496  ! DECIMAL DIGITS. THE INPUT P3 SHOULD SATISFY ABS(P3-1/2)<.499999. IF
2500  ! P3 IS IN [0,1) BUT THE ABOVE INEQUALITY IS NOT SATISFIED, THEN THE
2501  ! OUTPUT R MAY NOT BE CORRECT TO 6-DECIMAL DIGITS. A MAXIMUM OF 30 ITERA-
2505  ! TIONS IS ALLOWED. IF E5=1 THEN 30 ITERATIONS HAVE OCCURRED. IF E4=1
2506  ! THEN P3 IS IN (0,1) BUT NOT IN [1(-6),.999999]. IF P3 IS NOT IN [0,1]
2510  ! THEN A VALUE OF -1E-99 IS RETURNED FOR R. SUBROUTINES USED ARE ELLCOV,
2511  ! ELLRC, AND GAMINV.
2515  DEF FNInvellcov(P3,Hx,Hy,Hz,Sx,Sy,Sz,E4,E5)
2520    COM X(43),Y(43),A6(10),B6(10),Q9,V,A7,P4,Ve,T6,P1,T9
2525    K0=Hx*Hx+Hy*Hy+Hz*Hz
2530    K2=SQR(K0)
2535    S=MAX(Sx,Sy,Sz)
2540    A4=.797884560803! SQR(2/PI)
2545    R3=E5=E4=0
2550    W4=Sx*Sx+Sy*Sy+Sz*Sz
2555    IF ABS(P3-.5)<=.499999 THEN 2595
2560    IF ABS(P3-.5)<=.5 THEN 2575
2565    R3=-1E99  !****"P3 IS UNACCEPTABLE"
2570    RETURN R3
2575    E4=1  !****P3 NOT IN [1E-6,.999999].
2580    IF P3=0 THEN RETURN 0
2585    IF P3<1 THEN 2595
2590    RETURN 1E99
2595    Q9=0
2600    X8=.0000005
2605    IF P3>.00001 THEN 2620
2610    X8=5E-10
2615    GOTO 2650
2620    IF P3>.001 THEN 2635
2625    X8=.000000025
2630    GOTO 2650
2635    IF P3<.999995 THEN 2650
2640    X8=.00000005
2645    IF P3>.9999975 THEN X8=1E-8  !NEW
2650    P5=P3*S/A4
2655    Q4=MAX(3*P3*Sx*Sy*Sz/A4,P5*P5*P5)
2660    P4=0
2665    IF P3>=.5 THEN 2685
2670    C2=10*X8*(1-P3)
2675    R3=Q4^(1/3)
2680    GOTO 2700
2685    R3=K2
2690    IF Q4>K2*K2*K2 THEN R3=Q4^(1/3)
2695    C2=10*X8*P3
2700    Rmin=Q4=R3
2705    P4=FNEllcov(R3,Hx,Hy,Hz,Sx,Sy,Sz,K0,K2,S)
2710    IF P4>=P3 THEN RETURN R3
2715    P5=P4-P3
2720    FOR J=1 TO 10
2725      IF P3<=A6(J) THEN 2735
```

```
2730     NEXT J
2735     R3=K2+B6(J)*S
2740     Rmax=Q5=R3
2745     T6=0
2750     P4=FNEllcov(R3,Hx,Hy,Hz,Sx,Sy,Sz,K0,K2,S)
2755     IF P4<=P3 THEN RETURN R3
2760     P6=P4-P3
2765     I6=Q4
2770     I7=Q5
2775     ! ************GRUBB'S ESTIMATE FOR R3*****************
2780     D2=0
2785     D3=1.1
2790     IF P3<=.8 THEN 2810
2795     D3=1.25
2800     IF P3<=.9 THEN 2810
2805     D3=1.9
2810     D4=P4=P3
2815     M0=1+K0/W4     !1+(Hx*Hx+Hy*Hy+Hz*Hz)/W4
2820     Wx=Sx*Sx/W4
2825     Wy=Sy*Sy/W4
2830     Wz=Sz*Sz/W4
2835     Gx=Hx/Sx
2840     Gy=Hy/Sy
2845     Gz=Hz/Sz
2850     V4=2*(Wx*Wx*(1+2*Gx*Gx)+Wy*Wy*(1+2*Gy*Gy)+Wz*Wz*(1+2*Gz*Gz))
2855     V5=8*(Wx*Wx*Wx*(1+3*Gx*Gx)+Wy*Wy*Wy*(1+3*Gy*Gy)+Wz*Wz*Wz*(1+3*Gz*Gz))
2860     W2=.5*V5/V4
2865     V5=V5*V5/(V4*V4*V4)
2870     R4=R3
2875     IF P4=1 THEN P4=.999999999
2880     R3=W4*((FNGaminv(4/V5,X,0,P4,1-P4,Ierr)-4/V5)*W2+M0)
2885     IF (R3<=Q4*Q4) OR (R3>=Q5*Q5) THEN 3090
2890     R3=SQR(R3)
2895     IF ABS(R3-R4)<5E-9*R3 THEN 3090
2900     P4=FNEllcov(R3,Hx,Hy,Hz,Sx,Sy,Sz,K0,K2,S)
2905     I8=P4
2910     IF P4>0 THEN Q9=Q9+1
2915     IF P4>5E-9 THEN 2940
2920     IF R3<=Q4 THEN 3090
2925     Q4=R3
2930     P5=P4-P3
2935     GOTO 3090
2940     W5=P4-P3
2945     IF ABS(W5)<=X8 THEN RETURN R3
2950     IF W5<0 THEN 3020
2955     Q5=R3
2960     I7=Q5
2965     P6=W5
2970     IF D2<0 THEN 3175
2975     IF Q9>2 THEN 3090
2980     IF I8>10*P3 THEN 3090
2985     IF (P3>.01) OR (P4=1) THEN 3005
2990     IF ABS(W5)>.1*P3 THEN 3005
```

```
2995    P4=D4=P3-2*W5
3000    GOTO 3010
3005    P4=D4=D4^D3
3010    D2=1
3015    GOTO 2870
3020    Q4=R3
3025    I6=Q4
3030    P5=W5
3035    D5=2-D3
3040    IF D2>0 THEN 3175
3045    IF (Q9>4) OR (I8<.01*P3) THEN 3090
3050    IF (P3>.01) OR (P4=0) THEN 3070
3055    IF ABS(W5)>.1*P3 THEN 3070
3060    P4=D4=P3-2*W5
3065    GOTO 3075
3070    P4=D4=D4^D5
3075    D2=-1
3080    GOTO 2870
3085    ! *************ESTIMATE FOR R3 USING CUBE***************
3090    P11=P1=FNEllrc(Hx,Hy,Hz,Sx,Sy,Sz,P3,0,Q5,T6)
3095    R4=R3
3100    IF T6<=Ve THEN 3110     ! USED FOR MAX. OF T6.
3105    Ve=T6                   ! USED FOR MAX. OF T6.
3110    IF P1<=Q4 THEN 3130
3115    R3=P1
3120    GOSUB 3740
3125    IF P4>=0 THEN RETURN P1
3130    P1=1.732051*P1
3135    IF P1<Q5 THEN Q5=P1
3140    IF (P3<=.2) AND (P11=Q4) THEN 3160
3145    R4=R3
3150    R3=.5*(Q4+Q5)
3155    GOSUB 3740        ! COMPUTES ELLCOV(R3) AND MAKES PROPER STORAGES.
3160    IF ABS(P4)<=X8 THEN 3660
3165    IF (I8<=5E-9) OR (ABS(Q5-Q4)>.1*R3) THEN 3145
3170    IF (ABS(R3-R4)<=.005*R3) OR (ABS(P6-P5)<=.001*P3) THEN 3190
3175    R4=R3
3180    R3=Q4-(Q4-Q5)/(P5-P6)*P5      ! REGULA-FALSI
3185    GOTO 3200
3190    R4=R3
3195    R3=.5*(Q4+Q5)
3200    GOSUB 3740
3205    IF ABS(P4)<=X8 THEN 3660
3210    IF ABS(Q5-Q4)<=.000001*R3 THEN 3315
3215    ! ***************STEPPING PROCEDURE****************
3220    I9=.2*(Q5-Q4)
3225    P7=P5
3230    R3=Q4
3235    IF (P3<.2) OR (P6+P5>0) THEN 3255
3240    I9=-I9
3245    P7=P6
3250    R3=Q5
```

```
3255    FOR J2=1 TO 4
3260      R4=R3
3265      R3=R3+I9
3270      T9=3270
3275      GOSUB 3740
3280      IF ABS(P4)<=X8 THEN 3660
3285      IF Q9>30 THEN 3725
3290      IF P4*P7>0 THEN 3305
3295      IF (P3<.2) AND (ABS(I8/P3-5.05)>4.95) THEN 3220
3300      GOTO 3310
3305    NEXT J2
3310    IF (ABS(I7-I6)>.005*R3) OR (ABS(P6-P5)>.001*P3) OR (I8>5E-9) THEN 3335
3315    R3=.5*(Q4+Q5)  ! R3=(2*Q4+Q5)/3
3320    GOTO 3345
3325    ! ***************KING'S PROCEDURE*****************
3330    R4=R3
3335    R3=Q5-P6*(Q5-Q4)/(P6-P5)
3340    T9=3400
3345    IF Q9>=30 THEN 3725
3350    P4=FNEllcov(R3,Hx,Hy,Hz,Sx,Sy,Sz,K0,K2,S)
3355    IF P4<>0 THEN Q9=Q9+1
3360    P4=P4-P3
3365    IF ABS(P4)<=X8 THEN 3660
3370    IF P4>0 THEN 3390
3375    I6=R3
3380    I3=-P4
3385    GOTO 3400
3390    I7=R3
3395    I4=P4
3400    IF I7<=I6 THEN 3555
3405    IF ABS(I7-I6)<=.0000005*R3 THEN 3660
3410    IF (ABS(R4-R3)<=.000001*R3) AND (ABS(P4)<=C2) THEN 3660
3415    IF Q9>20 THEN 3605
3420    IF P4*P6>0 THEN 3455
3425    A6=Q5
3430    Q5=Q4
3435    Q4=A6
3440    A6=P5
3445    P5=P6
3450    P6=A6
3455    P5=P5*P6/(P6+P4)
3460    P6=P4
3465    Q5=R3
3470    R4=R3
3475    R3=Q5-(Q4-Q5)*P6/(P5-P6)
3480    T9=3480
3485    IF Q9>=30 THEN 3725
3490    P4=FNEllcov(R3,Hx,Hy,Hz,Sx,Sy,Sz,K0,K2,S)
3495    IF P4<>0 THEN Q9=Q9+1
3500    P4=P4-P3
3505    IF (ABS(P4)<=X8) AND (P5<>-P3) AND (P6<>1-P3) THEN 3660
3510    IF P4>0 THEN 3530
3515    I6=R3
3520    I3=-P4
3525    GOTO 3540
```

```
3530    I7=R3
3535    I4=P4
3540    IF ABS(I7-I6)<=.0000005*R3 THEN 3660
3545    IF Q9>20 THEN 3605
3550    IF I7>I6 THEN 3570
3555    R3=I6
3560    IF ABS(I4)<ABS(I3) THEN R3=I7
3565    RETURN R3
3570    IF (ABS(R4-R3)<=.000001*R3) AND (ABS(P4)<=C2) THEN 3660
3575    IF P4*P6>=0 THEN 3455
3580    Q4=Q5
3585    Q5=R3
3590    P5=P6
3595    P6=P4
3600    GOTO 3330
3605    Q4=I6
3610    Q5=I7
3615    R4=R3
3620    R3=.5*(Q4+Q5)
3625    T9=3685
3630    GOSUB 3740
3635    IF ABS(Q5-Q4)<=.000001*R3 THEN 3660
3640    IF ABS(P4)<=X8 THEN 3660
3645    IF Q9>30 THEN 3725
3650    GOTO 3615
3655    ! ***************EXITS****************8
3660    IF P4*P6<0 THEN 3680
3665    IF P4*P5>0 THEN 3700
3670    R3=R3-(Q4-R3)*P4/(P5-P4)
3675    RETURN R3
3680    R3=R3-(Q5-R3)*P4/(P6-P4)
3685    RETURN R3
3690    R3=.5*(Q4+Q5)
3695    RETURN R3
3700    IF P4<0 THEN 3715
3705    R3=MIN(R3,R4)
3710    RETURN R3
3715    R3=MAX(R3,R4)
3720    RETURN R3
3725    E5=1      !R3 GIVEN AFTER 30 ITERATIONS.
3730    RETURN R3
3735    ! ********* A SUBROUTINE FOR P4**************
3740    P4=FNEllcov(R3,Hx,Hy,Hz,Sx,Sy,Sz,K0,K2,S)
3745    IF P4>0 THEN Q9=Q9+1
3750    I8=P4
3755    P4=P4-P3
3760    IF P4>0 THEN 3790
3765    Q4=R3
3770    I6=R3
3775    P5=P4
3780    I3=-P4
3785    RETURN
```

