CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

EXCHANGEABLE RANDOM MEASURES IN THE PLANE

by

Olav Kallenberg

Technical Report No. 242

September 1988

161. R.L. Karandikar, On the Feynman-Kac formula and its applications to filtering theory. Oct. 86. *Appl. Math. Optimization*, to appear.

162. R.L. Taylor and T.-C. Hu, Strong laws of large numbers for arrays of rowwise independent random elements, Nov. 86. *Intern. J. Math. & Math. Sci.*, 10, 1987, 805-814.

163. M. O'Sullivan and T.R. Fleming, Statistics for the two-sample survival analysis problem based on product limit estimators of the survival functions, Nov. 86.

164. F. Avram, On bilinear forms in Gaussian random variables, Toeplitz matrices and Parseval's relation, Nov. 86.

165. D.B.H. Cline, Joint stable attraction of two sums of products, Nov. 86. *J. Multivariate Anal.*, 25, 1988, 272-285.

166. R.J. Wilson, Model fields in crossing theory—a weak convergence perspective, Nov. 86. *Adv. in Appl. Probability*, 20, 1988, to appear.

167. D.B.H. Cline, Consistency for least squares regression estimators with infinite variance data, Dec. 86.

168. L.L. Campbell, P.H. Wittke, G.D. Swanson, The distribution of amplitude and continuous phase of a sinusoid in noise, Nov. 86. *IEEE Trans. Information Theory*, to appear.

169. B.G. Nguyen, Typical cluster size for 2-dim percolation processes, Dec. 86. *J. Statist. Physics*, to appear.

170. H. Oodaira, Freidlein-Wentzell type estimates for a class of self-similar processes represented by multiple Wiener integrals, Dec. 86.

171. J. Nolan, Local properties of index-β stable fields, Dec. 86. *Ann. Probability*, 1988, to appear.

172. R. Menich and R.F. Serfozo, Optimality of shortest queue routing for dependent service stations, Dec. 86.

173. F. Avram and M.S. Taqqu, Probability bounds for M-Skorohod oscillations, Dec. 86.

174. F. Moricz and R.L. Taylor, Strong laws of large numbers for arrays of orthogonal random variables, Dec. 86. *Math. Nachr.*, 138, 1988, 243-250.

175. G. Kallianpur and V. Perez-Abreu, Stochastic evolution equations driven by nuclear space valued martingales, Apr. 87. *Appl. Math. Optimization*, 17, 1988, 237-272.

176. E. Merzbach, Point processes in the plane, Feb. 87. *Acta Appl. Mathematicae*, 1988, to appear.

177. Y. Kasahara, M. Maejima and W. Vervaat, Log fractional stable processes, March 87. *Stoch. Proc. Appl.*, 1989, to appear.

178. G. Kallianpur, A.G. Miamee and H. Niemi, On the prediction theory of two parameter stationary random fields, March 87. *J. Multivariate Anal.*, 1988, to appear.

179. R. Brigola, Remark on the multiple Wiener integral, Mar. 87.

180. R. Brigola, Stochastic filtering solutions for ill-posed linear problems and their extension to measurable transformations, Mar. 87.

181. G. Samorodnitsky, Maxima of symmetric stable processes, Mar. 87.

182. H.L. Hurd, Representation of harmonizable periodically correlated processes and their covariance, Apr. 87.

183. H.L. Hurd, Nonparametric time series analysis for periodically correlated processes, Apr. 87.

184. T. Mori and H. Oodaira, Freidlin-Wentzell estimates and the law of the iterated logarithm for a class of stochastic processes related to symmetric statistics, May 87.

185. R.F. Serfozo, Point processes, May 87. *Operations Research Handbook on Stochastic Processes*, to appear.

186. Z.D. Bai, W.Q. Liang and W. Vervaat, Strong representation of weak convergence, June 87.

187. O. Kallenberg, Decoupling identities and predictable transformations in exchangeability, June, 87.

188. O. Kallenberg, An elementary approach to the Daniell-Kolmogorov theorem and some related results, June 87. *Math. Nachr.*, to appear.

189. G. Samorodnitsky, Extrema of skewed stable processes, June 87.*Stochastic Proc. Appl.*, to appear.

190. D. Nualart, M. Sanz and M. Zakai, On the relations between increasing functions associated with two-parameter continuous martingales, June 87.

191. F. Avram and M. Taqqu, Weak convergence of sums of moving averages in the α-stable domain of attraction, June 87.

192. M.R. Leadbetter, Harald Cramér (1893-1985), July 87. *ISI Review*, 56, 1988, 89-97.

193. R. LePage, Predicting transforms of stable noise, July 87.

194. R. LePage and B.M. Schreiber, Strategies based on maximizing expected log, July 87.

195. J. Rosinski, Series representations of infinitely divisible random vectors and a generalized shot noise in Banach spaces, July 87.

196. J. Szulga, On hypercontractivity of α-stable random variables, $0<\alpha<2$, July 87.

197. I. Kuznezova-Sholpo and S.T. Rachev, Explicit solutions of moment problems I, July 87. *Probability Math. Statist.*, 10, 1989, to appear.

198. T. Hsing, On the extreme order statistics for a stationary sequence, July 87.

199. T. Hsing, Characterization of certain point processes, Aug. 87. *Stochastic Proc. Appl.* 26, 1987, 297-316.

200. J.P. Nolan, Continuity of symmetric stable processes, Aug. 87. *J. Multivariate Analysis*, 1988, to appear.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY: NA	3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for Public Release; Distribution Unlimited
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE: NA	
4. PERFORMING ORGANIZATION REPORT NUMBER(S): Technical Report No. 242	5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 88-1198
6a. NAME OF PERFORMING ORGANIZATION: University of North Carolina Center for Stochastic Processes / 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION: AFOSR/NM
6c. ADDRESS (City, State and ZIP Code): Statistics Department, CB #3260, Phillips Hall, Chapel Hill, NC 27599-3260	7b. ADDRESS (City, State and ZIP Code): Bldg. 410, Bolling AFB, DC 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION: AFOSR / 8b. OFFICE SYMBOL (If applicable): NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: F49620 85C 0144
8c. ADDRESS (City, State and ZIP Code): Bldg. 410, Bolling AFB, DC 20332-6448	10. SOURCE OF FUNDING NOS.: PROGRAM ELEMENT NO. 6.1102F; PROJECT NO. 2304; TASK NO. A6; WORK UNIT NO. leave blnk

11. TITLE (Include Security Classification): Exchangeable random measures in the plane

12. PERSONAL AUTHOR(S): Kallenberg, O.

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
preprint	FROM 1/9/88 TO 31/8/89	1988, September	65

16. SUPPLEMENTARY NOTATION: N/A

17. COSATI CODES: FIELD	GROUP	SUB. GR.	18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
XX	XXXXXXXXX	XXX	Key words and phrases: Separate and joint exchangeability; ergodic distributions; Poisson processes; uniform random variables.

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

A random measure ξ on $[0,1]^2$, $[0,1]\times R_+$ or R_+^2 is said to be separately exchangeable, if its distribution is invariant under arbitrary Lebesgue measure preserving transformations in the two coordinates, and jointly exchangeable if ξ is defined on $[0,1]^2$ or R_+^2, and its distribution is invariant under mappings by a common measure preserving transformation in both directions. In each case, we derive a general representation of ξ in terms of independent Poisson processes and i.i.d. random variables.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL: Major Brian W. Woodruff	22b. TELEPHONE NUMBER (Include Area Code): (202767-5026 / 22c. OFFICE SYMBOL: AFOSR/NM

DD FORM 1473, 83 APR EDITION OF 1 JAN 73 IS OBSOLETE.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

EXCHANGEABLE RANDOM MEASURES IN THE PLANE

By Olav Kallenberg[1]

Suggested running head: Exchangeable Random Measures

Author's mailing address: 120 Mathematics Annex
Auburn University
Auburn, Alabama 36849

Author's telephone #: (205)826-5111

1/ Department of Mathematics ACA, Auburn University, Auburn, Alabama 36849

EXCHANGEABLE RANDOM MEASURES IN THE PLANE*

By Olav Kallenberg

Auburn University
and
University of North Carolina at Chapel Hill

Abstract

A random measure ξ on $[0,1]^2$, $[0,1]\times R_+$ or R_+^2 is said to be separately exchangeable, if its distribution is invariant under arbitrary Lebesgue measure preserving transformations in the two coordinates, and jointly exchangeable if ξ is defined on $[0,1]^2$ or R_+^2, and its distribution is invariant under mappings by a common measure preserving transformation in both directions. In each case, we derive a general representation of ξ in terms of independent Poisson processes and i.i.d. random variables.

KEY WORDS: Separate and joint exchangeability; ergodic distributions; Poisson processes; uniform random variables.

*/ Research supported by NSF Grant DMS-8703804, and by the Air Force Office of Scientific Research Contract F49620 85 C 0144.

1. INTRODUCTION AND MAIN RESULTS

A random measure ξ defined on the product of two diffuse Polish measure spaces (X,λ) and (Y,μ) is said to be <u>separately exchangeable,</u> if its distribution is invariant under arbitrary measure preserving transformations of X and Y, i.e. if $\xi h^{-1} \overset{d}{=} \xi$ for any measurable mapping h of the form $h(x,y)=f(x)g(y)$, with $\lambda f^{-1}=\lambda$ and $\mu g^{-1}=\mu$. If $(X,\lambda)=(Y,\mu)$, and if the stated invariance is only required to hold for functions of the form $h(x,y)=f(x)f(y)$ with $\lambda f^{-1}=\lambda$, we shall say instead that ξ is <u>jointly exchangeable.</u> Of these two notions, separate exchangeability is clearly the strongest. An even stronger condition is that of <u>complete exchangeability,</u> where $\xi h^{-1} \overset{d}{=} \xi$ is required for any measurable h with $(\lambda\times\mu)h^{-1}=\lambda\times\mu$.

The main purpose of ~~the present~~ this paper is to derive <u>de Finetti-type representations</u> of arbitrary separately or jointly exchangeable ~~random~~ measures ξ. By this is meant representations of the distributions of ξ as unique mixtures (convex combinations) of so called <u>extreme</u> exchangeable distributions. The existence of such integral representations is essentially a consequence of ~~the~~ <u>general theory</u> (cf. Maitra[15], Dynkin[3], and Section 12 in Aldous[2]), so ~~our~~ so the author's main point is to describe the extreme measures explicitly.

Through suitable Borel isomorphisms from the two spaces, one may easily reduce the problem to the special case when X and Y are real intervals, equipped with corresponding restrictions lambda λ and mu μ of Lebesgue measure (henceforth always denoted by lambda λ). Depending on whether X and Y are finite or infinite, there are essentially only five different cases to examine, namely those of separate exchangeability on $[0,1]^2$, $R_+\times[0,1]$ or R_+^2, and of joint exchangeability on $[0,1]^2$ or R_+^2. The general representations in these five fundamental cases will be given in our main Theorems 1-5, stated later in this section.

The corresponding one-dimensional case has been studied extensively in Kallenberg[7,9], and the one-dimensional representation theorems will in fact play a basic role in the present paper. Those results will be presented in

Keywords: ergodic distributions, random variables. (KR)

Section 3 below, in extended forms suitable for our present needs, and with new and simpler proofs. To establish the representations on R_+^2, we shall further need some extensions of the representation theorems for exchangeable arrays, where the original results are due to Aldous[1,2] and Hoover[5,6] (see also Kallenberg[11,12]). In this context, the required extensions are provided in Section 4.

Cases of continuous parameter multivariate exchangeability were first mentioned briefly in Kallenberg[8] and in Section 15 of Aldous[2]. Our Theorem 4 essentially confirms a conjecture by Aldous[2], p.139, about the general form of an extreme, separately exchangeable <u>counting</u> random measure on R_+^2 (though Aldous' statement appears somewhat unclear, and his convergence criteria are wrong). Analogous problems for continuous two-parameter processes with separately or jointly exchangeable increments have been studied extensively by Kallenberg[11] and Hestir[4].

To state our main representation theorems, recall that all notions of exchangeability are henceforth with respect to Lebesgue measure λ on $[0,1]$ or R_+, and that an exchangeable distribution is <u>extreme</u> by definition, if it admits only the trivial representation as a mixture of exchangeable laws. By saying that a random object ξ has an a.s. <u>representation</u> $f(\eta)$, we mean that there exists some random element (r.e.) η, possibly defined on some extension of the original probability space $(\Omega,\mathcal{F},P)$, such that $\xi=f(\eta)$ holds a.s. Thus no claim is made about uniqueness or even measurability of η. Note that no extension as above is needed, if the probability space is already rich enough to support an independent random variable (r.v.) with a uniform distribution on $[0,1]$ ($U(0,1)$, cf. Lemma 1).

By a <u>unit rate Poisson process</u> η on R_+^d, we shall mean a Poisson random measure in the sense of Kallenberg[9] with intensity measure λ^d. We shall further say that η is <u>formed</u> by the sequence $\alpha_1,\alpha_2,\ldots$, if $\eta=\sum\delta_{\alpha_j}$, where δ_a denotes a unit mass at a (Dirac measure). Since the <u>atom positions</u> α_j are only determined up to a random permutation of indices, to say that a r.e. ζ is independent of

(α_j) is clearly stronger than saying that ζ is independent of η. To avoid misunderstandings, we further stress the distinction between the phrases 'independent sequence' and 'sequence of independent ...', where independence in the former expression is between the sequence and all previously mentioned r.e.'s, and in the latter between the elements in the sequence. Thus the somewhat awkward phrase 'an independent sequence of independent r.e.'s' is required to express both.

Our five main representation theorems may now be stated, in order of increasing depth and complexity, starting with the relatively elementary representations on the unit square. We shall use the notation λ_D for the measure along the diagonal D in $[0,1]^2$ or R_+^2 with projections λ on the diagonal axes. Thus $\lambda_D(B)=\lambda\{x;\ (x,x)\in B\}$ for Borel sets B in $[0,1]^2$ or R_+^2.

Theorem 1. A random measure ξ on $[0,1]^2$ is separately exchangeable, iff it has an a.s. representation

$$\xi = \sum_i \sum_j \alpha_{ij}\delta_{\tau_i,\tau_j'} + \sum_j \{\beta_j(\delta_{\tau_j}\times\lambda) + \beta_j'(\lambda\times\delta_{\tau_j'})\} + \gamma\lambda^2, \tag{1.1}$$

for some R_+-valued r.v.'s $\alpha_{ij},\beta_i,\beta_j',\gamma$, $i,j\in N$, and some independent set of independent U(0,1) r.v.'s τ_i,τ_j', $i,j\in N$. Moreover, the former set of r.v.'s may be chosen to be non-random, iff $P\xi^{-1}$ is extreme.

Theorem 2. A random measure ξ on $[0,1]^2$ is jointly exchangeable, iff it has an a.s. representation

$$\xi = \sum_i \sum_j \alpha_{ij}\delta_{\tau_i,\tau_j} + \sum_j \{\beta_j(\delta_{\tau_j}\times\lambda) + \beta_j'(\lambda\times\delta_{\tau_j})\} + \gamma\lambda^2 + \vartheta\lambda_D, \tag{1.2}$$

for some R_+-valued r.v.'s $\alpha_{ij},\beta_j,\beta_j',\gamma,\vartheta$, $i,j\in N$, and some independent sequence of independent U(0,1) r.v.'s $\tau_1,\tau_2,\ldots$ Moreover, the former set of r.v.'s may be chosen to be non-random, iff $P\xi^{-1}$ is extreme.

Theorem 3. A random measure ξ on $R_+\times[0,1]$ is separately exchangeable, iff it has an a.s. representation

$$\xi = \sum_i \sum_j f_j(\alpha,\vartheta_i)\delta_{\sigma_i,\tau_j} + \sum_i \sum_k g_k(\alpha,\vartheta_i)\delta_{\sigma_i,\rho_{ik}} + \sum_i h(\alpha,\vartheta_i)(\delta_{\sigma_i}\times\lambda) + \sum_j \beta_j(\lambda\times\delta_{\tau_j}) + \gamma\lambda^2, \tag{1.3}$$

for some measurable functions $f_j, g_k, h\colon R_+^2 \to R_+$, $j,k\in N$, some R_+-valued r.v.'s α, β_j, γ, $j\in N$, some independent set of independent $U(0,1)$ r.v.'s τ_j and ρ_{ik}, $i,j,k\in N$, and some independent sequence of random vectors (σ_i,ϑ_i), $i\in N$, which form a unit rate Poisson process on R_+^2. Moreover, the r.v.'s α, β_j and γ may be chosen to be non-random, iff $P\xi^{-1}$ is extreme.

Theorem 4. A random measure ξ on R_+^2 is separately exchangeable, iff it has an a.s. representation

$$\begin{aligned}\xi = {}& \sum_i \sum_j f(\alpha,\vartheta_i,\vartheta_j',\zeta_{ij})\delta_{\tau_i,\tau_j'} + \sum_k \ell(\alpha,\eta_k)\delta_{\rho_k,\rho_k'} + \gamma\lambda^2 \\ & + \sum_i \sum_k \left\{ g(\alpha,\vartheta_i,\chi_{ik})\delta_{\tau_i,\sigma_{ik}} + g'(\alpha,\vartheta_i',\chi_{ik}')\delta_{\sigma_{ik}',\tau_i'} \right\} \\ & + \sum_i \left\{ h(\alpha,\vartheta_i)(\delta_{\tau_i}\times\lambda) + h'(\alpha,\vartheta_i')(\lambda\times\delta_{\tau_i'}) \right\},\end{aligned} \tag{1.4}$$

for some measurable functions $f\colon R_+^4 \to R_+$, $g,g'\colon R_+^3 \to R_+$ and $h,h',\ell\colon R_+^2 \to R_+$, some R_+-valued r.v.'s α and γ, some independent set of independent $U(0,1)$ r.v.'s ζ_{ij}, $i,j\in N$, and some independent set of independent sequences of random vectors $((\tau_i,\vartheta_i),\ i\in N)$, $((\tau_i',\vartheta_i'),\ i\in N)$, $((\sigma_{jk},\chi_{jk}),\ k\in N)$, $((\sigma_{jk}',\chi_{jk}'),\ k\in N)$, $j\in N$, and $((\rho_k,\rho_k',\eta_k),\ k\in N)$, which form unit rate Poisson processes on R_+^2 or R_+^3, respectively. Moreover, α and γ may be chosen to be non-random, iff $P\xi^{-1}$ is extreme.

Theorem 5. A random measure ξ on R_+^2 is jointly exchangeable, iff it has an a.s. representation

$$\begin{aligned}\xi = {}& \sum_i \sum_j f(\alpha,\vartheta_i,\vartheta_j,\zeta_{ij})\delta_{\tau_i,\tau_j} + \beta\lambda_D + \gamma\lambda^2 \\ & + \sum_i \sum_k \left\{ g(\alpha,\vartheta_i,\chi_{ik})\delta_{\tau_i,\sigma_{ik}} + g'(\alpha,\vartheta_i,\chi_{ik})\delta_{\sigma_{ik},\tau_i} \right\} \\ & + \sum_i \left\{ h(\alpha,\vartheta_i)(\delta_{\tau_i}\times\lambda) + h'(\alpha,\vartheta_i)(\lambda\times\delta_{\tau_i}) \right\}\end{aligned} \tag{1.5}$$

$$+ \sum_k \{ \ell(\alpha,\eta_k)\delta_{\rho_k,\rho_k'} + \ell'(\alpha,\eta_k)\delta_{\rho_k',\rho_k} \},$$

for some measurable functions $f: R_+^4 \to R_+$, $g,g': R_+^3 \to R_+$ and $h,h',\ell,\ell': R_+^2 \to R_+$, some R_+-valued r.v.'s α,β,γ, some independent set of independent $U(0,1)$ r.v.'s $\zeta_{ij}=\zeta_{ji}$, $1\le i\le j$, and some independent set of independent sequences of random vectors $((\tau_i,\vartheta_i),\ i\in N)$, $((\sigma_{jk},\chi_{jk}),\ k\in N)$, $j\in N$, and $((\rho_k,\rho_k',\eta_k),\ k\in N)$, which form unit rate Poisson processes on R_+^2 and R_+^3, respectively. Moreover, α, β and γ may be chosen to be non-random, iff $P\xi^{-1}$ is extreme.

In most previous work, the various notions of exchangeability for random measures have actually been defined in the formally weaker sense of invariance under permutations of the increments. Thus for random measures ξ on $S=[0,1]^2$, $R_+\times[0,1]$ or R_+^2, the space S is divided into an arbitrary regular grid of dyadic squares

$$A_{ij}^{(n)} = [(i-1)2^{-n},\ i2^{-n}) \times [(j-1)2^{-n},\ j2^{-n}), \quad i,j=1,2,\ldots, \tag{1.6}$$

and one requires the associated arrays of increments $\xi_{ij}^{(n)}=\xi A_{ij}^{(n)}$, $i,j=1,2,\ldots$, to be separately or jointly exchangeable, in the sense that

$$(\xi_{\pi_i,\pi_j'}^{(n)}) \stackrel{d}{=} (\xi_{ij}^{(n)}) \quad \text{or} \quad (\xi_{\pi_i,\pi_j}^{(n)}) = (\xi_{ij}^{(n)}), \tag{1.7}$$

for each $n\in N$, and for any finite permutations π and π' of the two index sets.

An intermediate version is to consider the array of restrictions of ξ to the squares $A_{ij}^{(n)}$. More precisely, we may define for each $n\in N$ an array of random measures $\tilde{\xi}_{ij}^{(n)}$ on $[0,1]^2$ by

$$\tilde{\xi}_{ij}^{(n)}(dsdt) = \xi((i-1+ds)2^{-n} \times (j-1+dt)2^{-n}),\ s,t\in[0,1],\ i,j=1,2,\ldots, \tag{1.8}$$

and require the condition in (1.7) to hold with the $\xi_{ij}^{(n)}$ replaced by $\tilde{\xi}_{ij}^{(n)}$. This is clearly equivalent to restricting f and g in our original definition to the sub-classes of λ-preserving transformations, which only permute a finite number of disjoint dyadic intervals of equal length, while leaving the remaining set invariant.

The possibilities seem bewildering, but fortunately the different ways of defining exchangeability for random measures turn out to be equivalent. In the one-dimensional case, this was noted already in Kallenberg[9], Lemma 9.0, and for higher dimensions a simple proof will be furnished in Section 8 below. The equivalence in the two-dimensional case will also follow from our proofs of the main results, since these will be based on the third definition above, the one involving transformations of permutation type.

In this way, a slightly greater generality will thus be attained for free. However, our main reason for the chosen approach is to make certain general results from the abstract theory apply. Here one considers random elements ξ in some Polish space S, equipped with its Borel σ-field $\mathcal{S}$ and a countable group $\mathcal{T}$ of measurable transformations of S, and one says that ξ (or its distribution $P\xi^{-1}$) is $\mathcal{T}$-<u>exchangeable</u>, if $T\circ\xi \overset{d}{=} \xi$ for every $T\in\mathcal{T}$. A set $I\in\mathcal{S}$ is said to be $\mathcal{T}$-<u>invariant</u>, if $T^{-1}I=I$ for all $T\in\mathcal{T}$, and the class of all $\mathcal{T}$-invariant sets form a sub-σ-field of $\mathcal{S}$, the so called $\mathcal{T}$-<u>invariant</u> σ-<u>field</u> $\mathcal{J}$. One says that ξ or $P\xi^{-1}$ is $\mathcal{T}$-<u>ergodic</u>, if $\xi^{-1}\mathcal{J}$ is P-<u>trivial</u>, i.e. if $P\{\xi\in I\}$ equals 0 or 1 for every $I\in\mathcal{J}$.

