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Consider a Brownian motion with a downward drift of rate a. Its maximum over all time has the exponential distribution with parameter 2a. Our aim is to study this maximum as a stochastic process indexed by a. That process is related to the convex majorant of the standard Brownian motion and, through the latter, to a Poisson random measure. This connection is exploited to obtain various distributional results. The results are of interest in queueing theory. Keywords,

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SUNSET OVER BROWNISTAN*

by

ERHAN ÇINLAR

Princeton University
Department of Civil Engineering and Operations Research
School of Engineering and Applied Science
Princeton, New Jersey 08544

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Abstract

Consider a Brownian motion with a downward drift of rate a. Its maximum over all time has the exponential distribution with parameter $2a$. Our aim is to study this maximum as a stochastic process indexed by a. That process is related to the convex majorant of the standard Brownian motion and, through the latter, to a Poisson random measure. This connection is exploited to obtain various distributional results. The results are of interest in queueing theory.

considered by NEWELL [4] and by COFFMAN, KADOTA, and SHEPP [2], the latter viewing the model as that of storage allocation in computer memory.

Let $Q_t(n)$ be the random variable that is 0 or 1 according as the stall n is empty or occupied at time t. The random vector $Q_t = (Q_t(1), Q_t(2), \dots)$ is the state of the system at time t. The process $\{\sum_n Q_t(n);\ t \geq 0\}$ is the queue size process in an M/M/∞ system; it is regenerative, and 0 is a regeneration state for it. It follows that the vector $(0, 0, \dots)$ is a regeneration state for $\{Q_t;\ t \geq 0\}$ and that the latter has an equilibrium distribution. Let Q be a random vector (of zeros and ones) whose law is that equilibrium distribution.

The distribution of $\sum_n Q(n)$ is Poisson with mean λ. The distribution of $\sum_1^m Q(n)$ is the equilibrium distribution of the queue size process in the M/M/m/m system with arrival rate λ and service rate 1; thus, that distribution is the conditional distribution of $\sum_n Q(n)$ given that $\sum_n Q(n) \leq m$. Other than these facts and a few conclusions that can be drawn from them by elementary probabilistic considerations, there is not much known about the distribution of Q.

For $a > 0$, let $\lambda^{1/4} Y_\lambda(a,t)$ be the number of empty stalls at time t among those labeled with $n < \lambda - a\lambda^{3/4}$. ALDOUS [1] has shown that the process $\{Y_\lambda(a,t);\ a > 0,\ t \geq 0\}$ converges weakly, as $\lambda \to \infty$, to a process $\{Y(a,t);\ a > 0,\ t \geq 0\}$, which he identified and showed that, in the limit as $t \to \infty$, converges weakly to the process

$$Y(a) = \max_{s \geq 0} (\sqrt{2} B_s - as), \qquad a > 0,$$

where B is the standard Brownian motion. He calls $\{Y(a);\ a > 0\}$ the exponential process, after the well-known fact that $Y(a)$ has the exponential distribution with mean $1/a$ for each a.

Our main contribution is to supply the probability law of Y in simpler terms. For this purpose we choose to work with

$$(1.1) \qquad Z_a = \frac{1}{\sqrt{2}} Y(\sqrt{2}\, a) = \max_{t \geq 0} (B_t - at), \qquad a > 0,$$

and let D_a be the last time t at which $B_t - at$ touches its zenith Z_a, that is, (1.2) $$D_a \;=\; \sup\{t : B_t - at = Z_a\}, \qquad a > 0.$$

It turns out that D_a is the left-derivative of Z at a and, thus, is related to the density of empty stalls in the parking lot, in equilibrium, around $\lambda - a\lambda^{3/4}$ for large λ.

The next section contains a few simple geometric observations. First we relate the process (D, Z) to the convex majorant of the Brownian motion B. Using the hard results of GROENEBOOM [3] and PITMAN [5] about the latter, we are able to express D and Z in terms of a Poisson random measure on $(0,\infty)\times(0,\infty)$. It follows, in particular, that D has non-stationary independent increments, and that (D, Z) is a non-homogeneous Markov process.

The process $a \rightarrow Z_a$ is continuous, concave, and decreases from its limit $+\infty$ at $a = 0+$ to its limit 0 at ∞. Therefore, its "hitting time" process

(1.3) $$A_z \;=\; \inf\{a : Z_a < z\}, \qquad z > 0,$$

is the functional inverse of Z. It turns out that the process A has the same probability law as Z. This observation is also put in the next section.