```
3790    Q5=R3
3795    I7=R3
3800    P6=P4
3805    I4=P4
3810    RETURN
3815  FNEND
3820  ! ********************ELLCOV********************************
3825 ! THIS PROGRAM IS CALLED "ELLCOV". (X,Y,Z) IS AN ELEMENT OF A CARTESIAN
3826 ! COORDINATE SYSTEM. ELLCOV RETURNS THE PROBABILITY OF A SHOT FALLING UNDER
3830 ! AN ELLIPSOIDAL NORMAL DISTRIBUTION, IN A SPHERE WITH CENTER (Hx,Hy,Hz)
3831 ! AND RADIUS R3. THE DISTRIBUTION HAS MEAN ZERO AND STANDARD DEVIATIONS Sx,
3835 ! Sy,Sz IN THE X,Y,Z DIRECTIONS, RESPECTIVELY. THE INPUT PARAMETERS ARE
3836 ! R3,Hx,Hy,Hz,Sx,Sy,Sz,K0,K2,S, WHERE K0=Hx*Hx+Hy*Hy+Hz*Hz, K2=SQR(K0)
3837 ! AND S=MAX(Sx,Sy,Sz).
3840 ! THE OUTPUT P IS GENERALLY ACCURATE TO AT LEAST 6-DECIMAL DIGITS.
3841 ! SUBROUTINES USED ARE CIRCV, DAWS, DXDAWS, DXERF, DXERF3, ELLCV, EQSIG,
3845 ! ERF, ERF3, ERFC, ERFC1, FN.
3850  DEF FNEllcov(R3,Hx,Hy,Hz,Sx,Sy,Sz,K0,K2,S)
3855    COM X(43),Y(43),A6(10),B6(10),Q9,V,A7,P4,Ve,T6,P1,T9
3860    C5=6.02738    !5.6123
3865    C6=.398942280402     !1/SQR(2*PI)
3870    C7=.707106781187     !SQR(.5)
3875    C8=1.41421356237
3880    F2=4.262
3885    C10=.999999998334    !ERF(F2)
3890    B1=.564189583548     !1/SQR(PI)
3895    A4=.797884560803     !SQR(2/PI)
3900    Z4=.000000025
3905    H=ABS(Hx)
3910    K=ABS(Hy)
3915    L=ABS(Hz)
3920    S1=Sx
3925    S2=Sy
3930    S3=Sz
3935    P4=0
3940    T5=0
3945    H1=1.5*Sx*Sy*Sz*Z4/C6
3950    K7=R3*R3*R3
3955    IF K7<=H1 THEN RETURN P4
3960    A3=(K2-R3)*C7/S
3965    A5=EXP(-A3*A3)
3970    IF K2<R3 THEN 3985
3975    IF K7*A5<=H1 THEN RETURN P4
3980    GOTO 4010
3985    P4=FNEqsig(R3,K0,K2,S,A3,A5,A4,C7)  ! ELLCOV FOR EQUAL SIGMAS
3990    V=-3
3995    IF P4<1-Z4 THEN 4010
4000    P4=1
4005    RETURN P4
4010    IF (Sx<>Sy) AND (Sx<>Sz) AND (Sy<>Sz) THEN 4305
4015    IF (Sx<>Sy) OR (Sy<>Sz) THEN 4035
4020    IF R3>K2 THEN RETURN P4
4025    P4=FNEqsig(R3,K0,K2,S,A3,A5,A4,C7)
4030    RETURN P4
```

```
4035    IF Sx=Sy THEN 4090
4040    IF Sx=Sz THEN 4070
4045    S3=Sx
4050    S1=Sz
4055    L=ABS(Hx)
4060    H=ABS(Hz)
4065    GOTO 4090
4070    S2=Sz
4075    S3=Sy
4080    K=ABS(Hz)
4085    L=ABS(Hy)
4090    IF H+K<>0 THEN 4280
4095    ! ************S1=S2 AND H=K=0*********************
4100    L1=ABS((S1*S1-S3*S3)/(S1*S1))
4105    L2=SQR(L1)
4110    L4=(L+R3)*C7/S3
4115    L5=(L-R3)*C7/S3
4120    IF L5>F2 THEN RETURN P4
4125    W9=-1
4130    J3=EXP(-L5*L5)
4135    S0=C7/(S3*L2)
4140    IF L=0 THEN 4240
4145    IF R3<L THEN 4160
4150    P4=FNErf(L4)-FNErf3(L5,J3)
4155    GOTO 4170
4160    W9=EXP(-2*R3*L/(S3*S3))
4165    P4=J3*(FNErfc1(L5,1)-W9*FNErfc1(L4,1))
4170    F1=(L+L1*R3)*S0
4175    S0=(L-L1*R3)*S0
4180    IF S1<S3 THEN 4220
4185    V=-5
4190    IF S0<0 THEN 4210
4195    IF W9<0 THEN W9=EXP(-2*L*R3/(S3*S3))
4200    P4=.5*(P4-J3/L2*(FNErfc1(S0,1)-W9*FNErfc1(F1,1)))
4205    RETURN P4
4210    P4=.5*(P4-1/L2*EXP(-.5*(R3*R3-L*L/L1)/(S1*S1))*(FNErf(F1)-FNErf(S0)))
4215    RETURN P4
4220    V=-6
4225    IF W9<0 THEN W9=EXP(-2*L*R3/(S3*S3))
4230    P4=.5*P4-B1*J3/L2*(FNDaws(F1)-W9*FNDaws(S0))
4235    RETURN P4
4240    IF S3>S1 THEN 4260
4245    V=-5.5
4250    P4=FNErf3(L4,J3)-L4*EXP(-.5*R3*R3/(S1*S1))*FNDxerf3(L4*L2,-1)
4255    RETURN P4
4260    V=-6.5
4265    P4=FNErf3(L4,J3)-2*B1*J3*L4*FNDxdaws(L4*L2)
4270    RETURN P4
4275    ! *****************S1=S2*****************
4280    V=1
4285    D=SQR(H*H+K*K)
4295    GOTO 4415
```

```
4300    ! ****************H=K=0*******************
4305    IF (H+K=0) AND (ABS(S1/S2-50000.000005)<49999.999995) THEN 4385
4310    IF (H+L=0) AND (ABS(S1/S3-50000.000005)<49999.999995) THEN 4355
4315    IF (K+L<>0) OR (ABS(S2/S3-50000.000005)>49999.999995) THEN 4510
4320    L1=S1
4325    S1=S3
4330    S3=L1
4335    L1=H
4340    H=L
4345    L=L1
4350    GOTO 4385
4355    L1=S2
4360    S2=S3
4365    S3=L1
4370    L1=K
4375    K=L
4380    L=L1
4385    IF S1>S2 THEN 4405
4390    W1=S1
4395    S1=S2
4400    S2=W1
4405    D=S2/S1
4410    V=0
4415    X1=C7/S3
4420    A5=X1*(L-R3)
4425    IF A5>F2 THEN RETURN P4
4430    W9=1
4435    IF A5>-F2 THEN 4475
4440    A5=-F2
4445    B5=F2
4450    X2=L*X1
4455    IF X2>=F2 THEN 4825
4460    W9=0
4465    B5=X2
4470    GOTO 4825
4475    B5=X1*L
4480    IF B5>F2 THEN 4495
4485    W9=0
4490    GOTO 4825
4495    B5=F2
4500    GOTO 4825
4505    ! **************GENERAL CASE***********************
4510    V=2
4515    H(1)=H
4520    H(2)=K
4525    H(3)=L
4530    S(1)=S1
4535    S(2)=S2
4540    S(3)=S3
4545    ! **************FIX ORDER INTEGRATION***************
4550    T(1)=MAX(S1,S2)
4555    T(2)=MAX(S2,S3)
4560    T(3)=MAX(S3,S1)
```

```
4565    W8(1)=S1/S2
4570    IF W8(1)>1 THEN W8(1)=1/W8(1)
4575    W8(2)=S2/S3
4580    IF W8(2)>1 THEN W8(2)=1/W8(2)
4585    W8(3)=S3/S1
4590    IF W8(3)>1 THEN W8(3)=1/W8(3)
4595    J=3
4600    IF W8(1)>W8(2) THEN 4645
4605    IF W8(1)<>W8(2) THEN 4620
4610    IF T(1)<T(2) THEN J=1
4615    GOTO 4670
4620    J=1
4625    IF W8(2)>W8(3) THEN 4670
4630    IF W8(2)<>W8(3) THEN 4660
4635    IF T(2)<T(3) THEN J=2
4640    GOTO 4670
4645    IF W8(1)>W8(3) THEN 4670
4650    IF W8(1)<>W8(3) THEN 4660
4655    IF T(1)>T(3) THEN 4670
4660    J=2
4665    ! ***********FIND LIMITS OF INTEGRATION**************
4670    X1=C7/S(J)
4675    A5(J)=(H(J)-R3)*X1
4680    IF A5(J)>=F2 THEN RETURN P4
4685    W8(J)=1
4690    IF A5(J)>-F2 THEN 4730
4695    A5(J)=-F2
4700    B5(J)=F2
4705    X2=H(J)*X1
4710    IF X2>=F2 THEN 4755
4715    B5(J)=X2
4720    W8(J)=0
4725    GOTO 4755
4730    B5(J)=H(J)*X1
4735    IF B5(J)>F2 THEN 4750
4740    W8(J)=0
4745    GOTO 4755
4750    B5(J)=F2
4755    IF J=3 THEN 4810
4760    IF J=2 THEN 4790
4765    H=H(3)
4770    L=H(1)
4775    S1=S(3)
4780    S3=S(1)
4785    GOTO 4810
4790    L=H(2)
4795    S3=S(2)
4800    K=H(3)
4805    S2=S(3)
4810    A5=A5(J)
4815    B5=B5(J)
4820    W9=W8(J)
```

```
4822    ! ************SPECIAL CASE*************
4825    Py=0
4830    IF A5>-F2 THEN 4915
4835    K7=L-C8*S3*B5
4840    T8=L-C8*S3*A5
4845    K7=R3*R3-K7*K7
4850    T8=R3*R3-T8*T8
4855    IF (K7<0) OR (T8<0) THEN 4915
4860    K7=SQR(K7)
4865    T8=SQR(T8)
4870    IF ABS(T8-K7)>K7*1E-8 THEN 4915
4875    V=-9+V
4880    IF V<>-7 THEN 4895
4885    P4=C10*FNEllcv(.5*(K7+T8),H,K,S1,S2)
4890    RETURN P4
4895    P4=C10*FNCircv(.5*(K7+T8),D,S1,V)
4900    RETURN P4
4910    ! *********START OF GAUSSIAN INTEGRATION***********
4915    K7=MAX(S1,S2)
4920    T8=C5*K7+SQR(H*H+K*K)
4925    T8=R3*R3-T8*T8
4930    IF T8<0 THEN 4985
4935    A3=SQR(T8)
4940    T8=(L-A3)/(C8*S3)
4945    IF T8>=B5 THEN 4985
4950    B5=T8
4955    K7=(L+A3)/(C8*S3)
4960    IF K7<4.4 THEN 4975
4965    Py=P4=.5*FNErfc(T8)
4970    GOTO 4985
4975    Py=P4=.5*(FNErf(K7)-FNErf(T8))
4980    IF (ABS(B5-A5)<=ABS(B5+A5)*1E-10) OR (B5<=A5) THEN RETURN P4
4985    A3=A5
4990    FOR I=-12 TO 12
4995      IF I=0 THEN 5020
5000      T5=.5*((B5-A5)*SGN(I)*X(31+ABS(I))+B5+A5)
5005      X2=FNFn(T5,R3,H,K,L,S1,S2,S3,V,D,W9)
5010      IF X2>1E-10 THEN 5025
5015      A3=T5
5020    NEXT I
5025    IF X2<1E-7 THEN 5115
5030    A5=A8=A3
5035    IF (T5-A5)*X2<=1E-9 THEN 5115
5040    T8=A5
5045    K7=(T5-A5)*(1-X(43))        !.5*K7=.5*(T5+A5-(T5-A5)*X(43))-A5
5050    IF (ABS(K7)<=ABS(A5)*1E-10) OR (K7<=0) THEN 5115
5055    T8=T8+K7
5060    X3=FNFn(T8,R3,H,K,L,S1,S2,S3,V,D,W9)   ! COMPUTATION OF INTEGRAND
5065    IF X3>1E-10 THEN 5080
5070    A8=T8
5075    GOTO 5055
5080    A3=A5=A8
5085    IF X3<=1E-7 THEN 5115
5090    IF (T5-A5)*X2<=1E-9 THEN 5115
```