In this abstract setting, it is known (cf. Aldous[2]) that the distribution Q of an arbitrary $\mathcal{T}$-exchangeable r.e. ξ in S has a unique integral representation in terms of extreme points, and that the latter are identical with the $\mathcal{T}$-ergodic distributions. Furthermore, the conditional distributions

$$Q[\cdot\,|\mathcal{J}]\circ\xi = P[\xi\in\cdot\,|\xi^{-1}\mathcal{J}] \tag{1.9}$$

are a.s. ergodic, so the de Finetti-type representation of Q is formally obtained simply by taking expectations in (1.9).

It should now be clear why the third of the proposed definitions is the most appropriate one for our needs. The class of arbitrary measure preserving transformations f (or of their tensor products $f\times g$ or $f\times f$) is not a group, simply because f is usually not invertible. Moreover, the class of such mappings is uncountable. On the other hand, the elementary definition based on permutations of increments over square lattices is not suitable either, since it is stated in terms of transformations of certain functions of the random measure ξ, rather than for transformations of ξ itself. Only the last definition is useful, in the sense of fitting into the abstract framework.

To see this, all we need to verify is that random measures on an arbitrary Euclidean rectangle A may be regarded as random elements in a suitable Polish space. For this purpose, we take S to be the set of all locally finite Borel measures on A, and endow S with the σ-field $\mathcal{S}$ generated by all coordinate mappings $\mu \to \mu B$, $\mu \in S$, where B is an arbitrary Borel set in A. Then $\mathcal{S}$ is also generated by the vague topology on S, and the latter is known to be Polish (cf. Kallenberg[9], pp. 12 and 170).

For the reasons just mentioned, we shall henceforth (except in Proposition 1) take the notions of exchangeability, ergodicity and invariance for random measures to be defined with respect to the group of λ-preserving transformations of $[0,1]$ or R_+ which only permute a finite number of disjoint dyadic intervals of equal length, in the sense described before. In particular, the terms 'ergodic' and 'extreme' may then be taken as synonymous.

As already mentioned, the proofs of our main results require some representation formulas and other structural properties in the one-dimensional case, as well as for two-dimensional exchangeable arrays. These are provided in Sections 3 and 4, respectively. In Section 2, we collect a variety of abstract results, including a general ergodicity criterion, and a device for automatic extension of most representation formulas from the ergodic to the general case. After this preparation, the main results will be proved in Sections 5-7. The final Section 8 is devoted to the before-mentioned comparison between the different notions of exchangeability, to criteria for convergence of the series in the main representation formulas, and to a discussion of some related questions of uniqueness.

Several auxiliary results in this paper may be of some independent interest, in which case their statements are often slightly more general than actually needed for the main proofs. In fact, the author's main motivation for the present work was to develop some general techniques and a deeper understanding within the area of multivariate exchangeability, rather than just provide some rigorous proofs of certain representation formulas, whose statements may be intuitively rather obvious anyway.

2. SOME ABSTRACT LEMMAS

Our aim in this section is to establish some abstract results of varying difficulty which will be needed to prove the main results of the paper. Some of the results in this section may be of independent interest, such as Lemma 3 which yields an automatic extension of our representation formulas from the extreme to the general case, and Lemmas 4-5 where we state some useful conditions for ergodicity and propose a related approach to the construction of directing r.e.'s.

Our first result is the simple 'coupling' Lemma 1.1 from Kallenberg[10], which we restate here for the reader's convenience.

Lemma 1. *Let ξ and η be r.e.'s in some Polish spaces* S *and* T, *such that* $\xi \overset{d}{=} f(\eta)$ *for some measurable mapping* $f\colon T \to S$. *Then there exists some measurable mapping* $g\colon S \times [0,1] \to T$, *such that whenever* ϑ *is a* U(0,1) *r.v. independent of* ξ, *the r.e.* $\eta' = g(\xi, \vartheta)$ *satisfies* $\xi = f(\eta')$ *a.s. and* $\eta' \overset{d}{=} \eta$.

A typical application of this result is to obtain a.s. representations of random elements from their distributional properties. For example, a random measure with the same distribution as ξ in (1.1) has itself an a.s. representation of this form, on a suitable extension of the original probability space.

The representations of the extreme distributions in Theorems 1 and 2 are both *parametric*, in the sense that the general ergodic distribution is specified by an array of real parameters. The situation in Theorems 3-5 appears to be very different, since here even measurable functions appear as parameters. (This is also true for the basic representations of Aldous[1,2] and Hoover[5,6] for exchangeable arrays.) However, we shall show that the latter representations may be restated in parametric form, which is useful for proving ergodicity criteria and extensions to the non-ergodic case (cf. Kallenberg[11]).

Lemma 2. *Given a σ-finite measure space* (S,μ) *and a Polish space* T, *there exists a measurable mapping* $F\colon [0,1] \times S \to T$, *such that any measurable function* $f\colon S \to T$ *agrees a.e.* μ *with* $F(c,\cdot)$ *for some* $c \in [0,1]$.

Proof. By a Borel isomorphism, we may reduce to the case when T is a Borel subset of $[0,1]$. Next we fix a complete orthonormal system $\varphi_1,\varphi_2,\ldots$ in $L^2(S,\mu)$ with arbitrarily specified versions, and define a jointly measurable version of the function

$$h(a,s) = \sum_{k=1}^{\infty} a_k \varphi_k(s), \quad a=(a_k)\in\ell^2,\ s\in S,$$

by taking the limit of partial sums along the index sequence

$$m_n = \inf\Big\{r\in N;\ \sum_{k>r} a_k^2 < 2^{-n}\Big\}, \quad n\in N,$$

whenever that sequence converges, and putting $h(a,s)=0$ on the exceptional μ-nullset. For any $f\in L^2(S,\mu)$, there is then an $a\in\ell^2$ with $h(a,\cdot)=f$ a.e. μ, obtained by taking $a_k=\int\varphi_k f d\mu$ for each k. In particular, $h(a,\cdot)\in T$ a.e. when f is T-valued, so by an obvious modification of h, we get a measurable function $H: \ell^2\times S\to T$ with the same property as h. Next note that ℓ^2 is Borel isomorphic to a Borel subset of $[0,1]$. Hence there exists a measurable mapping g of $[0,1]$ onto ℓ^2, and it remains to take $F(c,\cdot)=H(g(c),\cdot)$ for all $c\in[0,1]$. □

The last lemma will now be used to prove a generalized version of Lemma 2.2 in Kallenberg[11], which is going to be our main tool for extending representation formulas from the ergodic to the general case.

Lemma 3. Fix four Polish spaces S, T, U, V, a measurable mapping $F: T\times U^\infty\to V$, and some r.e.'s τ in T and $\sigma_1,\sigma_2,\ldots$ in S. Let $\mathcal{G}$ denote the class of measurable functions from S to U, and consider a r.e. ξ in V and some σ-field $\mathcal{J}\subset\mathcal{F}$, such that

$$P[\xi\in\cdot\,|\,\mathcal{J}] \in \{P(F(\tau,(g\circ\sigma_j)))^{-1};\ g\in\mathcal{G}\}\ \text{a.s.} \tag{2.1}$$

Then there exist some measurable function $G: [0,1]\times S\to U$, some U(0,1) r.v. α, and some independent random sequence $(\tau',\sigma_1',\sigma_2',\ldots) \overset{d}{=} (\tau,\sigma_1,\sigma_2,\ldots)$, such that

$$\xi = F(\tau',(G(\alpha,\sigma_j'))) \ \text{a.s.} \tag{2.2}$$

Proof. Define on S the probability measure $\mu=\sum 2^{-j}P\sigma_j^{-1}$. By Lemma 2, there exists some measurable function $H: [0,1]\times S\to U$, such that every $g\in\mathcal{G}$ agrees a.e. μ with $H(c,\cdot)$ for some $c\in[0,1]$. Hence $g(\sigma_j)=H(c,\sigma_j)$ a.s. for each j, so (2.1) yields

$$P[\xi\in\cdot\,|\,\mathcal{T}]\in\{P(F(\tau,(H(c,\sigma_j)))^{-1},\ c\in[0,1]\}.$$

By Lemma 2.2 in Kallenberg[11], there exist some $\mathcal{T}$-measurable r.v. γ in $[0,1]$, and an independent random sequence $(\tau',\sigma_1',\sigma_2',\ldots)\overset{d}{=}(\tau,\sigma_1,\sigma_2,\ldots)$, such that

$$\xi=F(\tau',(H(\gamma,\sigma_j'))) \quad \text{a.s.}$$

Now $\gamma\overset{d}{=}p(\alpha_0)$, where α_0 is $U(0,1)$ while p is the inverse distribution function of γ, and by Lemma 1 we may then choose α $U(0,1)$ and independent of τ' and (σ_j'), such that $\gamma=p(\alpha)$ a.s. But then

$$\xi=F(\tau',(H(p(\alpha),\sigma_j'))) \quad \text{a.s.,}$$

and (2.2) follows if we take $G(a,\cdot)=H(p(a),\cdot)$. $\square$

The next result will be our main tool to characterize extremality. Though stated formally for representations of parametric type, it extends immediately, via Lemma 2, to the wider class of representations containing arbitrary measurable functions.

Lemma 4. *Fix four Polish spaces* S,T,U,V, *a measurable mapping* $f\colon T\times U\to S$, *a r.e.* ϑ *in* U, *and an independent* $U(0,1)$ *r.v.* γ. *Let* $\mathcal{M}$ *denote the set of all convex combinations of measures* $m_t=P\xi_t^{-1}$ *with* $\xi_t=f(t,\vartheta)$, $t\in T$. *Then each* m_t *is extreme in* $\mathcal{M}$, *provided there exist some measurable mappings* $g\colon S\to V$ *and* $h\colon T\to V$, *such that*

$$g\circ\xi_t=h(t) \quad \text{a.s.,} \quad t\in T, \tag{2.3}$$

$$h(s)=h(t) \implies m_s=m_t, \quad s,t\in T. \tag{2.4}$$

This holds in particular, if there exists some measurable mapping $F\colon S\times[0,1]\to S$, *such that* ξ_t *and* $\xi_t'=F(\xi_t,\gamma)$ *are i.i.d. for every* $t\in T$.

Proof. Fix $t\in T$, and assume that $m_t=\int m_s\mu(ds)$ for some probability measure μ on T. Then $\xi_t\overset{d}{=}\xi_\tau$, where τ is a r.e. in T which is independent of ϑ with distribution μ. If f and g exist with the stated properties, it follows from

(2.3) and Fubini's theorem that $h(t)=h(\tau)$ a.s., so $m_t=m_\tau$ a.s. by (2.4), which means that $m_s=m_t$ for $s\in T$ a.e. μ. Hence m_t is extreme.

Assuming instead that F exists with the stated properties, we define

$$g(s)=P(F(s,\gamma))^{-1},\ s\in S;\quad h(t)=m_t,\ t\in T. \tag{2.5}$$

Then (2.4) is trivially true, while (2.3) is obtained through the chain of a.s. equalities

$$g\circ\xi_t = P[F(\xi_t,\gamma)\in\cdot\,|\,\xi_t] = P[\xi_t'\in\cdot\,|\,\xi_t] = P\xi_t'^{-1} = P\xi_t^{-1} = m_t = h(t),$$

where the first relation holds by (2.4), Fubini's theorem, and the fact that ϑ and γ are independent. □

A related problem of general importance (cf. Kallenberg[7,9,11]) is to find a so called <u>directing r.e.</u> ρ associated with an exchangeable r.e. ξ, with the properties that ρ is a.s. ξ-measurable and invariant, and such that the distributions of ξ and ρ determine each other uniquely. Here such an object will be obtained, under the hypotheses of Lemma 4. The result applies immediately to most representations in this paper.

<u>Lemma 5. Assume in Lemma 4 that $\mathcal{M}$ consists of all exchangeable distributions with respect to some countable group $\mathcal{T}$ of measurable transformations of S, and let g and h be such as stated. Then every $\mathcal{T}$-exchangeable r.e. ξ in S admits an a.s. representation $\xi=f(\tau,\eta)$ for some r.e. $\eta\overset{d}{=}\vartheta$ in U and some independent r.e. τ in T. Moreover, $\rho=h(\tau)$ is a directing r.e. for ξ.</u>

<u>Proof.</u> By Lemma 2.2 in Kallenberg[11], the first statement is even true with τ a $\xi^{-1}\mathcal{J}$-measurable r.e., where $\mathcal{J}$ denotes the $\mathcal{T}$-invariant σ-field in S. By (2.3) and Fubini's theorem, we get for any τ

$$\rho = h(\tau) = g(\xi)\quad \text{a.s.}, \tag{2.6}$$

which shows that ρ is a.s. independent of the choice of τ. Choosing τ to be $\xi^{-1}\mathcal{J}$-measurable, we get a $\xi^{-1}\mathcal{J}$-measurable version of ρ, which is clearly invariant.

From (2.6) it is further seen that the distribution of ξ determines that of ρ. To prove the reverse statement, let ξ' be another $\mathcal{T}$-exchangeable r.e., say with a.s. representation $\xi'=f(\tau',\eta')$, and assume that $h(\tau)\overset{d}{=}h(\tau')$. Then there exists by Lemma 1 some r.e. $\tau''\overset{d}{=}\tau'$ with $h(\tau'')=h(\tau)$ a.s., so $m_\tau=m_{\tau''}\overset{d}{=}m_{\tau'}$ by (2.4), and it follows by Fubini's theorem that $P\xi^{-1}=Em_\tau=Em_{\tau'}=P\xi'^{-1}$. □

The remainder of this section is devoted to some rather technical results, which will be of frequent use in subsequent sections.

Lemma 6. Let S and S' be two measurable spaces, endowed with groups $\mathcal{T}$ and $\mathcal{T}'$ of measurable transformations, and fix a measurable mapping $f: S\to S'$ with

$$\{f\circ T;\ T\in\mathcal{T}\}=\{T'\circ f;\ T'\in\mathcal{T}'\}. \tag{2.7}$$

Further assume that ξ is a $\mathcal{T}$-exchangeable [$\mathcal{T}$-ergodic] r.e. in S. Then the r.e. $f\circ\xi$ in S' is $\mathcal{T}'$-exchangeable [$\mathcal{T}'$-ergodic].

Proof. Assume that ξ is $\mathcal{T}$-exchangeable, fix $T'\in\mathcal{T}'$, and choose $T\in\mathcal{T}$ with $f\circ T=T'\circ f$. Then

$$T'\circ f\circ\xi=f\circ T\circ\xi\overset{d}{=}f\circ\xi,$$

which shows that $f\circ\xi$ is $\mathcal{T}'$-exchangeable.

Next we note that the invariant σ-fields $\mathcal{J}$ and $\mathcal{J}'$ in S and S' satisfy $f^{-1}\mathcal{J}'\subset\mathcal{J}$. In fact, letting $I'\in\mathcal{J}'$ and $T\in\mathcal{T}$ be arbitrary, and choosing $T'\in\mathcal{T}'$ with $T'\circ f=f\circ T$, we get $T^{-1}f^{-1}I'=f^{-1}T'^{-1}I'=f^{-1}I'$. If ξ is $\mathcal{T}$-ergodic, we hence obtain

$$P\{f\circ\xi\in I'\}=P\{\xi\in f^{-1}I'\}=0 \text{ or } 1,\quad I'\in\mathcal{J}',$$

which shows that $f\circ\xi$ is $\mathcal{T}'$-ergodic. □

Lemma 7. Fix three measurable spaces S, S' and S'', endowed with classes $\mathcal{T},\mathcal{T}'$ and $\mathcal{T}''$ of measurable transformations, and a measurable mapping $f: S'\times S''\to S$. Assume that, for every $T\in\mathcal{T}$, there exist some $T'\in\mathcal{T}'$ and some family $T''_x\in\mathcal{T}''$, $x\in S'$, such that $T''_x y$ is product measurable in $(x,y)\in S'\times S''$, and moreover

$$T\circ f(x,y)=f(T'x,\ T''_x y),\qquad x\in S',\ y\in S''. \tag{2.8}$$

Let the r.e.'s ξ in S' and η in S'' be independent and exchangeable with respect to $\mathcal{T}'$ and $\mathcal{T}''$, respectively. Then $f(\xi,\eta)$ is a $\mathcal{T}$-exchangeable r.e. in S.

Proof. Fix $T\in\mathcal{T}$, and choose $T'\in\mathcal{T}'$ and $T''_x\in\mathcal{T}''$, $x\in S'$, with the stated properties. Then $T\circ f(\xi,\eta)=f(T'\circ\xi,\ T''_\xi\circ\eta)$, so it is enough to show for any bounded measurable function $g\colon S'\times S''\to R$ that $Eg(T'\circ\xi,\ T''_\xi\circ\eta)=Eg(\xi,\eta)$. To see this, let μ and ν denote the distributions of ξ and η. Using Fubini's theorem and the independence and exchangeability of ξ and η, we get

$$\begin{aligned} Eg(T'\circ\xi,\ T''_\xi\circ\eta) &= \int\mu(dx)\int g(T'x,T''_xy)\nu(dy) = \int\mu(dx)\int g(T'x,y)\nu(dy) \\ &= \int\nu(dy)\int g(T'x,y)\mu(dx) = \int\nu(dy)\int g(x,y)\mu(dx) = Eg(\xi,\eta). \end{aligned}$$ □

Before stating the next two results, we need to introduce the notions of separate or joint exchangeability and ergodicity for random elements in product spaces. (The double meaning of these terms in the context of random arrays or measures should cause no confusion.) Thus assume that $\xi=(\xi_1,\ldots,\xi_n)$ is a r.e. in $S_1\times\ldots\times S_n$, where each S_k is equipped with a class $\mathcal{T}_k$ of measurable transformations. Then ξ is said to be separately exchangeable or ergodic, if it is exchangeable or ergodic with respect to the class of transformations $\mathcal{T}_1\times\ldots\times\mathcal{T}_n =\{T_1\times\ldots\times T_n;\ T_1\in\mathcal{T}_1,\ldots,\ T_n\in\mathcal{T}_n\}$. The notions of joint exchangeability or ergodicity are only defined when $S_1=\ldots=S_n$ and $\mathcal{T}_1=\ldots=\mathcal{T}_n=\mathcal{T}$, in which case the generating class of transformations is $\mathcal{T}^{(n)}=\{T\times\ldots\times T;\ T\in\mathcal{T}\}$.

The two results we need involving these notions may now be stated. Their proofs are easy, so we shall only prove the second one.

Lemma 8. For $k\in\{1,\ldots,n\}$, let S_k be a Polish space endowed with a countable group $\mathcal{T}_k$ of measurable transformations and the associated invariant σ-field $\mathcal{J}_k$, and let $\xi=(\xi_1,\ldots,\xi_n)$ be a r.e. in $S_1\times\ldots\times S_n$. Then ξ is separately exchangeable, iff each component is conditionally exchangeable, given all the others, and also iff the ξ_k are conditionally independent and ergodic exchangeable, given $\xi_1^{-1}\mathcal{J}_1\vee\ldots\vee\xi_n^{-1}\mathcal{J}_n$. In that case, ξ_1 is conditionally independent of $(\xi_2,\ldots,\xi_n)$, given $\xi_1^{-1}\mathcal{J}_1$, so the ξ_k are mutually independent when at least n-1 of them are ergodic. They are further all ergodic, iff ξ is so.

Lemma 9. Fix a Polish space S, endowed with a countable group $\mathcal{T}$ of measurable transformations. Let ξ and η be two r.e.'s in S, such that ξ is $\mathcal{T}$-exchangeable, while η is a.s. conditionally $\mathcal{T}$-exchangeable, given ξ, and assume that the pair (ξ,η) is jointly $\mathcal{T}$-ergodic. Then ξ and η are independent ergodic $\mathcal{T}$-exchangeable.

Proof. The exchangeability of η follows through averaging from the corresponding conditional property. Next note that, if $I \subset S$ is measurable invariant, then $I \times S$ and $S \times I$ are both $\mathcal{T}^{(2)}$-invariant, so the joint ergodicity of (ξ,η) implies $P\{\xi \in I\} = P\{(\xi,\eta) \in I \times S\} = 0$ or 1, and similarly for $P\{\eta \in I\}$, which shows that ξ and η are ergodic. In particular, the distribution of η is extreme exchangeable, so the decomposition $P\eta^{-1} = E\,P[\eta \in \cdot \mid \xi]$ must be trivial, which means that $P[\eta \in \cdot \mid \xi] = P\eta^{-1}$ a.s. Hence ξ and η are independent. □

The last result in this section will only be needed in Section 6, in order to characterize extremality for exchangeable random measures on $[0,1] \times R_+$, by means of the last condition in Lemma 4. Recall that a kernel on a measurable space $(S,\mathcal{S})$ is a mapping $K: S \times \mathcal{S} \to R_+$, such that $K(s,\cdot)$ is a measure for each $s \in S$, while $K(\cdot,B)$ is $\mathcal{S}$-measurable for each $B \in \mathcal{S}$.

Lemma 10. Fix three Polish spaces S, U, V, and an index set T, some measurable mappings $f_t: U \times V \to S$, $t \in T$, two independent r.e.'s, α in U and η in V, and an independent U(0,1) r.v. ϑ. Put

$$\xi_t = f_t(\alpha,\eta), \quad t \in T, \tag{2.9}$$

and assume that

$$P[\xi_t \in \cdot \mid \alpha] = K(\xi_t;\cdot) \quad \text{a.s.,} \quad t \in T, \tag{2.10}$$

for some kernel K on S. Then there exist some measurable mapping $g: S \times [0,1] \to S$, and for each $t \in T$ some r.e. $\eta_t \overset{d}{=} \eta$ independent of (ξ_t,α), such that

$$\xi_t' \equiv g(\xi_t,\vartheta) = f_t(\alpha,\eta_t) \quad \text{a.s.,} \quad t \in T. \tag{2.11}$$

Proof. Let us first reduce the discussion, through a Borel isomorphism, to the case when S=R. Then (2.10) may be stated in the form

$$H_t(\alpha,x) \equiv P[\xi_t \leq x \mid \alpha] = K(\xi_t,(-\infty,x]) \equiv G(\xi_t,x), \quad x\in R, \quad \text{a.s.}, \quad t\in T, \tag{2.12}$$

where G and the H_t are distribution functions in the last argument. Since G and the H_t are clearly product measurable, so are the corresponding right-continuous inverses g and h_t, and (2.12) shows that the r.e.'s $\xi_t'=g(\xi_t,\alpha)$ satisfy

$$\xi_t' = h_t(\alpha,\gamma) \quad \text{a.s.}, \quad t\in T. \tag{2.13}$$

Since α and γ are independent, (2.13) yields by the definition of h_t

$$(\xi_t',\alpha) \stackrel{d}{=} (\xi_t,\alpha), \quad t\in T. \tag{2.14}$$

By (2.9), this is also true with ξ_t repleced by $f_t(\alpha,\eta')$, where $\eta' \stackrel{d}{=} \eta$ and independent of (α,γ,η), and with this change, (2.13) shows that η becomes independent of both sides of (2.14), so we get

$$(\xi_t',\alpha,\eta) \stackrel{d}{=} (f_t(\alpha,\eta'),\alpha,\eta), \quad t\in T.$$

Applying Lemma 1 for each t, we conclude that there exist some random triples

$$(\tilde{\alpha}_t,\tilde{\eta}_t,\tilde{\eta}_t') \stackrel{d}{=} (\alpha,\eta,\eta'), \quad t\in T, \tag{2.15}$$

satisfying

$$(\xi_t',\alpha,\eta) = (f_t(\tilde{\alpha}_t,\tilde{\eta}_t'),\tilde{\alpha}_t,\tilde{\eta}_t) \quad \text{a.s.}, \quad t\in T.$$

In particular, $\tilde{\alpha}_t=\alpha$ and $\tilde{\eta}_t=\eta$ a.s., so (2.11) holds with $\eta_t=\tilde{\eta}_t'$. Moreover, (2.15) shows that η_t is independent of (α,η), and hence also of (ξ_t,α). □

3. SOME ONE-DIMENSIONAL RESULTS

The proofs of our main theorems depend on some results about random measures on a product space, which are exchangeable in one coordinate only. In particular, the first three lemmas of this section provide the basic characterizations of such random measures. These results generalize the characterizations of random measures on $[0,1]$ or R_+, given in Kallenberg[7,9], and could in fact have been derived from the latter. However, we prefer to prove them afresh, using a new and more elementary method. Also of some independent interest are the next two results, Lemmas 14-15, which relate the exchangeability of a marked point process to properties of the associated sequences of random times and marks. The remaining results of this section are more technical, and tailored to fit our special needs in the subsequent sections.