The last section is devoted to computational issues. We compute the joint distribution of D_a and Z_a and also the transition function of the Markov process (D, Z).

2. ZENITH PROCESS

The problem with the definition (1.1) of Z_a is that it suggests re-drawing the path $t \to B_t - at$ if we wish to vary a. The following observation circumvents the problem:

$$(2.1) \qquad Z_a = \inf\{x > 0 : x + at > B_t \text{ for all } t \geq 0\}.$$

Obviously, this is a re-wording of (1.1), but the mental picture it suggests is much more convenient for manipulating a: the line $t \to Z_a + at$ is the infimum of all lines of slope a that never touch B. This picture is drawn in Figure 1 below.

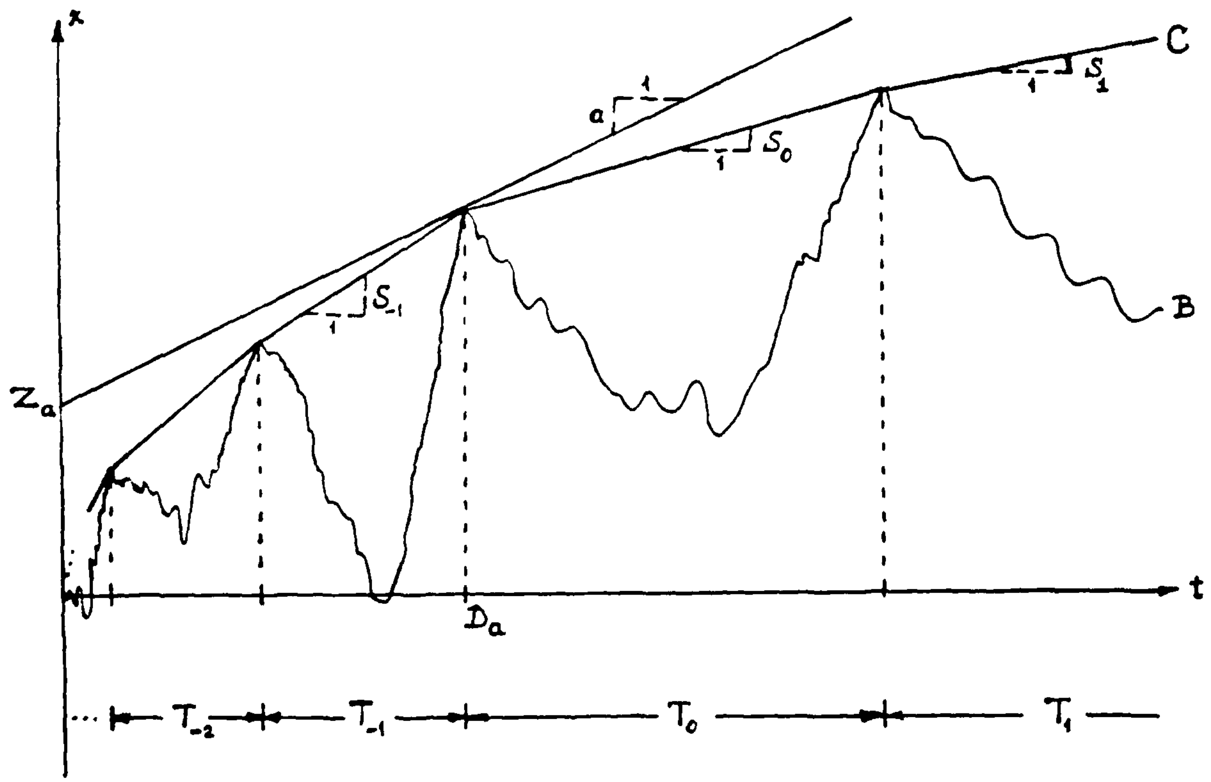

FIGURE 1

Let C denote the convex majorant of B, that is, the minimal convex path that dominates B (see Figure 1 again). Note that Z_a and D_a are determined by C: the line $t \to Z_a + at$ is the infimum of all lines of slope a that never touch C, and D_a is the last time at which that line touches C. In fact, for fixed $a > 0$, D_a is almost surely the only time t with $C_t = Z_a + at$.