```
5095 ! ************GAUSSIAN RULE. Pz GIVES THE INTEGRAL FROM
5096 !                A5 TO T5 OF THE INTEGRAND F(X)=FNFn(X).***********
5100   CALL Gquad(A5,T5,Ierr,1,Pz,E,12,N2,R3,H,K,L,S1,S2,S3,V,D,W9)
5105   A5=T5
5110   GOTO 5130
5115   Pz=0
5120   A5=A3
5125 ! ************GAUSSIAN RULE. P4 GIVES THE INTEGRAL FROM
5126 !                A5 TO B5 OF THE INTEGRAND F(X)=FNFn(X).***********
5130   CALL Gquad(A5,B5,Ierr,1,P4,E,12,N2,R3,H,K,L,S1,S2,S3,V,D,W9)
5135   P4=B1*(P4+Pz)+Py
5140   IF P4>1 THEN P4=1
5145   RETURN P4
5150 FNEND
5155 ! *****************EQSIG**************************************
5160 ! FNEqsig COMPUTES ELLCOV FOR EQUAL SIGMAS.
5165 ! SUBROUTINES NEEDED: ERF, ERF3, ERFC1, REXP.
5170 DEF FNEqsig(R3,K0,K2,S,A3,A5,A4,C7)
5175   IF K0<>0 THEN 5195
5180   V=-3
5185   P4=FNErf3(-A3,A5)-A4*R3*A5/S
5190   RETURN P4
5195   K1=(R3+K2)*C7/S
5200   P5=2*R3*K2/(S*S)
5205   Q5=FNRexp(-P5)
5210   IF R3>K2 THEN 5230
5215   Q4=FNErfc1(K1,1)
5220   P4=A5*(FNErfc1(A3,1)-Q4+P5*Q5*Q4)
5225   GOTO 5235
5230   P4=FNErf(K1)-FNErf3(A3,A5)
5235   P4=.5*P4-A4*A5*R3*Q5/S
5240   V=-3.5
5245   RETURN P4
5250 FNEND
5255 ! *********************GAMINV*********************************
5260 ! SUBPROGRAM FUNCTION GAMINV(A,X,X0,P,Q,IERR)
5265 ! INVERSE INCOMPLETE GAMMA RATIO FUNCTION
5270 ! GIVEN A>0 AND NONNEGATIVE P AND Q WHERE P+Q=1, THEN X IS OBTAINED
5271 ! WHERE P(A,X)=P AND Q(A,X)=Q.
5275 ! SCHRODER ITERATION IS EMPLOYED. THE ROUTINE ATTEMPTS TO COMPUTE X
5276 ! TO 10 SIGNIFICANT DIGITS.
5280 ! X IS A VARIABLE. IF P=0 THEN X IS ASSIGNED THE VALUE 0, AND IF Q=0
5281 ! THEN X IS SET TO THE LARGEST FLOATING POINT NUMBER AVAILABLE.
5285 ! OTHERWISE, GAMINV ATTEMPTS TO OBTAIN A SOLUTION FOR P(A,X)=P AND
5286 ! Q(A,X)=Q. IF THE ROUTINE IS SUCCESSFUL THE SOLUTION IS IN X.
5290 ! X0 IS AN OPTIONAL INITIAL APPROXIMATION FOR X. IF THE USER DOES NOT
5291 ! WISH TO GIVE AN INITIAL APPROXIMATION, THEN SET X0 <= 0.
5295 ! Ierr IS A VARIABLE THAT REPORTS THE STATUS OF THE RESULTS. WHEN THE
5296 ! ROUTINE TERMINATES, Ierr HAS ONE OF THE FOLLOWING VALUES:
5300 !   Ierr=0     THE SOLUTION WAS OBTAINED. ITERATION WAS NOT USED.
5305 !   Ierr>0     THE SOLUTION WAS OBTAINED. Ierr ITERATIONS WERE DONE.
```

```
5310 !    Ierr=-2     (INPUT ERROR) A<=0.
5315 !    Ierr=-3     X IS TOO SMALL. X=0 GIVEN.
5320 !    Ierr=-4     (INPUT ERROR) P+Q # 1.
5325 !    Ierr=-6     20 ITERATIONS DONE. MOST RECENT VALUE OF X GIVEN.
5326 !                CANNOT OCCUR IF X0 <= 0.
5330 !    Ierr=-7     ITERATION FAILED. NO VALUE FOR X IS GIVEN. THIS MAY
5331 !                OCCUR WHEN X IS APPROXIMATELY 0.
5335 !    Ierr=-8     A VALUE FOR X HAS BEEN OBTAINED, BUT GAMINV IS NOT
5336 !                CERTAIN OF ITS ACCURACY. ITERATION CANNOT BE DONE
5340 !                IN THIS CASE. IF X0 <= 0, THIS CAN OCCUR ONLY WHEN P
5341 !                OR Q IS APPROXIMATELY 0. IF X0 > 0 THEN THIS CAN OCCUR
5345 !                WHEN A IS EXCEEDINGLY CLOSE TO X AND A IS VERY LARGE
5346 !                (SAY A > 1E20).
5350 ! CODE TAKEN FROM THE FORTRAN VERSION BY AL MORRIS.
5355 ! SUBROUTINES NEEDED: GAMMAI, GRATIO, ALNREL, GAMLN, RCOMP, GAMLN1
5360 DEF FNGaminv(A,X,X0,P,Q,Ierr)
5365   C=.577215664902
5370   Ln10=2.302585
5375   A0=3.31125922109
5380   A1=11.6616720289
5385   A2=4.28342155967
5390   A3=.213623493716
5395   B1=6.61053765625
5400   B2=6.4069159776
5405   B3=1.27364489782
5410   B4=3.61170810188E-2
5415   Eps0(1)=1E-10
5420   Eps0(2)=1E-8
5425   Amin(1)=500
5430   Amin(2)=100
5435   Bmin(1)=1E-28
5440   Bmin(2)=1E-13
5445   Dmin(1)=1E-6
5450   Dmin(2)=1E-4
5455   Emin(1)=2E-3
5460   Emin(2)=6E-3
5465   Tol=1E-5
5470   E=1E-12
5475   Xmin=1E-99
5480   Xmax=9.999999999E99
5485   X=0
5490   IF A<=0 THEN 6405
5495   T=-.5+P+Q-.5
5500   E2=E+E
5505   IF ABS(T)>10*E THEN 6425
5510   Ierr=0
5515   IF P=0 THEN RETURN X
5520   IF Q=0 THEN 6365
5525   IF A=1 THEN 6375
5530   Amax=4E-11/(E*E)
5535   Iop=1
5540   IF E>1E-10 THEN Iop=2
5545   Eps=Eps0(Iop)
5550   Xn=X0
5555   IF X0>0 THEN 6070
```

```
5560  ! ****INITAIL APPROXIMATION Xn OF X FOR A<1.****
5565    IF A>1 THEN 5795
5570    G=1/FNGammai(.5+A+.5)
5575    Qg=Q*G
5580    IF Qg=0 THEN 6470
5585    B=Qg/A
5590    IF Qg>.6*A THEN 5735
5595    IF (A>=.3) OR (B<.35) THEN 5620
5600    T=EXP(-(B+C))
5605    U=T*EXP(T)
5610    Xn=T*EXP(U)
5615    GOTO 6070
5620    IF B>=.45 THEN 5735
5625    IF B=0 THEN 6470
5630    Y=-LOG(B)
5635    S=.5-A+.5
5640    Z=LOG(Y)
5645    T=Y-S*Z
5650    IF B<.15 THEN 5665
5655    Xn=Y-S*LOG(T)-LOG(1+S/(T+1))
5660    GOTO 6220
5665    IF B<=.01 THEN 5685
5670    U=((T+2*(3-A))*T+(2-A)*(3-A))/((T+(5-A))*T+2)
5675    Xn=Y-S*LOG(T)-LOG(U)
5680    GOTO 6220
5685    C1=-S*Z
5690    C2=-S*(1+C1)
5695    C3=S*((.5*C1+(2-A))*C1+(2.5-1.5*A))
5700    C4=-S*(((C1/3+(2.5-1.5*A))*C1+((A-6)*A+7))*C1+((11*A-46)*A+47)/6)
5705    C5=-S*((((-C1/4+(11*A-17)/6)*C1+((-3*A+13)*A-13))*C1+.5*(((2*A-25)*A+72)
*A-61))*C1+(((25*A-195)*A+477)*A-379)/12)
5710    Xn=(((C5/Y+C4)/Y+C3)/Y+C2)/Y+C1+Y
5715    IF A>1 THEN 6190
5720    IF B>Bmin(Iop) THEN 6220
5725    X=Xn
5730    RETURN X
5735    IF B*Q>1E-8 THEN 5750
5740    Xn=EXP(-(Q/A+C))
5745    GOTO 5770
5750    IF P<=.9 THEN 5765
5755    Xn=EXP((FNAlnrel(-Q)+FNGamln1(A))/A)
5760    GOTO 5770
5765    Xn=EXP(LOG(P*G)/A)
5770    IF Xn=0 THEN 6415
5775    T=.5+(.5-Xn/(.5+A+.5))
5780    Xn=Xn/T
5785    GOTO 6070
5790  ! *****INITIAL APPROXIMATION FOR A > 1.*****
5795    IF Q<=.5 THEN 5810
5800    W=LOG(P)
5805    GOTO 5815
5810    W=LOG(Q)
5815    T=SQR(-2*W)
5820    S=T-(((A3*T+A2)*T+A1)*T+A0)/((((B4*T+B3)*T+B2)*T+B1)*T+1)
```

```
5825    IF Q>.5 THEN S=-S
5830    Rta=SQR(A)
5835    S2=S*S
5840    Xn=A+S*Rta+(S2-1)/3+S*(S2-7)/(36*Rta)-((3*S2+7)*S2-16)/(810*A)+S*((9*S2+
256)*S2-433)/(38880*A*Rta)
5845    Xn=MAX(Xn,0)
5850    IF A<Amin(Iop) THEN 5870
5855    X=Xn
5860    D=.5-X/A+.5
5865    IF ABS(D)<=Dmin(Iop) THEN RETURN X
5870    IF P<=.5 THEN 5940
5875    IF Xn<3*A THEN 6220
5880    Y=-(W+FNGamln(A))
5885    IF A<2 THEN 5920
5890    GOTO 6220
5895    D=MAX(2,A*(A-1))
5900    IF Y<Ln10*D THEN 5920
5905    S=.5-A+.5
5910    Z=LOG(Y)
5915    GOTO 5685
5920    T=-.5+A-.5
5925    Xn=Y+T*LOG(Xn)-FNAlnrel(-T/(Xn+1))
5930    Xn=Y+T*LOG(Xn)-FNAlnrel(-T/(Xn+1))
5935    GOTO 6220
5940    Ap1=A+1
5945    IF Xn>.7*Ap1 THEN 6075
5950    W=W+FNGamln(Ap1)
5955    IF Xn>.15*Ap1 THEN 6010
5960    Ap2=A+2
5965    Ap3=A+3
5970    X=EXP((W+X)/A)
5975    X=EXP((W+X-LOG(1+X/Ap1*(1+X/Ap2)))/A)
5980    X=EXP((W+X-LOG(1+X/Ap1*(1+X/Ap2)))/A)
5985    X=EXP((W+X-LOG(1+X/Ap1*(1+X/Ap2*(1+X/Ap3))))/A)
5990    Xn=X
5995    IF Xn>.01*Ap1 THEN 6010
6000    IF Xn<=Emin(Iop)*Ap1 THEN RETURN X
6005    GOTO 6075
6010    Apn=Ap1
6015    T=Xn/Apn
6020    Sum=1+T
6025    Apn=Apn+1
6030    T=T*Xn/Apn
6035    Sum=Sum+T
6040    IF T>1E-4 THEN 6025
6045    T=W-LOG(Sum)
6050    Xn=EXP((Xn+T)/A)
6055    Xn=Xn*(1-(A*LOG(Xn)-Xn-T)/(A-Xn))
6060    GOTO 6075
6065  ' ****SCHRODER ITERATION USING P--REF. VOL.1 OF HENRICI, 1974, (P.529),
6070    IF P>.5 THEN 6220
6075    IF P<=1E10*Xmin THEN 6455
6080    Am1=-.5+A-.5
6085    IF A<=Amax THEN 6100
```