For the basic characterizations, we shall need to consider random measures ξ on product spaces S of the form $[0,1]\times K$ or $R_+ \times K$, where K is Polish. We shall then assume that K admits a complete metrization, such that ξB is a finite r.v. for every (metrically) bounded Borel set $B\subset S$. Such a random measure ξ is said to be exchangeable, if $\xi f^{-1} \stackrel{d}{=} \xi$ for all transformations f in the first coordinate which preserve Lebesgue measure λ, or equivalently, for the subclass of transformations which permute finitely many disjoint dyadic intervals of equal length. As explained in Section 1, we shall carry out all proofs under the second and formally weaker definition. The equivalence of the two definitions will then follow from the form of the representations. In particular, the notions of invariance and ergodicity are tacitly assumed to be defined with respect to the smaller class of transformations.

By a K-marked point process on $[0,1]$ or R_+ we shall mean an integer valued random measure ξ on $[0,1]\times K$ or $R_+\times K$, respectively, such that $\xi(\{t\}\times K)=0$ or 1 for all t. By analogy, we may further say that ξ is a K-marked diffuse random measure on $[0,1]$ or R_+, if it is a random measure on the corresponding product space satisfying $\xi(\{t\}\times K)\equiv 0$. In either case, ξ is said to have independent increments, if $\xi B_1,\ldots,\xi B_m$ are independent for any collection of Borel sets $B_1,\ldots,B_m$ with disjoint projections on $[0,1]$ or R_+.

Lemma 11. Fix a Polish space K, and let ξ be a K-marked diffuse random measure on $[0,1]$ or R_+. Then ξ is exchangeable, iff $\xi = \lambda\times\eta$ a.s. for some random measure η on K. In that case, ξ is extreme iff η is a.s. non-random, i.e. iff ξ has independent increments.

Proof. Let ξ be exchangeable. In order to prove that ξ has the stated form, we may clearly assume that ξ is defined on $[0,1]\times K$. Then the projection $\eta = \xi([0,1]\times\cdot)$ onto K is invariant, so the exchangeability of ξ is preserved under conditioning on η, and we may assume by Lemma 3 that η is non-random. By a monotone class argument, it is then enough to show, for any bounded Borel set $B\subset K$, that $\xi(\cdot\times B)=(\eta B)\lambda$ a.s., which reduces the discussion to the case of diffuse random measures ξ on $[0,1]$ with fixed total mass m. In that case, the exchangeability of ξ implies $E\xi = m\lambda$. Moreover, the product moment $E(\xi B)(\xi C)$ for disjoint dyadic intervals $B,C\subset[0,1]$ is seen to depend on λB and λC only, so for dyadic rectangles A outside the diagonal D we get

$$E\,\xi^2 A = c\,\lambda^2 A, \tag{3.1}$$

for some constant $c\geq 0$. The last relation extends by a monotone class argument to arbitrary Borel sets $A\subset[0,1]^2\setminus D$, and since $\xi^2 D=0$ when ξ is diffuse, (3.1) must in fact be true for arbitrary $A\subset[0,1]^2$. In particular, we get $c=m^2$ by taking $A=[0,1]^2$. But then we get for A of the form B^2 that $E(\xi B)^2 = m^2(\lambda B)^2$, so $\mathrm{Var}(\xi B)=0$, and therefore $\xi B = m\lambda B$ a.s. Hence $\xi = m\lambda$ a.s., as asserted. Conversely, any random measure of the form $\lambda\times\eta$ is invariant and hence trivially exchangeable. The last assertion follows easily from Lemma 4, plus the fact that a random variable is independent of itself iff it is a.s. non-random. $\square$

Lemma 12. Fix a Polish space K, and let ξ be a K-marked point process on $[0,1]$. Then ξ is exchangeable, iff

$$\xi = \sum_{j<\nu} \delta_{\tau_j,\beta_j} \quad \text{a.s.,} \tag{3.2}$$

<u>for some $\overline{N}$-valued r.v. ν, some K-valued r.e.'s β_j, $j<\nu$, and some independent sequence of independent U(0,1) r.v.'s $\tau_1,\tau_2,\ldots$ Moreover, the distributions of ξ and $\beta=\sum\delta_{\beta_j}$ determine each other uniquely, and ξ is extreme iff β is a.s. non-random, in which case ν and the β_j may be chosen to be constants.</u>

<u>Proof.</u> Let ξ be exchangeable. Since the projection $\beta=\xi([0,1]\times\cdot)$ is invariant, the exchangeability of ξ is preserved under conditioning on β, so we may assume that β is non-random. Let $b_1,b_2,\ldots$ be the atom positions of β, and write $m_k=\beta\{b_k\}$ and $\xi_k=\xi(\cdot\times\{b_k\})$. Fixing the indices $j_1,\ldots,j_n$, it is clear that the product moment $E(\xi_{j_1}B_1)\ldots(\xi_{j_n}B_n)$ for disjoint dyadic intervals $B_1,\ldots,B_n\subset[0,1]$ will only depend on $\lambda B_1,\ldots,\lambda B_n$. Hence we get by linearity, for any dyadic rectangle $A\subset[0,1]^n$ outside the union D_n of all diagonal sets,

$$E(\xi_{j_1}\times\ldots\times\xi_{j_n})A = c\,\lambda^n A, \tag{3.3}$$

where $c\geq 0$ is a constant depending on $j_1,\ldots,j_n$. By a monotone class argument, (3.3) extends to arbitrary Borel sets $A\subset D_n^c$. Now (3.3) is equivalent to

$$E(\xi_1^{n_1}\times\ldots\times\xi_d^{n_d})A = c'\,\lambda^{n_1+\ldots+n_d}A, \tag{3.4}$$

with a constant $c'\geq 0$ depending on $n_1,\ldots,n_d$, and (3.4) extends to arbitrary Borel sets $A\subset D_{n_1}^c\times\ldots\times D_{n_d}^c$, since $\xi_1,\ldots,\xi_d$ have no atom sites in common. Taking $A=D_{n_1}^c\times\ldots\times D_{n_d}^c$, we get in particular

$$c' = \prod_{k=1}^{d}(m_k)_{n_k} = \prod_{k=1}^{d}\left(m_k(m_k-1)\cdots(m_k-n_k+1)\right). \tag{3.5}$$

Next we note that (3.4) remains valid with c' as in (3.5), if we replace the ξ_k by independent sample processes η_k (cf. Kallenberg[9]) with the same total masses m_k. Thus

$$E(\xi_1^{n_1}\times\cdots\times\xi_d^{n_d})A = E(\eta_1^{n_1}\times\cdots\times\eta_d^{n_d})A,\quad A\subset D_{n_1}^c\times\ldots\times D_{n_d}^c,$$

which extends to arbitrary Borel sets $A\subset[0,1]^{n_1+\ldots+n_d}$, since the ξ_k and η_k are simple point processes (cf. Krickeberg[14]). Hence the sequences (ξ_k) and (η_k) have the same product moment measures of all orders, and since each ξ_k and η_k is bounded by a constant, the joint distributions must be the same. Thus ξ has the same distribution as the sum in (3.2), and the a.s. representation then follows by Lemma 1.

Conversely, any point process of the form (3.2) is clearly exchangeable. The remaining statements follow easily from Lemmas 4 and 5. $\square$

To state the next result, recall that if ξ and η are random measures on some Polish space S, then ξ is said to be a Cox process directed by η, if conditionally on η, ξ is a.s. a Poisson process on S with intensity measure η (cf. Kallenberg[9]). In this case, the distributions of ξ and η are known to determine each other uniquely. Note also that, if $\xi' \stackrel{d}{=} \xi$, then Lemma 1 yields the existence of some $\eta' \stackrel{d}{=} \eta$, such that ξ' is a Cox process directed by η'.

Lemma 13. Fix a Polish space K, and let ξ be a K-marked point process on R_+. Then ξ is exchangeable, iff there exists some random measure η on K, such that ξ is a Cox process directed by $\lambda \times \eta$. In that case, the distributions of ξ and η determine each other uniquely, and ξ is extreme iff η is a.s. non-random, which happens iff ξ has independent increments.

Proof. Let ξ be exchangeable. Then Lemma 12 applies to the restrictions ξ_t of ξ to the sets $[0,t] \times K$, so writing $\beta_t = \xi([0,t] \times \cdot)$, it is clear that β_t is a p-thinning of $\beta_{t/p}$ for arbitrary $t > 0$ and $p \in (0,1)$. Hence each β_t is a Cox process (cf. Corollary 8.5 in Kallenberg[9]), and Lemma 12 then shows that the same thing is true for each ξ_t, with a directing random measure of the form $\lambda \times \eta_t$. But then $\eta_s \stackrel{d}{=} \eta_t$ for any $s,t > 0$, so ξ itself must be a Cox process directed by some random measure $\lambda \times \eta$ with $\eta \stackrel{d}{=} \eta_1$. Since a version of η is measurably determined by ξ through the law of large numbers, the remaining assertions follow easily by Lemma 4 and Kolmogorov's 0-1 law. $\square$

The next lemma uses the notion of separate exchangeability for random elements in product spaces, introduced in Section 2.

Lemma 14. *Fix a Polish space* K, *and consider a* K-*marked point process* ξ *on* $[0,1]$ *or* R_+ *of the form*

$$\xi = \sum_{j=1}^{n} \delta_{\tau_j, \alpha_j}, \tag{3.6}$$

where $n \in \bar{N}$ *is fixed, while* $\tau_1 < \tau_2 < \ldots$ *Write* $\eta = \xi(\cdot \times K)$ *and* $\alpha = (\alpha_1, \alpha_2, \ldots)$. *Then* ξ *is exchangeable iff the pair* (η, α) *is separately exchangeable, and in that case,* ξ *is ergodic iff* η *and* α *are independent and ergodic.*

Proof. Let us first take ξ to be defined on R_+, so that $n=\infty$. Assume that η and α are independent and ergodic exchangeable. Then η is homogeneous Poisson by Lemma 13, while α is i.i.d., so writing $c=E\eta[0,1]$ and $\mu=P\alpha_1^{-1}$, it is seen that ξ is Poisson with intensity measure $c\lambda \times \mu$, and hence ergodic exchangeable by Lemma 13. Conversely, if ξ is ergodic exchangeable, it must be Poisson with an intensity measure of the form $c\lambda \times \mu$, with $c>0$ and μ a probability measure on K. Since η and α are measurable functions of ξ, their joint distribution is determined by that of ξ, and hence must be the same as before. Thus η and α are independent ergodic exchangeable in this case.

If ξ is instead defined on $[0,1]$, then $n<\infty$ by Lemma 12, so assuming η and α to be independent exchangeable, it may be seen directly from (3.6) that even ξ is exchangeable. In this case η is automatically ergodic, and if even α is assumed to be ergodic, then $\xi([0,1]\times \cdot)=\sum \delta_{\alpha_j}$ is a.s. non-random, so ξ will be ergodic by Lemma 12. Conversely, the distribution of an ergodic exchangeable process ξ is determined by the non-random measure $\beta=\xi([0,1]\times \cdot)$ on K, and since every β can be written as $\sum \delta_{\beta_j}$ for some ergodic exchangeable sequence $(\beta_1, \ldots, \beta_n)$ in K, the previous uniqueness argument shows that η and α are again independent ergodic exchangeable.

In both cases, it follows by Lemma 8 that ξ is ergodic exchangeable, iff (η, α) is separately ergodic exchangeable. Hence we obtain the first assertion by conditioning on the invariant σ-fields for ξ and (η, α), respectively. $\square$

We shall need the following simple corollary:

Lemma 15. Let $\tau_1,\ldots,\tau_n$ be independent U(0,1) r.v.'s, write $\eta=\sum\delta_{\tau_i}$, and define $\varkappa=(\varkappa_1,\ldots,\varkappa_n)$ as the a.s. unique permutation of $(1,\ldots,n)$, such that $\tau_{\varkappa_1}\le\cdots\le\tau_{\varkappa_n}$. Then η and $\varkappa$ are independent ergodic exchangeable.

Proof. Write $\sigma_j=\tau_{\varkappa_j}$, and note that the marked point process

$$\xi=\sum_{i=1}^{n}\delta_{\tau_i,i}=\sum_{j=1}^{n}\delta_{\sigma_j,\varkappa_j}$$

is ergodic exchangeable on $[0,1]$. Then apply Lemma 14. □

The remainder of this section is devoted to a study of random measures ξ on $R_+\times[0,1]$ which are exchangeable along R_+, i.e. such that $\xi h^{-1}\overset{d}{=}\xi$ for every function h of the form $h(x,y)=f(x)$, where f is a measure preserving transformation of R_+. The general representation of such random measures ξ can be easily deduced from Lemmas 11 and 13. However, we shall only concentrate on certain specific features, which will be important for the proofs of Theorems 3-5.

As before, we shall in fact assume in the proofs that the measure preserving transformations f above are of the special type which permute finitely many disjoint dyadic intervals of equal length. Note in particular that ergodicity is always defined with respect to this restricted class of transformations. Let us further agree to denote by $\mathcal{M}[0,1]$ the Polish space of finite measures on $[0,1]$, and to write ψ for the function $1-e^{-x}$ on R_+.

Lemma 16. A random measure ξ on $R_+\times[0,1]$ is ergodic exchangeable along R_+, iff ξ has stationary independent increments along R_+. In that case, define

$$c_t=E\psi\circ\xi([0,1]\times\{t\}),\qquad t\in[0,1]. \tag{3.7}$$

Then the sets $T_\varepsilon=\{t\in[0,1];\ c_t>\varepsilon\}$ are finite for all $\varepsilon>0$, so T_0 is countable. Moreover, ξ is a.s. such that

$$\xi\{(s',t)\}\,\xi\{(s'',t)\}=0,\qquad 0\le s'<s'',\quad t\in[0,1]\setminus\bar{T}_0. \tag{3.8}$$

Proof. Assume ξ to be ergodic exchangeable, define $M=\{s\ge 0;\ \xi(\{s\}\times[0,1])>0\}$, and let ζ denote the restriction of ξ to $M^c\times[0,1]$. Then ζ is again ergodic exchangeable by Lemma 6, and since $\zeta(\cdot\times[0,1])$ is diffuse, Lemma 11 shows that $\zeta=\lambda\times\mu$ a.s., for some fixed measure μ on $[0,1]$. Let us further define a point process η on R_+ with marks in $\mathcal{M}[0,1]\setminus\{0\}$, by putting

$$\eta=\sum_{s\in M}\delta_{s,\xi(\{s\}\times\cdot)}.$$

Then even η is ergodic exchangeable by Lemma 6, so Lemma 13 shows that η is stationary Poisson. From the structure of ζ and η, it is clear that ξ has stationary independent increments along R_+. Conversely, any such random measure ξ is trivially exchangeable, and the ergodicity of ξ follows easily from the Hewitt-Savage 0-1 law.

Next we write $\xi_t=\xi([0,1]\times\{t\})$ and $q_t=\psi^{-1}(c_t)$, for $t\in[0,1]$. If $c_t>\varepsilon>0$, we get by Chebyshev's inequality and (3.7),

$$P\{2\xi_t>\varepsilon\}\geq P\{2\xi_t>q_t\}\geq 1-\exp(q_t/2)E\exp(-\xi_t)=\psi(q_t/2)\geq\psi(\varepsilon/2),$$

so if T_ε contains an infinite sequence $t_1,t_2,\ldots$, we get by Fatou's lemma

$$0=P\{\xi[0,1]^2=\infty\}\geq P\{2\xi_{t_n}>\varepsilon \text{ i.o.}\}\geq\limsup_{n\to\infty}P\{2\xi_{t_n}>\varepsilon\}\geq\psi(\varepsilon/2)>0,$$

which is impossible and shows that T_ε is finite.

To prove the last assertion, define

$$A_s=\{t\in[0,1]\setminus T_0:\ \xi([0,s]\times\{t\})>0\},\qquad s\geq 0.$$

We then have to prove, for any fixed rational $s\geq 0$, that $\xi((s,\infty)\times A_s)=0$ a.s. Since ξ has independent increments, this follows formally by the computation

$$E\,\xi((s,\infty)\times A_s)=E\sum_{t\in A_s}E\,\xi((s,\infty)\times\{t\})=0.$$

To justify the use of Fubini's theorem in the first step, recall that there exist some random variables ν in $\bar{N}$ and $\tau_1,\tau_2,\ldots$ in $[0,1]$, all measurable with respect to $\xi([0,s]\times\cdot)$, such that $A_s=\{\tau_k;\ k<\nu\}$ (cf. Lemma 2.3 in Kallenberg[(9)]). It remains to notice that $\mu\{s\}$ is jointly measurable in $(\mu,s)\in\mathcal{M}[0,1]\times[0,1]$, as may be seen by a simple approximation argument. □

Lemma 17. Let ξ be a random measure on $R_+\times[0,1]$ which is exchangeable along R_+, define

$$\rho_t = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \psi \circ \xi((k-1,k]\times\{t\}), \quad t\in[0,1], \tag{3.9}$$

and put $M=\{t\in[0,1];\ \rho_t>0\}$. Then $M\subset\{\tau_j\}$ a.s., for some ξ-measurable r.v.'s $\tau_1,\tau_2,\ldots$ in $[0,1]$, satisfying $\rho_{\tau_1}\geq\rho_{\tau_2}\geq\cdots\to 0$ a.s., and such that $\tau_i=\tau_j\neq 0$ implies $i=j$. Moreover, ξ is a.s. such that

$$\xi\{(s',t)\}\,\xi\{(s'',t)\} = 0, \qquad 0\leq s'<s'',\ t\in[0,1]\setminus M. \tag{3.10}$$

Proof. If ξ is ergodic, the law of large numbers yields $\rho_t=c_t$ a.s. for each $t\in[0,1]$, where c_t is given by (3.7), and the last statement of Lemma 16 shows that the exceptional P-nullset may be taken to be independent of t. In particular, the set $\{t\in[0,1];\ \rho_t>\varepsilon\}$ is a.s. finite for every $\varepsilon>0$.

In the general case, we note that M is contained in the set

$$M' = \{t\in[0,1];\ \xi(R_+\times\{t\})\neq 0\} = \bigcup_{n=1}^{\infty}\{t\in[0,1];\ \xi([0,n]\times\{t\})>0\}.$$

By Lemma 2.3 in Kallenberg[9], there exist some distinct ξ-measurable r.v.'s $\sigma_1,\sigma_2,\ldots$ in $[0,1]$, such that $M'\subset\{\sigma_j\}$, and we note that even the quantities $\beta_j=\rho_{\sigma_j}$ become ξ-measurable. It is now obvious how to define the τ_j recursively, by suitable ordering of the σ_j according to the sizes of β_j.

The last assertion is clearly equivalent to

$$1\{\beta_j=0\}\,\xi([0,s]\times\{\sigma_j\})\,\xi((s,\infty)\times\{\sigma_j\}) = 0, \qquad s\in Q_+,\ j\in N.$$

But this holds in the ergodic case by Lemma 16, and in general it then follows by conditioning on the invariant σ-field. ∎

We finally record a simple result, stated in terms of the shift operators θ_t along R_+, defined in an obvious way on the class of measures on $R_+\times[0,1]$.

Lemma 18. Let the random measure ξ on $R_+\times[0,1]$ have conditionally stationary independent increments along R_+, given some σ-field $\mathcal{G}$, and fix arbitrary measurable mappings $f\colon R_+\times[0,1]\to R_+$ and $h\colon R_+\to R_+$. Then

$$E[h(\xi f)\,|\,\mathcal{G}] = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} h((\theta_k\circ\xi)f) \quad \text{a.s.} \tag{3.11}$$

Proof. If $\mathcal{G}=\{\Omega,\emptyset\}$, then (3.11) holds by the ergodic theorem and Kolmogorov's 0-1 law. In general, it then follows by conditioning on $\mathcal{G}$. ∎

4. NESTED EXCHANGEABLE ARRAYS

Our aim in this section in to prove a representation theorem for certain sequences of separately or jointly exchangeable arrays, which will be needed for the proofs of Theorems 4 and 5 in Section 7.

To prepare for this, recall from Aldous[1,2] and Hoover[5,6] (cf. Kallenberg[11]) that an array $X=(X_{ij};\ i,j\in N)$ of random vectors in R^d is separately ergodic exchangeable (under finite permutations of indices), iff it admits a representation

$$X_{ij} = f(\alpha_i,\beta_j,\gamma_{ij}) \quad \text{a.s.}, \qquad i,j\in N, \tag{4.1}$$

in terms of some measurable function $f\colon [0,1]^3 \to R^d$ and some set of independent $U(0,1)$ r.v.'s $\alpha_i,\beta_j,\gamma_{ij}$, $i,j\in N$. Similarly, X is jointly ergodic exchangeable, iff (4.1) holds with $\alpha_i \equiv \beta_i$, $\gamma_{ij} \equiv \gamma_{ji}$ and $\gamma_{ii} \equiv 0$, for some function f as above and some independent $U(0,1)$ r.v.'s α_i and γ_{ij}, $1 \le i < j$. In both cases, we shall call the f in (4.1) a <u>representing function</u> for X.