It is known (see GROENEBOOM [3] for instance) that C is continuous and piecewise linear. The countable collection of its vertices has, almost surely, only one accumulation point, namely (0,0). Fix an $a > 0$; note that $(D_a, Z_a + aD_a)$ is a vertex; let $T_0, T_1, \ldots$ be the lengths of successive intervals of linearity going to the right from D_a; let $T_{-1}, T_{-2}, \ldots$ be those to the left; and let S_i be the slope of C over the interval whose length is denoted by T_i. The following major result was obtained by GROENEBOOM [3]; a simpler proof using the excursions of B may be found in PITMAN [5].

(2.2) THEOREM. The pairs (S_i, T_i), $i \in Z$, form a Poisson random measure N on $(0,\infty) \times (0,\infty)$ whose mean measure is

$$\nu(ds, dt) = \frac{ds}{s} \gamma_s(dt), \tag{2.3}$$

where γ_s is the gamma distribution with shape index 1/2 and scale parameter $s^2/2$ (the corresponding mean is $1/s^2$).

The probability law of a Poisson random measure is determined by its mean measure. Thus, the following specifies the probability law of (D, Z). For computational purposes, the representations given here for D_a and Z_a are the key starting points.

(2.4) PROPOSITION. For each $a > 0$,

$$D_a = \int_{[a,\infty)\times(0,\infty)} N(ds, dt)\, t, \tag{2.5}$$

$$Z_a = \int_a^\infty D_s\, ds = \int_{[a,\infty)\times(0,\infty)} N(ds, dt)\, (s-a)\, t. \tag{2.6}$$

The process D has non-stationary independent increments. The process (D, Z) is a temporally non-homogeneous Markov process.

PROOF. First note that (see Figure 1)

$$D_a = \sum_i T_i \, 1_{[a,\infty)}(S_i),$$

$$Z_a + a\, D_a = B(D_a) = \sum_i S_i \, T_i \, 1_{[a,\infty)}(S_i).$$

Expressed in terms of the Poisson random measure N, these become (2.5) and (2.6). The remaining statements are immediate from the independence of the restrictions of N to disjoint Borel sets.

Figure 2 below shows the qualitative features of D: it is piecewise constant, left-continuous, and decreases from its limit $+\infty$ at $a = 0+$ to its limit 0 at $+\infty$. It follows from (2.6) that Z is continuous, concave, piecewise linear, and decreases from its limit $+\infty$ at $a = 0+$ to 0 at $+\infty$.