```
6090    D=.5-Xn/A+.5
6095    IF ABS(D)<=E2 THEN 6455
6100    IF Ierr>=20 THEN 6435
6105    Ierr=Ierr+1
6110    CALL Gratio(A,Xn,Pn,Qn,0)
6115    IF (Pn=0) OR (Qn=0) THEN 6455
6120    R=FNRcomp(A,Xn)
6125    IF R=0 THEN 6455
6130    T=(Pn-P)/R
6135    W=.5*(Am1-Xn)
6140    IF (ABS(T)<=.1) AND (ABS(W*T)<=.1) THEN 6165
6145    X=Xn*(1-T)
6150    IF X<=0 THEN 6445
6155    D=ABS(T)
6160    GOTO 6190
6165    H=T*(1+W*T)
6170    X=Xn*(1-H)
6175    IF X<=0 THEN 6445
6180    IF (ABS(W)>=1) AND (ABS(W)*T*T<=Eps) THEN RETURN X
6185    D=ABS(H)
6190    Xn=X
6195    IF D>Tol THEN 6085
6200    IF D<=Eps THEN RETURN X
6205    IF ABS(P-Pn)<=Tol*P THEN RETURN X
6210    GOTO 6085
6215  ! ****SCHRODER ITERATION USING Q******
6220    IF Q<=1E10*Xmin THEN 6455
6225    Am1=-.5+A-.5
6230    IF A<=Amax THEN 6245
6235    D=.5-Xn/A+.5
6240    IF ABS(D)<=E2 THEN 6455
6245    IF Ierr>=20 THEN 6435
6250    Ierr=Ierr+1
6255    CALL Gratio(A,Xn,Pn,Qn,0)
6260    IF (Pn=0) OR (Qn=0) THEN 6455
6265    R=FNRcomp(A,Xn)
6270    IF R=0 THEN 6455
6275    T=(Q-Qn)/R
6280    W=.5*(Am1-Xn)
6285    IF (ABS(T)<=.1) AND (ABS(W*T)<=.1) THEN 6310
6290    X=Xn*(1-T)
6295    IF X<=0 THEN 6445
6300    D=ABS(T)
6305    GOTO 6335
6310    H=T*(1+W*T)
6315    X=Xn*(1-H)
6320    IF X<=0 THEN 6445
6325    IF (ABS(W)>=1) AND (ABS(W)*T*T<=Eps) THEN RETURN X
6330    D=ABS(H)
6335    Xn=X
6340    IF D>Tol THEN 6230
6345    IF D<=Eps THEN RETURN X
6350    IF ABS(Q-Qn)<=Tol*Q THEN RETURN X
6355    GOTO 6230
```

```
6360  ! *****SPECIAL CASES*******
6365    X=Xmax
6370    RETURN X
6375    IF Q<.9 THEN 6390
6380    X=-FNAlnrel(-P)
6385    RETURN X
6390    X=-LOG(Q)
6395    RETURN X
6400  ! *****ERROR RETURN********
6405    Ierr=-2
6410    RETURN X
6415    Ierr=-3
6420    RETURN X
6425    Ierr=-4
6430    RETURN X
6435    Ierr=-6
6440    RETURN X
6445    Ierr=-7
6450    RETURN X
6455    X=Xn
6460    Ierr=-8
6465    RETURN X
6470    X=Xmax
6475    Ierr=-8
6480  FNEND
6485    ! ******************GRATIO***********************************
6490    REM SUBROUTINES NEEDED: ERF,ERFC1, GAM1, GAMMAI, RLOG, REXP.
6495  SUB Gratio(A,X,Ans,Qans,Ind)
6500    DIM A(3),B(3),E(3),X(3)
6505    DIM W(20),D(13),D1(12),D2(10),D3(8),D4(6),D5(4),D6(2)
6510    E9=5E-12
6515    E=5E-13
6520    A(1)=5E-13
6525    A(2)=5E-7
6530    A(3)=5E-4
6535    B(1)=20
6540    B(2)=14
6545    B(3)=10
6550    E(1)=2.5E-4
6555    E(2)=.025
6560    E(3)=.14
6565    X(1)=31
6570    X(2)=17
6575    X(3)=9.7
6580    Ln10=2.30258509299
6585    Rpi=.398942280401
6590    Spi=1.77245385091
6595    Th=.333333333333
6600    D(1)=8.33333333333E-2
6605    D(2)=-1.48148148148E-2
6610    D(3)=1.15740740741E-3
6615    D(4)=3.52733686067E-4
6620    D(5)=-1.78755144033E-4
6625    D(6)=3.91926317852E-5
```

```
6630    D(7)=-2.18544851068E-6
6635    D(8)=-1.85406221072E-6
6640    D(9)=8.29671134095E-7
6645    D(10)=-1.76659527368E-7
6650    D(11)=6.70785354340E-9
6655    D(12)=1.02618097842E-8
6660    D(13)=-4.38203601845E-9
6665    D10=-1.85185185185E-3
6670    D1(1)=-3.47222222222E-3
6675    D1(2)=2.64550264550E-3
6680    D1(3)=-9.90226337449E-4
6685    D1(4)=2.05761316872E-4
6690    D1(5)=-4.01877572016E-7
6695    D1(6)=-1.80985503345E-5
6700    D1(7)=7.64916091608E-6
6705    D1(8)=-1.61209008946E-6
6710    D1(9)=4.64712780281E-9
6715    D1(10)=1.37863344692E-7
6720    D1(11)=-5.75254560352E-8
6725    D1(12)=1.19516285998E-8
6730    D20=4.1335978836E-3
6735    D2(1)=-2.68132716049E-3
6740    D2(2)=7.71604938272E-4
6745    D2(3)=2.00938786008E-6
6750    D2(4)=-1.07366532264E-4
6755    D2(5)=5.29234488291E-5
6760    D2(6)=-1.27606351886E-5
6765    D2(7)=3.4235787341E-8
6770    D2(8)=1.37219573091E-6
6775    D2(9)=-6.29899213838E-7
6780    D2(10)=1.42806142061E-7
6785    D30=6.49434156379E-4
6790    D3(1)=2.29472093621E-4
6795    D3(2)=-4.69189494395E-4
6800    D3(3)=2.67720632063E-4
6805    D3(4)=-7.56180167188E-5
6810    D3(5)=-2.39650511387E-7
6815    D3(6)=1.10826541153E-5
6820    D3(7)=-5.67495282699E-6
6825    D3(8)=1.42309007324E-6
6830    D40=-8.61888290917E-4
6835    D4(1)=7.8403922172E-4
6840    D4(2)=-2.99072480303E-4
6845    D4(3)=-1.46384525788E-6
6850    D4(4)=6.64149821547E-5
6855    D4(5)=-3.96836504718E-5
6860    D4(6)=1.13757269707E-5
6865    D50=-3.36798553366E-4
6870    D5(1)=-6.97281375837E-5
6875    D5(2)=2.77275324496E-4
6880    D5(3)=-1.99325705162E-4
6885    D5(4)=6.79778047794E-5
6890    D60=5.31307936464E-4
6895    D6(1)=-5.92166437354E-4
6900    D6(2)=2.70878209672E-4
6905    D70=3.44367606892E-4
```

```
6910    IF (A<0) OR (X<0) THEN 8055
6915    IF (A=0) AND (X=0) THEN 8055
6920    IF A*X=0 THEN 8020
6925    I=Ind+1
6930    IF (I<>1) AND (I<>2) THEN I=3
6935    Acc=MAX(A(I),E)
6940    E0=E(I)
6945    X0=X(I)
6950  ! *******SELECT APPROPRIATE ALGORITHM******
6955    IF A>=1 THEN 6995
6960    IF A=.5 THEN 7980
6965    IF X<1.1 THEN 7370
6970    T1=A*LOG(X)-X
6975    U=A*EXP(T1)
6980    IF U=0 THEN 8040
6985    R=U*(1+FNGam1(A))
6990    GOTO 7605
6995    IF A>=B(I) THEN 7050
7000    IF (A>X) OR (X>=X0) THEN 7035
7005    Ta=A+A
7010    M=INT(Ta)
7015    IF Ta<>M THEN 7035
7020    J=INT(.5*M)
7025    IF A=J THEN 7510
7030    GOTO 7535
7035    T1=A*LOG(X)-X
7040    R=EXP(T1)*FNGammai(A)
7045    GOTO 7115
7050    L=X/A
7055    IF L=0 THEN 8025
7060    S=.5-L+.5
7065    Z=FNRlog(L)
7070    IF Z>=228/A THEN 8015
7075    Y=A*Z
7080    R1=SQR(A)
7085    IF ABS(S)<=E0/R1 THEN 7830
7090    IF ABS(S)<=.4 THEN 7700
7095    T=1/(A*A)
7100    T1=(((.75*T-1)*T+3.5)*T-105)/(A*1260)
7105    T1=T1-Y
7110    R=Rpi*R1*EXP(T1)
7115    IF R=0 THEN 8020
7120    IF X<=MAX(A,Ln10) THEN 7140
7125    IF X<X0 THEN 7605
7130    GOTO 7255
7135  ! ******TAYLOR SERIES FOR P/R*****
7140    Apn=A+1
7145    T=X/Apn
7150    W(1)=T
7155    FOR N=2 TO 20
7160      Apn=Apn+1
7165      T=T*(X/Apn)
7170      IF T<=1E-3 THEN 7190
7175      W(N)=T
7180    NEXT N
```

```
7185    N=20
7190    S=T
7195    Tol=.5*Acc
7200    Apn=Apn+1
7205    T=T*(X/Apn)
7210    S=S+T
7215    IF T>Tol THEN 7200
7220    FOR M1=N-1 TO 1 STEP -1
7225      S=S+W(M1)
7230    NEXT M1
7235    Ans=R/A*(1+S)
7240    Qans=.5-Ans+.5
7245    SUBEXIT
7250  ! *******ASYMPTOTIC EXPANSION******
7255    A1=A-1
7260    T=A1/X
7265    W(1)=T
7270    FOR N=2 TO 20
7275      A1=A1-1
7280      T=T*(A1/X)
7285      IF ABS(T)<=1E-3 THEN 7305
7290      W(N)=T
7295    NEXT N
7300    N=20
7305    S=T
7310    IF ABS(T)<=Acc THEN 7335
7315    A1=A1-1
7320    T=T*(A1/X)
7325    S=S+T
7330    GOTO 7310
7335    FOR M=N-1 TO 1 STEP -1
7340      S=S+W(M)
7345    NEXT M
7350    Qans=R/X*(1+S)
7355    Ans=.5-Qans+.5
7360    SUBEXIT
7365  ! *******TAYLOR SERIES FOR P(A,X)/X**A******
7370    A1=3
7375    C=X
7380    S=X/(A+3)
7385    Tol=3*Acc/(A+1)
7390    A1=A1+1
7395    C=-C*(X/A1)
7400    T=C/(A+A1)
7405    S=S+T
7410    IF ABS(T)>Tol THEN 7390
7415    J=A*X*((S/6-.5/(A+2))*X+1/(A+1))
7420    Z=A*LOG(X)
7425    H=FNGam1(A)
7430    G=1+H
7435    IF X<.25 THEN 7450
7440    IF A<X/2.59 THEN 7475
7445    GOTO 7455
7450    IF Z>-.13394 THEN 7475
```

```
7455     W=EXP(Z)
7460     Ans=W*G*(.5-J+.5)
7465     Qans=.5-Ans+.5
7470     SUBEXIT
7475     L=FNRexp(Z)
7480     W=.5+L+.5
7485     Qans=(W*J-L*Z)*G-H
7490     IF Qans<0 THEN 8040
7495     Ans=.5-Qans+.5
7500     SUBEXIT
7505   ! ******FINITE SUMS FOR Q WHEN A>=1 AND 2*A IS AN INTEGER.*******
7510     S=EXP(-X)
7515     T=S
7520     K=1
7525     C=0
7530     GOTO 7560
7535     T1=SQR(X)
7540     S=FNErfc1(T1,0)
7545     T=EXP(-X)/(Spi*T1)
7550     K=0
7555     C=-.5
7560     FOR N=K TO J-1
7565       C=C+1
7570       T=X*T/C
7575       S=S+T
7580     NEXT N
7585     Qans=S
7590     Ans=.5-Qans+.5
7595     SUBEXIT
7600   ! ********CONTINUED FRACTION EXPANSION********
7605     Tol=MAX(5*E9,Acc)
7610     A21=1
7615     A2=1
7620     B21=X
7625     B2=X+.5-A+.5
7630     C=1
7635     A21=X*A2+C*A21
7640     B21=X*B2+C*B21
7645     Amo=A21/B21
7650     C=C+1
7655     C1=C-A
7660     A2=A21+C1*A2
7665     B2=B21+C1*B2
7670     Ano=A2/B2
7675     IF ABS(Ano-Amo)>=Tol*Ano THEN 7635
7680     Qans=R*Ano
7685     Ans=.5-Qans+.5
7690     SUBEXIT
7695   ! *******GENERAL TEMME EXPNSION***********
7700     IF (ABS(S)<=2*E9) AND (A*E9*E9>3.28E-3) THEN 8055
7705     C=EXP(-Y)
7710     W=.5*FNErfc1(SQR(Y),1)!********************
7715     U=1/A
7720     Z=SQR(Z+Z)
7725     IF L<1 THEN Z=-Z
7730     IF I>=2 THEN 7785
```