Let us next consider a sequence of arrays $X^{(n)}$ with R^3-valued entries of the special form

$$X_{ij}^{(n)} = (Y_{ij}^{(n)},\ U_i^{(n)},\ V_j^{(n)}), \qquad i,j\in N, \tag{4.2}$$

and let $A_1 \subset A_2 \subset \ldots$ and $B_1 \subset B_2 \subset \ldots$ be Borel sets in R. We shall say that the $X^{(n)}$ are <u>nested</u> with respect to the sequences (A_n) and (B_n), if $X^{(m)}$ can be obtained from $X^{(n)}$ for any $m<n$ by selection of all rows and columns with $U_i^{(n)}\in A_m$ and $V_j^{(n)}\in B_m$. Formally,

$$X_{ij}^{(m)} = X_{\kappa_i,\kappa_j'}^{(n)}, \qquad i,j\in N, \tag{4.3}$$

where

$$\kappa_i = \inf\Big\{k\in N;\ \sum_{j=1}^{k} 1_{A_m}(U_j^{(n)})=i\Big\},\quad \kappa_i' = \inf\Big\{k\in N;\ \sum_{j=1}^{k} 1_{B_m}(V_j^{(n)})=i\Big\},\quad i\in N. \tag{4.4}$$

It is easily verified that, if $X^{(n)}$ is separately ergodic exchangeable, then so is $X^{(m)}$ for each $m<n$. In this case, the probability that a fixed row or column of $X^{(n)}$ will be included in $X^{(m)}$ is given by a_m/a_n or b_m/b_n, respectively, where the sequences $1=a_1\le a_2\le \ldots$ and $1=b_1\le b_2\le \ldots$ are defined by

$$a_n^{-1} = P\{U_i^{(n)}\in A_1\}, \qquad b_n^{-1} = P\{V_j^{(n)}\in B_1\}, \qquad n\in N. \tag{4.5}$$

Similar statements hold for jointly ergodic exchangeable arrays, with R^2-valued entries of the form

$$X_{ij}^{(n)} = (Y_{ij}^{(n)}, U_i^{(n)}), \quad i,j\in N, \tag{4.6}$$

which are assumed to be nested with respect to a single sequence of Borel sets $A_1 \subset A_2 \subset \ldots$ in R. Thus we take in this case $V_i^{(n)} \equiv U_i^{(n)}$ and $B_n \equiv A_n$, so that $\varkappa_i' = \varkappa_i$ and $b_n = a_n$.

In both cases it is clear that, if f_n is a representing function for $X^{(n)}$ and if $m<n$, then the function

$$f_m(x,y,z) = f_n((a_m/a_n)x,\ (b_m/b_n)y,\ z), \quad x,y,z\in[0,1], \tag{4.7}$$

is a representing function for $X^{(m)}$. This suggests that there should exist a function $f\colon R_+^2\times[0,1]\to R^3$ [or R^2], such that the arrays $X^{(n)}$ have representing functions

$$f_n(x,y,z) = f(a_n x,\ b_n y,\ z), \quad x,y,z\in[0,1],\ n\in N. \tag{4.8}$$

Lemma 19. Consider a sequence of separately ergodic exchangeable arrays $X^{(n)}$ as in (4.2), which are nested with respect to some sequences (A_n) and (B_n) of Borel sets in R, and define the sequences (a_n) and (b_n) by (4.5). Then there exists a measurable function $f\colon R_+^2\times[0,1]\to R^3$, such that the arrays $X^{(n)}$ have representing functions f_n given by (4.8). The corresponding statement holds for nested sequences of jointly ergodic exchangeable arrays of the form (4.6).

Unfortunately, no simple proof of this result seems to exist. One might try to construct a sequence of representing functions f_n by successive extensions, such that (4.7) becomes fulfilled in each step. It turns out, however, that no extension f_{n+1} of a given representing function f_n for $X^{(n)}$ with the desired properties need to exist. Relying on Hoover's equivalence criterion[5] for the case of a fixed array (cf. Kallenberg[12] and Proposition 3 below), one might then try a modified extension procedure, where two randomization variables are added in each step, so that f_n becomes a function on $[0,1]^{2n+1}$. Since this becomes rather complicated, we prefer a direct approach, mimicking the standard proofs of (4.1), as presented in Aldous[1,2] and Kallenberg[11].

Proof. We shall only consider the separately exchangeable case, the joint case being similar. By the Daniell-Kolmogorov theorem, we may extend each $X^{(n)}$ to an exchangeable array indexed by Z^2, and by another application of the same theorem, we may do this simultaneously for all n, in such a way that the extended sequence $X=(X^{(n)})$ becomes nested in the obvious sense. Let X^- denote the restriction of X to the index set $Z_-^2=\{(i,j);\ i,j\leq 0\}$. Let us further write $X_i^{(n)}=\{X_{ij}^{(n)};\ j\leq 0\}$, $i,n\in N$, and for $n\leq m$, let $X_i^{(m,n)}$ denote the subsequence of elements $X_{ij}^{(m)}$ with $V_j^{(m)}\in A_n$. If instead $m<n$, we put $X_i^{(m,n)}=X_{\varkappa_i}^{(n)}$, where $\varkappa_i$ is given by (4.4). Finally put $\alpha_i^{(m)}=(X_i^{(m,n)};\ n\in N)$ and $\alpha^{(m)}=(\alpha_i^{(m)};\ i\in N)$. The arrays $\beta_j^{(m)}$ and $\beta^{(m)}$ are defined in the same way, but with the roles of indices i and j interchanged. Fixing $m\leq n$ and letting the $\varkappa_i$ and $\varkappa_j'$ be given by (4.4), it is clear from the definitions that

$$\alpha_i^{(m)}=\alpha_{\varkappa_i}^{(n)},\quad \beta_j^{(m)}=\beta_{\varkappa_j'}^{(n)},\quad i,j\in N. \tag{4.9}$$

The argument in Aldous[1,2] applies to the array $(X_{ij}^{(m)}, \alpha_i^{(m)}, \beta_j^{(m)}, X^-;\ i,j\in N)$ for fixed $m\in N$, and shows that the sequences $\alpha^{(m)}$ and $\beta^{(m)}$ are conditionally independent and i.i.d., given X^-, that the $X_{ij}^{(m)}$ are conditionally independent, given $(X^-, \alpha^{(m)}, \beta^{(m)})$, and that

$$P[X_{ij}^{(m)}\in\cdot\,|X^-, \alpha^{(m)}, \beta^{(m)}]=K_m(X^-,\alpha_i^{(m)},\beta_j^{(m)};\cdot)\ \text{a.s.},\quad i,j\in N, \tag{4.10}$$

for some kernel K_m independent of i and j. We can actually choose K_m to be independent even of m. To see this, it is enough to show, for fixed $n>m$ and $i,j\in N$, that

$$P[X_{ij}^{(m)}\in\cdot\,|X^-, \alpha_i^{(m)}, \beta_j^{(m)}]=K_n(X^-, \alpha_i^{(m)}, \beta_j^{(m)};\cdot)\quad \text{a.s.}, \tag{4.11}$$

which by (4.4) and (4.9) is equivalent to

$$P[X_{\varkappa_i,\varkappa_j'}^{(n)}\in\cdot\,|X^-, \alpha_{\varkappa_i}^{(n)}, \beta_{\varkappa_j'}^{(n)}]=K_n(X^-, \alpha_{\varkappa_i}^{(n)}, \beta_{\varkappa_j'}^{(n)};\cdot)\quad \text{a.s.} \tag{4.12}$$

Now (4.12) is in fact true with $\varkappa_i$ and $\varkappa_j'$ replaced by any finite stopping times $\varkappa$ and $\varkappa'$ with respect to the sequences $\alpha^{(n)}$ and $\beta^{(n)}$, respectively, as may be seen if we replace (4.12) by its integrated version, split each side into a double sum, corresponding to the partition of Ω into sets $\{(\varkappa,\varkappa')=(k,k')\}$, and apply relation (4.10) termwise.

By a standard procedure, we may next construct a measurable function $F: R^\infty \times R^\infty \times R^\infty \times [0,1] \to R^3$, where R^∞ symbolizes the state spaces of the arrays X^-, $\alpha_i^{(m)}$ and $\beta_j^{(m)}$, such that for any $U(0,1)$ r.v. γ,

$$P\{F(x,y,z,\gamma)\in\cdot\} \stackrel{=}{\cdot} K(x,y,z|\cdot) \equiv K_m(x,y,z|\cdot). \tag{4.13}$$

By Lemma 1, relations (4.10) and (4.13), and the conditional independence of the $X_{ij}^{(m)}$, there exist for each $m\in N$ some independent $U(0,1)$ r.v.'s $\gamma_{ij}^{(m)}$, independent even of X^-, $\alpha^{(m)}$ and $\beta^{(m)}$, such that

$$X_{ij}^{(m)} = F(X^-, \alpha_i^{(m)}, \beta_j^{(m)}, \gamma_{ij}^{(m)}) \quad \text{a.s.,} \quad i,j,m\in N. \tag{4.14}$$

Since the arrays $X^{(n)}$ are ergodic by hypothesis, their restrictions to N^2 are independent of X^-, so conditioning on X^- leaves their joint distribution invariant. Note also that the $\gamma_{ij}^{(m)}$ remain conditionally independent $U(0,1)$ for fixed m, independently of $\alpha^{(m)}$ and $\beta^{(m)}$. By Lemma 1, we may then redefine the quantities on the right of (4.14), such that the joint distribution of all r.e.'s in (4.14) will agree with some fixed conditional distribution with the stated properties. Thus we may henceforth take X^- to be a constant array, and assume for each $m\in N$ that the sequences $\alpha^{(m)}$ and $\beta^{(m)}$ are i.i.d. and mutually independent, while the $\gamma_{ij}^{(m)}$ are independent $U(0,1)$ and independent also of $\alpha^{(m)}$ and $\beta^{(m)}$. Note that (4.14) now reduces to

$$X_{ij}^{(m)} = f(\alpha_i^{(m)}, \beta_j^{(m)}, \gamma_{ij}^{(m)}) \text{ a.s.,} \quad i,j,m\in N, \tag{4.15}$$

for some measurable function $f: R^\infty \times R^\infty \times [0,1] \to R^3$.

From the definitions of $\alpha_i^{(m)}$ and $\beta_j^{(m)}$, it is clear that

$$U_i^{(m)} = \pi_1 \circ \alpha_i^{(m)}, \qquad V_j^{(m)} = \pi_2 \circ \beta_j^{(m)}, \qquad i,j,m\in N, \tag{4.16}$$

for suitable projections $\pi_1, \pi_2: R^\infty \to R$. Defining

$$A_i' = \pi_1^{-1} A_i, \quad B_j' = \pi_2^{-1} B_j, \quad i,j\in N, \tag{4.17}$$

we may then rewrite (4.4) in the form

$$\kappa_i = \inf\{k\in N;\ \sum_{j=1}^{k} 1_{A_m'}(\alpha_j^{(m)})=i\}, \quad \kappa_i' = \inf\{k\in N;\ \sum_{j=1}^{k} 1_{B_m'}(\beta_j^{(m)})=i\}, \quad i\in N. \tag{4.18}$$

Letting μ_m and ν_m denote the distributions of $\alpha_i^{(m)}$ and $\beta_j^{(m)}$, respectively, and using the strong Markov property for i.i.d. sequences, we may conclude from (4.9) and (4.18) that

$$\mu_m = \mu_n[\cdot\,|A'_m], \quad \nu_m = \nu_n[\cdot\,|B'_m], \qquad m<n. \tag{4.19}$$

Hence there exist some σ-finite measures μ and ν on R^∞, such that

$$\mu_n = \mu[\cdot\,|A'_n], \quad \nu_n = \nu[\cdot\,|B'_n], \quad n\in N. \tag{4.20}$$

By (4.5), (4.16) and (4.17), we may normalize μ and ν in such a way that

$$\mu(A'_n) = a_n, \quad \nu(B'_n) = b_n, \qquad n\in N. \tag{4.21}$$

Imbedding R^∞ into R and using (4.20) and (4.21), we may easily construct a pair of measurable mappings $g,h\colon R_+ \to R$, satisfying

$$\lambda\{s\in[0,a_n];\ g(s)\in\cdot\} = a_n\mu_n, \ \lambda\{s\in[0,b_n];\ h(s)\in\cdot\} = b_n\nu_n, \quad n\in N. \tag{4.22}$$

Hence we get, for any U(0,1) r.v. ϑ,

$$g(a_n\vartheta) \overset{d}{=} \alpha_i^{(n)}, \quad h(b_n\vartheta) \overset{d}{=} \beta_j^{(n)}, \qquad i,j,n\in N. \tag{4.23}$$

Using (4.15), (4.23) and Lemma 1, we may conclude that $X^{(n)}$ has representing function

$$f_n(x,y,z) = f(g(a_n x),\ h(b_n y),\ z), \quad x,y,z\in[0,1]. \tag{4.24}$$

Thus the assertion of the lemma holds with f replaced by $f(g(\cdot),h(\cdot),\cdot)$. □

5. EXCHANGEABILITY IN A SQUARE

The aim of this section is to prove the first two main theorems of the paper, and further to derive some auxiliary results about exchangeable random measures in the unit square, which will be needed in subsequent sections. Throughout this section, the notions of exchangeability, invariance and ergodicity for random measures are defined with respect to the groups of measure preserving transformations which permute a finite number of disjoint dyadic intervals of equal length. For convenience, we shall often write $A\mu$ for the restriction of a measure μ to some measurable set A, i.e. $A\mu=\mu(A\cap\cdot)$.

<u>Proof of Theorem 1.</u> Consider a separately exchangeable random measure ξ on $[0,1]^2$. In order to prove that ξ has a representation as in (1.1), it is enough by Lemma 3 to assume that ξ is ergodic, and to establish the representation formula (1.1) with non-random coefficients. In that case, we define

$$M_1 = \{s\in[0,1];\ \xi(\{s\}\times[0,1])>0\},\quad M_2 = \{t\in[0,1];\ \xi([0,1]\times\{t\})>0\}, \tag{5.1}$$

and conclude from Lemma 6 that $(M_1^c\times[0,1])\xi$ is exchangeable in the first coordinate and $([0,1]\times M_2^c)\xi$ in the second. Hence Lemma 11 yields

$$(M_1^c\times[0,1])\xi = \lambda\times\eta_2,\quad ([0,1]\times M_2^c)\xi = \eta_1\times\lambda,\quad \text{a.s.}, \tag{5.2}$$

where

$$\eta_2 = \xi(M_1^c\times\cdot),\quad \eta_1 = \xi(\cdot\times M_2^c). \tag{5.3}$$

In particular,

$$(M_1^c\times M_2^c)\xi = (\eta_1 M_1^c)\lambda^2 = (\eta_2 M_2^c)\lambda^2\quad \text{a.s.}, \tag{5.4}$$

where the coefficients on the right must be a.s. constant, say equal to $c\geq 0$, since ξ is ergodic. The measure $c\lambda^2$ is invariant, so $\xi-c\lambda^2$ is again ergodic exchangeable, and we may henceforth assume that $\xi(M_1^c\times M_2^c)=0$.

In that case, it is seen from (5.2) that ξ has a representation

$$\xi = \sum_i\sum_j \alpha_{ij}\delta_{\sigma_i,\sigma_j'} + \sum_j\{\beta_j(\delta_{\sigma_j}\times\lambda) + \beta_j'(\lambda\times\delta_{\sigma_j'})\}\quad \text{a.s.}, \tag{5.5}$$

for some ξ-measurable r.v.'s $\alpha_{ij},\beta_i,\beta_j'\geq 0$ and $\sigma_i,\sigma_j'\in[0,1]$, $i,j\in N$, where the latter are a.s. distinct. If we take the sequences

$$r_i = \beta_i + \sum_j\alpha_{ij},\quad r_j' = \beta_j' + \sum_i\alpha_{ij},\qquad i,j\in N, \tag{5.6}$$

to be non-decreasing, they become a.s. invariant functions of ξ, and hence a.s. non-random, since ξ is ergodic. Then so are the index sets

$$J = \{j\in N;\ r_j > 0\}, \qquad J' = \{j\in N;\ r'_j > 0\}. \tag{5.7}$$

For definiteness, we may further assume that $\sigma_i < \sigma_j$ when $i < j$ with $r_i = r_j > 0$, and similarly for the σ'_j.

Independently of ξ, we introduce some independent $U(0,1)$ r.v.'s ϑ_i and ϑ'_j, $i\in J$ and $j\in J'$, and form the point processes

$$\zeta_1 = \sum_i \delta_{\sigma_i, r_i, \vartheta_i}, \qquad \zeta_2 = \sum_j \delta_{\sigma'_j, r'_j, \vartheta'_j} \tag{5.8}$$

on $[0,1]\times(0,\infty)\times[0,1]$. By a straightforward application of Lemma 7, the triple (ζ_1,ξ,ζ_2) is seen to be separately exchangeable, in the sense that

$$(\zeta_1 f_1^{-1},\ \xi(f_1\times f_2)^{-1},\ \zeta_2 f_2^{-1}) \overset{d}{=} (\zeta_1,\xi,\zeta_2), \tag{5.9}$$

for any transformations f_1 and f_2 of $[0,1]$ which permute disjoint dyadic intervals of equal length. Here the transformations of ζ_1 and ζ_2 are of course in the first coordinate.

The distribution of (ζ_1,ξ,ζ_2) is a mixture of ergodic exchangeable distributions Q, and since ξ is ergodic, it retains its distribution under a.e. Q. Note also that the projections

$$\zeta'_1 = \sum_i \delta_{r_i, \vartheta_i}, \qquad \zeta'_2 = \sum_j \delta_{r'_j, \vartheta'_j} \tag{5.10}$$

of ζ_1 and ζ_2 onto $(0,\infty)\times[0,1]$ are invariant under the transformations in (5.9), and hence a.s. non-random under a.e. Q. Finally, the r.v.'s ϑ_i and ϑ'_j are clearly a.s. distinct under a.e. Q. Fixing a measure Q with the above properties, it is clear from Lemma 1 that we may redefine the r.v.'s ϑ_i and ϑ'_j, such that the distribution of (ζ_1,ξ,ζ_2) becomes Q.

Since ζ'_1 and ζ'_2 are a.s. non-random, they may be written in the form

$$\zeta'_1 = \sum_{i\in J} \delta_{r_i, c_i}, \qquad \zeta'_2 = \sum_{j\in J'} \delta_{r'_j, c'_j} \quad \text{a.s.}, \tag{5.11}$$

for some fixed numbers c_i and c'_j in $[0,1]$. Comparing with (5.10), it is clear that there exist some random permutations $(\varkappa_i)$ of J and $(\varkappa'_j)$ of J', such that

$$c_i = \vartheta_{\varkappa_i}, \qquad c'_j = \vartheta'_{\varkappa'_j} \quad \text{a.s.}, \qquad i\in J,\ j\in J'. \tag{5.12}$$

Defining

$$\tau_i = \sigma_{\varkappa_i}, \quad \tau'_j = \sigma'_{\varkappa'_j}, \qquad i\in J,\ j\in J', \tag{5.13}$$

and noting that the pair (ζ_1,ζ_2) is jointly exchangeable, we may conclude from (5.8) and Lemma 12 that the r.v.'s τ_i and τ'_j are independent and U(0,1).

To finish the proof of (1.1), it remains to show that the r.v.'s

$$a_{ij} = \alpha_{\varkappa_i,\varkappa'_j}, \quad b_i = \beta_{\varkappa_i}, \quad b'_j = \beta'_{\varkappa'_j}, \qquad i\in J,\ j\in J', \tag{5.14}$$

are a.s. non-random. But this follows from the ergodicity of (ζ_1,ξ,ζ_2), since the quantities in (5.14) are a.s. invariant functions of (ζ_1,ξ,ζ_2), in the sense of the transformations in (5.9). For example, $a_{ij}=\xi\{(\tau_i,\tau'_j)\}$ a.s., where τ_i and τ'_j are a.s. uniquely determined by the relations

$$\zeta_1\{(\tau_i,r_i,c_i)\} = \zeta_2\{(\tau'_j,r'_j,c'_j)\} = 1 \quad \text{a.s.,} \quad i\in J,\ j\in J'.$$

Conversely, it is obvious from (1.1) that any random measure ξ of this form is separately exchangeable. It remains to prove that ξ is ergodic when (1.1) holds with non-random coefficients $a_{ij},b_i,b'_j,c\geq 0$, $i,j\in N$. By Lemma 4, it is then enough to construct another random measure $\tilde{\xi}$, as a fixed measurable function of ξ and some independent U(0,1) r.v. ϑ, such that ξ and $\tilde{\xi}$ are independent with the same distribution. Since the last term in (1.1) is non-random and measurably determined by ξ, it may be omitted for the sake of simplicity.

We construct $\tilde{\xi}$ by letting the r.v.'s $\alpha_{ij},\beta_i,\beta'_j,\ \sigma_i,\sigma'_j$, $i,j\in N$, be defined as in the preceding argument, introducing an independent set of independent U(0,1) r.v.'s $\tilde{\sigma}_i,\tilde{\sigma}'_j$, $i,j\in N$, and putting

$$\tilde{\xi} = \sum_i\sum_j \alpha_{ij}\delta_{\tilde{\sigma}_i,\tilde{\sigma}'_j} + \sum_j\{\beta_j(\delta_{\tilde{\sigma}_j}\times\lambda) + \beta'_j(\lambda\times\delta_{\tilde{\sigma}'_j})\}. \tag{5.15}$$

Comparing (1.1) and (5.5) (the former with coefficients a_{ij},b_i,b'_j, $i\in J$ and $j\in J'$), it is clear that there exist some random permutations $(\varkappa_i)$ of J and $(\varkappa'_j)$ of J', satisfying (5.13) and (5.14). Defining

$$\tilde{\tau}_i = \tilde{\sigma}_{\varkappa_i}, \quad \tilde{\tau}'_j = \tilde{\sigma}'_{\varkappa'_j}, \qquad i\in J,\ j\in J', \tag{5.16}$$

we may then rewrite (5.15) in the form

$$\tilde{\xi} = \sum_i\sum_j a_{ij}\delta_{\tilde{\tau}_i,\tilde{\tau}'_j} + \sum_j\{b_j(\delta_{\tilde{\tau}_j}\times\lambda) + b'_j(\lambda\times\delta_{\tilde{\tau}'_j}) \quad \text{a.s.} \tag{5.17}$$

To prove that $\tilde{\xi}$ has the desired properties, it is hence enough to show that the r.v.'s $\tilde{\tau}_i$ and $\tilde{\tau}'_j$ are independent U(0,1) and independent of ξ. But this is equivalent to showing that they are conditionally independent U(0,1), given ξ, which follows via (5.16) from the corresponding property for the variables $\tilde{\sigma}_i$ and $\tilde{\sigma}'_j$. □

Proof of Theorem 2. Assume that ξ is a jointly ergodic exchangeable random measure on $[0,1]^2$. Letting D denote the diagonal in $[0,1]^2$, we define $\xi_D = (D\xi)(\cdot \times [0,1])$. By Lemma 6, the diffuse component of ξ_D is ergodic exchangeable along with ξ, and by Lemma 11 it is then a.s. of the form $c\lambda$, for some constant $c \geq 0$. Hence the diffuse component of $D\xi$ is a.s. of the form $c\lambda_D$, in agreement with (1.2). To simplify the writing, we may henceforth assume that $c=0$.

Next we note that ξ is separately exchangeable on every product set $B \times B^c$, such that either B or B^c is a dyadic interval. Applying Theorem 1 with different choices of B, it follows easily that

$$(M_1^c \times [0,1])(D^c\xi) = \lambda \times \eta_2, \quad ([0,1] \times M_2^c)(D^c\xi) = \eta_1 \times \lambda \quad \text{a.s.,} \tag{5.18}$$

with M_1 and M_2 as in (5.1), for some random measures η_1 and η_2 on $[0,1]$. But

$$(D\xi)(M_1^c \times [0,1]) = (D\xi)([0,1] \times M_2^c) = 0 \quad \text{a.s.} \tag{5.19}$$

since $D\xi$ is a.s. diffuse, and therefore (5.2) and (5.3) remain valid in the present context. In particular, it is seen as before that $(M_1^c \times M_2^c)\xi = c'\lambda^2$ a.s. for some constant $c' \geq 0$, and we may henceforth take $c'=0$, for convenience.