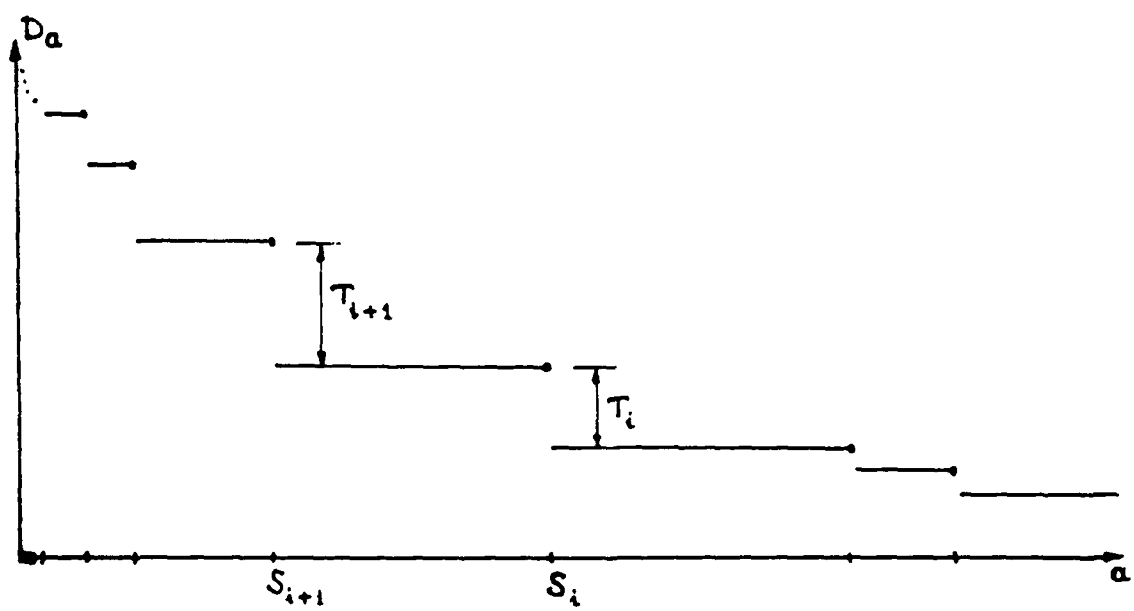

FIGURE 2

It was noted by ALDOUS [1] that, for each $c > 0$, the process $(c^2 D_{ca}, cZ_{ca})_{a>c}$ has the same probability law as (D, Z). This can be seen from the preceding characterization: $a \to c^2 D_{ca}$ jumps at the points S_i/c by the amounts $c^2 T_i$; the pairs $(S_i/c, c^2 T_i)$ form a Poisson random measure that has the same mean measure as N; hence, $a \to c^2 D_{ca}$ has the same law as D.

We end this section with an observation on the process

$$(2.7) \qquad A_z = \inf\{a : Z_a < z\}, \qquad z > 0.$$

Obviously, $z \to A_z$ is the functional inverse of the one-to-one mapping $a \to Z_a$ of $(0,\infty)$ onto $(0,\infty)$. It follows that the qualitative picture of A is exactly that of Z. In particular, A is piecewise linear and

$$(2.8) \qquad \hat{D}_z = \lim_{\varepsilon \downarrow 0} \frac{A_{z+\varepsilon} - A_z}{\varepsilon} = \frac{1}{D(A_z)}, \qquad z > 0.$$

The process $\hat{D}$ is piecewise constant, right-continuous, decreasing.

(2.9) PROPOSITION. The process $(\hat{D}, A)$ has the same probability law as the process (D, Z). In particular, the collection $\{Z(S_i);\ i \in \mathbf{Z}\}$ has the same law as the collection $\{S_i;\ i \in \mathbf{Z}\}$; they form Poisson random measures on $(0,\infty)$ with mean measure ds/s.

PROOF. We put (2.7) and (2.1) together and manipulate:

$$\begin{aligned} A_z &= \inf\{a : \inf\{x : x + at > B_t \text{ for all } t\} < z\} \\ &= \inf\{a : z + at > B_t \quad \text{for all } t\} \\ &= \inf\{a : a + zu > u B_{1/u} \quad \text{for all } u\}. \end{aligned}$$

This shows that A is the zenith process associated with the process $\{uB_{1/u};\ u \geq 0\}$, just as Z is the zenith process associated with B. Since $(uB_{1/u})$ is a standard Brownian motion like B, it follows that A has the same probability law as Z. This proves the first statement, since $\hat{D}$ is the derivative of $-A$ and D is the derivative of $-Z$.

The points S_i are the jump locations of D, and the points $Z(S_i)$ are those of $\hat{D}$. This proves the second statement.

3. ENTRANCE LAW AND TRANSITION FUNCTION

We derive the distribution of the random variable (D_a, Z_a) and the transition function of the process (D, Z). The computations rest on the characterization given by Proposition (2.4) and on the well-known formula for the Laplace functional of the Poisson random measure N with mean measure ν:

$$(3.1) \qquad E \exp - \int N(dx) f(x) = \exp - \int \nu(dx) \left(1 - e^{-f(x)}\right)$$

for every positive Borel function f on $(0,\infty) \times (0,\infty)$.

(3.2) PROPOSITION. For each $a > 0$,

$$(3.3) \qquad E \exp(-pD_a - qZ_a) = \frac{2a}{a + q + \sqrt{a^2 + 2p}}, \qquad p \geq 0,\ q \geq 0;$$

$$(3.4) \qquad P\{D_a \in dt,\ Z_a \in dz\} = dt dz \frac{2az\ e^{-(z+at)^2/2t}}{\sqrt{2\pi t^3}}, \qquad t > 0,\ z > 0.$$

In particular,

$$(3.5) \qquad P\{Z_a \in dz\} = dz\, 2ae^{-2az}, \quad P\{D_a \in dt\} = dt \int_t^\infty du \frac{ae^{-a^2u/2}}{\sqrt{2\pi u^3}}.$$