```
7735    IF ABS(S)<=1E-3 THEN 7865
7740    C0=(((((((((((D(13)*Z+D(12))*Z+D(11))*Z+D(10))*Z+D(9))*Z+D(8))*Z+D(7))*
Z+D(6))*Z+D(5))*Z+D(4))*Z+D(3))*Z+D(2))*Z+D(1))*Z-Th
7745    C1=((((((((((D1(12)*Z+D1(11))*Z+D1(10))*Z+D1(9))*Z+D1(8))*Z+D1(7))*Z+D1
(6))*Z+D1(5))*Z+D1(4))*Z+D1(3))*Z+D1(2))*Z+D1(1))*Z+D10
7750    C2=(((((((((D2(10)*Z+D2(9))*Z+D2(8))*Z+D2(7))*Z+D2(6))*Z+D2(5))*Z+D2(4))
*Z+D2(3))*Z+D2(2))*Z+D2(1))*Z+D20
7755    C3=(((((((D3(8)*Z+D3(7))*Z+D3(6))*Z+D3(5))*Z+D3(4))*Z+D3(3))*Z+D3(2))*Z+
D3(1))*Z+D30
7760    C4=(((((D4(6)*Z+D4(5))*Z+D4(4))*Z+D4(3))*Z+D4(2))*Z+D4(1))*Z+D40
7765    C5=(((D5(4)*Z+D4(3))*Z+D5(2))*Z+D5(1))*Z+D50
7770    C6=(D6(2)*Z+D6(1))*Z+D60
7775    T=((((((D70*U+C6)*U+C5)*U+C4)*U+C3)*U+C2)*U+C1)*U+C0
7780    GOTO 7940
7785    IF I>2 THEN 7815
7790    C0=(((((D(6)*Z+D(5))*Z+D(4))*Z+D(3))*Z+D(2))*Z+D(1))*Z-Th
7795    C1=(((D1(4)*Z+D1(3))*Z+D1(2))*Z+D1(1))*Z+D10
7800    C2=D2(1)*Z+D20
7805    T=(C2*U+C1)*U+C0
7810    GOTO 7940
7815    T=((D(3)*Z+D(2))*Z+D(1))*Z-Th
7820    GOTO 7940
7825  ! ***********TEMME EXPANSION FOR L=1***********
7830    IF A*E9*E9>3.28E-3 THEN 8055
7835    C=.5-Y+.5
7840    W=(.5-SQR(Y)*(.5+(.5-Y/3))/Spi)/C
7845    U=1/A
7850    Z=SQR(Z+Z)
7855    IF L<1 THEN Z=-Z
7860    IF I>=2 THEN 7910
7865    C0=((((((D(7)*Z+D(6))*Z+D(5))*Z+D(4))*Z+D(3))*Z+D(2))*Z+D(1))*Z-Th
7870    C1=(((((D1(6)*Z+D1(5))*Z+D1(4))*Z+D1(3))*Z+D1(2))*Z+D1(1))*Z+D10
7875    C2=((((D2(5)*Z+D2(4))*Z+D2(3))*Z+D2(2))*Z+D2(1))*Z+D20
7880    C3=(((D3(4)*Z+D3(3))*Z+D3(2))*Z+D3(1))*Z+D30
7885    C4=(D4(2)*Z+D4(1))*Z+D40
7890    C5=(D5(2)*Z+D5(1))*Z+D50
7895    C6=D6(1)*Z+D60
7900    T=((((((D70*U+C6)*U+C5)*U+C4)*U+C3)*U+C2)*U+C1)*U+C0
7905    GOTO 7940
7910    IF I>2 THEN 7935
7915    C0=(D(2)*Z+D(1))*Z+C0
7920    C1=D1(1)*Z+D10
7925    T=(D20*U+C1)*U+C0
7930    GOTO 7940
7935    T=D(1)*Z-Th
7940    IF L<1 THEN 7960
7945    Qans=C*(W+Rpi*T/R1)
7950    Ans=.5-Qans+.5
7955    SUBEXIT
7960    Ans=C*(W-Rpi*T/R1)
7965    Qans=.5-Ans+.5
7970    SUBEXIT
```

```
7975 ! *********SPECIAL CASES************
7980   IF X>=.25 THEN 8000
7985   Ans=FNErf(SQR(X))
7990   Qans=.5-Ans+.5
7995   SUBEXIT
8000   Qans=FNErfc1(SQR(X),0)
8005   Ans=.5-Qans+.5
8010   SUBEXIT
8015   IF ABS(S)<=2*E9 THEN 8055
8020   IF X>A THEN 8040
8025   Ans=0
8030   Qans=1
8035   SUBEXIT
8040   Ans=1
8045   Qans=0
8050   SUBEXIT
8055   Ans=2
8060 SUBEND
8065 ! *******************ALNREL************************
8070 DEF FNAlnrel(A)
8075   P1=-1.29418923022E0
8080   P2=.405303492862
8085   P3=-1.78874546012E-2
8090   Q1=-1.62752256355E0
8095   Q2=.747811014038
8100   Q3=-8.45104217946E-2
8105   IF ABS(A)>.375 THEN 8130
8110   T=A/(A+2)
8115   T2=T*T
8120   W=(((P3*T2+P2)*T2+P1)*T2+1)/(((Q3*T2+Q2)*T2+Q1)*T2+1)
8125   RETURN 2*T*W
8130   X=.5+A+.5
8135   RETURN LOG(X)
8140 FNEND
8145 ! ******************GAMMAI************************************
8150 ! Gammai=1/(GAMMA FUNCTION OF A). ! NEEDS GAM1.
8155 DEF FNGammai(A)
8160   IF A>=20 THEN 8245
8165   E=5E-13
8170   I=INT(A)
8175   I1=A-I
8180   Gammai=1
8185   IF I1<>0 THEN 8200
8190   I6=0
8195   GOTO 8220
8200   I6=-.5+I1-.5
8205   H=FNGam1(I1)  !I6=I1-.5-.5
8210   Gammai=I1+I1*H
8215   IF I=0 THEN RETURN Gammai
8220   I2=I-1
8225   I6=I6+1
8230   Gammai=Gammai/I6
8235   IF I6<I2 THEN 8225
8240   RETURN Gammai
```

```
8245    IF A<69.5 THEN 8260
8250    Gammai=0
8255    RETURN Gammai
8260    B=.918938533205
8265    I1=1/A
8270    I2=I1*I1
8275    I6=(.5-A)*LOG(A)+A-B-((1/1260*I2-1/360)*I2+1/12)*I1
8280    Gammai=EXP(I6)
8285    RETURN Gammai
8290  FNEND
8295  ! ***************RCOMP***************
8300  ! COMPUTATION OF X^A*EXP(-X)/GAMMA(A)
8305  ! SUBROUTINES NEEDED: GAM1, GAMMAI, RLOG.
8310  DEF FNRcomp(A,X)
8315    Rt2pin=.398942280401 !1/SQR(2*PI)
8320    Rcomp=0
8325    IF A>=20 THEN 8360
8330    T=A*LOG(X)-X
8335    IF A>=1 THEN 8350
8340    Rcomp=A*EXP(T)*(1+FNGam1(A))
8345    RETURN Rcomp
8350    Rcomp=EXP(T)*FNGammai(A)
8355    RETURN Rcomp
8360    U=X/A
8365    IF U=0 THEN RETURN Rcomp
8370    T=1/(A*A)
8375    T1=(((.75*T-1)*T+3.5)*T-105)/(1260*A)
8380    T1=T1-A*FNRlog(U)
8385    Rcomp=Rt2pin*SQR(A)*EXP(T1)
8390    RETURN Rcomp
8395  FNEND
8400  ! ***************COMPUTES Gamln(My)=LOG(GAMMA(My)***************
8405  ! NEEDS GAMLN1
8410  DEF FNGamln(My)
8415    J5=.418938533205  !J5=.5*LOG(2*PI-1)
8420    J0=8.33333333333E-2
8425    J1=-2.77777777777E-3
8430    J2=7.93650663184E-4
8435    J3=-5.95156336429E-4
8440    J4=8.20756370354E-4
8445    IF My>.8 THEN 8460 !Lg(My)
8450    Gamln=FNGamln1(My)-LOG(My)!SUBROUTINE FOR LOG(GAMMA(My+1))
8455    RETURN Gamln
8460    IF My>=2.25 THEN 8480
8465    Ny=-.5+My-.5
8470    Gamln=FNGamln1(Ny)
8475    RETURN Gamln
8480    IF My>=15 THEN 8535
8485    Ny=INT(My-1.25)
8490    Ay=My
8495    Wy=1
```

```
8500    FOR Iy=1 TO Ny
8505      Ay=Ay-1
8510      Wy=Ay*Wy
8515    NEXT Iy
8520    Ayy=Ay-1
8525    Gamln=FNGamln1(Ayy)+LOG(Wy)
8530    RETURN Gamln
8535    Ayy=1/(My*My)
8540    Wy=((((J4*Ayy+J3)*Ayy+J2)*Ayy+J1)*Ayy+J0)/My
8545    Gamln=J5+Wy+(My-.5)*(LOG(My)-1)
8550    RETURN Gamln
8555  FNEND
8560  ! *************COMPUTES Rlog=L-1+LOG(L) FOR L>0.******************
8565  DEF FNRlog(L)
8570    B3=.333333333333
8575    F2=.2
8580    F3=1.42857142857E-1
8585    F4=1.11111111111E-1
8590    F5=9.09090909091E-2
8595    F6=5.88434334173E-2  !**F6=.695-1-LOG(.695).**
8600    F7=4.56512608816E-2  !**F7=4/3-1-LOG(4/3).**
8605    S1=.5-L+.5
8610    IF (L<.548) OR (L>1.681) THEN 8685 !**SUBROUTINE FOR R2=L-1-LOG(L).**
8615    IF L<.83667 THEN 8640
8620    IF L>1.1547 THEN 8655
8625    R2=-S1
8630    W1=0
8635    GOTO 8665
8640    R2=(L-.695)/.695
8645    W1=F6-R2*.305
8650    GOTO 8665
8655    R2=-.75*S1-.25
8660    W1=F7+R2*B3
8665    R2=R2/(R2+2)
8670    T=R2*R2
8675    Rlog=2*T*(1/(.5-R2+.5)-R2*((((F5*T+F4)*T+F3)*T+F2)*T+B3))+W1
8680    RETURN Rlog
8685    Rlog=-S1-LOG(L)
8690    RETURN Rlog
8695  FNEND
8700  ! *************GAM1=1/GAMMA(A+1)-1, -.5<=A<=1.5.***************
8705  DEF FNGam1(A)
8710    DIM P(7),Q(5),R(9)
8715    P(1)=.577215664902
8720    P(2)=-.409078193006
8725    P(3)=-.230975380858
8730    P(4)=5.97275330452E-2
8735    P(5)=7.66968181649E-3
8740    P(6)=-5.14889771324E-3
8745    P(7)=5.89597428611E-4
8750    Q(1)=1
8755    Q(2)=.427569613095
8760    Q(3)=.158451672430
8765    Q(4)=2.61132021441E-2
```

```
8770    Q(5)=4.23244297897E-3
8775    R(1)=-.422784335098
8780    R(2)=-.771330383816
8785    R(3)=-.244757765222
8790    R(4)=.118378989872
8795    R(5)=9.30357293360E-4
8800    R(6)=-1.18290993445E-2
8805    R(7)=2.23047661158E-3
8810    R(8)=2.66505979059E-4
8815    R(9)=-1.32674909766E-4
8820    S1=.273076135304
8825    S2=5.59398236957E-2
8830    T=A
8835    D=A-.5
8840    IF D>0 THEN T=D-.5
8845    IF T<>0 THEN 8860
8850    Gam1=0
8855    RETURN Gam1
8860    IF T<0 THEN 8895
8865    Gam1=((((((P(7)*T+P(6))*T+P(5))*T+P(4))*T+P(3))*T+P(2))*T+P(1))/((((Q(5)
*T+Q(4))*T+Q(3))*T+Q(2))*T+Q(1))
8870    IF D>0 THEN 8885
8875    Gam1=A*Gam1
8880    RETURN Gam1
8885    Gam1=T/A*(-.5+Gam1-.5)
8890    RETURN Gam1
8895    Gam1=((((((((R(9)*T+R(8))*T+R(7))*T+R(6))*T+R(5))*T+R(4))*T+R(3))*T+R(2)
)*T+R(1))/((S2*T+S1)*T+1)
8900    IF D>0 THEN 8915
8905    Gam1=A*(.5+Gam1+.5)
8910    RETURN Gam1
8915    Gam1=T*Gam1/A
8920    RETURN Gam1
8925  FNEND
8930  ! *****************REXP=(EXP(X)-1)/X*****************
8935  DEF FNRexp(X)
8940    P1=9.1404191482E-10
8945    P2=2.3808236104E-2
8950    Q1=-.499999999086
8955    Q2=.107141568981
8960    Q3=-1.19041179761E-2
8965    Q4=5.95130811186E-4
8970    IF ABS(X)>.15 THEN 8985
8975    Rexp=((P2*X+P1)*X+1)/((((Q4*X+Q3)*X+Q2)*X+Q1)*X+1)
8980    RETURN Rexp
8985    W=EXP(X)
8990    IF X>0 THEN 9005
8995    Rexp=(W-.5-.5)/X
9000    RETURN Rexp
9005    Rexp=W*(.5+(.5-1/W))/X
9010    RETURN Rexp
9015  FNEND
```