As before, we may write ξ in the form (5.5), except that now we take $\sigma'_j = \sigma_j$ for all j. We may further assume that the sequence

$$r_j = \beta_j + \beta'_j + \sum_i (\alpha_{ij} + \alpha_{ji}), \quad j \in N,$$

is non-decreasing, and define $J = \{j \in N;\ r_j > 0\}$. From this point on, the proof follows closely that for Theorem 1, so we omit the details. □

The two representation Theorems 1 and 2 have many interesting consequences. Here we shall merely single out a few facts which will be needed for the proofs of Theorems 4 and 5. The first of these relates the three notions of joint, separate and complete exchangeability for random measures ξ on $[0,1]^2$, and may be of some independent interest. For convenience, we shall write

$$\tilde{\xi}(ds\,dt)=\xi(dt\,ds), \quad \xi_s(dt)=\xi(\{s\}\times dt), \quad \tilde{\xi}_t(ds)=\xi(ds\times\{t\}), \tag{5.20}$$

and define $\mathcal{M}_1=\{a\delta_t;\ a\geq 0,\ t\in[0,1]\}$.

Lemma 20. *For any random measure* ξ *on* $[0,1]^2$, *conditions* (i)-(iii) *below are equivalent:*

(i) ξ *is completely [ergodic] exchangeable,*

(ii) ξ *is separately [ergodic] exchangeable, and w.p.1,* $\xi_s, \tilde{\xi}_s \in \mathcal{M}_1$ *for all* s,

(iii) ξ *is jointly [ergodic] exchangeable,* $\xi D=0$ *a.s., and w.p.1,* $\xi_s+\tilde{\xi}_s\in\mathcal{M}_1$ *and* $\xi_s\wedge\tilde{\xi}_s=0$ *for all* s.

Moreover, condition (iv) *below implies* (v), *where*

(iv) ξ *is jointly exchangeable,* $\xi D=0$ *a.s., and w.p.1,* $\xi_s+\tilde{\xi}_s\in\mathcal{M}_1$ *for all* s,

(v) $\xi \overset{d}{=} \tilde{\xi}$, *and* $(\xi,\tilde{\xi})$ *is completely exchangeable in* $W_1=\{(s,t);\ 0\leq s\leq t\leq 1\}$.

Proof. Since clearly (i)$\Rightarrow$(ii)$\Rightarrow$(iii), it is enough to show that (iii)$\Rightarrow$(i) and (iv)$\Rightarrow$(v). The two proofs are very similar, so we shall only prove the latter implication. Thus assume that ξ satisfies (iv). Since the conditions in (v) are stable under convex combination of distributions, we may add the hypothesis that ξ be ergodic. By Theorem 2, ξ must then be of the form

$$\xi = \sum_j \{a_j\delta_{\sigma_j,\tau_j} + b_j\delta_{\tau_j,\sigma_j}\} + c\lambda^2, \tag{5.21}$$

for some constants $a_j,b_j,c\geq 0$, $j\in N$, and some independent U(0,1) r.v.'s σ_j,τ_j, $j\in N$. The conditions in (v) are further stable under addition of independent random measures, as well as under monotone convergence, so we may consider each term in (5.21) separately. The case $\xi=c\lambda^2$ being trivial, it remains to take

$$\xi = a\delta_{\sigma,\tau} + b\delta_{\tau,\sigma}, \tag{5.22}$$

for some constants $a,b\geq 0$ and some independent U(0,1) r.v.'s σ and τ. In this case,

$$(\xi,\tilde{\xi}) = (a,b)\delta_{\sigma,\tau} + (b,a)\delta_{\tau,\sigma}, \tag{5.23}$$

so $\xi \overset{d}{=} \tilde{\xi}$ follows from the fact that $(\sigma,\tau) \overset{d}{=} (\tau,\sigma)$. To prove that $(\xi,\tilde{\xi})$ is completely exchangeable in W_1, it suffices to note that the pair (σ,τ) is uniformly distributed in $[0,1]^2$, so that (σ,τ) is conditionally uniform in W_1, given $\{\sigma<\tau\}$, while (τ,σ) is conditionally uniform in W_1, given $\{\tau<\sigma\}$. The remaining possibility $\sigma=\tau$ has of course zero probability. □

For subsequent proofs, we shall also need the following two rather technical results. In the present context, the notions of separate or joint exchangeability of a vector valued random measure $\xi=(\xi^1,\ldots,\xi^4)$ on $[0,1]^2$ refer to the two coordinates of the unit square, rather than to the four components of the vector. Thus we are requiring conditions of the form

$$(\xi^1(f\times g)^{-1}, \ \ldots, \ \xi^4(f\times g)^{-1}) \overset{d}{=} (\xi^1,\ldots,\xi^4). \tag{5.24}$$

where $f=g$ in the case of joint exchangeability. We shall further use $\mathcal{M}_1$ and the notation of (5.20), in their versions for vector valued measures.

<u>Lemma 21.</u> <u>Let</u> $\xi=(\xi^1,\xi^2,\xi^3,\xi^4)$ <u>be a separately exchangeable</u> R_+^4-<u>valued random measure on</u> $[0,1]^2$, <u>whose components are a.s. mutually singular, and assume that w.p.1, for every</u> $s\in[0,1]$,

$$\xi_s^1+\xi_s^4\neq 0 \Rightarrow \xi_s\in\mathcal{M}_1 \quad \underline{\text{and}} \quad \tilde{\xi}_s^2+\tilde{\xi}_s^4\neq 0 \Rightarrow \tilde{\xi}_s\in\mathcal{M}_1. \tag{5.25}$$

<u>Then</u> ξ^4 <u>is conditionally separately exchangeable, given</u> (ξ^1,ξ^2,ξ^3), <u>while</u> ξ^1 [<u>or</u> $\tilde{\xi}^2$] <u>is conditionally exchangeable in the first coordinate, given</u> (ξ^2,ξ^3) [<u>or</u> (ξ^1,ξ^3), <u>respectively</u>].

<u>Proof.</u> Since any a.s. property of ξ is preserved under conditioning, and since the asserted properties are stable under convex combinations of distributions for ξ, we may reduce to the ergodic case through conditioning on the invariant σ-field for ξ. In that case, ξ has a representation as in Theorem 1, but with non-random R_+^4-valued coefficients. From (5.25) plus the hypothesis of singularity, it is clear that ξ^4 and (ξ^1,ξ^2,ξ^3) are represented in terms of disjoint and hence independent sets of r.v.'s τ_j and τ_j', so ξ^4 and (ξ^1,ξ^2,ξ^3) must be independent. Thus the conditional distribution of ξ^4, given (ξ^1,ξ^2,ξ^3), agrees with the unconditional one, and the first assertion follows.

Next conclude from (5.25) and the singularity hypothesis that the sets of r.v.'s τ_j in the representations for ξ^1 and (ξ^2,ξ^3) are disjoint and therefore independent. Hence ξ^1 must be of the form

$$\xi^1 = \sum_i a_i \delta_{\tau_i,\sigma_i} + \sum_i b_i(\lambda\times\delta_{\sigma_i}) + c\lambda^2 \quad \text{a.s.,} \tag{5.26}$$

where the r.v.'s τ_i are independent U(0,1) and independent of (ξ^2,ξ^3) and all the σ_i. By (5.26) and Fubini's theorem, it follows that ξ^1 is conditionally exchangeable in the first coordinate, given (ξ^2,ξ^3) and the σ_i, and the assertion for ξ^1 follows by the chain rule for conditional expectations. The same proof applies in the case of $\tilde{\xi}^2$. □

In stating the next result, we shall use the further notation $\bar{\xi} = \xi + \tilde{\xi}$ and $\bar{\xi}_s = \xi_s + \tilde{\xi}_s$.

Lemma 22. *Let $\xi=(\xi^1,\xi^2,\xi^3,\xi^4)$ be a jointly exchangeable R_+^4-valued random measure on $[0,1]^2$ with $(\xi^1+\xi^2+\xi^4)D=0$ a.s., and such that $\xi^1+\tilde{\xi}^2$, $\tilde{\xi}^1+\xi^2$, $\bar{\xi}^3$ and $\bar{\xi}^4$ are a.s. mutually singular. Further assume that w.p.1*

$$\xi_s^1 + \tilde{\xi}_s^2 + \bar{\xi}_s^4 \neq 0 \;\Rightarrow\; \bar{\xi}_s^1 + \bar{\xi}_s^2 + \bar{\xi}_s^3 + \bar{\xi}_s^4 \in \mathcal{M}_1, \qquad s\in[0,1]. \tag{5.27}$$

Then ξ^4 is conditionally jointly exchangeable, given (ξ^1,ξ^2,ξ^3), while $(\xi^1,\tilde{\xi}^2)$ is conditionally exchangeable in the first coordinate, given ξ^3.

Proof. As in the preceding proof, we may assume that ξ is ergodic, and hence has a representation as in Theorem 2, but with non-random R_+^4-valued coefficients. The first assertion then follows as before, while the second one is obtained from the fact that $(\xi^1,\tilde{\xi}^2)$ has a representation

$$(\xi^1,\tilde{\xi}^2) = \sum_i (a_i,a_i')\delta_{\tau_i,\sigma_i} + \sum_i (b_i,b_i')(\lambda\times\delta_{\sigma_i}) + (c,c')\lambda^2 \quad \text{a.s.,} \tag{5.28}$$

where the coefficients are constant vectors, while the r.v.'s τ_i and σ_i are independent U(0,1), and such that the τ_i are independent even of ξ^3. □

6. EXCHANGEABILITY IN A STRIP

The purpose of this section is to prove Theorem 3, which characterizes the class of separately exchangeable random measures on the strip $R_+ \times [0,1]$, and its subclass of ergodic measures. Though Theorem 3 is not needed formally for the proofs of Theorems 4 and 5, its demonstration in this section will prepare the reader for the partly similar but more subtle arguments required for Theorems 4 and 5.

For the technical reasons explained earlier, we shall still take the notions of exchangeability, invariance and ergodicity for random measures to be defined with respect to the groups of measure preserving transformations of R_+ or $[0,1]$ which only permute finitely many disjoint dyadic intervals of equal length.

<u>Proof of Theorem 3.</u> Assume that ξ is a separately exchangeable random measure on $R_+ \times [0,1]$. To prove that ξ can be represented as in (1.3), we may assume by Lemma 3 that ξ is ergodic, and prove instead that (1.3) holds with α, γ and the β_j non-random. In that case clearly h and all the f_j and g_k reduce to functions of one variable only, which we denote by the same letters, for the sake of economy.

In analogy with (5.1), we introduce the countable random sets

$$M_1 = \{s \in R_+;\ \xi(\{s\} \times [0,1]) > 0\},\quad M_2 = \{t \in [0,1];\ \xi(R_+ \times \{t\}) \neq 0\}. \tag{6.1}$$

Applying Theorem 1 to the restrictions of ξ to the rectangles $[0,n] \times [0,1]$, it is seen as in (5.2) that

$$(M_1^c \times [0,1])\xi = \lambda \times \eta_2,\qquad (R_+ \times M_2^c)\xi = \eta_1 \times \lambda \qquad \text{a.s.,} \tag{6.2}$$

for suitable random measures η_1 on R_+ and η_2 on $[0,1]$. In particular, we get $(M_1^c \times M_2^c)\xi = c\lambda^2$ a.s. for some constant $c \geq 0$, which yields the last term in (1.3). In the sequel, we may assume that $c=0$.

Since ξ is exchangeable along R_+, we may next conclude from Lemma 17 that, with ρ_t given by (3.9), the random set $M=\{t \in [0,1];\ \rho_t > 0\}$ is a.s. covered by some distinct $[0,1]$-valued r.v.'s $\tau_1, \tau_2, \ldots$, such that the associated sequence

$r_j = \rho_{\tau_j}$, $j \in N$, is non-increasing. Since the r_j are clearly invariant functions of ξ, they must be a.s. non-random because of the ergodicity of ξ, and the same thing is then true for the index set $J = \{j \in N;\ r_j > 0\}$.

Proceeding as in the proof of Theorem 1, we may next use Lemma 7 to construct an auxiliary point process

$$\zeta = \sum_{j \in J} \delta_{\tau_j, r_j, \chi_j} \doteq \sum_{j \in J} \delta_{\tau'_j, r_j, c_j}, \tag{6.3}$$

with a.s. distinct marks χ_j in $[0,1]$, $j \in N$, and such that the pair (ξ, ζ) is ergodic exchangeable, in the sense of the mappings $(\xi,\zeta) \to (\xi(f_1 \times f_2)^{-1}, \zeta f_2^{-1})$. Here f_1 and f_2 are measure preserving transformations on R_+ and $[0,1]$, respectively, of the special permutation type. The second expression in (6.3) is obtained from the first one, if we apply a suitable random permutation (κ_j) of J, to make the quantities $c_j = \chi_{\kappa_j}$ a.s. non-random. Note that the points $\tau'_j = \tau_{\kappa_j}$ will be measurably determined a.s. by (ξ, ζ). From (6.2) we conclude that

$$\xi((M_1^c \cap \cdot) \times \{\tau'_j\}) = b_j \lambda \quad \text{a.s.}, \quad j \in J, \tag{6.4}$$

for some random variables $b_j \geq 0$, and since the latter are clearly a.s. invariant functions of (ξ, ζ), they must be a.s. non-random.

Next we note that $\# M_1$ is invariant and hence a.s. constant, so that we can write $M_1 = \{\sigma_i\}$ for some sequence of a.s. distinct r.v.'s σ_i. By (6.2), we may further write w.p.1, simultaneously for all i,

$$\xi(\{\sigma_i\} \times \cdot) = \sum_j \alpha_{ij} \delta_{\tau'_j} + \sum_k \gamma_{ik} \delta_{\rho_{ik}} + \beta'_i \lambda, \tag{6.5}$$

in terms of suitable ξ-measurable r.v.'s $\alpha_{ij}, \gamma_{ik}, \beta'_i \geq 0$, and $\rho_{ik} \in [0,1]$. Note that the τ'_j and ρ_{ik} may be taken to be a.s. distinct for fixed i, and that we may assume $\gamma_{i1} \geq \gamma_{i2} \geq \ldots$ a.s. In fact, Lemma 17 shows that we may choose the entire collection of r.v.'s ρ_{ik}, with arbitrary i and k, to be a.s. distinct and different from the τ'_j.

We shall now use the quantities in (6.5) to construct a marked point process $\eta = \sum \delta_{\sigma_i, \vartheta_i}$ along R_+, where the marks ϑ_i are defined by

$$\vartheta_i = (\mu_i, \beta'_i, (\alpha_{ij};\ j \in J), (\gamma_{ik},\ k \in N)), \tag{6.6}$$

with

$$\mu_i = \xi(\{\sigma_i\}\times[0,1]) = \beta_i' + \sum_j \alpha_{ij} + \sum_k \gamma_{ik}. \tag{6.7}$$

Thus the space of marks is given by

$$K = (0,\infty)\times R_+ \times R_+^J \times R_+^N, \tag{6.8}$$

and is certainly Polish. Our only reason for including the redundant mark μ_i in (6.6) is to make sure that η will be finite on bounded sets, in any natural complete metrization of K. We note that the coefficients α_{ij}, γ_{ik} and β_i' in (6.5) may be measurably recovered from the marks ϑ_i in (6.6), through suitable projections which we denote by f_j, g_k and h. Thus

$$\alpha_{ij} = f_j(\vartheta_i), \quad \gamma_{ik} = g_k(\vartheta_i), \quad \beta_i' = h(\vartheta_i). \tag{6.9}$$

From the separate ergodic exchangeability of (ξ,ζ), we may conclude by means of Lemma 6 that η is ergodic exchangeable along R_+. Indeed, the hypothesis of Lemma 6 is fulfilled in the present case, since η, when regarded as a function of (ξ,ζ), is clearly invariant under measure preserving transformations of the special permutation type along $[0,1]$. Thus Lemma 13 shows that η is Poisson, with an intensity measure of the form $\lambda\times\nu$. By the invariance of η, it is further seen that the exchangeability of (ξ,ζ) along $[0,1]$ remains conditionally true, given η. In fact, the sequence of pairs (σ_i,ϑ_i) may clearly be chosen to be η-measurable, in which case the exchangeability of (ξ,ζ) along $[0,1]$ is even true under conditioning with respect to the random elements σ_i and ϑ_i.

By Lemma 6, the conditional exchangeability of (ξ,ζ) carries over to the pair of marked point processes (ζ,ζ') with ζ' given by

$$\zeta' = \sum_i\sum_k \delta_{\rho_{ik},\gamma_{ik},\sigma_i}, \tag{6.10}$$

where the second summation extends over indices k with $\gamma_{ik}>0$. Using Lemmas 12, 3 and 1, we may conclude that (6.10) remains a.s. true with the r.v.'s ρ_{ik} replaced by some ρ_{ik}', such that all the τ_j' and ρ_{ik}' are independent U(0,1) and independent of η. Comparing the two versions of (6.10), it is clear that even (6.5) remains true with ρ_{ik} replaced by ρ_{ik}'. This shows that ξ has the a.s. representation

$$\xi=\sum_i\sum_j f_j(\vartheta_i)\delta_{\sigma_i,\tau'_j}+\sum_i\sum_k g_k(\vartheta_i)\delta_{\sigma_i,\rho'_{ik}}+\sum_i h(\vartheta_i)(\delta_{\sigma_i}\times\lambda)+\sum_j b_j(\lambda\times\delta_{\tau'_j}),\quad(6.11)$$

in agreement with (1.3). Note also the r.v.'s in (6.11) have the desired joint distribution, except that η is a Poisson process on the 'wrong' space $R_+\times K$.

To attain a complete conformity with Theorem 3, we add an extraneous point ∂ to K, and note that a measurable mapping $T: R_+\to K_\partial=K\cup\{\partial\}$ exists, such that $\lambda T^{-1}=\nu$ on K. The induced mapping T' of R_+^2 into $R_+\times K_\partial$ will then transform any unit rate Poisson process on R_+^2 into a Poisson process on $R_+\times K$ with intensity $\lambda\times\nu$. By Lemma 1, there further exist some pairs of r.v.'s (σ'_i,ϑ'_i), $i\in N$, which are independent of all τ'_j and ρ'_{ik}, and form a unit rate Poisson process η' on R_+^2 satisfying $\eta' T'^{-1}=\eta$ a.s. on $R_+\times K$. Defining

$$f'_j=f_j\circ T,\quad g'_k=g_k\circ T,\quad h'=h\circ T,\qquad(6.12)$$

with the added convention that $f_j(\partial)\equiv g_k(\partial)\equiv h(\partial)=0$, it is clear that (6.11) remains true, with the objects f_j, g_k, h, σ_i and ϑ_i replaced by their 'primed' counterparts f'_j, g'_k, h', σ'_i and ϑ'_i, reprectively. This completes the proof of the representation (1.3).

In the other direction, it is clear that a random measure ξ of the form (1.3) is separately exchangeable. The ergodicity of ξ when α and the β_j are constants will follow from Lemma 4, if we can only produce an independent copy $\xi''=F(\xi,\gamma)$ of ξ, with γ a U(0,1) r.v. independent of ξ, where the measurable mapping F is not allowed to depend on the particular functions f_j, g_k, h and coefficients β_j occurring in (1.3). Our construction of ξ'' will proceed in two steps, where we first construct a random measure $\xi'=G(\xi,\gamma')$ from ξ and an independent U(0,1) r.v. γ', such that ξ' has again a representation (1.3), with the same functions and coefficients as for ξ and the same r.v.'s τ_j, but with the set of r.v.'s σ_i, ϑ_i and ρ_{ik} replaced by a new collection $\sigma'_i,\vartheta'_i,\rho'_{ik}$, $i,k\in N$, which have the same joint distribution but are independent of ξ and the τ_j. As before, the mapping G is not allowed to depend on which particular functions and coefficients occur in (1.3). The existence of such a mapping is guaranteed by Lemma 10 (with $\alpha=(\tau_j)$), where the crucial condition (2.10) holds by Lemma 18, since ξ has conditionally stationary independent increments, given the sequence (τ_j).

The second step in our construction of ξ'' uses the method already employed in case of Theorem 1. Thus we first introduce a sequence of r.v.'s γ_j, $j\in J$, defined as the τ_j of Lemma 17, and note that the γ_j are measurably determined by ξ'. Comparing with the representation (1.3) for ξ', it is clear that $\tau_j = \gamma_{\kappa_j}$ a.s., $j\in J$, for some random permutation (κ_j) of J. Let us next introduce a sequence of independent U(0,1) r.v.'s $\gamma_1', \gamma_2', \ldots$, independent of everything else, and define the new random measure

$$\xi'' = \xi' + \sum_j \{\xi'(\cdot \times \{\gamma_j\}) \times (\delta_{\gamma_j'} - \delta_{\gamma_j})\}. \tag{6.13}$$

Then ξ'' has a.s. the same representation (1.3) as ξ', except that each τ_j is replaced by the corresponding quantity $\tau_j' = \gamma_{\kappa_j}'$. Moreover, (6.13) exhibits ξ'' as a fixed measurable function of ξ' and $\gamma_1', \gamma_2', \ldots$ Representing $\gamma', \gamma_1', \gamma_2', \ldots$ as functions of a single U(0,1) r.v. γ independent of ξ, it is then clear that $\xi'' = F(\xi, \gamma)$ for a suitable fixed F.

To check the distributional properties, note that the quantities τ_j' are conditionally independent and U(0,1), given ξ and all the σ_i', ϑ_i', ρ_{ik}' and κ_j. In other words, they are independent U(0,1) and independent of ξ and all those variables. From this we conclude that ξ'' is independent of ξ, and that the two arrays $(\tau_j, \sigma_i, \vartheta_i, \rho_{ik};\ i,j,k\in N)$ and $(\tau_j', \sigma_i', \vartheta_i', \rho_{ik}';\ i,j,k\in N)$ have the same joint distribution. Since these are the r.v's occurring in the representations (1.3) for ξ and ξ'', even the latter are equally distributed. This completes the proof of Theorem 3. $\square$

7. EXCHANGEABILITY IN A QUADRANT

In this section, we shall prove the last two of our main results, those which characterize separate or joint exchangeability for random measures on R_+^2. Much of the preparatory work has already been done in previous sections. Yet it will be convenient to isolate some of the main steps in the form of lemmas, which may easily be put together in the end to furnish complete proofs of the two theorems. Since the arguments for separately and jointly exchangeable measures are rather similar, we shall treat the two cases in parallel.

Our first lemma characterizes the component of ξ which has independent increments, globaly or in some suitably restricted sense. Until further notice, we shall use the notation of (5.20), and we shall write $\mathcal{M}_1=\{a\delta_t;\ a,t\in R_+\}$. Recall that, throughout this section, the notions of exchangeability, ergodicity, invariance, etc., are to be understood in the sense of arbitrary measure preserving transformations of R_+, which permute a finite number of disjoint dyadic intervals of equal length. Define $D=\{(s,t)\in R_+^2;\ s=t\}$ and $W=\{(s,t)\in R_+^2;\ s\le t\}$.

<u>Lemma 23.</u> <u>Let</u> ξ <u>be a random measure on</u> R_+^2. <u>Then conditions</u> (i)-(iii) <u>below are equivalent:</u>

(i) ξ <u>is separately ergodic exchangeable, and w.p.1,</u> $(\xi_s+\tilde{\xi}_s)R_+<\infty$ <u>for all</u> $s\ge 0$;

(ii) ξ <u>has stationary independent increments;</u>

(iii) $\xi=\sum f(\alpha_i)\delta_{\sigma_i,\tau_i}+c\lambda^2$ <u>a.s., for some constant</u> $c\ge 0$, <u>some measurable function</u> $f\colon R_+\to R_+$, <u>and some random triples</u> $(\sigma_i,\tau_i,\alpha_i)$, $i\in N$, <u>which form a unit rate Poisson process on</u> R_+^3.