PROOF. Fix $a > 0$, $p \geq 0$, $q \geq 0$. In view of (2.5) and (2.6),

$$pD_a + qZ_a = \int_{[a,\infty)\times(0,\infty)} N(ds, dt)\ (pt + q(s-a)t).$$

Using (3.1) and the form of the mean measure ν given by (2.3), the Laplace transform (3.3) is obtained via elementary calculus. To invert the Laplace transform, first write it as

$$\int_0^\infty dz \ e^{-qz} \ 2ae^{-az} \ e^{-z\sqrt{a^2+2p}}$$

and then recall that $e^{-z\sqrt{2r}}$, $r \geq 0$, is the Laplace transform of H_z, the first time a standard Brownian motion hits the level z, that is,

$$e^{-z\sqrt{a^2+2p}} = \int_0^\infty dt \cdot \frac{z\, e^{-z^2/2t}}{\sqrt{2\pi t^3}} \ e^{-(p+a^2/2)t} .$$

The rest is trivial.

(3.6) REMARK. Although (3.4) is explicit and shades of exponential and stable distributions can be felt, it does not seem well-suited for probabilistic thinking. The following representation is better, especially for Monte-Carlo methods. For $a > 0$,

$$a^2 D_a = X\,(1-\sqrt{U}\,)^2, \quad aZ_a = X\sqrt{U}\;(1-\sqrt{U}\,),$$

where X and U are independent, U has the uniform distribution on (0,1), and X has the gamma distribution with shape index 3/2 and scale parameter 1/2.

The following specifies the joint Laplace transform of any number of increments of Z (upon taking $f = p_1 1_{A_1} + \ldots + p_n 1_{A_n}$ with $A_1, \ldots, A_n$ disjoint intervals).

(3.7) PROPOSITION. For any positive Borel function f on $(0,\infty)$,

$$E \ \exp \int_{(0,\infty)} f(a)\, dZ_a = \exp - \int_0^\infty ds \ \left(\frac{1}{s} - \frac{1}{\sqrt{s^2 + 2\bar{f}(s)}}\right)$$

where $\bar{f}(s)$ is the Lebesgue integral of f over $(0,s)$.

PROOF. Note that

$$\int f(a)\, dZ_a \;=\; -\int f(a)\, D_a\, da = -\int N(ds,dt)\ \bar{f}(s)t\ ,$$

and use (3.1).

As mentioned in Proposition (2.4), the process D has non-stationary independent increments, and the process (D, Z) is a non-homogeneous parameter Markov process. Let

$$(3.8)\qquad P_{ab}(t,x\ ;\ du, dy) \;=\; P\{D_b \in du\, , Z_b \in dy \mid D_a = t, Z_a = x\}$$

for $0 < b < a$, $0 < t < u$, $0 < x < y$ (in our zeal to deal with positive random variables, we choose to work with the parameters in decreasing order). In view of (2.5) and (2.6),

$$(3.9)\qquad P_{ab}(t,x\ ;\ du, dy) \;=\; P\{t + U \in du\ ,\ x + (a-b)t + Y \in dy\}\ ,$$

where

$$(3.10)\qquad U \;=\; \int_{[b,a)\times(0,\infty)} N(ds,dt)t\ , \qquad Y = \int_{[b,a)\times(0,\infty)} N(ds,dt)\ \ (s-b)t\ .$$

The joint Laplace transform of U and Y can be obtained from (3.1) as in the first step of the proof of (3.2):

$$(3.11)\qquad E\ e^{-pU-qY} \;=\; \frac{b}{a}\cdot\frac{a+q+\sqrt{a^2+2p+2(a-b)q}}{b+q+\sqrt{b^2+2p}}$$

$$= \frac{b}{a} + \frac{a-b}{a}\cdot\frac{1}{2}\cdot\frac{2b}{b+q+\sqrt{b^2+2p}}$$

$$+ \frac{a-b}{a}\cdot\frac{1}{2}\ \frac{2b}{b+q+\sqrt{b^2+2p}}\cdot\frac{1}{a-b}\int_b^a \frac{(c+q)\,dc}{\sqrt{(c+q)^2+2p-2bq-q^2}}\ .$$

Inverting this is tedious but manageable. It yields the following for the distribution φ of the pair

(U, Y):

$$(3.12) \qquad \varphi = \frac{b}{a}\,\delta_{(0,0)} + \left(1 - \frac{b}{a}\right)\left(\frac{1}{2}\,\lambda_b + \frac{1}{2}\,\lambda_b * \frac{\mu_{bb} - \mu_{ab}}{a - b}\right)$$

where the asterisque denotes convolution, δ_x is the Dirac measure at x, λ_b is the distribution of (D_b, Z_b) specified by (3.4), and

$$(3.13) \qquad \mu_{ab}(dt, dy) = \frac{e^{-b^2t/2}\,dt}{\sqrt{2\pi t^3}}\,\delta_{(a-b)t}(dy), \qquad t > 0,\ y > 0 .$$