```
9020  ! *********************ERF(X)*********************
9025  DEF FNErf(X)
9030    X2=X*X
9035    IF ABS(X)>.5 THEN 9050
9040    Erf=X*(((-3.56098437018E-2*X2+6.99638348862)*X2+21.9792616183)*X2+242.66
7955231)/(((X2+15.0827976304)*X2+91.1649054045)*X2+215.05887587)
9045    RETURN Erf
9050    K4=ABS(X)
9055    IF K4>4 THEN 9085
9060    K3=((((-6.08581519597E-6*K4+.564371606864)*K4+4.26772010709)*K4+14.57189
85969)*K4+26.0947469561)*K4+22.8989928517
9065    K3=K3/(((((K4+7.56884822936)*K4+26.2887957588)*K4+50.2732028638)*K4+51.9
335706876)*K4+22.8989857499)
9070    Erf=1-EXP(-X2)*K3
9075    IF X<0 THEN RETURN -Erf
9080    RETURN Erf
9085    Erf=1
9090    IF K4>=4.883 THEN 9115
9095    K6=1/X2
9100    B1=.564189583548! 1/SQR(PI)
9105    Erf=(B1-K6*(((3.24319519278E-2*K6+.243911029489)*K6+.119903955268)*K6+.0
12130827639)/(((K6+1.43771227937)*K6+.489552441961)*K6+4.30026643453E-2))/K4
9110    Erf=1-EXP(-X2)*Erf
9115    IF X<0 THEN RETURN -Erf
9120    RETURN Erf
9125  FNEND
9130  ! ****************ERFC1(X)******************
9135  ! IF Ind=0 THEN Erfc1=ERFC(X); ELSE Erfc1=EXP(X*X)*ERFC(X).
9140  DEF FNErfc1(X,Ind)
9145    X2=X*X
9150    IF ABS(X)>.5 THEN 9170
9155    Erfc1=.5-X*(((-3.56098437018E-2*X2+6.99638348862)*X2+21.9792616183)*X2+2
42.667955231)/(((X2+15.0827976304)*X2+91.1649054045)*X2+215.05887587)+.5
9160    IF Ind<>0 THEN RETURN EXP(X2)*Erfc1
9165    RETURN Erfc1
9170    K4=ABS(X)
9175    IF K4>4 THEN 9205
9180    K3=((((-6.08581519597E-6*K4+.564371606864)*K4+4.26772010709)*K4+14.57189
85969)*K4+26.0947469561)*K4+22.8989928517
9185    Erfc1=K3/(((((K4+7.56884822936)*K4+26.2887957588)*K4+50.2732028638)*K4+5
1.9335706876)*K4+22.8989857499)
9190    IF Ind=0 THEN 9245
9195    IF X<0 THEN RETURN 2*EXP(X2)-Erfc1
9200    RETURN Erfc1
9205    IF X<-4.89 THEN 9260
9210    IF (X>=14.9891301) AND (Ind=0) THEN RETURN 0
9215    K6=1/X2
9220    B1=.564189583548! 1/SQR(PI)
9225    Erfc1=(B1-K6*(((3.24319519278E-2*K6+.243911029489)*K6+.119903955268)*K6+
.012130827639)/(((K6+1.43771227937)*K6+.489552441961)*K6+4.30026643453E-2))/K4
9230    IF Ind=0 THEN 9245
9235    IF X<0 THEN RETURN 2*EXP(X2)-Erfc1
9240    RETURN Erfc1
9245    Erfc1=EXP(-X2)*Erfc1
9250    IF X<0 THEN RETURN 2-Erfc1
9255    RETURN Erfc1
```

```
9260    IF Ind<>0 THEN RETURN 2*EXP(X2)
9265    RETURN 2
9270  FNEND
9275  ! *********COMPUTES Gamln1=LOG(GAMMA(1+Ayy), -.2<=Ayy<=1.25*********
9280  DEF FNGamln1(Ayy)
9285    G0=.577215664902   !SUBROUTINE FOR LOGGAMMA(My)=Lg
9290    G1=.844203922187
9295    G2=-.168860593647
9300    G3=-.780427615534
9305    G4=-.402055799310
9310    G5=-6.73562214326E-2
9315    G6=-2.71935708323E-3
9320    H1=2.88743195474E0
9325    H2=3.12755088915E0
9330    H3=1.56875193295E0
9335    H4=.361951990101
9340    H5=3.25038868254E-2
9345    H6=6.67465618796E-4
9350    M0=.422784335098
9355    M1=.848044614535
9360    M2=.565221050692
9365    M3=.156513060487
9370    M4=1.70502484023E-2
9375    M5=4.97958207639E-4
9380    V1=1.24313399878E0
9385    V2=.548042109832
9390    V3=.10155218744
9395    V4=7.13309612391E-3
9400    V5=1.1616547599E-4
9405    IF Ayy>=.60 THEN 9425
9410    Gamln1=((((((G6*Ayy+G5)*Ayy+G4)*Ayy+G3)*Ayy+G2)*Ayy+G1)*Ayy+G0)/((((((H6
*Ayy+H5)*Ayy+H4)*Ayy+H3)*Ayy+H2)*Ayy+H1)*Ayy+1)
9415    Gamln1=-Ayy*Gamln1
9420    RETURN Gamln1
9425    Ny=Ayy-.5-.5
9430    Gamln1=(((((M5*Ny+M4)*Ny+M3)*Ny+M2)*Ny+M1)*Ny+M0)/(((((V5*Ny+V4)*Ny+V3)*
Ny+V2)*Ny+V1)*Ny+1)
9435    Gamln1=Ny*Gamln1
9440    RETURN Gamln1
9445  FNEND
9450  ! ************************DAWS(X)*******************
9455  !  Daws(X) = EXP(-X*X)* INTEGRAL FROM 0 TO X OF EXP(T*T) DT.
9456  !  REF: MATH OF COMP. 1970, PP. 171-178.  DAWSON'S INTEGRAL.
9460  DEF FNDaws(K1)
9465    Daws=K1*K1
9470    IF ABS(K1)>2.5 THEN 9490
9475    K4=((((-2.38594565696E-2*Daws+1.67795116189)*Daws-15.1982152422)*Daws+41
9.67290228)*Daws-1284.05832279)*Daws+10832.6558873
9480    Daws=K1*K4/(((((Daws+19.9422336364)*Daws+219.728331833)*Daws+1489.435572
42)*Daws+5937.71276935)*Daws+10832.6558772)
9485    RETURN Daws
9490    IF ABS(K1)>3.5 THEN 9520
9495    K4=88.7619386764/(Daws+3.47393742586)
9500    K4=13.8216341182/(Daws-12.6817901598+K4)
```

```
9505     K4=37.3902763512/(Daws+4.07068101667+K4)
9510     Daws=(.214221965778/(Daws-4.91605365741+K4)+.500652754437)/K1
9515     RETURN Daws
9520     IF ABS(K1)>=5 THEN 9545
9525     K4=17.761778077/(Daws-4.1083233770)
9530     K4=-.53714272981/(Daws-10.385024821+K4)
9535     Daws=(.24924632421/(Daws-1.5853935006+K4)+.50000965199)/K1
9540     RETURN Daws
9545     K4=-4.12544065608/(Daws-11.1952164237)
9550     K4=-2.48787658804/(Daws-4.67312022141+K4)
9555     Daws=.5*(1+(.749999190567/(Daws-2.50017116686+K4)+.500000001674)/Daws)/K
1
9560     RETURN Daws
9565   FNEND
9570   ! *****************ELLCV*********************
9575   ! THIS PROGRAM IS CALLED "ELLCV". IT SUPPLIES
9576   ! THE ELLIPTICAL COVERAGE FUNCTION:Ellcv=P(R,H,K,S1,S2).
9580   ! Ellcv DENOTES THE PROBABILITY OF A SHOT, NORMALLY DISTRIBUTED WITH
9585   ! MEAN (0,0) AND  STANDARD DEVIATIONS S1,S2 IN THE X AND Y DIRECTIONS,
9590   ! RESPECTIVELY, FALLING IN A CIRCLE IN THE XY-PLANE OF RADIUS R AND
9591   ! CENTERED AT (H,K). THE INPUT IS R,H,K,S1,S2. THE OUTPUT IS Ellcv.
9595   ! PROGRAM IS SET FOR 8-DECIMAL-DIGIT ACCURACY IN Ellcv.
9600   ! ELLCV USES ERFC, ERF WITH 12 DIGIT RELATIVE ACCURACY.
9605   ! SOURCES: NWL REPORT #1710, AUG.1960. MATH OF COMP. OCT. 1961,
9606   ! PP. 375,382. NSWC REPORT #83-13, NOVEMBER, 1982.
9610   DEF FNEllcv(R,H,K,S1,S2)
9615     COM X(*),Y(*)
9620     B1=.564189583548! 1/SQR(PI)
9625     Z3=.00000005*S1*S2
9630     IF R*R<=Z3 THEN RETURN 0
9635     A=6.027    !6 !5.6123
9640     B=1.41421356237 ! SQR(2)
9645     B2=36       !31.49791129  !B2=A*A
9650     H2=H*H+K*K
9655     H8=ABS(H)
9660     K8=ABS(K)
9665     D=MAX(S1,S2)
9670     T=R-A*D
9675   ! ***PROCEED TO SEE IF P=0 OR P=1
9680     IF T<0 THEN 9695
9685     IF T*T<H2 THEN 9695
9690     RETURN 1
9695     IF R-H8+A*S1<=0 THEN RETURN 0
9700     IF R-K8+A*S2<=0 THEN RETURN 0
9705     S0=SQR(H2)
9710     IF S0<=R THEN 9730
9715     D=(S0-R)*(S0-R)/(D*D)
9720     IF R*R*EXP(-.5*D)>Z3 THEN 9730
9725     RETURN 0
9730     IF S1<>S2 THEN 9750
9735     H8=S0
9740     K8=0
9745     IF R-H8+A*S1<=0 THEN RETURN 0
9750     IF H8*K8=0 THEN 9855
```

```
9755   H9=H8/S1
9760   K9=K8/S2
9765   D=H9*H9+K9*K9
9770   Z2=R/S2
9775   IF D<=B2 THEN 9855
9780   Z1=R+A*MIN(S1,S2)
9785   IF H2<Z1*Z1 THEN 9855
9790   Q=S2/S1
9795   Q1=Q*Q
9800   F=Q1*H9*H9+K9*K9
9805   Z1=Z2*Z2*F/D
9810   Z=D-Z1-B2
9815   IF Z<0 THEN 9825
9820   IF Z*Z-4*Z1*B2>=0 THEN RETURN 0
9825   T1=H8*H8+Q1*K8*K8
9830   Z8=B2*S1*S1*T1/H2
9835   R2=R*R
9840   Y=H2-R2-Z8
9845   IF Y<0 THEN 9855
9850   IF Y*Y-4*R2*Z8>=0 THEN RETURN 0
9855   Z8=0     !****FIND LIMITS OF INTEGRATION
9860   Z=K8+A*S2
9865   H3=K8-A*S2
9870   S0=S1
9875   S9=S2
9880   Z=R-Z
9885   H5=0
9890   D1=0
9895   IF Z>=0 THEN D1=SQR(Z/R)
9900   IF H3>=0 THEN 9915
9905   E3=1-D1
9910   GOTO 9925
9915   E3=SQR(1-H3/R)-D1
9920   H5=1
9925   IF Z8<>0 THEN 9965
9930   Z8=1
9935   F=E3
9940   T=D1
9945   Z=H8+A*S1
9950   H6=H5
9955   H3=H8-A*S1
9960   GOTO 9880
9965   IF F>=E3 THEN 10010
9970   E3=F
9975   D1=T
9980   S9=S1
9985   Z8=H8
9990   S0=S2
9995   H8=K8
10000  K8=Z8
10005  H5=H6
10010  E3=.5*E3 !***BEGIN GAUSSIAN INTEGRATION*****
10015  N=3
10020  IF (H8-R)-3.5*S0) OR (K8-R)-3.5*S9) THEN N=E3*R*(.34/S0+1/(.025*ABS(R-K8
)+5*S9))
```

```
10025   Z2=R/(B*S9)
10030   R8=R/(B*S0)
10035   H8=H8/(B*S0)
10040   K8=K8/(B*S9)
10045   IF N<2.75 THEN 10065
10050   J=31
10055   N1=12
10060   GOTO 10155
10065   IF N<1.35 THEN 10085
10070   J=21
10075   N1=10
10080   GOTO 10155
10085   IF N<.75 THEN 10105
10090   J=13
10095   N1=8
10100   GOTO 10155
10105   IF N<.35 THEN 10125
10110   J=7
10115   N1=6
10120   GOTO 10155
10125   IF N<.15 THEN 10145
10130   J=3
10135   N1=4
10140   GOTO 10155
10145   J=0
10150   N1=3
10155   Z=0
10160   Y=B1*E3*R8
10165   K9=0      !1.04E-8
10170   H9=2      !1.9999999702
10175   G3=0
10180   IF K8=0 THEN 10390
10185   Z3=0
10190   FOR I=-N1 TO N1
10195     IF I=0 THEN 10370
10200     T=E3*(SGN(I)*X(J+ABS(I))+1)+D1
10205     T9=T*T
10210     T1=R8*(1-T9)
10215     T2=(H8-T1)*(H8-T1)
10220     T4=EXP(-T2)
10225     IF H8<>0 THEN 10240
10230     T4=T4+T4
10235     GOTO 10255
10240     IF H5<>0 THEN 10255
10245     T2=(H8+T1)*(H8+T1)
10250     T4=T4+EXP(-T2)
10255     IF Z=0 THEN 10270
10260     K5=H9
10265     GOTO 10365
10270     Z1=Z2*T*SQR(2-T9)
10275     K1=K8-Z1
10280     IF K1<=-A THEN 10300
10285     Erfc=FNErfc(K1)
10290     K5=Erfc
10295     GOTO 10315
```