<u>So are the following conditions</u> (iv)-(vi):

(iv) ξ <u>is jointly ergodic exchangeable</u>, $\xi D=0$ <u>a.s., and w.p.1,</u> $(\xi_s+\tilde{\xi}_s)R_+<\infty$ <u>for all</u> $s\ge 0$;

(v) $\tilde{\xi}\stackrel{d}{=}\xi$, <u>and</u> $(\xi,\tilde{\xi})$ <u>has stationary independent increments in</u> W;

(vi) $\xi=\sum\{f(\alpha_i)\delta_{\sigma_i,\tau_i}+g(\alpha_i)\delta_{\tau_i,\sigma_i}\}+c\lambda^2$ <u>a.s., for some constant</u> $c\ge 0$, <u>some measurable functions</u> $f,g\colon R_+\to R_+$, <u>and some random triples</u> $(\sigma_i,\tau_i,\alpha_i)$, $i\in N$, <u>which form a unit rate Poisson process on</u> R_+^3.

Proof. Assume (i). Then Lemma 17 shows that, w.p.1, $\xi_s, \tilde{\xi}_s \in \mathcal{M}_1$ for all $s \geq 0$, so ξ is completely exchangeable by Lemma 20. Since conversely every completely exchangeable random measure is trivially separately exchangeable, it follows from the ergodicity=extremality in (i) that ξ is even ergodic=extreme in the sense of complete exchangeability. Hence (ii) follows by the obvious two-dimensional extensions of Lemmas 11 and 13. Assuming (ii), we may next conclude from the same two-dimensional results that

$$\xi = c\lambda^2 + \sum_i \vartheta_i \delta_{\sigma_i, \tau_i} \quad \text{a.s.,} \tag{7.1}$$

for some constant $c \geq 0$ and some random triples $(\sigma_i, \tau_i, \vartheta_i)$, $i \in N$, which form a Poisson process on R_+^3 with intensity of the form $\lambda^2 \times \nu$. Choosing $f: R_+ \to R_+$ to be measurable with $\lambda f^{-1} = \nu$ on $R_+ \setminus \{0\}$, it is then clear that ξ has the same distribution as the expression in (iii), and the corresponding a.s. representation follows by Lemma 1. Next (iii) implies that ξ is separately exchangeable, and that w.p.1, $(\xi_s + \tilde{\xi}_s) R_+ < \infty$ for all $s \geq 0$. To see that ξ is ergodic, we note that every invariant function η of ξ is also an invariant function of the sequence $\xi_1, \xi_2, \ldots$, where ξ_k denotes the restriction of the translated measure $\xi(\cdot + (k,0))$ to the strip $[0,1] \times R_+$. Since the ξ_k are i.i.d., η is a.s. non-random by the Hewitt-Savage 0-1 law, and ξ is ergodic. This proves that (i)-(iii) are equivalent.

Now assume instead that (iv) holds. Then even $\xi + \tilde{\xi}$ is jointly exchangeable by Lemma 6, so $\xi + \tilde{\xi}$ is separately exchangeable on every set of the form $(a,b) \times ([0,a] \cup [b,\infty))$ with $a < b$. Thus Lemma 17 shows that, w.p.1, the restriction of $\xi_s + \tilde{\xi}_s$ to $[0,a] \cup [b,\infty)$ belongs to $\mathcal{M}_1$ for all $s \in (a,b)$. Since this holds simultaneously for all rational a and b, outside some fixed P-nullset, we get w.p.1, $\{s\}^c (\xi_s + \tilde{\xi}_s) \in \mathcal{M}_1$ for all $s \geq 0$. Since $\xi D = 0$ a.s., it follows that w.p.1 $\xi_s + \xi_s \in \mathcal{M}_1$ for all s. We may then conclude from Lemma 20 that $\xi \overset{d}{=} \tilde{\xi}$, and that the pair $(\xi, \tilde{\xi})$ is completely exchangeable in W.

From the obvious extensions of Lemmas 11 and 13 to W, it follows that the diffuse component of $(\xi, \tilde{\xi})$ is a.s. of the form $(\gamma, \tilde{\gamma})\lambda^2$ on W, where γ and $\tilde{\gamma}$ are suitable r.v.'s, while the marked point process η which describes the purely atomic part of $(\xi,\tilde{\xi})$ is Cox on W with a directing random measure of the form $\lambda^2 \times \nu$, where ν is a random measure on $R_+^2 \setminus \{0\}$. Now γ, $\tilde{\gamma}$ and ν can a.s. be determined, via the law of large numbers, as jointly invariant measurable functions of ξ, so by the ergodicity in (i), these quantities must be a.s. non-random. In that case, $(\xi,\tilde{\xi})$ has independent increments in W, and (v) follows.

Assuming (v), we get as before a representation of $(\xi,\tilde{\xi})$ in terms of quantities γ, $\tilde{\gamma}$ and ν, which may again be taken to be non-random. The condition $\xi \overset{d}{=} \tilde{\xi}$ implies that $(\xi,\tilde{\xi}) \overset{d}{=} (\tilde{\xi},\xi)$, so $\gamma = \tilde{\gamma}$, while ν is symmetric with respect to D. In particular, the diffuse component of ξ equals $\gamma\lambda^2$ a.s., in agreement with (vi), and we may henceforth assume that ξ is purely atomic. Let us next choose two measurable functions f,g: $R_+ \to R_+$, such that $\lambda(f,g)^{-1} = \nu/2$ on $R_+^2 \setminus \{0\}$. Introducing a unit rate Poisson process η on R_+^3, and writing $\tilde{\eta}(ds\,dt\,\cdot) = \eta(dt\,ds\,\cdot)$, we define

$$\xi' = \int f(du)\eta(\cdot \times du) + \int g(du)\tilde{\eta}(\cdot \times du), \tag{7.2}$$

so that

$$\begin{aligned}(\xi',\tilde{\xi}') &= \int (f,g)(du)\eta(\cdot \times du) + \int (g,f)(du)\tilde{\eta}(\cdot \times du) \\ &= \int (x,y)(\zeta + \zeta')(\cdot \times dxdy),\end{aligned} \tag{7,3}$$

where

$$\zeta = \eta(1 \times (f,g))^{-1}, \quad \zeta' = \tilde{\eta}(1 \times (g,f))^{-1}. \tag{7.4}$$

Here ζ and ζ' are both Poisson processes on $R_+^2 \times (R_+^2 \setminus \{0\})$ with the same intensity $\lambda^2 \times \nu/2$, and since they are further independent on $W \times (R_+^2 \setminus \{0\})$, even their sum must be Poisson on the latter set, with intensity given by $\lambda^2 \times \nu$. Thus $(\xi',\tilde{\xi}') \overset{d}{=} (\xi,\tilde{\xi})$ on W by (7,3), so $\xi' \overset{d}{=} \xi$ on R_+^2, and then Lemma 1 shows that even ξ has a representation as in (7.2). Hence (v) implies (vi).

Let us finally assume (vi). Then the second and third statements in (iv) are obvious, while the joint exchangeability of ξ follows from the fact that $\eta = \sum \delta_{\sigma_i,\tau_i,\alpha_i}$ is jointly exchangeable in the first two coordinates. To prove the ergodicity of ξ in the sense of (iv), we note that the distribution of ξ can be measurably constructed from ξ through the law of large numbers, since ξ is dissociated (cf. Aldous[1,2]), and hence that Lemma 4 applies with $h(t)=\pi_t$. Thus (vi) implies (iv), so even the last three conditions are equivalent. ∎

To state the next result, we associate with a given random measure ξ on R_+^2 the sets

$$M_1 = \{s\geq 0;\ \xi(\{s\}\times R_+)=\infty\},\quad M_2 = \{t\geq 0;\ \xi(R_+\times\{t\})=\infty\}, \tag{7.5}$$

$$M = M_1\cup M_2\cup\{s\geq 0;\ \xi\{(s,s)\}>0\}, \tag{7.6}$$

and introduce the decomposition of ξ into four components

$$\xi_1=(M_1^c\times M_2)\xi,\quad \xi_2=(M_1\times M_2^c)\xi,\quad \xi_3=(M_1\times M_2)\xi,\quad \xi_4=(M_1^c\times M_2^c)\xi. \tag{7.7}$$

It is easy to check that the ξ_i are measurable functions of ξ. The following result describes the joint distribution of the ξ_i when ξ is separately exchangeable, and states the corresponding result for the jointly exchangeable case.

Lemma 24. Let ξ be a separately ergodic exchangeable random measure on R_+^2, and define $\xi_1,\ldots,\xi_4$ by (7.7). Then there exist some random measures α_1, α_2 on R_+ and ν_1,ν_2 on $R_+\times(R_+\setminus\{0\})$, such that ξ_1 and ξ_2 have a.s. diffuse components $\lambda\times\alpha_1$ and $\alpha_2\times\lambda$, respectively, while their associated point processes η_1, η_2 of jump positions and sizes are Cox and directed by $\lambda\times\nu_1$ and $\nu_2\times\lambda$. Moreover, ξ_4 and $(\alpha_1,\alpha_2,\nu_1,\nu_2,\xi_3)$ are both separately ergodic exchangeable, and ξ_4 is independent of (ξ_1,ξ_2,ξ_3), while η_1 is conditionally independent of (α_1,ξ_2,ξ_3), given ν_1, and η_2 is conditionally independent of (α_2,ξ_1,ξ_3), given ν_2.

Assume instead that ξ is jointly ergodic exchangeable on R_+^2, and define $\xi_1,\ldots,\xi_4$ by (7.7), but with M_1 and M_2 replaced by M. Then there exist some random measures α_1, α_2 on R_+ and ν on $R_+\times(R_+^2\setminus\{0\})$, such that ξ_1 and ξ_2 have a.s. diffuse components $\lambda\times\alpha_1$ and $\alpha_2\times\lambda$, respectively, while the point process η of atom positions and sizes associated with $(\xi_1,\tilde{\xi}_2)$ is Cox and directed by $\lambda\times\nu$. Moreover, ξ_4 and $(\alpha_1,\alpha_2,\nu,\xi_3)$ are both jointly ergodic exchangeable, and ξ_4 is independent of (ξ_1,ξ_2,ξ_3), while η is conditionally independent of $(\alpha_1,\alpha_2,\xi_3)$, given ν.

In these statements, the separate or joint exchangeability of $(\alpha_1,\alpha_2,\nu_1,\nu_2,\xi_3)$ or $(\alpha_1,\alpha_2,\nu,\xi_3)$ is defined as in (5.9), with the functions f_1 and f_2 taken to be equal in the joint case. For an atomic random measure $\xi=\sum\alpha_j\delta_{\tau_j}$, the associated point process of atom positions and sizes is given by $\eta=\sum\delta_{\tau_j,\alpha_j}$.

Proof. Assume that ξ is separately ergodic exchangeable. Then so are $\xi_1,\ldots,\xi_4$, as well as their diffuse and atomic components, by Lemma 6. Applying Lemma 17 to ξ_1 and $\tilde{\xi}_2$, we may conclude that, w.p.1, $\xi_1(\{s\}\times\cdot)\in\mathcal{M}_1$ and $\xi_2(\cdot\times\{t\})\in\mathcal{M}_1$ for all $s,t\geq 0$. Thus Lemmas 11 and 13 yield the stated forms of the diffuse and atomic components of ξ_1 and ξ_2, in terms of some random measures α_1, α_2, ν_1 and ν_2. Since the latter are measurably determined by ξ, through the law of large numbers, we may conclude by another application of Lemma 6 that even $(\alpha_1,\alpha_2,\nu_1,\nu_2,\xi_3)$ is separately ergodic exchangeable.

By Lemmas 6 and 17, the hypotheses of Lemma 21 are fulfilled for the restriction of $(\xi_1,\ldots,\xi_4)$ to an arbitrary square $[0,a]^2$, so by martingale convergence as $a\to\infty$, we may conclude that ξ_4 is conditionally separately exchangeable, given (ξ_1,ξ_2,ξ_3), while ξ_1 is conditionally exchangeable in the first coordinate, given (ξ_2,ξ_3). Since ξ_4 is ergodic, it follows from the first statement, as in case of Lemma 9, that ξ_4 is independent of (ξ_1,ξ_2,ξ_3). Since (α_1,ν_1) is invariant under measure preserving transformations in the first coordinate, the exchangeability of ξ_1 remains valid under conditioning with respect to $(\alpha_1,\nu_1,\xi_2,\xi_3)$. Even η_1 is then conditionally exchangeable in the first coordinate, given $(\alpha_1,\nu_1,\xi_2,\xi_3)$. By Lemma 13 plus the law of large numbers, η_1 must then be conditionally Poisson with intensity $\lambda\times\nu_1$, just as under conditioning with respect to ν_1. Thus η_1 is conditionally independent of (α_1,ξ_2,ξ_3), given ν_1, and the same argument shows that η_2 is conditionally independent of (α_2,ξ_1,ξ_3), given ν_2.

Assuming instead that ξ is jointly ergodic exchangeable, and using M instead of M_1 and M_2 in the definitions of $\xi_1,\ldots,\xi_4$, it is seen from Lemma 6 that ξ_4 and $(\xi_1,\tilde{\xi}_2)$ as well as their diffuse and atomic parts are jointly ergodic exchangeable. Proceeding as in the proof of Lemma 23, one may easily check that the hypotheses of Lemma 22 are fulfilled for the restrictions of $\xi_1,\ldots,\xi_4$ to an arbitrary square $[0,a]^2$. In particular, $(\xi_1,\tilde{\xi}_2)$ must then be exchangeable in the first coordinate, and w.p.l., $(\xi_1+\tilde{\xi}_2)(\{s\}\times\cdot)\in\mathcal{M}_1$ for all s, so using Lemmas 11 and 13, we may conclude as before that ξ_1 and ξ_2 have a.s. diffuse components of the form $\lambda\times\alpha_1$ and $\alpha_2\times\lambda$, respectively, while η is Cox with directing random measure of the form $\lambda\times\nu$. It is further clear from Lemma 6 and the law of large numbers that even $(\alpha_1,\alpha_2,\nu,\xi_3)$ is jointly ergodic exchangeable. The independence and conditional independence assertions may be proved as in the separately exchangeable case, except that Lemma 22 should now be used instead of Lemma 21. □

The structure of the last component in the preceding decomposition of ξ was essentially analyzed already in Lemma 23. We proceed to derive representations for the sequences $(\alpha_1,\alpha_2,\nu_1,\nu_2,\xi_3)$ and $(\alpha_1,\alpha_2,\nu,\xi_3)$ occurring in Lemma 24. As a first step, we shall then consider marked point processes on R_+^2 of the form

$$\xi = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\delta_{\sigma_i,\tau_j,\alpha_{ij}}, \tag{7.8}$$

where $\sigma_1<\sigma_2<\ldots$ and $\tau_1<\tau_2<\ldots$, while the α_{ij} take their values in some Polish space K. Here we may write

$$A = (\alpha_{ij};\ i,j\in N),\quad \eta=\sum_{i=1}^{\infty}\delta_{\sigma_i},\quad \zeta=\sum_{j=1}^{\infty}\delta_{\tau_j}, \tag{7.9}$$

and note that the array A and the simple point processes η and ζ are uniquely and measurably determined by ξ.

Lemma 25. Fix a Polish space K, and let ξ be a separately ergodic exchangeable point process on $R_+^2\times K$ of the form (7.8), where $\sigma_1<\sigma_2<\ldots$ and $\tau_1<\tau_2<\ldots$ Then the random objects A, η and ζ in (7.9) are independent, and A is separately ergodic exchangeable, while η and ζ are homogeneous Poisson processes on R_+.

Assume instead that ξ is jointly ergodic exchangeable, and that $\eta=\zeta$. Then A and η are independent, and A is jointly ergodic exchangeable, while η is homogeneous Poisson.

Proof. Assume that ξ is separately ergodic exchangeable. Then η and ζ are ergodic exchangeable by Lemma 6, so both are homogeneous Poisson by Lemma 13.

Since ζ is invariant under measure preserving transformations in the first coordinate, ξ remains conditionally exchangeable in that coordinate, given ζ. Using Lemmas 8 and 14 plus the fact that η is ergodic, we may conclude that η is conditionally homogeneous Poisson and independent of A, given ζ. Hence η has the same conditional distribution, given A and ζ, and since η is ergodic it follows that η is independent of A and ζ. Applying the same argument to ζ, we may conclude that A, η and ζ are independent.

From Lemma 14 it is further seen that A is separately exchangeable. To see that A is even ergodic, let $\mathcal{J}$ denote the invariant σ-field induced by A, and conclude from Lemma 7 that ξ remains separately exchangeable, conditionally on $\mathcal{J}$. Since ξ is ergodic, it follows that ξ is independent of $\mathcal{J}$, so $\mathcal{J}$ must be independent of itself and hence trivial.

In the jointly exchangeable case, we first note that η is ergodic exchangeable by Lemma 6, and hence must be homogeneous Poisson by Lemma 13. Next we introduce the random objects

$$\nu_a = \eta[0,a], \quad \eta^a = [0,a]\eta, \quad \eta_a^b = (a,b]\eta, \qquad 0 < a < b \leq \infty, \tag{7.10}$$

$$A_a = (\alpha_{ij};\ i,j \leq \nu_a), \quad A^{(n)} = (\alpha_{ij};\ i,j \leq n), \qquad a > 0,\ n \in \mathbb{N}. \tag{7.11}$$

Applying Theorem 2 and Lemma 15 to the restriction of ξ to a square $[0,a]^2$, it is seen that A_a is conditionally jointly exchangeable, given ν_a, while η^a is conditionally exchangeable and independent of A_a, given ν_a. The first statement implies that $A^{(n)}$ is jointly exchangeable, conditionally on the event $\{\nu_a \geq n\}$, and from this we obtain the joint exchangeability of A by letting $a \to \infty$ and then $n \to \infty$.

The second statement shows that η^a is conditionally exchangeable, given ν_a and A_a, so if $0 < a < b < \infty$, it is clear that η^a is conditionally exchangeable, given ν_a, η_a^b and A_b. Here we may let $b \to \infty$, and conclude by martingale convergence that η^a is conditionally exchangeable, given ν_a, η_a^∞ and A. Taking conditional expectations, given A, and letting $a \to \infty$, it follows that η is conditionally exchangeable, given A, and since η is ergodic, it must then be independent of A. It remains to show that A is ergodic. But this follows by same argument as in the separately exchangeable case. □

Our next aim is to combine the results of the last lemma with those of Section 4, to obtain explicit representations of certain exchangeable marked point processes ξ on R_+^2, in terms of suitable Poisson processes and i.i.d. sequences. Since the projection of ξ onto R_+^2 is no longer assumed to be locally finite, we shall need to introduce an extra mark in each coordinate, so that ξ will take the form

$$\xi = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \delta_{\sigma_i, \tau_j, \alpha_i, \beta_j, \gamma_{ij}}. \tag{7.12}$$

Here the marks α_i and β_j are assumed to be R_+-valued, while the γ_{ij} take their values in some Polish space K. We shall further assume that the random measures

$$\eta_n = \sum_{i=1}^{\infty}\{\delta_{\sigma_i};\ \alpha_i \le n\}, \quad \zeta_n = \sum_{j=1}^{\infty}\{\delta_{\tau_j};\ \beta_j \le n\}, \qquad n \in N, \tag{7.13}$$

are locally finite simple point processes on R_+. Note that the notions of exchangeability and ergodicity for ξ are defined with respect to transformations of the first two components σ_i and τ_j in (7.12).

<u>Lemma 26.</u> <u>Fix a Polish space</u> K, <u>and let</u> ξ <u>be a separately ergodic exchangeable point process on</u> $R_+^4 \times K$ <u>of the form</u> (7.12), <u>such that the</u> η_n <u>and</u> ζ_n <u>in</u> (7.13) <u>are locally finite simple point processes. Then there exist some measurable mappings</u> $f, g: R_+ \to \bar{R}_+$ <u>and</u> $h: R_+^2 \times [0,1] \to K$, <u>some independent</u> U(0,1) <u>r.v.'s</u> γ'_{ij}, $i,j \in N$, <u>and some independent pair of independent sequences</u> $((\sigma'_i, \alpha'_i);\ i \in N)$ <u>and</u> $((\tau'_j, \beta'_j);\ j \in N)$, <u>which form unit rate Poisson processes on</u> R_+^2, <u>such that</u> (7.12) <u>holds a.s. on</u> $R_+^4 \times K$ <u>with</u> σ_i, τ_j, α_i, β_j <u>and</u> γ_{ij} <u>replaced by</u> σ'_i, τ'_j, $f(\alpha'_i)$, $g(\beta'_j)$ <u>and</u> $h(\alpha'_i, \beta'_j, \gamma'_{ij})$, <u>respectively.</u>

<u>Assume instead that</u> ξ <u>is jointly ergodic exchangeable, and that</u> (7.12) <u>holds with</u> $\sigma_i = \tau_i$ <u>and</u> $\alpha_i = \beta_i$. <u>Then there exist some measurable mappings</u> $f: R_+ \to \bar{R}_+$ <u>and</u> $h: R_+^2 \times [0,1] \to K$, <u>some independent</u> U(0,1) <u>r.v.'s</u> $\gamma'_{ij} = \gamma'_{ji}$, $1 \le i < j$, <u>and some</u>

independent sequence $((\sigma_i', \tau_i');\ i\in N)$, which forms a unit rate Poisson process on R_+^2, such that (7.12) holds a.s. on $R_+^4 \times K$ with $\sigma_i = \tau_i$, $\alpha_i = \beta_i$ and γ_{ij} replaced by σ_i', $f(\alpha_i')$ and $h(\alpha_i', \alpha_j', \gamma_{ij}')$, respectively, where $\gamma_{ii}' = 0$.