Putting the distribution φ of (U, Y) into (3.9) yields an explicit expression for the transition function P_{ab}. As a by-product, we have the joint distribution of

$$U = D_b - D_a, \quad Y = Z_b - Z_a - (b - a) D_a .$$

Noting that D_a is independent of (U, Y), one can obtain the distribution of $(D_b - D_a,\ Z_b - Z_a)$ among other things.

However, it is clear that such results are of limited use because of their complexity. Overall, the computational complexity is caused by a confluence of two incompatible operations, addition and multiplication: look at the form (2.3) of the mean measure ν; the Haar measure ds/s indicates that the natural group operation on the jump points S_i is multiplication, whereas the jump amounts T_i are additive.

Of course, it is easy to transform the Poisson random measure N into one with a nicer intensity: define f to be the mapping $(s, t) \to (\log s,\ s^2 t)$; then the image of N under f is the Poisson random measure $\hat{N} = N f^{-1}$ on $(-\infty, \infty) \times (0, \infty)$ with mean measure $du\,\gamma(dv)$ where γ is the gamma distribution with shape and scale parameters equal to 1/2. But, then, expressions for D_a and Z_a in terms of $\hat{N}$ have to undo the transformation, and there is no gain at the end. Using $p = \log a$ to index the processes involved (and working with $\hat{Z}_p = Z(e^a)$) does not help either.

On the other hand, it is easy to describe the construction of the path of (D, Z) over the interval $(0,a]$. This may be useful for Monte-Carlo purposes.

First, we observe that the conditional distribution of S_{i+1} given $(S_i, S_{i-1}, \ldots)$ is the uniform distribution on $(0,S_i)$. Thus, to construct (D, Z) over $(0,a]$, we start with U and X described in Remark (3.6) and generate D_a and Z_a. Then, we let $U_1, U_2, \ldots$ be i.i.d. uniform on $(0,1)$, let $X_1, X_2, \ldots$ be i.i.d. Gaussian with mean 0 and variance 1, and set

$$(3.14) \qquad S_o = a \ , \quad S_i = aU_1 U_2 \cdots U_i \ , \quad T_i = (\frac{X_i}{S_i})^2 \ , \qquad i = 1,2,\ldots \ .$$

With these, define

$$(3.15) \qquad D(S_o) = D_a \ , \quad D(S_i) = D(S_{i-1}) + T_i \ , \qquad i \geq 1 \ ,$$

$$(3.16) \qquad Z(S_o) = Z_a \ , \quad Z(S_i) = Z(S_{i-1}) + (S_{i-1} - S_i) \cdot D(S_{i-1}) \ , \qquad i \geq 1 \ .$$

Then, $(D_b)_{b\in(0,a]}$ is the left-continuous piecewise constant path whose value at S_i is $D(S_i)$, and $(Z_b)_{b\in(0,a]}$ is the continuous piecewise linear path whose value at S_i is $Z(S_i)$. Incidentally, (S_i), $(S_i, D(S_i))$, $(S_i, D(S_i), Z(S_i))$ are all Markov chains.

REFERENCES

[1] D. ALDOUS. Some interesting processes arising as heavy traffic limits in an M/M/∞ storage process. *Stochastic Processes and Their Applications*, **22** (1986), 291-313.

[2] E. G. COFFMAN, T. T. KADOTA, and L. A. SHEPP. A stochastic model of fragmentation in dynamic storage allocation. *SIAM J. Computing*, **14** (1985), 416-425.

[3] P. GROENEBOOM. The concave majorant of Brownian motion. *Ann. Probab.* **11** (1983), 1016-1027.

[4] G. F. NEWELL. The M/M/∞ sservice system with ranked servers in heavy traffic. *Lecture Notes in Econ. and Math. Systems*, vol. 231. Springer-Verlag, New York, 1984.

[5] J. W. PITMAN. Remarks on the convex minorant of Brownian motion. In *Seminar on Stochastic Processes 1982*, pp. 219-227. Birkhäuser, Boston, 1983.