```
10300     Z=1
10305     K5=H9
10310     GOTO 10365
10315     IF Z3=0 THEN 10330
10320     K5=K5-K9
10325     GOTO 10365
10330     K1=K8+Z1
10335     IF K1>=A THEN 10355
10340     Erfc=FNErfc(K1)
10345     K5=K5-Erfc
10350     GOTO 10365
10355     K5=K5-K9
10360     Z3=1
10365     G3=G3+K5*T4*T*Y(J+ABS(I))
10370   NEXT I
10375   Ellcv=Y*G3
10380   IF Ellcv>1 THEN RETURN 1
10385   RETURN Ellcv
10390   FOR I=-N1 TO N1
10395     IF I=0 THEN 10370
10400     T=E3*(SGN(I)*X(J+ABS(I))+1)+D1
10405     T9=T*T
10410     T1=R8*(1-T9)
10415     T2=(H8-T1)*(H8-T1)
10420     T4=EXP(-T2)
10425     IF H5<>0 THEN 10440
10430     T2=(H8+T1)*(H8+T1)
10435     T4=T4+EXP(-T2)
10440     IF Z=0 THEN 10455
10445     K5=H9
10450     GOTO 10365
10455     K1=-Z2*T*SQR(2-T9)
10460     IF K1<=-A THEN 10480
10465     Erfc=FNErf(-K1)      !FNErfc(K1)
10470     K5=2*Erfc      !2*(Erfc-1)
10475     GOTO 10365
10480     Z=1
10485     K5=H9
10490     GOTO 10365
10495   FNEND
10500   ! ****************ERFC(X)*****************
10505 DEF FNErfc(X)
10510   K6=X*X
10515   IF ABS(X)>.5 THEN 10525
10520   RETURN .5-X*(((-3.560984370l8E-2*K6+6.99638348862)*K6+21.9792616183)*K6+
242.667955231)/(((K6+15.0827976304)*K6+91.1649054045)*K6+215.05887587)+.5
10525   K4=ABS(X)
10530   IF K4>4 THEN 10550
10535   Erfc=((((-6.08581519597E-6*K4+.564371606864)*K4+4.26772010709)*K4+14.571
8985969)*K4+26.0947469561)*K4+22.8989928517
10540   Erfc=EXP(-K6)*Erfc/(((((K4+7.56884822936)*K4+26.2887957588)*K4+50.273202
8638)*K4+51.9335706876)*K4+22.8989857499)
```

```
10545   GOTO 10580
10550   IF X<-4.89 THEN RETURN 2
10555   IF X>=14.9891301 THEN RETURN 0
10560   K6=1/K6
10565   B1=.564189583548! 1/SQR(PI)
10570   Erfc=(B1-K6*(((3.24319519278E-2*K6+.243911029489)*K6+.119903955268)*K6+.
012130827639)/(((K6+1.43771227937)*K6+.489552441961)*K6+4.30026643453E-2))/K4
10575   Erfc=EXP(-X*X)*Erfc
10580   IF X<0 THEN RETURN 2-Erfc
10585   RETURN Erfc
10590 FNEND
10595 ! ****************CIRCV*****************
10600 ! THIS SUBROUTINE IS CALLED CIRCV. INPUT: R,D,S,V.
10605 ! IF V#0. OUTPUT: Circv=CIRCULAR COVERAGE FUNCTION. Circv GIVES THE
10606 ! PROBABILITY OF A SHOT FALLING, UNDER A NORMAL DISTRIBUTION WITH
10610 ! MEAN (0,0) AND EQUAL STANDARD DEVIATIONS, S, IN A CIRCLE OF RADIUS R
10611 ! OFFSET A DISTANCE D FROM (0,0).
10615 ! IF V=0, OUTPUT: Circv=GENERALIZED CIRCULAR ERROR FUNCTON. Circv GIVES
10616 ! THE PROBABLITY OF A SHOT FALLING, UNDER A NORMAL DISTRIBUTION WITH
10620 ! MEAN (0,0) AND STANDARD DEVIATIONS S1 AND S, IN A CIRCLE OF RADIUS R
10621 ! CENTERED AT (0,0). D = S1/S <= 1.
10625 ! FOR V=0, WITH S1 = 0, S1#S, Circv=ERF(R/(SQR(2)*S)).
10630 ! SUBROUTINES NEEDED: ERFC2, ERF.
10635 ! THE PARIAL DERIVATIVE OF Circv WITH RESPECT TO R IS CONTAINED WITHIN THE
10636 ! ROUTINE AND CAN BE EXTRACTED IF DESIRED.
10640 ! SOURCES: MATH OF COMP APRIL 1961,PP169,173 AND OCT.1961, PP 375,382.
10641 ! NWL REPORT #1768, JAN. 1962. NSWC REPORT#83-13, NOV. 1982.
10645 ! IEEE TRANS. INFO. TH. APRIL 1965, P. 312.
10650 ! PROGRAM IS SET FOR APPROXIMATELY EIGHT DECIMAL DIGIT ACCURACY.
10655 DEF FNCircv(R,D,S,V)
10660   IF R=0 THEN RETURN 0
10665   B1=.564189583548! 1/SQR(PI)
10670   C7=.707106781187
10675   C1=1E-8
10680   R1=R/S
10685   IF V<>0 THEN 10740
10690   IF ABS(D-.5)<=.5 THEN 10700
10695   PRINT "ERROR: D>1 OR D<0, IN CIRCV."
10700   D1=D
10705   R1=R/S
10710   IF D1<>0 THEN 10720
10715   RETURN FNErf(R1*C7)     ! V=0 AND D=S1=0.
10720   T=.5*R1/D1
10725   R1=T*(1+D1)
10730   D1=T*(1-D1)
10735   GOTO 10745
10740   D1=D/S
10745   A1=R1-D1
10750   K1=ABS(A1)
10755   IF K1<6.02738 THEN 10770   !5.6123
10760   IF A1<0 THEN RETURN 0
10765   RETURN 1
```

```
10770   T=R1*D1
10775   T3=.5*R1*R1
10780   B=.5*D1*D1
10785   N=0
10790   IF T>7 THEN 10930
10795   T1=C7*T-1
10800   T2=T3*B
10805   S0=EXP(-T3-B)
10810   S1=EXP(-B)
10815   IF T3>.0005 THEN 10830
10820   S1=S1*T3*(.5+((-T3/24+1/6)*T3-.5)*T3+.5)
10825   GOTO 10835
10830   S1=S1-S0
10835   S2=S0
10840   T0=S1
10845   N=N+1
10850   M=1/N
10855   S0=T2*M*M*S0
10860   T0=B*M*T0-S0
10865   S1=S1+T0
10870   S2=S2+S0
10875   IF T1>N THEN 10845
10880   IF T0>C1 THEN 10895
10885   P=S1
10890   GOTO 11080
10895   N=N+1
10900   M=1/N
10905   S0=T2*M*M*S0
10910   T0=B*M*T0-S0
10915   S1=S1+T0
10920   S2=S2+S0
10925   GOTO 10880
10930   T1=2*ABS(T3-B)
10935   K1=K1*C7
10940   T3=1/(T+T)
10945   T2=SQR(T3)
10950   S1=.5*A1*A1
10955   S2=EXP(-S1)
10960   S0=B1*T2*S2
10965   T0=(R1+D1)*C7*T2*FNErfc2(K1,S2)
10970   T2=S1*T3
10975   T3=.5*T3
10980   S1=T0
10985   S2=S0
10990   N=N+2
10995   M=N-1
11000   K1=M/N
11005   S0=K1*T3*S0
11010   T0=T1*S0-T2*K1*T0
11015   S0=M*S0
11020   S1=S1+T0
11025   S2=S2+S0
11030   IF T0-C1>0 THEN 10990
11035   IF S0-C1<=0 THEN 11075
11040   T1=ABS(S0)
```

```
11045   N=N+2
11050   M=N-1
11055   S0=M*M*T3*S0/N
11060   IF ABS(S0)>=T1 THEN 11075
11065   S2=S2+S0
11070   GOTO 11035
11075   P=.5*ABS(1+SGN(A1)-S2-SGN(A1)*S1)
11080   IF V<>0 THEN 11090
11085   P=ABS(P+P+S2-1)
11090   IF P>1 THEN RETURN 1
11095   RETURN P !***EXIT***
11100 FNEND
11105 ! ***************ERFC2****************
11110 ! Erfc=ERFC(X). Exp=EXP(-X*X) IF AVAILABLE, OTHERWISE Exp=-1.
11115 DEF FNErfc2(X,Exp)
11120   K6=X*X
11125   IF ABS(X)>.5 THEN 11135
11130   RETURN .5-X*(((-3.56098437018E-2*K6+6.99638348862)*K6+21.9792616183)*K6+
242.667955231)/(((K6+15.0827976304)*K6+91.1649054045)*K6+215.05887587)+.5
11135   K4=ABS(X)
11140   IF K4>4 THEN 11165
11145   IF Exp<0 THEN Exp=EXP(-K6)
11150   Erfc=((((-6.08581519597E-6*K4+.564371606864)*K4+4.26772010709)*K4+14.571
8985969)*K4+26.0947469561)*K4+22.8989928517
11155   Erfc=Exp*Erfc/(((((K4+7.56884822936)*K4+26.2887957588)*K4+50.2732028638)
*K4+51.9335706876)*K4+22.8989857499)
11160   GOTO 11200
11165   IF X<-4.89 THEN RETURN 2
11170   IF X>=14.9891301 THEN RETURN 0
11175   IF Exp=-1 THEN Exp=EXP(-K6)
11180   K6=1/K6
11185   B1=.564189583548! 1/SQR(PI)
11190   Erfc=(B1-K6*(((3.24319519278E-2*K6+.243911029489)*K6+.11990395526B)*K6+.
012130827639)/(((K6+1.43771227937)*K6+.489552441961)*K6+4.30026643453E-2))/K4
11195   Erfc=Exp*Erfc
11200   IF X<0 THEN RETURN 2-Erfc
11205   RETURN Erfc
11210 FNEND
11215 ! ***************GQUAD*****************
11220 ! Gquad IS A QUADRATURE ROUTINE WHICH USES GAUSSIAN MULTIPLIERS TO OBTAIN
11221 ! THE INTEGRAL I FROM A TO B OF Fn(T), (SEE NEXT SUBPROGRAM.)
11225 ! A,B : ARE LOWER AND UPPER LIMITS OF INTEGRATION.
11230 ! Ierr : IF 0 THEN ANSWER O.K. IF 1 THEN MAXIMUM NO. OF SUBDIVISIONS USED.
11235 ! M : MAXIMUM NO. OF EQUAL SUBDIVISIONS OF [A,B] DESIRED WITH GAUSSIAN
11236 ! APPLIED ON EACH SUBDIVISION.
11240 ! I1 : VALUE OF INTEGRAL STORED IN I1.
11245 ! E : |I1-I0| < E SAYS DIFFERENCE BETWEEN TWO CONSECUTIVE CALULATIONS FOR
11246 ! I ARE LESS THAN E.
11250 ! N : 2N = ORDER OF GAUSSIAN MULTIPLIERS USED. ONE MAY SPECIFY N=3,4,6,8,
11251 ! 10,12. ONLY THESE VALUES OF N ALLOWED.
11255 ! X(*),Y(*) : STORED VALUES OF GAUSSIAN ABSCISSAS AND MULTIPLIERS ON
11256 ! [-1,1].
11260 ! J : NO. OF EQUAL SUBDIVISIONS USED. J <=M.
```