Proof. Assume that ξ is separately ergodic exchangeable. By Lemma 25, the point processes η_n and ζ_n are homogeneous Poisson, say with intensities a_n and b_n, respectively. Consider n so large that $a_n \wedge b_n > 0$. Then we may write

$$\eta_n = \sum_{i=1}^{\infty} \delta_{\sigma_{ni}}, \quad \zeta_n = \sum_{j=1}^{\infty} \delta_{\tau_{nj}}, \quad n\in N, \tag{7.14}$$

where $\sigma_{n1} < \sigma_{n2} < \ldots$ and $\tau_{n1} < \tau_{n2} < \ldots$ Here

$$\sigma_{ni} = \sigma_{\kappa_{ni}}, \quad \tau_{nj} = \tau_{\kappa'_{nj}}, \quad i,j,n\in N, \tag{7.15}$$

for suitable random indices κ_{ni} and κ'_{nj}, and we may introduce the corresponding random marks

$$\alpha_{ni} = \alpha_{\kappa_{ni}}, \quad \beta_{nj} = \beta_{\kappa'_{nj}}, \quad \gamma_{nij} = \gamma_{\kappa_{ni}, \kappa'_{nj}}, \quad i,j,n\in N, \tag{7.16}$$

and form the arrays

$$X_{ij}^{(n)} = (\alpha_{ni}, \beta_{nj}, \gamma_{nij}), \quad i,j,n\in N. \tag{7.17}$$

By Lemmas 6 and 25, the arrays $X^{(n)}$ are separately ergodic exchangeable, and they are further nested in the sense of Section 4. Hence there exist by Lemma 19 some measurable function $F\colon R_+^2 \times [0,1] \to R_+^2 \times K$ and some r.v.'s α'_{ni}, β'_{nj}, γ'_{nij}, $i,j,n\in N$, which are independent for fixed n and uniformly distributed on the intervals $[0,a_n]$, $[0,b_n]$ and $[0,1]$, respectively, such that

$$X_{ij}^{(n)} = F(\alpha'_{ni}, \beta'_{nj}, \gamma'_{nij}) \quad \text{a.s.}, \quad i,j,n\in N. \tag{7.18}$$

Since α_{ni} does not depend on j, and similarly for β_{nj}, we may rewrite (7.18) in the form

$$\alpha_{ni} = f(\alpha'_{ni}),\ \beta_{nj} = g(\beta'_{nj}),\ \gamma_{nij} = h(\alpha'_{ni}, \beta'_{nj}, \gamma'_{nij}) \text{ a.s., } i,j,n\in N, \tag{7.19}$$

for some measurable functions $f,g\colon R_+ \to \bar{R}_+$ and $h\colon R_+^2 \times [0,1] \to K$ (cf. Lemma 2.4 in Kallenberg[12]). For definiteness, we may take

$$f(x) = \infty \text{ for } x > \sup_n a_n, \quad g(y) = \infty \text{ for } y > \sup_n b_n. \tag{7.20}$$

Since $X^{(n)}$, η_n and ζ_n are independent for fixed n by Lemma 25, the sequences (σ_{ni}) and (τ_{nj}) are independent, and we may take the array $(\alpha'_{ni}, \beta'_{nj}, \gamma'_{nij})$, $i,j\in N$, to be independent of all σ_{ni} and τ_{nj}. The σ_{ni} form a Poisson process with constant intensity a_n, while the α'_{ni} are independent and uniform on $[0,a_n]$, so the pairs $(\sigma_{ni},\alpha'_{ni})$, $i\in N$, form a unit rate Poisson process on $R_+\times[0,a_n]$ (cf. Kallenberg[9]). Similarly, the pairs (τ_{ni},β'_{nj}), $j\in N$, form a unit rate Poisson process on $R_+\times[0,b_n]$, and the two sequences are mutually independent and independent of all γ'_{nij}, $i,j\in N$.

Let us now introduce r.v.'s σ'_i, α'_i, τ'_j, β'_j, γ'_{ij}, $i,j\in N$, on an arbitrary probability space with the stated joint distribution, and define ξ' as the double sum in (7.12), but with $\sigma_i,\tau_j,\alpha_i,\beta_j$ and γ_{ij} replaced by σ'_i, τ'_j, $f(\alpha'_i)$, $g(\beta'_j)$ and $h(\alpha'_i,\beta'_j,\gamma'_{ij})$, respectively. Then $\xi'\overset{d}{=}\xi$ on $R_+^2\times[0,n]^2\times K$ for every n, so a monotone class argument shows that in fact $\xi'\overset{d}{=}\xi$ on $R_+^4\times K$. By Lemma 1, we may then redefine the r.v.'s σ'_i, τ'_j, α'_i, β'_j, γ'_{ij} with the same joint distribution, such that $\xi'=\xi$ holds a.s. This completes the proof in the separately exchangeable case. The representation in the jointly exchangeable case is obtained in a similar way. $\square$

We shall further need a simple technical fact:

<u>Lemma 27.</u> <u>Let</u> K <u>be a kernel from</u> R_+ <u>into some Polish space</u> S, <u>and let</u> $\partial\notin S$. <u>Then there exists some measurable mapping</u> $f\colon R_+^2\to S\cup\{\partial\}$, <u>such that</u>

$$\lambda(f(u,\cdot))^{-1} = K(u,\cdot) \ \underline{\text{on}}\ S, \quad u\geq 0. \tag{7.21}$$

<u>Proof.</u> Use a Borel isomorphism to reduce to the case when $S=R_+$. Then take

$$f(u,v) = \begin{cases} \inf\{t\geq 0;\ K(u,[0,t])>v\}, & v<K(u,R_+), \\ \partial, & v\geq K(u,R_+). \end{cases} \tag{7.22}$$

$\square$

<u>Proof of Theorem 4.</u> Assume that ξ is separately ergodic exchangeable, and define $\xi_1,\ldots,\xi_4$ by (7.7). Then Lemma 24 shows that ξ_4 is also separately ergodic exchangeable and is independent of (ξ_1,ξ_2,ξ_3). Moreover, ξ_4 satisfies condition (i) of Lemma 23, and hence must have a representation as in condition (iii) of the same lemma, corresponding to the second and third terms of (1.4). For convenience, we may henceforth assume that $\xi_4=0$.

Next we conclude from Lemma 17 that

$$M_1 = \{s\geq 0;\ \rho_s > 0\},\quad M_2 = \{t\geq 0;\ \rho'_t > 0\}\ \text{ a.s.,} \tag{7.23}$$

where

$$\rho_s = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \psi\circ\xi_s((k-1,k]),\quad s\geq 0, \tag{7.24}$$

$$\rho'_t = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \psi\circ\tilde{\xi}_t((k-1,k]),\quad t\geq 0, \tag{7.25}$$

and that there exist some sequences of a.s. distinct r.v.'s $\sigma_1,\sigma_2,\ldots$ and $\sigma'_1,\sigma'_2,\ldots$, such that $M_1\subset\{\sigma_i\}$ and $M_2\subset\{\sigma'_j\}$ a.s. The cardinalities of M_1 and M_2 are invariant functions of ξ and therefore a.s. constant, so by Lemmas 6 and 13 it is clear that each M_i is a.s. empty or a.s. infinite. In the latter case, we may assume that

$$M_1 = \{\sigma_i\}\ \text{ or }\ M_2 = \{\sigma'_j\}\ \text{ a.s.,} \tag{7.26}$$

and define

$$\rho_i = \rho_{\sigma_i},\quad \rho'_j = \rho'_{\sigma'_j},\quad i,j\in N. \tag{7.27}$$

We further recall from Lemma 17 that the sets $\{s\geq 0;\ \rho_s > \varepsilon\}$ and $\{t\geq 0;\ \rho'_t>\varepsilon\}$ are a.s. locally finite for every $\varepsilon>0$.

Let α, α', ν and ν' denote the measures α_2, α_1, ν_2 and ν_1 of Lemma 24. Since ξ_1 is supported by $R_+\times M_2$ and ξ_2 by $M_1\times R_+$, it is clear from the law of large numbers that α and α' are a.s. supported by M_1 and M_2, respectively, while ν and ν' are a.s. supported by $M_1\times(0,\infty)$ and $M_2\times(0,\infty)$. We define for $i,j\in N$

$$\alpha_i=\alpha\{\sigma_i\},\ \alpha'_j=\alpha'\{\sigma'_j\},\ \nu_i=\nu(\{\sigma_i\}\times\cdot),\ \nu'_j=\nu'(\{\sigma'_j\}\times\cdot),\ \gamma_{ij}=\xi_3\{(\sigma_i,\sigma'_j)\}, \tag{7.28}$$

and introduce the marked point process

$$\zeta = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\delta_{\sigma_i,\sigma'_j,\rho_i,\rho'_j,\alpha_i,\alpha'_j,\nu_i,\nu'_j,\gamma_{ij}}. \tag{7.29}$$

If M_1 and M_2 are a.s. infinite, it is seen from Lemma 6 that ζ is separately ergodic exchangeable in the first two coordinates. Hence there exist by Lemma 26 some measurable functions $f\colon R_+^2\times[0,1]\to R_+$ and $h,h'\colon R_+\to R_+$, some kernels G and G' from R_+ to $(0,\infty)$, some independent $U(0,1)$ r.v.'s ζ_{ij}, $i,j\in N$, and some independent pair of independent random sequences $((\tau_i,\vartheta_i);\ i\in N)$ and $((\tau'_j,\vartheta'_j);\ j\in N)$, which form unit rate Poisson processes on R_+^2, such that (7.29) remains a.s. true with $\sigma_i, \sigma'_j, \alpha_i, \alpha'_j, \nu_i, \nu'_j$ and γ_{ij} replaced by τ_i, τ'_j, $h(\vartheta_i)$, $h'(\vartheta'_j)$, $G(\vartheta_i)$, $G'(\vartheta'_j)$ and $f(\vartheta_i, \vartheta'_j, \zeta_{ij})$, respectively. In view of (7.28), we thus have, a.s. for $i,j\in N$,

$$\alpha\{\tau_i\}=h(\vartheta_i),\ \alpha'\{\tau'_j\}=h'(\vartheta'_j),\ \nu(\{\tau_i\}\times\cdot)=G(\vartheta_i),\ \nu'(\{\tau'_j\}\times\cdot)=G'(\vartheta'_j),$$
$$\xi_3\{(\tau_i,\tau'_j)\}=f(\vartheta_i,\vartheta'_j,\zeta_{ij}). \tag{7.30}$$

If instead only M_1 is infinite while $M_2=\emptyset$ a.s., then $\xi_1=\xi_3=0$ a.s., and we may consider in place of ζ in (7.29) the marked point process

$$\zeta_1 = \sum_{i=1}^{\infty} \delta_{\sigma_i,\rho_i,\alpha_i,\nu_i}, \tag{7.31}$$

which by Lemma 6 is ergodic exchangeable in the first coordinate. Hence Lemma 13 shows that ζ_1 is homogeneous Poisson, so there must exist some measurable function $h: R_+\to R_+$, some kernel G from R_+ to $(0,\infty)$, and some unit rate Poisson process on R_+^2 with atom positions (τ_i,ϑ_i), $i\in N$, such that (7.31) remains true with σ_i, α_i and ν_i replaced by τ_i, $h(\vartheta_i)$ and $G(\vartheta_i)$, respectively. Thus (7.30) still holds in this case, with τ'_j, ϑ'_j and ζ_{ij} as before, and with $f=h'=G'=0$. The same argument applies to the case when $M_1=\emptyset$ while M_2 is infinite a.s. Finally, (7.30) holds with $f=h=h'=G=G'=0$ when $M_1=M_2=\emptyset$ a.s.

By Lemma 27, there exist some measurable functions $g,g': R_+^2\to R_+$, such that

$$\lambda(g(x,\cdot))^{-1} = G(x), \quad \lambda(g'(x,\cdot))^{-1} = G'(x) \quad \text{on } (0,\infty), \quad x\in R_+. \tag{7.32}$$

Let us further introduce some mutually independent random sequences $((\sigma_{jk},\chi_{jk});\ k\in N)$ and $((\sigma'_{jk},\chi'_{jk});\ k\in N)$, $j\in N$, independent of the r.v.'s τ_i, τ'_j, ϑ_i, ϑ'_j, ζ_{ij}, $i,j\in N$, such that each sequence forms a unit rate Poisson process on R_+^2. Define

$$\xi'_1 = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} g'(\vartheta'_j,\chi'_{jk})\delta_{\sigma'_{jk},\tau'_j} + \sum_{j=1}^{\infty} h'(\vartheta'_j)(\lambda\times\delta_{\tau'_j}), \tag{7.33}$$

$$\xi'_2 = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} g(\vartheta_i,\chi_{ik})\delta_{\tau_i,\sigma_{ik}} + \sum_{i=1}^{\infty} h(\vartheta_i)(\delta_{\tau_i}\times\lambda). \tag{7.34}$$

From (7.30) it is clear that the diffuse components of ξ'_1 and ξ'_2 equal a.s. $\lambda\times\alpha_1$ and $\alpha_2\times\lambda$, respectively. Moreover, the point process η'_1 of atom positions and sizes associated with ξ'_1 is conditionally independent of (α_1,ξ'_2,ξ_3) and Poisson with intensity $\lambda\times\nu_1$, given the r.v.'s τ_i and ϑ_i. Thus η'_1 and (α_1,ξ'_2,ξ_3) remain conditionally independent, given ν_1. Similarly, the point process η'_2

associated with ξ_2' is conditionally independent of (α_2,ξ_1',ξ_3), given ν_2, and Poisson with intensity $\nu_2 \times \lambda$. Since these properties determine the conditional distribution of (ξ_1',ξ_2') given $(\alpha_1,\alpha_2,\nu_1,\nu_2,\xi_3)$, and are the same as for (ξ_1,ξ_2) by Lemma 24, it follows that $(\xi_1',\xi_2',\xi_3) \overset{d}{=} (\xi_1,\xi_2,\xi_3)$. By Lemma 1, we may then redefine the r.v.'s σ_{ik}, σ'_{jk}, χ_{ik} and χ'_{jk} with the same properties as before, such that $\xi_1'=\xi_1$ and $\xi_2'=\xi_2$ a.s. This completes the proof of (1.4) in the ergodic case. The representation formula extends immediately to the non-ergodic case, by means of Lemma 3.

Conversely, the separate exchangeability of a random measure ξ with representation (1.4) follows from the corresponding invariance properties of Poisson processes. If α and γ are a.s. non-random, then ξ is dissociated in the sense of Aldous[1,2], so its distribution can a.s. be reconstructed from a realization, via the law of large numbers. Hence ξ is ergodic in this case, by Lemma 4 with $h(t)\equiv m_t$. □

<u>Proof of Theorem 5.</u> Most of the argument is very similar to that of the preceding proof, so we shall only indicate the changes. Assume that ξ is jointly ergodic exchangeable. By Lemma 6, the diffuse mass along the diagonal D is ergodic exchangeable, so by Lemma 11 it is a.s. equal to a constant times λ_D. The remaining part of ξ_4 fulfills the condition (iv) of Lemma 23, and hence must be representable as in condition (vi). Thus ξ_4 gives rise to the second, third and last terms in (1.5), and by the independence assertion in Lemma 24, it remains to derive the representation for (ξ_1,ξ_2,ξ_3).

Excluding the trivial case when $M=\emptyset$ a.s., the only remaining possibility is when M is a.s. infinite. We then define

$$\rho_s = \xi\{(s,s)\} + \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \psi\circ(\xi_s+\tilde{\xi}_s)((k-1,k]), \quad s\geq 0, \tag{7.35}$$

and conclude from Lemma 17, applied to the restrictions of ξ and $\tilde{\xi}$ to sets of the form $(s,t)^c \times (s,t)$, that $M=\{s\geq 0;\ \rho_s>0\}$ a.s., and that the sets $\{s\geq 0;\ \rho_s>\varepsilon\}$ are a.s. locally finite for arbitrary $\varepsilon>0$. As for M_1 and M_2 before, we may choose a sequence of a.s. distinct r.v.'s $\sigma_1,\sigma_2,\ldots$, such that $M=\{\sigma_i\}$ a.s., and we shall put $\rho_i=\rho_{\sigma_i}$.

The random measures $(\alpha,\alpha')=(\alpha_2,\alpha_1)$ and ν in Lemma 24 are a.s. supported by M and $M\times(R_+^2\times\{0\})$, respectively. Let us write

$$\alpha_i=\alpha\{\sigma_i\},\ \alpha_i'=\alpha'\{\sigma_i\},\ \nu_i=\nu(\{\sigma_i\}\times\cdot),\ \gamma_{ij}=\xi_3\{(\sigma_i,\sigma_j)\},\ i,j\in N, \tag{7.36}$$

and define

$$\zeta=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\delta_{\sigma_i,\sigma_j,\rho_i,\rho_j,\alpha_i,\alpha_j',\nu_i,\gamma_{ij}}. \tag{7.37}$$

Then ζ is jointly ergodic exchangeable in the first two coordinates, by Lemma 6, so by Lemma 26 there exist some measurable mappings $f\colon R_+\times[0,1]\to R_+$ and $h,h'\colon R_+\to R_+$, some kernel G from R_+ to $R_+^2\setminus\{0\}$, some independent $U(0,1)$ r.v.'s $\zeta_{ij}=\zeta_{ji}$, $1\le i<j$, and some independent sequence of random vectors (τ_i,ϑ_i), $i\in N$, which forms a unit rate Poisson process on R_+^2, such that (7.37) remains a.s. true with σ_i, α_i, α_j', ν_i and γ_{ij} replaced by τ_i, $h(\vartheta_i)$, $h'(\vartheta_j)$, $G(\vartheta_i)$ and $f(\vartheta_i,\vartheta_j,\zeta_{ij})$, respectively, where $\zeta_{ii}=0$. Thus we have, a.s. for $i,j\in N$,

$$\alpha\{\tau_i\}=h(\vartheta_i),\ \alpha'\{\tau_i\}=h'(\vartheta_i),\ \nu(\{\tau_i\}\times\cdot)=G(\vartheta_i),\ \xi_3\{(\tau_i,\tau_j)\}=f(\vartheta_i,\vartheta_j,\zeta_{ij}). \tag{7.38}$$

By Lemma 27 there exist some measurable functions $g,g'\colon R_+^2\to R_+$, such that

$$\lambda(g'(x,\cdot),g(x,\cdot))^{-1}=G(x)\ \text{on}\ R_+^2\setminus\{0\},\qquad x\in R_+. \tag{7.39}$$

Let us further introduce some mutually independent random sequences $((\sigma_{ik},\chi_{ik});\ k\in N)$, $i\in N$, independent of all the $\tau_i,\vartheta_i,\zeta_{ij}$, $i,j\in N$, such that each sequence forms a unit rate Poisson process on R_+^2. Define ξ_1' and ξ_2' by (7.33) and (7.34), but with τ_j', ϑ_j', σ_{jk}' and χ_{jk}' replaced by τ_j, ϑ_j, σ_{jk} and χ_{jk}, respectively. Then $(\xi_1',\tilde\xi_2')$ has a.s. the diffuse component $\lambda\times(\alpha_1,\alpha_2)$. Moreover, the point process η' of atom positions and sizes of $(\xi_1',\tilde\xi_2')$ is conditionally independent of $(\alpha_1,\alpha_2,\xi_3)$ and Poisson $\lambda\times\nu$, given the r.v.'s τ_i and ϑ_i, and hence also given ν. Comparing with the properties of ξ_1 and ξ_2 in Lemma 24, it follows that $(\xi_1',\xi_2',\xi_3)\overset{d}{=}(\xi_1,\xi_2,\xi_3)$, and by Lemma 1 we can redefine the r.v.'s σ_{ik} and χ_{ik}, such that equality holds a.s. The proof may now be completed as in case of Theorem 4. $\square$

We conclude the section by remarking that Theorem 4 could also have been obtained as a corollary to Theorem 5. In fact, any separately exchangeable random measure ξ is also jointly exchangeable, and hence must have a representation as in (1.5). For the restriction of ξ to the set

$$A = \bigcup_m \bigcup_n \{[2m-1, 2m) \times [2n, 2n+1)\}, \tag{7.40}$$

we then get a representation of the form (1.4). Using the separate exchangeability plus Lemma 1, it follows that ξ itself has a representation (1.4). Our reason for giving a direct proof of Theorem 4 is that Theorem 5 is considerably deeper. In particular, one needs for its proof the representation theorem for (nested arrays of) jointly exchangeable arrays, which appears to be much harder than its counterpart for the separately exchangeable case (cf. Theorem 3.1 in Kallenberg[11]). Thus we did not want to burden the proof of the easier result by discussing complexities which are relevant only in a more general context.

8. CONCLUDING REMARKS

In this final section, we shall analyze the relationship between the various notions of exchangeability, give criteria for convergence of the series in the main theorems, and decide to what extent the functions occurring in the main representation formulas are unique. For the sake of brevity, our discussion in this section will be rather informal, with most proofs omitted or only briefly indicated.

1. Notions of exchangeability. As explained in the introduction, the notions of separate or joint exchangeability of a random measure ξ on R_+^2, $R_+ \times [0,1]$ or $[0,1]^2$ may be defined in terms of either

(i) the class of arbitrary measure preserving transformations of R_+ or $[0,1]$,

(ii) the subclass of transformations which permute finitely many disjoint dyadic intervals of equal length,

(iii) the array of increments of ξ with respect to an arbitrary regular dyadic square grid.

Formally, (i) gives the strongest and (iii) the weakest notion of exchangeability. However, the notions based on (ii) and (iii) are easily shown to be equivalent, and from the proofs of our main theorems, it is seen that all three notions are in fact equivalent. We shall indicate how this can be seen directly. Our argument has the virtue of applying without changes to higher dimensions, where no explicit representation formulas are known. (Given the methods and results of this paper, one may easily conjecture what the representations should be in higher dimensions, though the expected length and complexity of any rigorous proof seem rather discouraging.) Note that, in dimensions $d \geq 3$, there are also intermediate cases between separate and joint exchangeability to consider, namely one for each partition of the set of d coordinates, where a common transformation is used within each subset. We may refer collectively to these various notions of symmetry as multivariate exchangeability. A one-dimensional version of the following result was discussed in Lemma 9.0 of Kallenberg[9].

Proposition 1. For any notion of multivariate exchangeability of random measures on a Euclidean rectangle, the definitions based on (i), (ii) and (iii) are all equivalent.

The equivalence of (ii) and (iii) is an obvious consequence of basic uniqueness results for random measures (cf. Theorem 3.1 in Kallenberg[9]). To prove that (i) and (ii) are equivalent, it is clearly enough to consider random measures on a cube $[0,1]^d$. Our argument rests on a simple approximation result:

Lemma 28. Let f be a λ-preserving transformation of $[0,1]$. Then there exist some transformations $f_1, f_2, \ldots : [0,1] \to [0,1]$ of type (ii), such that $f_n \to f$ a.e. λ.

This is essentially a special case of a result for predictable transformations, proved in Section 5 of Kallenberg[10]. The next result is a simple consequence of Lemmas 11 and 12. Recall that a fixed measure μ is a supporting measure of a random measure ξ, if $\xi A=0$ a.s. iff $\mu A=0$ (cf. Kallenberg[9], p.103).

Lemma 29. Let ξ be an exchangeable random measure on $[0,1]$ or R_+, such that $P\{\xi \neq 0\} > 0$. Then λ is a supporting measure for ξ.

We shall finally need a simple exercise on weak convergence:

Lemma 30. Let ξ be a random measure on $S=[0,1]^d$, and let f and $f_1, f_2, \ldots$ be measurable transformations of S, such that $\xi\{f_n \not\to f\}=0$ a.s. Further assume that $\xi f_n^{-1} \overset{d}{=} \xi$ for all n. Then even $\xi f^{-1} \overset{d}{=} \xi$.

This holds since $\mu\{f_n \not\to f\}=0$ implies $\mu f_n^{-1} \xrightarrow{w} \mu f^{-1}$. It is now easy to prove the proposition. In fact, assume e.g. that the random measure ξ on $[0,1]^d$ is jointly exchangeable in the sense on (ii), and let f be an arbitrary λ-preserving transformation of $[0,1]$. By Lemma 28, we can choose functions $f_1, f_2, \ldots$ of type (ii), such that $f_n \to f$ a.e. λ. Let $A=\{f_n \to f\}$. Writing $f^d(x_1,\ldots,x_d)=f(x_1)\ldots f(x_d)$, it is clear that $f_n^d \to f^d$ on A^d. Now the d coordinate projections of ξ are again exchangeable in the sense of (ii), and have therefore supporting measure λ, by Lemma 29. Since $\lambda A^c=0$, we get $\xi(A^d)^c=0$ a.s., so $\xi\{f_n^d \not\to f^d\}=0$ a.s. Moreover, $\xi(f_n^d)^{-1} \overset{d}{=} \xi$. Thus Lemma 30 yields $\xi(f^d)^{-1} \overset{d}{=} \xi$, which means that ξ is jointly exchangeable even in the sense of (i). $\square$

2. *Convergence criteria.* Using the criteria for the existence of multiple Poisson integrals given in Kallenberg and Szulga[13], we may easily decide when the series in Theorems 3-5 converge a.s. Our results may be compared with the conjectures of Aldous[2], p.139, in the special case of separately exchangeable *counting* random measures on R_+^2.