```
11265 ! NEEDS X(*), Y(*) ARRAYS WHICH ARE STORED IN "GAUSS"; FUNCTION FN.
11270 SUB Gquad(A,B,Ierr,M,I1,E,N,N2,R3,H,K,L,S1,S2,S3,V,D,W9)
11275   COM X(*),Y(*)
11280   J1=Ierr=0
11285   IF N=3 THEN 11355
11290   J1=3
11295   IF N=4 THEN 11355
11300   J1=7
11305   IF N=6 THEN 11355
11310   J1=13
11315   IF N=8 THEN 11355
11320   J1=21
11325   IF N=10 THEN 11355
11330   J1=31
11335   IF N=12 THEN 11355
11340   PRINT "N AND J1 NOT RIGHT"
11345   BEEP
11350   STOP
11355   D3=B-A
11360   J=0
11365   I0=1E50
11370   J=J+1
11375   E1=A
11380   G=0
11385   D1=D3/J
11390   D2=D1/2
11395   K1=0
11400   K1=K1+1
11405   E0=E1
11410   E1=E0+D1
11415   E2=(E1+E0)*.5
11420   FOR I=-N TO N
11425     IF I=0 THEN 11445
11430     T=D2*(SGN(I)*X(J1+ABS(I)))+E2
11435     F=Y(J1+ABS(I))*FNFn(T,R3,H,K,L,S1,S2,S3,V,D,W9)
11440     G=G+F
11445   NEXT I
11450   IF K1<>J THEN 11400
11455   I1=D2*G
11460   IF ABS(I1-I0)<E THEN 11480
11465   I0=I1
11470   IF J<>M THEN 11370
11475   Ierr=1
11480   N2=N*J*(J+1)!NO. OF FCN EVALUATIONS.
11485 SUBEND
11490 ! ****************FN*****************
11495 ! NEEDED FOR GQUAD. NEEDS ELLCV, CIRCV.
11500 DEF FNFn(T5,R3,H,K,L,S1,S2,S3,V,D,W9)
11505   COM X(*),Y(*)
11510   C8=1.41421356237
11515   J1=0
11520   P=L-C8*S3*T5
11525   J1=EXP(-T5*T5)
11530   IF L<>0 THEN 11545
11535   J1=J1+J1
```

```
11540   GOTO 11565
11545   IF W9<>0 THEN 11565
11550   V2=2*L*P/(S3*S3)
11555   IF V2>27 THEN 11565
11560   J1=J1*(.5+EXP(-V2)+.5)
11565   R=SQR(R3*R3-P*P)
11570   IF V=2 THEN 11585
11575   P=FNCircv(R,D,S1,V)
11580   GOTO 11590
11585   P=FNEllcv(R,H,K,S1,S2)
11590   J1=J1*P
11595   RETURN J1
11600 FNEND
11605 ! **************ERF3***************
11610 ! Erf=ERF(X). IF AVAILABLE Exp=EXP(-X*X) ELSE Exp=-1.
11615 DEF FNErf3(X,Exp)
11620   X2=X*X
11625   IF ABS(X)>.5 THEN 11640
11630   Erf=X*(((-3.56098437018E-2*X2+6.99638348862)*X2+21.9792616183)*X2+242.66
7955231)/(((X2+15.0827976304)*X2+91.1649054045)*X2+215.05887587)
11635   RETURN Erf
11640   K4=ABS(X)
11645   IF K4>4 THEN 11680
11650   K3=((((-6.08581519597E-6*K4+.564371606864)*K4+4.26772010709)*K4+14.57189
85969)*K4+26.0947469561)*K4+22.8989928517
11655   K3=K3/(((((K4+7.56884822936)*K4+26.2887957588)*K4+50.2732028638)*K4+51.9
335706876)*K4+22.8989857499)
11660   IF Exp<0 THEN Exp=EXP(-X2)
11665   Erf=1-Exp*K3
11670   IF X<0 THEN RETURN -Erf
11675   RETURN Erf
11680   Erf=1
11685   IF K4>=4.883 THEN 11715
11690   K6=1/X2
11695   B1=.564189583548! 1/SQR(PI)
11700   Erf=(B1-K6*(((3.24319519278E-2*K6+.243911029489)*K6+.119903955268)*K6+.0
12130827639)/(((K6+1.43771227937)*K6+.489552441961)*K6+4.30026643453E-2))/K4
11705   IF Exp<0 THEN Exp=EXP(-X2)
11710   Erf=1-Exp*Erf
11715   IF X<0 THEN RETURN -Erf
11720   RETURN Erf
11725 FNEND
11730 ! **************ELLRC***************
11735 ! FINDS LENGTH OF SIDE OF CUBE WITH CENTER (H,K,L) WHICH CONTAINS P3 OF
11740 ! THE NORMAL DISTRIBUTION WITH MEAN 0 AND STANDARD DEVIATIONS (S1,S2,S3).
11745 ! FOR FNEllrc CHECK T6. IF T6>40 RESULT SUSPECT.
11750 ! SUBROUTINES NEEDED: ERF, ERF3, ERFC1.
11755 DEF FNEllrc(H,K,L,S1,S2,S3,P3,Rmin,Rmax,T6)
11760   T6=0
11765   IF (P3>1) OR (P3<0) THEN RETURN -1E99
11770   IF P3=1 THEN RETURN 1E99
11775   IF P3=0 THEN RETURN 0
11780   Sq=1.41421356237
11785   U1=1/(Sq*S1)
```

```
11790   U2=1/(Sq*S2)
11795   Sqpi=2.50662827463
11800   U3=1/(Sq*S3)
11805   Y1=1/(Sqpi*S1)
11810   Y2=1/(Sqpi*S2)
11815   Y3=1/(Sqpi*S3)
11820   R1=Rmax
11825   R0=Rmin
11830   T6=0
11835   A=.5*(R1+R0)
11840   GOSUB 11930
11845   IF (ABS(U)<.1*P3) OR (ABS(R1-R0)<.01*R0) THEN 11865
11850   GOTO 11835
11855   A=.5*(R1+R0)
11860   GOSUB 11930
11865   IF C0=0 THEN C0=EXP(-2*H*A/(S1*S1))
11870   IF C2=0 THEN C2=EXP(-2*K*A/(S2*S2))
11875   IF C4=0 THEN C4=EXP(-2*L*A/(S3*S3))
11880   B1=Y1*C1*(1+C0)*F2*F3+Y2*C3*(1+C2)*F1*F3+Y3*C5*(1+C4)*F1*F2
11885   IF B1<=0 THEN 11835
11890   A1=A-U/B1
11895   IF ABS(F-P3)<1E-10 THEN RETURN A1
11900   IF (ABS(A1-A)<5E-5*A1) AND (ABS(F-P3)<1E-3*P3) THEN RETURN A1
11905   IF ABS(A1-A)<=1E-8 THEN RETURN A1
11910   A=A1
11915   IF T6>40 THEN RETURN A1
11920   IF (A<R0) OR (A>R1) THEN 11855
11925   GOTO 11860
11930   T0=(H+A)*U1
11935   T1=(H-A)*U1
11940   T2=(K+A)*U2
11945   T3=(K-A)*U2
11950   T4=(L+A)*U3
11955   T5=(L-A)*U3
11960   C1=F=EXP(-T1*T1)
11965   C3=F=EXP(-T3*T3)
11970   C5=F=EXP(-T5*T5)
11975   C0=C2=C4=0
11980   IF T1<0 THEN 12005
11985   IF C1=0 THEN GOSUB 12085
11990   C0=EXP(-2*H*A/(S1*S1))
11995   F1=.5*C1*(FNErfc1(T1,1)-C0*FNErfc1(T0,1))
12000   GOTO 12010
12005   F1=.5*(FNErf(T0)-FNErf3(T1,C1))
12010   IF T3<0 THEN 12035
12015   IF C3=0 THEN GOSUB 12085
12020   C2=EXP(-2*K*A/(S2*S2))
12025   F2=.5*C3*(FNErfc1(T3,1)-C2*FNErfc1(T2,1))
12030   GOTO 12040
12035   F2=.5*(FNErf(T2)-FNErf3(T3,C3))
12040   IF T5<0 THEN 12065
12045   IF C5=0 THEN GOSUB 12085
12050   C4=EXP(-2*L*A/(S3*S3))
12055   F3=.5*C5*(FNErfc1(T5,1)-C4*FNErfc1(T4,1))
```

```
12060   GOTO 12070
12065   F3=.5*(FNErf(T4)-FNErf3(T5,C5))
12070   F=F1*F2*F3
12075   U=F-P3
12080   IF U>0 THEN 12095
12085   R0=A
12090   GOTO 12100
12095   R1=A
12100   T6=T6+1
12105   IF ABS(F-.5)=.5 THEN 11835
12110   RETURN
12115 FNEND
12120   ! ***********FNXerfc(X) = X*EXP(X*X)*ERFC(X)************
12125   DEF FNXerfc(X)
12130   X2=X*X
12135   IF ABS(X)>.5 THEN 12150
12140   Erfc1=.5-X*(((-3.56098437018E-2*X2+6.99638348862)*X2+21.9792616183)*X2+2
42.667955231)/(((X2+15.0827976304)*X2+91.1649054045)*X2+215.05887587)+.5
12145   RETURN X*EXP(X2)*Erfc1
12150   K4=ABS(X)
12155   IF K4>4 THEN 12180
12160   K3=((((-6.08581519597E-6*K4+.564371606864)*K4+4.26772010709)*K4+14.57189
85969)*K4+26.0947469561)*K4+22.8989928517
12165   Erfc1=K3/(((((K4+7.56884822936)*K4+26.2887957588)*K4+50.2732028638)*K4+5
1.9335706876)*K4+22.8989857499)
12170   IF X<0 THEN RETURN X*(2*EXP(X2)-Erfc1)
12175   RETURN X*Erfc1
12180   IF X<-4.900 THEN RETURN 2*X*EXP(X*X)
12185   K6=1/X2
12190   B1=.564189583548! 1/SQR(PI)
12195   Erfc1=(B1-K6*(((3.24319519278E-2*K6+.243911029489)*K6+.119903955268)*K6+
.012130827639)/(((K6+1.43771227937)*K6+.489552441961)*K6+4.30026643453E-2))/K4
12200   IF X<0 THEN RETURN X*(2*EXP(X2)-Erfc1)
12205   RETURN X*Erfc1
12210 FNEND
12215   ! ********FNDxerf3(X,Exp) = ERF(X)/X; EXP(-X*X) AVAILABLE IF Exp > 0.**
12220  DEF FNDxerf3(X,Exp)
12225   X2=X*X
12230   IF ABS(X)>.5 THEN 12245
12235   Erf=(((-3.56098437018E-2*X2+6.99638348862)*X2+21.9792616183)*X2+242.6679
55231)/(((X2+15.0827976304)*X2+91.1649054045)*X2+215.05887587)
12240   RETURN Erf
12245   K4=ABS(X)
12250   IF K4>4 THEN 12275
12255   K3=((((-6.08581519597E-6*K4+.564371606864)*K4+4.26772010709)*K4+14.57189
85969)*K4+26.0947469561)*K4+22.8989928517
12260   K3=K3/(((((K4+7.56884822936)*K4+26.2887957588)*K4+50.2732028638)*K4+51.9
335706876)*K4+22.8989857499)
12265   IF Exp<0 THEN Exp=EXP(-X2)
12270   RETURN (1-Exp*K3)/K4
```

```
12275   IF K4>=4.883 THEN RETURN 1/K4
12280   K6=1/X2
12285   B1=.564189583548! 1/SQR(PI)
12290   Erf=(B1-K6*(((3.24319519278E-2*K6+.243911029489)*K6+.119903955268)*K6+.0
12130827639)/(((K6+1.43771227937)*K6+.489552441961)*K6+4.30026643453E-2))/K4
12295   IF Exp<0 THEN Exp=EXP(-X2)
12300   RETURN (1-Exp*Erf)/K4
12305 FNEND
12310 ! **************DXDAWS(X)****************
12315 !  Daws(X) = 1/X*EXP(-X*X)* INTEGRAL FROM 0 TO X OF EXP(T*T) DT.
12316 !  REF: MATH OF COMP. 1970, PP. 171-178.  DAWSON'S INTEGRAL/X
12320   DEF FNDxdaws(K1)
12325   Daws=K1*K1
12330   IF ABS(K1)>2.5 THEN 12350
12335   K4=((((-2.38594565696E-2*Daws+1.67795116189)*Daws-15.1982152422)*Daws+41
9.67290228)*Daws-1284.05832279)*Daws+10832.6558873
12340   Daws=K4/(((((Daws+19.9422336364)*Daws+219.728331833)*Daws+1489.43557242)
*Daws+5937.71276935)*Daws+10832.6558772)
12345   RETURN Daws
12350   IF ABS(K1)>3.5 THEN 12380
12355   K4=88.7619386764/(Daws+3.47393742586)
12360   K4=13.8216341182/(Daws-12.6817901598+K4)
12365   K4=37.3902763512/(Daws+4.07068101667+K4)
12370   Daws=(.214221965778/(Daws-4.91605365741+K4)+.500652754437)/Daws
12375   RETURN Daws
12380   IF ABS(K1)>=5 THEN 12405
12385   K4=17.761778077/(Daws-4.1083233770)
12390   K4=-.53714272981/(Daws-10.385024821+K4)
12395   Daws=(.24924632421/(Daws-1.5853935006+K4)+.50000965199)/Daws
12400   RETURN Daws
12405   K4=-4.12544065608/(Daws-11.1952164237)
12410   K4=-2.48787658804/(Daws-4.67312022141+K4)
12415   Daws=.5*(1+(.749999190567/(Daws-2.50017116686+K4)+.500000001674)/Daws)/D
aws
12420   RETURN Daws
12425 FNEND
```

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