It is clearly enough to consider the ergodic cases, when the r.v.'s α, β, γ and $\beta_1, \beta_2, \ldots$ in (1.3)-(1.5) are constants. To simplify the notation, we may then delete α from our formulas, so that e.g. $f(\alpha, \vartheta_i, \vartheta_j', \zeta_{ij})$ in (1.4) will be replaced by $f(\vartheta_i, \vartheta_j', \zeta_{ij})$. For an arbitrary function f, we shall write $\hat{f} = f \wedge 1$. For the f_j, g_k and β_j in (1.3), we define

$$F = \sum f_j, \quad G = \sum g_k, \quad B = \sum \beta_j. \tag{8.1}$$

For functions $g: R_+^2 \to R_+$, we define

$$\lambda_D g = \int g(x,x)dx, \quad \lambda_1 g(y) = \lambda g(\cdot, y), \quad \lambda_2 g(x) = \lambda g(x, \cdot), \quad x,y \in R_+. \tag{8.2}$$

The function f in (1.4) and (1.5) is regarded as defined on $R_+^2 \times [0,1]$, and we put

$$\lambda_D f = \lambda_D f(\cdot,\cdot,0), \quad f_1(x) = \lambda^2 \hat{f}(x,\cdot,\cdot), \quad f_2(y) = \lambda^2 \hat{f}(\cdot,y,\cdot), \quad x,y \in R_+, \tag{8.3}$$

where λ^2 denotes Lebesgue measure on $R_+ \times [0,1]$.

Proposition 2. Consider formulas (1.3)-(1.5), *but with deleted* α *and with non-random* $\beta_1, \beta_2, \ldots$ *Then the series in* (1.3) *converge a.s. iff*

$$\lambda \hat{F} + \lambda \hat{G} + \lambda \hat{h} + B < \infty, \tag{8.4}$$

those in (1.4) *iff*

$$\lambda \hat{\ell} + \lambda \hat{h} + \lambda \hat{h}' < \infty, \quad \lambda(1 \wedge \lambda_2(\hat{g} + \hat{g}')) < \infty, \tag{8.5}$$

$$\lambda\{1 < f_i < \infty\} = \lambda\{1 < f_i\} < \infty \ \text{ for } i=1,2, \text{ and } \lambda^3[\hat{f};\ f_1 \vee f_2 \le 1] < \infty, \tag{8.6}$$

and those in (1.5) *iff* (8.5)-(8.6) *hold, and in addition*

$$\lambda \hat{\ell}' + \lambda_D \hat{f} < \infty. \tag{8.7}$$

For convenience, we collect the general facts we need about the a.s. convergence of random series and integrals. Let us denote integrals with respect to ξ, $\xi \times \eta$ and $\xi^2 = \xi \times \xi$ by ξf, $\xi\eta f$ and $\xi^2 f$, respectively. Put $\psi(x) = 1 - e^{-x}$.

<u>Lemma 31.</u> <u>Let</u> $\alpha_1,\alpha_2,\ldots$ <u>be independent</u> R_+<u>-valued r.v.'s, and let</u> ξ <u>and</u> η <u>be independent unit rate Poisson processes on</u> R_+. <u>Fix two measurable functions</u> $f\colon R_+\to R_+$ <u>and</u> $g\colon R_+^2\to R_+$, <u>and define</u> $g_1=\lambda_2\hat{g}$ <u>and</u> $g_2=\lambda_1\hat{g}$. <u>Then</u>

(a) $\sum\alpha_i<\infty$ <u>a.s. iff</u> $\sum E\hat{\alpha}_i<\infty$,

(b) $\xi f<\infty$ <u>a.s. iff</u> $\lambda\hat{f}<\infty$, <u>and</u> $E\psi(\xi f)=\psi(\lambda(\psi\circ f))$,

(c) $\xi\eta g<\infty$ <u>a.s. iff</u> $\lambda\{1<g_i<\infty\}=\lambda\{1<g_i\}<\infty$, $i=1,2$; $\lambda^2[\hat{g};\ g_1\vee g_2\le 1]<\infty$,

(d) $\xi^2 g<\infty$ <u>a.s. iff</u> $\xi\eta g<\infty$ <u>a.s. and</u> $\lambda_D\hat{g}<\infty$.

Here (a) is classical, while (b)-(d) are taken from Kallenberg and Szulga[13]. To prove Proposition 2, it suffices in view of the exchangeability to consider the restriction of ξ to $[0,1]^2$. In case of (1.3), we get

$$\xi[0,1]^2=\sum_i(F(\vartheta_i)+G(\vartheta_i)+h(\vartheta_i))1\{\sigma_i\le 1\}+B+\gamma, \tag{8.8}$$

and since the r.v.'s ϑ_i with $\sigma_i\le 1$ form a unit rate Poisson process on R_+, we get the convergence criterion (8.4) by using Lemma 31(b).

In case of (1.4), we get

$$\begin{aligned}\xi[0,1]^2=&\sum_i\sum_j f(\vartheta_i,\vartheta_j',\zeta_{ij})1\{\tau_i\vee\tau_j'\le 1\}+\sum_k \ell(\eta_k)1\{\rho_k\vee\rho_k'\le 1\}+\gamma\\ &+\sum_i\sum_k\{g(\vartheta_i,\chi_{ik})1\{\tau_i\vee\sigma_{ik}\le 1\}+g'(\vartheta_i',\chi_{ik}')1\{\tau_i'\vee\sigma_{ik}'\le 1\}\}\\ &+\sum_i\{h(\vartheta_i)1\{\tau_i\le 1\}+h'(\vartheta_i')1\{\tau_i'\le 1\}\}.\end{aligned} \tag{8.9}$$

By Lemma 31(b), the second and last sums converge a.s. iff $\lambda(\hat{\ell}+\hat{h}+\hat{h}')<\infty$. Conditioning on all the ϑ_i, ϑ_j', τ_i and τ_j', it is further seen from Lemma 31(a) that the first sum converges a.s. iff

$$\sum_i\sum_j\lambda\hat{f}(\vartheta_i,\vartheta_j',\cdot)1\{\tau_i\vee\tau_j'\le 1\}<\infty\quad\text{a.s.}, \tag{8.10}$$

which is equivalent to (8.6) by Lemma 31(c). By the same conditioning plus Lemma 31(a), the double sum involving g converges a.s. iff

$$\sum_i E\Big[1\wedge\sum_k g(\vartheta_i,\chi_{ik})1\{\sigma_{ik}\le 1\}\Big|\vartheta_i,\tau_i\Big]1\{\tau_i\le 1\}<\infty\quad\text{a.s.} \tag{8.11}$$

Since $\psi(x)\le\hat{x}\le 2\psi(x)$, this holds by the formula in Lemma 31(b), iff

$$\sum_i\{1\wedge\lambda[\psi\circ g(\vartheta_i,\cdot)]\}1\{\tau_i\le 1\}<\infty\quad\text{a.s.}, \tag{8.12}$$

which is equivalent, by the criterion in Lemma 31(b), to

$$\lambda(1\wedge\lambda_2(\psi\circ g))<\infty. \tag{8.13}$$

The last condition, together with the corresponding condition involving g', are equivalent to the remaining condition $\lambda(1\wedge\lambda_2(\hat{g}+\hat{g}'))<\infty$ in (8.5). This completes the proof in case of (1.4), and the argument for (1.5) is similar, except that statement (d) is needed instead of (c) in Lemma 31. □

3. Problems of uniqueness. Our aim in this final subsection is to examine to what extent the representations in Theorems 1-5 are unique. The corresponding problem for exchangeable arrays has been treated at length in Kallenberg[12], and since the present methods and statements are very similar, we shall only indicate some typical results, and omit all proofs. In particular, we shall restrict our attention to the ergodic case, in order to simplify the notation. As before, this allows us to omit the α-dependence from formulas (1.3)-(1.5). For a reader familiar with Kallenberg[12], the extensions to the general case should be obvious.

To simplify our statements, we shall only consider representations which are minimal. By this we mean that certain sets $J\subset N$ or $A\subset R_+$ associated with the representation should satisfy

$$\#J=\infty \;\Rightarrow\; J=N, \qquad \text{or} \quad \lambda A=\infty \;\Rightarrow\; \lambda A^c=0. \tag{8.14}$$

In case of (1.1), this should hold for the sets

$$J_1=\Big\{i\in N;\ \beta_i+\sum_{j=1}^{\infty}\alpha_{ij}>0\Big\},\quad J_2=\Big\{j\in N;\ \beta'_j+\sum_{i=1}^{\infty}\alpha_{ij}>0\Big\}, \tag{8.15}$$

and in case of (1.2) for the set

$$J=\Big\{j\in N;\ \beta_j+\beta'_j+\sum_{i=1}^{\infty}(\alpha_{ij}+\alpha_{ji})>0\Big\}. \tag{8.16}$$

In (1.3), we require (8.14) to hold for

$$J=\{j\in N;\ \beta_j+\lambda f_j>0\},\quad A=\Big\{h+\sum f_j+\sum g_k>0\Big\}, \tag{8.17}$$

in (1.4) for

$$\begin{aligned} &A_0=\{\ell>0\},\quad A=\{f_1+\lambda_2 g+h>0\},\quad A'=\{f_2+\lambda_2 g'+h'>0\},\\ &A_x=\{g(x,\cdot)>0\},\quad A'_x=\{g'(x,\cdot)>0\},\quad x\geq 0, \end{aligned} \tag{8.18}$$

and in (1.5) for

$$\Lambda_0 = \{\ell+\ell' > 0\}, \quad \Lambda = \{f_1 + f_2 + f_D + \lambda_2 g + \lambda_2 g' + h + h' > 0\}, \qquad (8.19)$$
$$\Lambda_x = \{(g+g')(x,\cdot) > 0\}, \quad x \geq 0.$$

The condition of minimality is no severe restriction, since any representation of the form (1.1)-(1.5) can easily be modified so as to become minimal. A further condition we may impose, without loss in generality, is that the sequence of functions g_k in (1.3) be non-increasing.

We now consider two random measures ξ and $\bar{\xi}$ on $[0,1]^2$, admitting representations as in (1.1) in terms of constants $\alpha_{ij}, \beta_i, \beta'_j, \gamma$, and $\bar{\alpha}_{ij}, \bar{\beta}_i, \bar{\beta}'_j, \bar{\gamma}$, respectively. Assume that there exist some permutations π and π' of N, such that

$$\alpha_{ij} = \bar{\alpha}_{\pi_i,\pi'_j}, \quad \beta_i = \bar{\beta}_{\pi_i}, \quad \beta'_j = \bar{\beta}'_{\pi'_j}, \quad \gamma = \bar{\gamma}, \quad i,j \in N. \qquad (8.20)$$

Then clearly $\xi \stackrel{d}{=} \bar{\xi}$. The same conclusion holds if ξ and $\bar{\xi}$ satisfy (1.2) with constants $\alpha_{ij}, \beta_i, \beta'_j, \gamma, \vartheta$, and $\bar{\alpha}_{ij}, \bar{\beta}_i, \bar{\beta}'_j, \bar{\gamma}, \bar{\vartheta}$, respectively, and there exists a permutation π of N satisfying

$$\alpha_{ij} = \bar{\alpha}_{\pi_i,\pi_j}, \quad \beta_i = \bar{\beta}_{\pi_i}, \quad \beta'_j = \bar{\beta}'_{\pi_j}, \quad \gamma = \bar{\gamma}, \; \vartheta = \bar{\vartheta}, \quad i,j \in N. \qquad (8.21)$$

Let us next assume that ξ and $\bar{\xi}$ are defined on $R_+ \times [0,1]$ and satisfy (1.3) (though without α), for some functions and constants $f_j, g_k, h, \beta_j, \gamma$, and $\bar{f}_j, \bar{g}_k, \bar{h}, \bar{\beta}_j, \bar{\gamma}$, respectively. Then $\xi \stackrel{d}{=} \bar{\xi}$, provided there exist some permutation π of N and some λ-preserving transformations T and $\bar{T}$ of R_+, such that a.e. λ for all $j,k \in N$,

$$f_j \circ T = \bar{f}_{\pi_j} \circ \bar{T}, \quad g_k \circ T = \bar{g}_k \circ \bar{T}, \quad h \circ T = \bar{h} \circ \bar{T}, \quad \beta_j = \bar{\beta}_{\pi_j}, \quad \gamma = \bar{\gamma}. \qquad (8.22)$$

If ξ and $\bar{\xi}$ are instead defined on R_+^2, and they satisfy (1.4) with functions and constants $f, \ell, \gamma, g, g', h, h'$, and $\bar{f}, \bar{\ell}, \bar{\gamma}, \bar{g}, \bar{g}', \bar{h}, \bar{h}'$, respectively, then $\xi \stackrel{d}{=} \bar{\xi}$ if there exist some λ-preserving transformations $X, \bar{X}, Y, \bar{Y}, U_x, \bar{U}_x, V_y, \bar{V}_y, T, \bar{T}$ of R_+ and $Z_{xy}, \bar{Z}_{xy}$ of $[0,1]$, $x,y \in R_+$, each jointly measurable in all arguments, such that a.e. λ,

$$f(X(x),Y(y),Z_{xy}(z)) = \bar{f}(\bar{X}(x),\bar{Y}(y),\bar{Z}_{xy}(z)),$$
$$g(X(x),U_x(u)) = \bar{g}(\bar{X}(x),\bar{U}_x(u)), \quad g'(Y(y),V_y(v)) = \bar{g}'(\bar{Y}(y),\bar{V}_y(v)), \qquad (8.23)$$
$$h \circ X = \bar{h} \circ \bar{X}, \quad h' \circ Y = \bar{h}' \circ \bar{Y}, \quad \ell \circ T = \bar{\ell} \circ \bar{T}, \quad \gamma = \bar{\gamma}.$$

The corresponding conditions in case of (1.5) is that there should exist some

λ-preserving transformations $X,\bar{X},U_x,\bar{U}_x,T,\bar{T}$ of R_+ and $Z_{xy},\bar{Z}_{xy}$ of $[0,1]$, $x,y\in R_+$, such that a.e. λ,

$$\begin{aligned} &f(X(x),X(y),Z_{xy}(z)) = \bar{f}(\bar{X}(x),\bar{X}(y),\bar{Z}_{xy}(z)), \\ &(g,g')(X(x),U_x(u)) = (\bar{g},\bar{g}')(\bar{X}(x),\bar{U}_x(u)), \ \beta=\bar{\beta}, \ \gamma=\bar{\gamma}, \\ &(h,h',f_D)\circ X = (\bar{h},\bar{h}',\bar{f}_D)\circ\bar{X}, \ (\ell,\ell')\circ T = (\bar{\ell},\bar{\ell}')\circ\bar{T}. \end{aligned} \tag{8.24}$$

The next result shows that the stated conditions for $\xi \overset{d}{=} \bar{\xi}$ are also necessary. We omit the proof, since the required arguments are very similar to those employed in Kallenberg[12].

Proposition 3. Consider two ergodic random measures ξ and $\bar{\xi}$, with minimal representations as in either one of the formulas (1.1)-(1.5). Assume also in case of (1.3) that $g_1 \geq g_2 \geq \cdots$ Then the stated conditions for $\xi \overset{d}{=} \bar{\xi}$ are both necessary and sufficient.

We remark that, in view of Lemma 1, any ergodic random measure ξ with a minimal representation as in (1.1)-(1.5) could also be represented in terms of any other set of functions and constants, which is equivalent in the sense of Proposition 2. Thus the latter result also tells us essentially to what extent the a.s. representations in (1.1)-(1.5) are unique.

Acknowledgements. The topic of continuous parameter multivariate exchangeability was born in discussions with David Aldous in 1979, when we happened to meet in Vancouver. Already at that time, a simple version of Lemma 23 was conjectured (cf. Kallenberg[8]). Most of the present work was done during visits to the Center for Stochastic Processes, Chapel Hill, and I am grateful to my hosts at the Statistics Department, UNC, for their great hospitality.

REFERENCES

1. Aldous, D.J. (1981). Representations for partially exchangeable arrays of random variables. J. Multivariate Anal. 11, 581-598.
2. Aldous, D.J. (1985). Exchangeability and related topics. In Hennequin, P.L. (ed.), École d'Été de Probabilités de Saint-Flour XIII - 1983, Lecture Notes in Mathematics, Vol. 1117, Springer, Berlin, pp. 1-198.
3. Dynkin, E.B. (1978). Sufficient statistics and extreme points. Ann. Probab. 6, 705-730.
4. Hestir, K. (1987). The Aldous representation theorem and weakly exchangeable non-negative definite arrays. Thesis, Statist. Dept., Univ. of California, Berkeley.
5. Hoover, D.N. (1979). Relations on probability spaces and arrays of random variables. Preprint, Institute for Advanced Study, Princeton.
6. Hoover, D.N. (1982). Row-column exchangeability and a generalized model for probability. In Koch, G. & Spizzichino, F. (eds.), Exchangeability in Probability and Statistics, North-Holland, Amsterdam, pp. 281-291.
7. Kallenberg, O. (1975). On symmetrically distributed random measures. Trans. Amer. Math. Soc. 202, 105-121.
8. Kallenberg, O. (1982). The stationarity - invariance problem. In Koch, G. & Spizzichino, F. (eds.), Exchangeability in Probability and Statistics, North-Holland, Amsterdam, pp. 293-296.
9. Kallenberg, O. (1986). Random Measures, 4th ed. Akademie-Verlag & Academic Press, Berlin-London.
10. Kallenberg, O. (1988). Spreading and predictable sampling in exchangeable sequences and processes. Ann. Probab. 16, 508-534.
11. Kallenberg, O. (1988). Some new representations in bivariate exchangeability. Probab. Th. Rel. Fields 77, 415-455.
12. Kallenberg, O. (1988). On the representation theorem for exchangeable arrays. J. Multivariate Anal. (to appear).

13. Kallenberg, O. and Szulga, J. (1989). Multiple integration with respect to Poisson and Lévy processes. Probab. Th. Rel. Fields, to appear.

14. Krickeberg, K. (1974). Moments of point processes. In Harding, E.F. & Kendall, D.G. (eds.), Stochastic Geometry, Wiley, New York, pp. 89-113.

15. Maitra, A. (1977). Integral representations of invariant measures. Trans. Amer. Math. Soc. 229, 209-225.

201. M. Marques and S. Cambanis, Admissible and singular translates of stable processes, Aug. 86.

202. O. Kallenberg, One-dimensional uniqueness and convergence criteria for exchangeable processes, Aug. 87. *Stochastic Proc. Appl.* 28, 1988, 159-183.

203. R.J. Adler, S. Cambanis and G. Samorodnitsky, On stable Markov processes, Sept. 87.

204. G. Kallianpur and V. Perez-Abreu, Stochastic evolution equations driven by nuclear space valued martingales, Sept. 87. *Appl. Math. Optimization*, 17, 1988, 237-272.

205. R.L. Smith, Approximations in extreme value theory, Sept. 87.

206. E. Willekens, Estimation of convolution tails, Sept. 87.

207. J. Rosinski, On path properties of certain infinitely divisible processes, Sept. 87.

208. A.H. Korezlioglu, Computation of filters by sampling and quantization, Sept. 87.

209. J. Bather, Stopping rules and observed significance levels, Sept. 87.

210. S.T. Rachev and J.E. Yukich, Convolution metrics and rates of convergence in the central limit theorem, Sept. 87. *Ann. Probability*, 1988, to appear.

211. M. Fujisaki, Normed Bellman equation with degenerate diffusion coefficients and its applications to differential equations, Oct. 87.

212. G. Simons, Y.C. Yao and X. Wu, Sequential tests for the drift of a Wiener process with a smooth prior, and the heat equation, Oct. 87.

213. R.L. Smith, Extreme value theory for dependent sequences via the Stein-Chen method of Poisson approximation, Oct. 87.

214. C. Houdré, A vector bimeasure integral with some applications, June 88 (Revised).

215. M.R. Leadbetter, On the exceedance random measures for stationary processes, Nov. 87.

216. M. Marques, A study on Lebesgue decomposition of measures induced by stable processes, Nov. 87 (*Dissertation*).

217. M.T. Alpuim, High level exceedances in stationary sequences with extremal index, Dec. 87. *Stochastic Proc. Appl.*, to appear.

218. R.F. Serfozo, Poisson functionals of Markov processes and queueing networks, Dec. 87.

219. J. Bather, Stopping rules and ordered families of distributions, Dec. 87.

220. S. Cambanis and M. Maejima, Two classes of self-similar stable processes with stationary increments, Jan. 88.

221. H.P. Hucke, G. Kallianpur and R.L. Karandikar, Smoothness properties of the conditional expectation in finitely additive white noise filtering, Jan. 88. *J. Multivariate Anal.*, to appear.

222. I. Mitoma, Weak solution of the Langevin equation on a generalized functional space, Feb. 88.

223. L. de Haan, S.I. Resnick, H. Rootzén and C. de Vries, Extremal behaviour of solutions to a stochastic difference equation with applications to arch-processes, Feb. 88.

224. O. Kallenberg and J. Szulga, Multiple integration with respect to Poisson and Lévy processes, Feb. 88. *Prob. Theor. Rel. Fields*, 1989, to appear.

225. D.A. Dawson and L.G. Gorostiza, Generalized solutions of a class of nuclear space valued stochastic evolution equations, Feb. 88.

226. G. Samorodnitsky and J. Szulga, An asymptotic evaluation of the tail of a multiple symmetric α-stable integral, Feb. 88. *Ann. Probability*, to appear.

227. J.J. Hunter, The computation of stationary distributions of Markov chains through perturbations, Mar. 88.

228. H.C. Ho and T.C. Sun, Limiting distribution of nonlinear vector functions of stationary Gaussian processes, Mar. 88.

229. R. Brigola, On functional estimates for ill-posed linear problems, Apr. 88.

230. M.R. Leadbetter and S. Nandagopalan, On exceedance point processes for stationary sequences under mild oscillation restrictions, Apr. 88.

231. S. Cambanis, J. P. Nolan and J. Rosinski, On the oscillation of infinitely divisible processes, Apr. 88.

232. G. Hardy, G. Kallianpur and S. Ramasubramanian, A nuclear space-valued stochastic differential equation driven by Poisson random measures, Apr. 88.

233. D.J. Daley, T. Rolski, Light traffic approximations in queues (II), May 88.

234. G. Kallianpur, I. Mitoma, R.L. Wolpert, Diffusion equations in duals of nuclear spaces, July 88.

235. S. Cambanis, Admissible translates of stable processes: A survey and some new models, July 88.

236. E. Platen, On a wide range exclusion process in random medium with local jump intensity, Aug. 88.

237. R.L. Smith, A counterexample concerning the extremal index, Aug. 88.

238. G. Kallianpur and I. Mitoma, A Langevin-type stochastic differential equation on a space of generalized functionals, Aug. 88.

239. C. Houdré, Harmonizability, V-boundedness, (2,P)-boundedness of stochastic processes, Aug. 1988.

240. G. Kallianpur, Some remarks on Hu and Meyer's paper and infinite dimensional calculus on finitely additive canonical Hilbert space, Sept. 88.